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ORIGINAL ARTICLE

New tuning rules for fractional PI$^\alpha$ controllers

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Abstract This paper describes a new tuning method for fractional PI$^\alpha$ controllers. The main theoretical contribution of the paper is the analytical solution of a nonlinear function minimization problem, which plays a central role in deriving the tuning formulae. These formulae take advantage of the fractional order α to offer an excellent tradeoff between dynamic performances and stability robustness. Finally, a position control is implemented to compare laboratory experiments with computer simulations. The comparison results show the good performance of the tuning formulae.

Keywords Robustness · PI$^\alpha$ controller design · Fractional system · Tuning methods · Symmetrical optimum

1 Introduction

Six decades after the seminal paper of Ziegler and Nichols (see [27]), proportional integral (PI) and proportional integral derivative (PID) controllers are still at the heart of control applications. However, in spite of their central role in control engineering, PID have attracted relatively little attention of research community (see [1]). Only recently, indeed, many authors have observed that the practice of PID can take considerable benefits from specific research in this field (see [1, 14]). Certainly, a better understanding of these control devices is highly desirable because it can contribute to enhance the quality of products and the efficiency of manufacturing.

In the context of controller design, the old Bode's idea of a reference optimal loop response is recently experiencing a revival of interest. The optimal response is an ideal open-loop asymptotic gain diagram, possibly including segments with slopes that, at least in principle, may assume any value. The controller design mainly consists in shaping the asymptotic gain diagram and, in particular, in choosing the slope of the segment crossing the frequency axis. In addition, the gain diagram must maintain this slope in a wide frequency interval around the crossover. So, the phase margin is constant in the same interval and stability robustness is guaranteed even for high gain variations. Clearly, the required slope corresponds to irrational functions of the type $(j\omega)^\nu$, where ν is real, which represent fractional integro-differential operators in the Laplace domain (see [15, 20]). Fractional calculus, indeed, is the framework underlying the fractional-order controllers (FOC) design and it is just a recent focus of interest of dynamic systems control (see [2, 24]).

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The application of the frequency response technique and of the fractional integro-differential operators to the controllers design dates back to [11]. Frequency analysis is also applied to fractional-order PID-like controllers which were introduced some years later, i.e., in the TID scheme proposed by [10], the proportional compensating unit of a classical PID device is replaced by an element referred to as a "tilt" compensator with transfer function equal to $s^{1/n}$ or $s^{-1/n}$ with n integer. The synthesis approach to the CRONE control proposed by [16] and [17] pursues the "fractal robustness" on the basis of a desired frequency template (see [18]). Also, the $PI^{\lambda}D^{\mu}$ controller, introduced by [21] in the time domain, is studied by [19] in the frequency domain. Recently, the subclass of the fractional-order controllers is still analyzed in the frequency domain to take advantage of the fractional order λ in process compensation (see [12]).

However, the design methods for PI^{λ} and $PI^{\lambda}D^{\mu}$ controllers are a recent research area (see [3, 5, 6, 13, 22, 23, 25]). The aim of this paper is to introduce a new approach to the design of PI^{λ} controllers, which is inspired to the classical "symmetrical optimum" method (SOM), which is very popular in the design of electromechanical systems and position control. This approach, which dates back to [8], is recently used by [26] for auto-calibration of conventional PID controllers.

This paper is organized as follows. Section 2 briefly reviews the reference models of the plant and the fractional controller. For convenience, the performance of the SOM is recalled and the symbols are also introduced. Section 3 derives the solution of a nonlinear equation which leads to the tuning method. Section 4 provides simulations and laboratory experiments confirming the performances of the PI^{α} controller. Section 5 gives concluding remarks.

2 The models of the plant and of the PI^{α} controller

The design approach proposed in this paper is inspired by the SOM (see [8, 26]). Despite being an old method, the SOM contains several ideas that have been widely developed in the years following its introduction. The most important one is to define the class of plants of interest in an effective manner. It is a common and widely-accepted by-product of the SOM to approximate many high-order processes, e.g., thermal and electromechanical systems (see [9, 26]) by an integration plus a first-order model. More precisely, let the plant be represented by

$$G_P(s) = K_{PL} \frac{\prod_j (1+\tau_j s)}{s \prod_i (1+\tau_i s)} e^{-Ls}. \tag{1}$$

