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ORIGINAL ARTICLE

Rotating flow of a third grade fluid in a porous space with Hall current

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Abstract This study investigates the rotating magnetohydrodynamic (MHD) flow of a third-grade fluid in a porous space. Modified Darcy's law has been utilized for the flow modeling. The Hall effects are taken into consideration. The basic equations governing the flow are reduced to a highly nonlinear ordinary differential equation. This equation has been solved analytically by employing the homotopy analysis method (HAM). The effects of the various interesting parameters on the velocity distribution have been discussed.

Keywords Rotating flow · Third-grade fluid · Porous space · HAM solution

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1 Introduction

It is well known that in technological applications, non-Newtonian fluids exhibiting a nonlinear relationship between the stresses and the rate of strain are more appropriate than Newtonian fluids. One of the important classes of non-Newtonian fluids is viscoelastic fluids for which one can reasonably hope to obtain analytic solutions. The governing equations for such fluids are of higher order, much more complicated, and subtle than the Newtonian fluids. These equations present various challenges to engineers, mathematicians, Numerical simulists and physicists alike. Even then, several workers [1–10] are now engaged in obtaining analytic solutions for flows of non-Newtonian fluids. Very limited attention has also been given to flows of non-Newtonian fluids in a rotating frame [11–16].

In all the above-mentioned studies, the flows have been considered in a nonporous space. Recently, the flow of non-Newtonian fluids in a porous space is a topic of special interest in many engineering applications. Examples of these applications are filtration processes, flows in biomechanics, packed bed reactors, geothermal engineering, insulation systems, ceramic processing, enhanced oil recovery, chromatography, and many others. Specifically, the rotating flows of non-Newtonian fluids through a porous medium are important in geophysical applications.

The main goal of the present study is to investigate the rotating flow of a third-grade fluid in a porous space. The fluid is electrically conducting in the presence of a uniform strong applied magnetic field. The induced magnetic field is neglected for small magnetic Reynolds number and the Hall effect is taken into account. No electric field is applied. The fluid is bounded by an insulated plate. The fluid motion is caused by suddenly moved plate. The relevant problem which governs the flow has been first modeled and then solved using homotopy analysis method (HAM) [17–30]. The convergence of the obtained solution is properly discussed. Graphs for the velocity components have also been plotted and discussed for various values of emerging parameters. To the best of our knowledge, such investigation for a rotating third-grade fluid in a porous domain is not available yet in the literature.

2 Development of the flow

Consider the hydromagnetic flow of an incompressible third-grade fluid bounded by an insulated plate at $z = 0$. The fluid occupying the space $z > 0$ fills the porous medium. The fluid has uniform properties and porous medium is isotropic and homogeneous. A uniform strong magnetic field $\mathbf{B}_0$ is applied in the z-direction. In the undisturbed state, both fluid and the plate are in a state of rigid body rotation with constant angular velocity $\Omega = \Omega\hat{k}$ ($\hat{k}$ is a unit vector parallel to the z-axis). The flow is driven by a sudden motion to the plate. The magnetic Reynolds number is taken small and hence the induced magnetic field is negligible. However, the Hall current effects are retained. In a porous space, the equations governing the motions and magnetic fields within the fluid may be written in a rotating coordinate system as

$$\rho[(\mathbf{V}\cdot\nabla)\mathbf{V} + 2\mathbf{\Omega}\times\mathbf{V} + \mathbf{\Omega}\times(\mathbf{\Omega}\times r)] = \mathrm{div}\mathbf{T} + \mathbf{J}\times\mathbf{B} + \mathbf{R}, \tag{1}$$

$$\mathrm{div}\mathbf{V} = 0, \tag{2}$$

$$\mathrm{div}\mathbf{B} = 0, \quad \mathrm{curl}\mathbf{B} = \mu_e\mathbf{J}, \quad \mathrm{Curl}\mathbf{E} = \mathbf{0}, \tag{3}$$

$$\mathbf{J} + \frac{\omega_e\tau_e}{B_0}(\mathbf{J}\times\mathbf{B}) = \sigma\left[\mathbf{E} + \mathbf{V}\times\mathbf{B} + \frac{1}{en_e}\nabla p_e\right]. \tag{4}$$

In above equations, $\mathbf{T}$ is the Cauchy stress tensor, ρ the fluid density, $\mathbf{V}$ the velocity, $\mathbf{r}$ the radial coordinate, and $\mathbf{R}$ the Darcy's resistance in the porous space. In Maxwell's Equations (3), the total magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, where $\mathbf{B}_0$ is the applied magnetic field and $\mathbf{b}$ the induced magnetic field. $\mathbf{E}$ is the total electric field, μ_e the magnetic permeability, and $\mathbf{J}$ the current density. Note that in the generalized Ohm's law (4), ω_e is the cyclotron frequency of electrons, τ_e the electron collision time, σ the electrical conductivity, e the electron charge, $1/en_e$ the Hall factor, n_e the number density of the electrons, and p_e the electron pressure. The ion-slip and applied voltage are neglected.

The constitutive equations for a third-grade fluid are

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{5}$$

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\mathbf{A}_2)\mathbf{A}_1. \tag{6}$$

Considering the thermodynamic conditions [31]

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad \beta_1 = \beta_2 = 0, \qquad \beta_3 \geq 0, \qquad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \tag{7}$$

we have

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_1^2 + \beta_3(\mathrm{tr}\mathbf{A}_2)\mathbf{A}_1, \tag{8}$$

where

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^\top, \qquad \mathbf{L} = \nabla\mathbf{V}, \tag{9}$$

$$\mathbf{A}_i = (\mathbf{V}\cdot\nabla)\mathbf{A}_{i-1} + \mathbf{A}_{i-1}\mathbf{L} + \mathbf{L}^\top\mathbf{A}_{i-1}, \quad i > 1. \tag{10}$$

In previous equations, ρ is the pressure, $\mathbf{I}$ and $\mathbf{S}$ are the identity and extra stress tensors, respectively, μ is the dynamic viscosity, $\mathbf{A}_i$ ($i = 1, 2, \ldots$) are the Rivlin–Ericksen tensors, $\mathbf{L}$ the velocity gradient, and $\mathbf{L}^\top$ the transpose of $\mathbf{L}$.

In porous space, the relationship between pressure drop and velocity is

$$\mathbf{\nabla} p = -\frac{\phi}{k_1}\left[\mu + 2\beta_3\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right]\mathbf{V}, \tag{11}$$

where ϕ and k_1, respectively, indicate the porosity and permeability of the porous space, and u and v are the velocity components. Since the pressure gradient in Eq. (11) is a measure of the resistance to the flow in the bulk of porous space and in Eq. (1), $\mathbf{R}$ is interpreted as the flow resistance offered by the solid matrix. Therefore, through Eq. (11), **R** satisfies [10]

$$\mathbf{R} = -\frac{\phi}{k_1}\left[\mu + 2\beta_3\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right]\mathbf{V}, \quad (12)$$

For the problem under consideration, the extra stress tensor and velocity are defined as follows:

$$\mathbf{S}(z) = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}, \quad (13)$$

$$\mathbf{V}(z) = (u(z), v(z), 0). \quad (14)$$

Clearly, Eq. (14) satisfies the incompressibility condition (2), and in scalar form, Eq. (1) gives

$$\begin{aligned} -2\Omega\rho\frac{du}{dz} &= \mu\frac{d^3u}{dz^3} + 2\beta_3\frac{d^2}{dz^2}\left[\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\frac{du}{dz}\right] - \frac{\phi}{k}\frac{d}{dz}\left[\mu u + 2\beta_3\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}u\right] \\ &\quad - \frac{\sigma B_0^2}{1 - im_0}\frac{du}{dz}, \quad (15) \\ 2\Omega\rho\frac{du}{dz} &= \mu\frac{d^3v}{dz^3} + 2\beta_3\frac{d^2}{dz^2}\left[\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\frac{du}{dz}\right] - \frac{\phi}{k}\frac{d}{dz}\left[\mu v + 2\beta_3\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}v\right] \\ &\quad - \frac{\sigma B_0^2}{1 - im_0}\frac{du}{dz}, \quad (16) \end{aligned}$$