The dead time L, the time constants τ_j and τ_i may be gathered into an equivalent time constant T_E as follows (see [8, 9]):

$$T_E = \sum_i \tau_i - \sum_j \tau_j + L, \tag{2}$$

so that the reference transfer function of the plant becomes:

$$G_P(s) = \frac{K_{PL}}{s(1+T_E s)}. \tag{3}$$

Hence, T_E may represent either the sum of uncompensable negligible time delays and time constants of the plant, or the small time constant which determines the closed-loop bandwidth necessary for tuning the controller. The SOM can be also applied to plants represented by:

$$G_{PL1}(s) = \frac{K_{PL1}}{(1+Ts)} \frac{\prod_j (1+\tau_j s)}{\prod_i (1+\tau_i s)} e^{-Ls}, \tag{4}$$

with $T \gg T_E$ and $T \gg \tau_i$. In this case, the transfer function (3) can be also assumed as reference, with $K_{PL} = K_{PL1}/T$ and T_E given by (2). Note that the gain K_{PL} is variable in many electrical drives applications.

Next, let us to briefly introduce the fractional controllers. According to the symbolism suggested by [15], consider the fractional integral of order α, of the generic function f:

$$\frac{d^{-\alpha} f}{dt^{-\alpha}} = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)\, d\tau}{(t-\tau)^{1-\alpha}}, \tag{5}$$

where Γ is the gamma function. The (5) is the Riemann–Liouville definition, written for the order $0 < \alpha \le 1$. For $\alpha = 1$, the (5) gives the ordinary integral of the generic function f. It can be also proved that (see [15]) :

$$\mathscr{L}\left\{\frac{d^{-\alpha} f}{dt^{-\alpha}}\right\} = \frac{F(s)}{s^{\alpha}}, \tag{6}$$

where $\mathscr{L}$ is the unilateral Laplace transform, and $F(s)$ is the Laplace transform of f. This makes the analysis of dynamical systems easier. In particular, it is also possible to define the PI$^\alpha$ controller as:

$$G_C(s) = K_C\left[1 + \frac{1}{(T_I s)^\alpha}\right], \tag{7}$$

where K_C and T_I are the gain and the integral constants, respectively. Of course, for $\alpha = 1$, (7) becomes the transfer function of an ordinary PI controller.

Now, introducing the non-dimensional frequency ν, with $s = \nu\,\omega_0$, $\omega_0 = 2\pi/T_0$ and $T_0 = 2\pi T_I$, with the position $K_P = K_{PL}T_I$, yields:

$$G(\nu) = G_P(\nu)G_C(\nu) = \frac{K_P K_C\,(1+\nu^\alpha)}{\nu^{1+\alpha}(1+\nu a^{-2})}, \tag{8}$$

where $T_I = a^2 T_E$ with $a > 1$. Moreover, if $\omega = u\,\omega_0$, then for $\omega = \omega_I = 1/T_I$ and $\omega = \omega_E = 1/T_E$ it follows $u_I = 1$ and $u_E = a^2$, respectively, and (8) gives:

$$G(ju) = \frac{K_P K_C\,[1+(ju)^\alpha]}{(ju)^{1+\alpha}(1+jua^{-2})}. \tag{9}$$

Equation (9) leads to a very simple formulation of the proposed design approach. In addition, with $\alpha = 1$ in (9), the classical tuning formula of the SOM can be rewritten with respect to the phase margin PM (design specification) as follows:

$$a = \frac{1+\sin(PM)}{\cos(PM)}. \tag{10}$$

The PM is reached at the frequency $\omega_{\max} = (aT_E)^{-1} = aT_I^{-1}$, and hence for $u_{\max} = a$. Since the SOM assumes $\omega_{\max} = \omega_{\mathrm{GC}}$, where ω_{GC} is the gain-crossover frequency, the condition $|G(ja)| = 1$ with $\alpha = 1$ gives:

$$K_C = \frac{a}{K_P}. \tag{11}$$

Note that, putting $PM \approx 37^\circ$ in (10), we obtain $a = 2.0057 \approx 2$, which is the value commonly used by the tuning rule. Moreover, u_E is one octave on the right of u_{GC}. With this standard tuning, the response to a command step shows a very steep normalized rise time, $t_{\mathrm{R}}/T_E = t_{\mathrm{R}}\omega_E = a^2 t_{\mathrm{R}}/T_I = a^2 t_{\mathrm{R}}\omega_I = 3.1$, and a short normalized settling time $t_{\mathrm{S}}/T_E = t_{\mathrm{S}}\omega_E = a^2 t_{\mathrm{S}}/T_I = a^2 t_{\mathrm{S}}\omega_I = 16.5$, but a large percentage overshoot $OS\% = 43.4\%$. However, the overshoot can be reduced by introducing an adequate filtering of the command signal (see [26]). So, the disadvantage of the SOM remains in a still rather unacceptable phase margin in the case of high sensitivity to variable plant gain K_P.

3 The proposed tuning method

Now, consider the PI$^\alpha$ controller (7) and use (9) with $0 < \alpha < 1$. Then:

$$\operatorname{Arg}(G(ju)) = \tan^{-1}\left[\frac{u^\alpha \sin(0.5\alpha\pi)}{1+u^\alpha\cos(0.5\alpha\pi)}\right] - \tan^{-1}\left[\frac{u}{a^2}\right] - 0.5(1+\alpha)\pi. \tag{12}$$

Given (12), determine the frequency $u_{\max}$ for which the value of $\operatorname{Arg}(G(ju))$ is maximum and impose $u_{\mathrm{GC}} = u_{\max}$, where u_{GC} is the gain-crossover frequency, $|G(ju_{\mathrm{GC}})| = 1$.

Hence, taking into account (9), $u_{\max}$ must satisfy:

$$\frac{\mathrm{d}}{\mathrm{d}u}\operatorname{Arg}(G(ju)) = 0, \tag{13}$$

and thus

$$\alpha u^{\alpha-1}\sin(0.5\alpha\pi)\left[1+\left(a^{-2}u\right)^2\right] - a^{-2}[1+u^{2\alpha}+2u^\alpha\cos(0.5\alpha\pi)] = 0. \tag{14}$$

The solution of this equation is not trivial. In many cases, a numerical algorithm and the support of an appropriate software are necessary. Here, however, the following new simple solution pattern is proposed.

Assume that the real parameter a can be chosen arbitrarily, provided that the constraint $a > 1$ holds. Moreover, let $u_{\max}$ be the value satisfying (14). If we define $Q = u/a^2$, we can choose a so that: $1 + Q_m^2 = \beta Q_m$, where Q_m corresponds to $u_{\max}$ and $\beta > 0$ is a new parameter depending on Q_m and hence on a only. This position also implies that Q_m must satisfy $q(Q) = Q^2 - \beta Q + 1 = 0$, i.e., it must coincide with one of the roots Q_1 and Q_2 of $q(Q) = 0$. Since, to have physical meaning, Q_1 and Q_2 must be both real and positive, this requires $\beta \geq 2$. Moreover, $q(Q)$ is a reciprocal polynomial, so that $Q_1 = Q_2^{-1}$. Consequently, if $Q_1 > 1$, then $Q_2 < 1$. Now, we assume that the un-modeled dynamics, due to time constants smaller than the equivalent parameter T_E, have little effects on the plant output, as long as the plant contains the bulk of its energy at low frequencies. If we design the controller to produce only low-frequency inputs for the plant, we obtain good closed-loop performances. In line with this strategy, we consider $u_{\max} < a^2 = u_E$, so that the controller introduces a positive phase lead over a frequency range below a^2. For this reason, only the root $Q_2 < 1$ is considered. However, there is a need for a tradeoff between using high-frequency inputs for quick response and avoiding high-frequency inputs for safety against un-modeled dynamics. This tradeoff is ruled by the frequency u_E, and hence by the parameter a, as it is further developed.

If $q(Q_m) = 0$ is satisfied, then using $u^\alpha = P$, Equation (14) becomes:

$$p(P) = P^2 + \gamma P + 1 = 0, \tag{15}$$

where $\gamma = 2\cos(0.5\alpha\pi) - \alpha\beta\sin(0.5\alpha\pi)$. Since the roots P_1 and P_2 of $p(P) = 0$ must be real and positive, it is necessary that $\gamma^2 - 4 \geq 0$ and $\gamma < 0$, so that $\gamma \leq -2$. Moreover, also $p(P)$ is a reciprocal polynomial, so that $P_1 = P_2^{-1}$. Now, consider two cases.