where $m_0 = w_e\tau_e$ is the Hall parameter. The boundary conditions are given by

$$\begin{aligned} &u = U, \quad v = 0 \quad \text{at } z = 0 \\ &u \to 0, \quad v \to 0 \quad \text{as } z \to \infty. \quad (17) \end{aligned}$$

Equations (15) and (16) can be combined in the following form

$$\begin{aligned} 2\Omega i\rho\frac{dF}{dz} &= \mu\frac{d^3F}{dz^3} + 2\beta_3\frac{d^2}{dz^2}\left\{\left(\frac{dF}{dz}\right)^2\frac{d\bar{F}}{dz}\right\} - \frac{\phi}{k}\frac{d}{dz}\left\{\mu F + 2\beta_3\frac{dF}{dz}\frac{d\bar{F}}{dz}F\right\} \\ &\quad - \frac{\sigma B_0^2}{1 - im_0}\frac{dF}{dz}, \quad (18) \end{aligned}$$

where

$$F = u + iv, \qquad \bar{F} = u - iv. \quad (19)$$

The boundary conditions (17) can be written as

$$F(0) = U, \qquad F(\infty) = 0. \quad (20)$$

We introduce the following dimensionless variables

$$z^* = \frac{\rho U}{\mu}z, \quad F^* = \frac{F}{U}, \quad \bar{F}^* = \frac{\bar{F}}{U}, \quad \Omega^* = \frac{\mu}{\rho U^2}\Omega$$

$$\beta = \frac{\beta_3\rho^2U^4}{\mu^3}, \qquad M^2 = \frac{\sigma B_0^2}{\rho\Omega}, \qquad K = \frac{\Omega\rho k}{\phi\mu}.$$

The problem under consideration now becomes

$$\begin{aligned} \frac{d^3F}{dz^3} &= \Omega\left\{\frac{M^2}{1 + m_0^2} + \left(2 - \frac{M^2m_0}{1 + m_0^2}\right)i\right\}\frac{dF}{dz} - 2\beta\frac{d^2}{dz^2}\left\{\left(\frac{dF}{dz}\right)^2\frac{d\bar{F}}{dz}\right\} \\ &\quad + \frac{\Omega}{K}\left\{\frac{dF}{dz} + 2\beta\frac{d}{dz}\left(\frac{dF}{dz}\frac{d\bar{F}}{dz}F\right)\right\}, \quad (21) \end{aligned}$$

$$F(0) = 1, \qquad f(\infty) = 0, \quad (22)$$

where the asterisks have been omitted for simplicity. Using

$$\eta = e^{-z} \quad (23)$$

the problem reduces to

$$\eta^3\frac{d^3F}{d\eta^3} + 3\eta^2\frac{d^2F}{d\eta^2} + \eta\frac{dF}{d\eta}$$

$$
\begin{aligned}
&= \Omega\eta\left\{\frac{M^2}{1+m_0^2} + \left(2-\frac{M^2m_0}{1+,m_0^2}\right)i\right\}\frac{dF}{d\eta} \\
&- 2\beta\eta^3\left[\begin{array}{c} 9\left(\frac{dF}{d\eta}\right)^2\frac{d\bar{F}}{d\eta} + 7\eta\frac{d}{d\eta}\left\{\left(\frac{dF}{d\eta}\right)^2\frac{d\bar{F}}{d\eta}\right\} \\ +\eta^2\frac{d^2}{d\eta^2}\left\{\left(\frac{dF}{d\eta}\right)^2\frac{d\bar{F}}{d\eta}\right\}\end{array}\right] \\
&+ \frac{\Omega}{K}\eta\left[\begin{array}{c}\frac{dF}{d\eta} + 2\beta\eta^2\frac{d}{d\eta}\left(\frac{dF}{d\eta}\frac{d\bar{F}}{d\eta}F\right) \\ +4\beta\eta\frac{dF}{d\eta}\frac{d\bar{F}}{d\eta}F\end{array}\right], \qquad (24)
\end{aligned}
$$