3.1 Case a

Let $\gamma = -2$. Then, with $C = 1 + \cos(0.5\alpha\pi)$ and $S = \sin(0.5\alpha\pi)$, the definition of γ leads to:

$$\beta = \hat{\beta} = \frac{2C}{\alpha S}, \tag{16}$$

with $0 < \alpha < 1$ and $\beta > 0$. The choice $\beta = \hat{\beta}$ leads to a particularly simple and effective algorithm. In this case, indeed, $\gamma = \hat{\gamma} = -2$, and the solutions of (15) are $\hat{P}_1 = \hat{P}_2 = 1$, so that it is $u_{\max} = \hat{u}_{\max} = 1$. Note also that, with a given α, $\hat{\beta}$ is determined by (16) and $q(Q) = 0$ can be solved. Let $\hat{Q}_2 = 1/a_0^2$ be the root where $\hat{Q}_2 < 1$, $a_0 > 1$. This equation also gives $a_0 = \sqrt{\hat{Q}_1}$ because $\hat{Q}_1 = \hat{Q}_2^{-1}$. Consequently, the integral constant of the controller is:

$$T_I = a^2 T_E = \hat{Q}_1 T_E. \tag{17}$$

At this point, it is possible to determine the gain $\hat{K}_C \hat{K}_P$ necessary to make the gain-crossover frequency u_{GC} equal to $\hat{u}_{\max}$. Namely, using $|G(j\hat{u}_{\max})| = |G(j1)| = 1$ gives:

$$\hat{K}_C \hat{K}_P = \sqrt{\frac{1 + a_0^4}{2a_0^4 C}} = \sqrt{\frac{1 + \hat{Q}_1^2}{2\hat{Q}_1^2 C}}. \tag{18}$$

Now, it is possible to determine the performances corresponding to the controller constants with the substitution of $\hat{u}_{\max} = 1$ and $a_0 = \sqrt{\hat{Q}_1}$ in $\mathrm{Arg}\,(G(ju))$. The maximum phase margin PM_α is uniquely determined as:

$$PM_\alpha = \tan^{-1}\left(\frac{S}{C}\right) - \tan^{-1}(\hat{Q}_2) + 0.5\pi(1 - \alpha). \tag{19}$$

As it is to be expected, larger values of PM_α correspond to smaller values of α. More precisely, for $\alpha = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$, PM_α assumes the values 80.1, 74.4, 68.3, 61.5, 54.1, 45.8, respectively.

3.2 Case b

Let $\gamma < -2$, which implies $\beta > \hat{\beta}$ because

$$\beta = \frac{2(C - 1) - \gamma}{\alpha S}. \tag{20}$$

Therefore, the roots P_1 and P_2 of (15) are real and positive and the former (say $P_1 > 1$) is the reciprocal of the latter (say $P_2 < 1$). Then, if α is assigned, $u_{\max 1} = \sqrt[\alpha]{P_1}$ and $u_{\max 2} = \sqrt[\alpha]{P_2}$ follow. Moreover, from (20) we may obtain the value of β which is necessary to compute $Q_2 < 1$ and $Q_1 = Q_2^{-1}$. However, the value $Q_2 < 1$ is considered here because of the constraint $u_{\max} < a^2$.

Note that the (20) with $\gamma = -2$ leads to $\beta = \hat{\beta}$ and $P_1 = P_2$ again. For each $\beta > \hat{\beta}$, two values P_1 and P_2 are obtained for P. However, the following considerations justify why only P_1 is considered for determining PM. Namely, consider:

$$\theta(P) = \tan^{-1}\left(\frac{P\sin(0.5\alpha\pi)}{1 + P\cos(0.5\alpha\pi)}\right), \tag{21}$$

which is a continuous, monotonically increasing function of the variable P. Then, for $P_2 < 1 < P_1$, $\theta(P)$

assumes higher (positive) values for $P = P_1$: $\theta(P_1) > \theta(1) > \theta(P_2)$. Hence, the value P_1 is chosen for achieving the objective of maximizing PM. Namely, using $S_1 = P_1 \sin(0.5\alpha\pi)$ and $C_1 = 1 + P_1 \cos(0.5\alpha\pi)$, the PM is given by:

$$PM_1 = \tan^{-1}\left(\frac{S_1}{C_1}\right) - \tan^{-1}(Q_2) + 0.5\pi(1-\alpha), \tag{22}$$

and the first term gives the major contribution to PM if P_1 is chosen.

For example, $\gamma = -2$ and $\alpha = 0.6$ lead to $PM_\alpha = 54.1^\circ$. Moreover, for each (smaller) value of γ, we obtain two points. In fact, a decrement in γ of 6% giving $\gamma = -2.12$ leads to $PM_1 = 58.3^\circ$ ($P_1 = 1.41$), $PM_2 = 49.7^\circ$ ($P_2 = 0.71$). Using P_1 only, we may now write $a_1^2 = \sqrt[\alpha]{P_1}/Q_2 = \sqrt[\alpha]{P_1} \cdot Q_1$ and determine the integral constant of the controller:

$$T_I = a_1^2 T_E = \sqrt[\alpha]{P_1} \cdot Q_1 \cdot T_E. \tag{23}$$

Note that the value of T_I given by (23) is $\sqrt[\alpha]{P_1}$ higher than the value corresponding to case a (see (17)).