$$F(1) = 1, \qquad F(0) = 0. \qquad (25)$$

It is worth pointing out that by setting $\phi = 0$ or $k \rightarrow \infty$, we get the governing problem for rotating flow of a third grade fluid in a nonporous space. Equation (24) is highly nonlinear and its HAM solution subject to the boundary conditions (25) can be sought in the next section.

3 HAM solution

For the HAM solution, we choose

$$F_0(\eta) = \eta, \qquad (26)$$

and the auxiliary linear operator

$$\mathcal{L}(f) = f'', \qquad (27)$$

satisfying

$$\mathcal{L}(C_1 + C_2\eta) = 0, \qquad (28)$$

in which C_1 and C_2 are arbitrary constants.

The deformation problem at the zeroth order satisfies

$$(1-p)\mathcal{L}[\hat{F}(\eta,p) - F_0(\eta)] = p\hbar\mathcal{N}[\hat{F}(\eta,p)], \qquad (29)$$

$$\hat{F}(0,p) = 0, \qquad \hat{F}(1,p) = 1, \qquad (30)$$

where $\hbar$ and $p \in [0, 1]$ are the auxiliary and embedding parameters, respectively, and

$$
\begin{aligned}
\mathcal{N}[\hat{F}(\eta,p)] &= \eta^3\frac{\partial^3\hat{F}(\eta,p)}{\partial\eta^3} + 3\eta^2\frac{\partial^2\hat{F}(\eta,p)}{\partial\eta^2} \\
&+ \eta\frac{\partial\hat{F}(\eta,p)}{\partial\eta} - \Omega\eta\left[\frac{M^2}{1+m_0^2} + \left(2-\frac{M^2m_0}{1+m_0^2}\right)i\right]\frac{\partial E(\eta,p)}{\partial\eta} \\
&+ 2\beta\eta^3\left[\begin{array}{c} 9\left(\frac{\partial\hat{F}(\eta,p)}{\partial\eta}\right)^2\frac{\partial\bar{\hat{F}}(\eta,p)}{\partial\eta} \\ +7\eta\frac{\partial}{\partial\eta}\left\{\left(\frac{\partial\hat{F}(\eta,p)}{\partial\eta}\right)^2\frac{\partial\bar{\hat{F}}(\eta,p)}{\partial\eta}\right\} \\ +\eta^2\frac{\partial^2}{\partial\eta^2}\left\{\left(\frac{\partial\hat{F}(\eta,p)}{\partial\eta}\right)^2\frac{\partial\bar{\hat{F}}(\eta,p)}{\partial\eta}\right\}\end{array}\right] \\
&- \frac{\Omega}{K}\eta\left[\frac{\partial\hat{F}(\eta,p)}{\partial\eta} + 2\beta\left\{\begin{array}{c}\eta^2\frac{\partial}{\partial\eta}\left(\frac{\partial\hat{F}(\eta,p)}{\partial\eta}\frac{\partial\bar{\hat{F}}(\eta,p)}{\partial\eta}F(\eta,p)\right) \\ +2\eta\left(\frac{\partial\hat{F}(\eta,p)}{\partial\eta}\frac{\partial\bar{\hat{F}}(\eta,p)}{\partial\eta}F(\eta,p)\right)\end{array}\right\}\right] \qquad (31)
\end{aligned}
$$

is the nonlinear differential operator. For $p = 0$ and $p = 1$, we have

$$\hat{F}(\eta,0) = F_0(\eta), \quad \hat{F}(\eta,1) = F(\eta). \qquad (32)$$

From the previous equation, we note that the derivation of p from 0 to 1 is continuous variation of $\hat{F}(\eta,p)$ from $F_0(\eta)$ to $F(\eta)$. Due to Taylor's theorem and Eq. (32), we can write

$$\hat{F}(\eta,p) = F_0(\eta) + \sum_{m=1}^{\infty} F_m(\eta)p^m \qquad (33)$$

in which

$$F_m(\eta) = \frac{1}{m!}\frac{\partial^m\hat{F}(\eta,p)}{\partial p^m}\bigg|_{p=0}.$$

Clearly, the convergence of the series (33) depends on the auxiliary parameter $\hbar$. Assume that $\hbar$ is selected such that the series (33) is convergent at $p = 1$, then due to Equation (32), we have

$$\hat{F}(\eta,p) = F_0(\eta) + \sum_{m=1}^{\infty} F_m(\eta). \qquad (34)$$

Differentiating m-times the zeroth-order deformation (29) with respect to p and then dividing them by $m!$ and finally setting $p = 0$, we have the mth-order deformation problem

$$\mathcal{L}[F_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar\mathcal{R}_m(\eta), \qquad (35)$$

$$F_m(0) = F_m(1) = 0, \qquad (36)$$

$$
\begin{aligned}
\mathcal{R}_m(\eta) &= \eta^3 F'''_{m-1}(\eta) + 3\eta^2 F''_{m-1}(\eta) + \eta F'_{m-1}(\eta) \\
&\quad - \Omega_\eta \left\{ \frac{M^2}{1+m_0^2} + \left(2 - \frac{M^2 m_0}{1+m_0^2}\right) i \right\} F'_{m-1}(\eta) \\
&\quad + 2\beta\eta^3 \left[\begin{array}{l}
9 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}'_l(\eta) \\
+7\eta \left\{ \begin{array}{l} 2\sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\bar{F}'_l(\eta) \\ + \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}''_l(\eta) \end{array} \right\} \\
+\eta^2 \left\{ \begin{array}{l} 2\sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\bar{F}'_l(\eta) \\ +2\sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'''_{k-l}(\eta)\bar{F}'_l(\eta) \\ +4\sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\bar{F}'_l(\eta) \\ + \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}'''_l(\eta) \end{array} \right\}
\end{array} \right] \\
&\quad - \frac{\Omega}{K}\eta \left[\begin{array}{l}
F'_{m-1}(\eta) + 2\beta\eta^2 \left\{ \begin{array}{l} \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta)F_l(\eta) \\ + \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}''_{k-l}(\eta)F_l(\eta) \\ + \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}'_l(\eta) \end{array} \right\} \\
+4\beta\eta \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta)F_l(\eta)
\end{array} \right]
\end{aligned} \tag{37}
$$

and

$$
\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{38}
$$

For the solution of the mth-order problem, we use the symbolic computation software MATH-EMATICA up to first few order of approximations and found that the solution of this problem is given by the following series

$$
F_m(\eta) = \sum_{n=0}^{4m+2} a_{m,n}\eta^n, \qquad m \ge 0. \tag{39}
$$

In order to obtain the recurrence formulae for the coefficients $a_{m,n}$ of $F_m(\eta)$, we substitute Eq. (39) in Eq. (35) and obtain for $m \ge 1$ and $0 \le n \le 7m+2$:

$$
a_{m,1} = \chi_m\chi_{4m-1}a_{m-1,1} - \sum_{n=0}^{4m+2} \frac{\Psi_{m,n}}{(n+1)(n+2)}, \tag{40}
$$

$$
a_{m,n} = \chi_m\chi_{4m-n}a_{m-1,n} + \frac{\Psi_{m,n-2}}{n(n-1)}, \quad 2 \le n \le 4m+2, \tag{41}
$$