Finally, from $|G(ju_{\max i})| = 1$ we may deduce that:

$$K_{C1} K_P = \frac{P_1^{1+1/\alpha}\sqrt{1+Q_2^2}}{\sqrt{1+P_1^2+2P_1(C-1)}}, \tag{24}$$

which yields the value K_{C1}. Clearly, the case b compares favorably with the previous case a because of the additional freedom improving the tradeoff between robustness and time-domain performances.

4 Simulation and experimental results

Since $s^{-\alpha}$ is irrational, it is difficult to directly implement this fractional operator in the time-domain simulations. Thus, to analyze the performance of the PI$^\alpha$ controller tuned with formulae (18) and (24), it is necessary to approximate $s^{-\alpha}$ with a rational integer order operator in the s-domain. Some research has been done in this area already; here, we consider the approximation of $s^{-\alpha}$ due to [4] and reported by [7] with a fifth-order rational transfer function. Using this approximation, Fig. 1 shows the Bode plots of

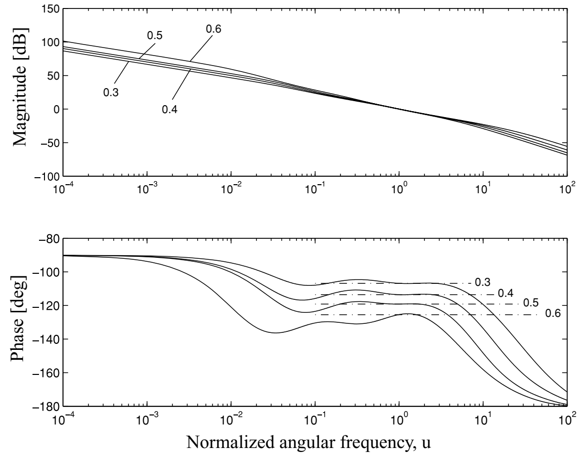

Fig. 1 Bode plots of $G(ju)$ for different values of parameter α

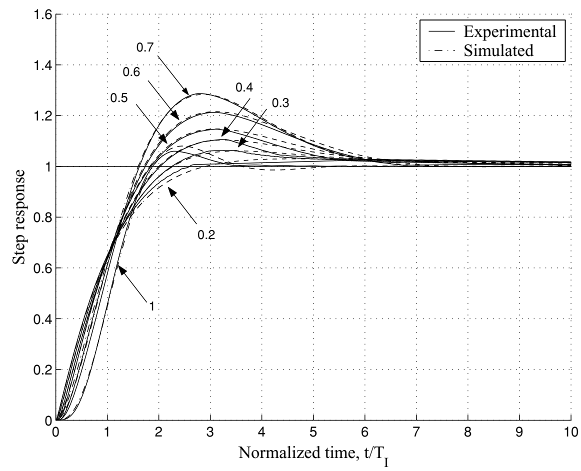

Fig. 2 Step response of closed-loop PI$^\alpha$ controlled system for different values of parameter α

$G(ju)$ for $\alpha = 0.3, 0.4, 0.5, 0.6$. In a wide frequency range around $u \approx 1$, phase plots are nearly flat and indicate that $\text{Arg}(G(ju))$ and hence PM_α are nearly constant. This behavior guarantees robust stability for wide variations in the plant gain. In addition, Fig. 2 shows the simulated step responses (dashed lines) of the closed-loop transfer function for $\alpha = i \cdot 0.1$, with $i = 2, \ldots, 7$. The values of a_0 and of $\hat{K}_C \hat{K}_P$ are determined by the tuning procedure (18). The step performances in Table 1 and Fig. 2 show that a proper choice of α can lead to a low percentage overshoot combined with an excellent PM (note that Table 1 reports PM values that are computed using the approximated fractional integrator). Namely, with $\alpha \leq 0.7$, the overshoots are considerably lower than 43%, the value

Table 1 Percentage overshoot ($OS\%$), normalized rise time (t_R/T_I), normalized settling time (t_S/T_I), phase margin (PM_α), as functions of parameter α