where the coefficient $\Psi_{m,n}$ is define by

$$
\Psi_{m,n} = \hbar \left[\begin{array}{l}
\chi_{4m-n} \left\{ \begin{array}{c} \chi_{n-1}d_{m-1,n-3} + 3\chi_n c_{m-1,n-2} \\ +\chi_{n+1}b_{m-1,n-1}\left(1 + \frac{\Omega}{K} - \Omega\left(\frac{M^2}{1+m_0^2} + \left(2 - \frac{M^2m_0}{1+m_0^2}\right)i\right)\right) \end{array} \right\} \\
+2\beta \left\{ \begin{array}{c} \chi_{n-1}(9\Delta1_{m,n-3} - \frac{\Omega}{K}(\Delta8_{m,n-3} + \Delta9_{m,n-3} + \Delta1_{m,n-3})) \\ +7\chi_{n-2}(2\Delta2_{m,n-4} + \Delta3_{m,n-4}) - \frac{2\Omega}{K}\chi_n\Delta10_{m,n-2} \\ +\chi_{n-3}(2\Delta4_{m,n-5} + 2\Delta5_{m,n-5} + 4\Delta6_{m,n-5} + \Delta7_{m,n-5}) \end{array} \right\}
\end{array} \right], \tag{42}
$$

in which the coefficients $\Delta1_{m,n}$ to $\Delta10_{m,n}$ for $m \ge 1$ and $0 \le n \le 4m+2$ are

$$
\Delta1_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{b}_{l,s}b_{k-l,p-s}b_{m-1-k,n-p},
$$

$$
\Delta2_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{b}_{l,s}c_{k-l,p-s}b_{m-1-k,n-p},
$$

$$
\Delta3_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{c}_{l,s}c_{k-l,p-s}b_{m-1-k,n-p},
$$

$$
\Delta4_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{b}_{l,s}c_{k-l,p-s}c_{m-1-k,n-p},
$$

$$
\Delta5_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{b}_{l,s}d_{k-l,p-s}b_{k-1-k,n-p},
$$

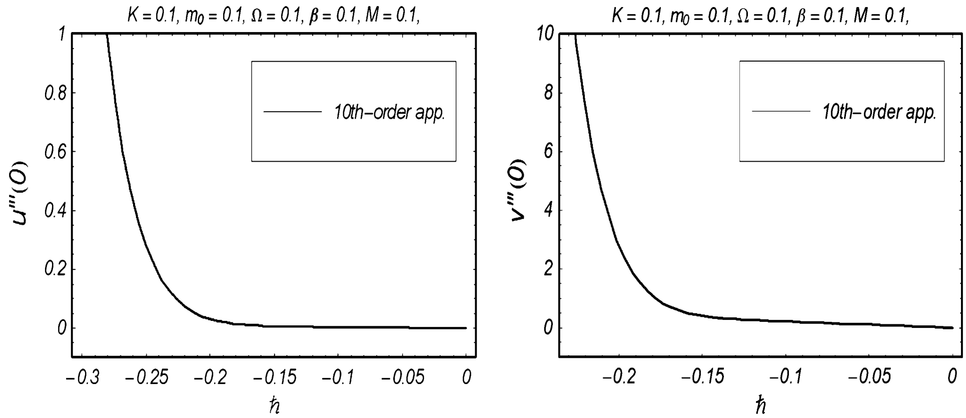
$$
\Delta6_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{c}_{l,s}c_{k-l,p-s}b_{m-1-k,n-p},
$$

$$
\Delta7_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{d}_{l,s}b_{k-l,p-s}b_{m-1-k,n-p},
$$

$$
\Delta8_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{b}_{l,s}a_{k-l,p-s}c_{m-1-k,n-p},
$$

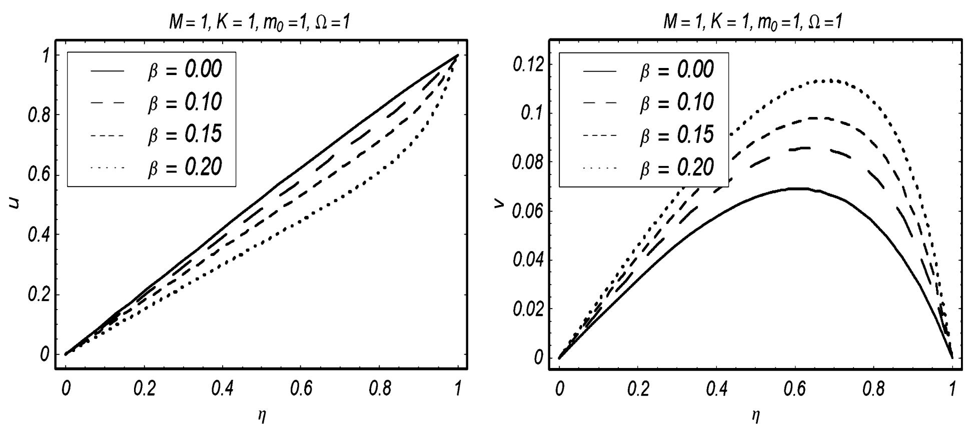
$$
\Delta9_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{c}_{l,s}a_{k-l,p-s}b_{m-1-k,n-p},
$$

$$
\Delta10_{m,n} = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}}\sum_{s=\max\{0,p-4k+4l-2\}}^{\min\{p,4l+2\}} \times \bar{b}_{l,s}a_{k-l,p-s}b_{m-1-k,n-p}
$$



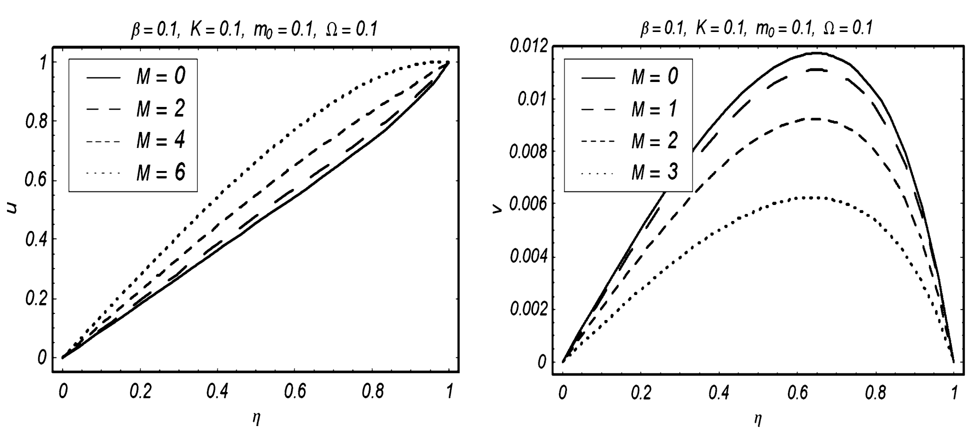
(a) (b)

Fig. 1 $\hbar$-curve for 10th order of approximation



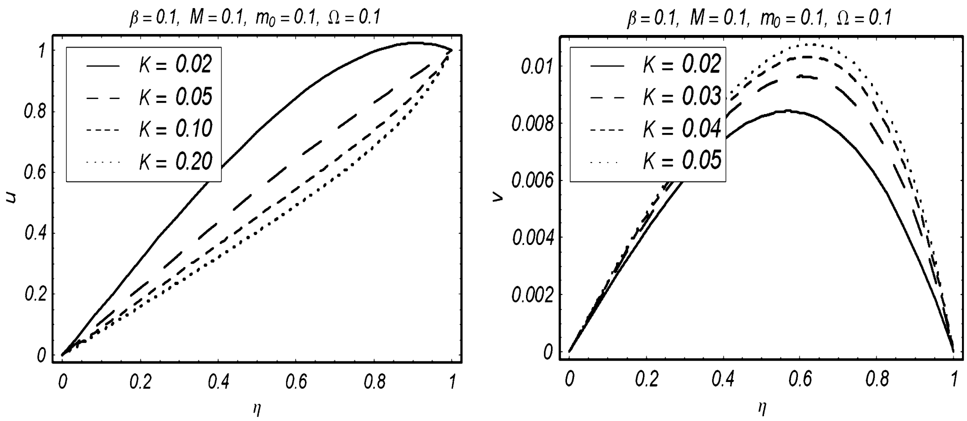
(a) (b)