	$OS\%$		t_R/T_I (0–100%)		t_S/T_I (2%)		
α	Simulation	Experimental	Simulation	Experimental	Simulation	Experimental	PM_α (°)
0.2	2.97	2.32	2.60	2.91	7.05	6.16	81.3
0.3	6.09	6.27	2.25	2.27	7.28	7.44	73.2
0.4	10.3	10.7	2.03	2.00	6.84	8.09	66.4
0.5	14.8	14.6	1.85	1.83	6.90	9.26	60.9
0.6	21.6	21.3	1.74	1.71	6.58	6.21	54.5
0.7	28.4	28.7	1.62	1.60	6.17	6.14	48.0
SO	43.4	43.8	0.53	0.48	4.11	2.65	37.0
SO (p.f.)	8.10	5.97	1.89	1.88	3.31	3.09	37.0

achieved by the SOM ($\alpha = 1$) using an ordinary PI controller. We also recall that, it is common practice of the classical SOM to reduce the overshoot to 8.1% by using a pre-filter (see [26]). Figure 2 also shows the closed-loop step response for $\alpha = 1$, with an ordinary PI tuned with the SOM using a pre-filter in the command channel. Simulation experiments confirm that the overshoot is 8.1% in this case. However, the PM is still 37° (see Table 1). To sum up, FOC conciliate a good dynamic performance and an improved robustness. The only drawback of FOC are their settling times, which are greater than the values obtained with a standard PI tuned with the SOM and the pre-filter.

To validate the tuning approach, we have also performed a laboratory experiment. The experimental setup consists of a nonlinear 370 W dc servomotor (AMIRA DR300), a power amplifier driving the plant, and a PC equipped with a floating point 250 MHz Motorola PPC dSPACE board (DS1104), which provides the position reference and runs the controllers. All routines run in discrete time with a 1 ms sampling period. The dc motor transfer function has been obtained through a frequency domain identification process, yielding:

$$G(s) = \frac{0.935}{s(1 + 0.124s)}. \tag{25}$$

A 1024 pulses incremental encoder gives the rotor position measurement. We implemented the control algorithms in the MATLAB/Simulink® environment. The dSPACE code generator compiles the Simulink program and then the real-time executable code is downloaded to the board memory. During motor operation, the board processor receives the feedback from the encoder and applies the appropriate control action to the power unit. The signals are processed using 16 bit A/D–D/A converters that are integrated in the dSPACE board.

Figure 2 and Table 1 compare the step responses obtained by simulation (dashed line) and by measured data (continuous line), for $\alpha = 0.2$–0.7. The tuning formula (18) is employed. The responses show a very good agreement between simulations and laboratory experiments. As Fig. 3 indicates for $\alpha = 0.4, 0.5, 0.6$, tuning formula (24) can improve the performance indexes. Namely, simulations and experiments agree each other also showing that, the greater is a_1, the smaller are the rise times and the settling times.

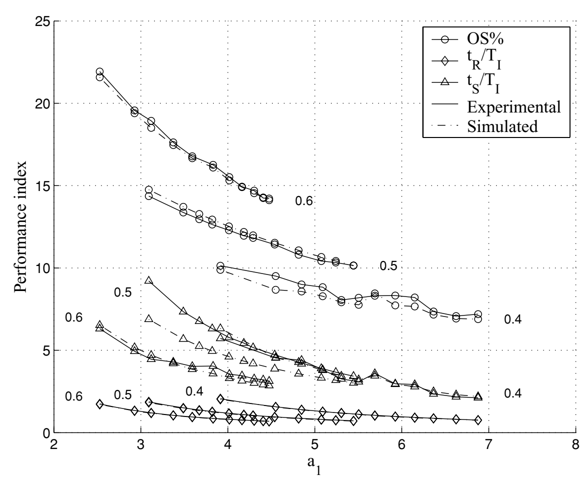

Fig. 3 Percentage overshoot ($OS\%$), normalized rise time (t_R/T_I), and normalized settling time (t_S/T_I) for different values of parameter α, from simulation and experimental results

5 Conclusion

We have developed a design approach for FOC, which is inspired by the SOM. Of course, the method leads to a phase diagram of the resulting open-loop frequency response which is no more symmetrical around the gain-crossover frequency. However, it is nearly flat in a wide range around the crossover. The parameter α also influences overshoot and rise time. Values in the range $0.4 \leq \alpha \leq 0.6$ assure a good tradeoff between robustness and dynamic performance.

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