Fig. 2 Variation of velocity profiles u and v with the change in parameter β for 10th-order approximation at $\hbar = -0.1$



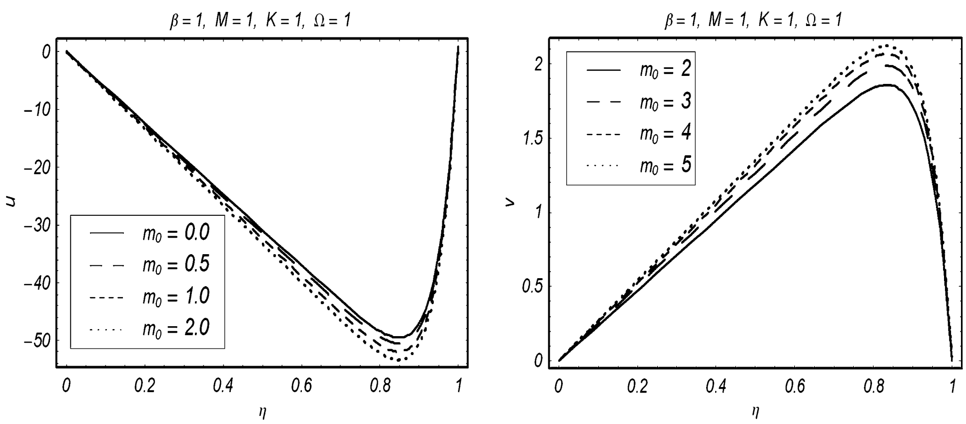
(a) (b)

Fig. 3 Variation of velocity profiles u and v with the change in parameter M for 10th-order approximation at $\hbar = -0.1$



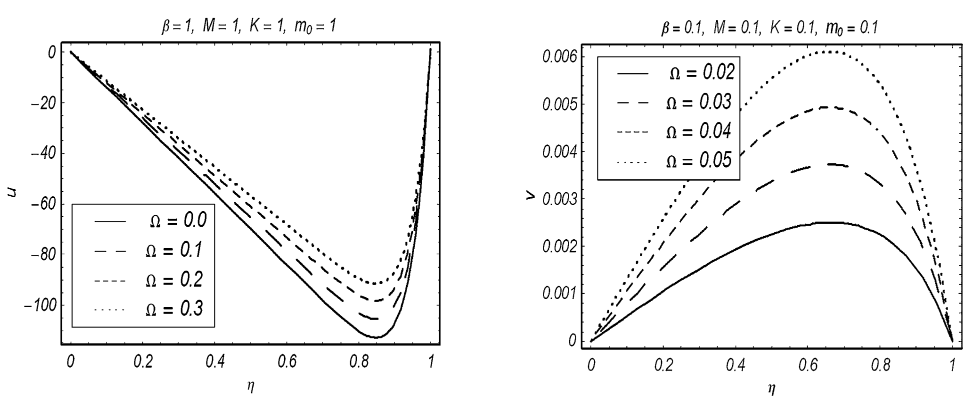
(a) (b)

Fig. 4 Variation of velocity fields u and v with the change in parameter K for 10th-order approximation at $\hbar = -0.1$



(a) (b)

Fig. 5 Variation of velocity profiles u and v with the change in parameter m_0 for 10th-order approximation at $\hbar = -0.1$



(a) (b)

Fig. 6 Variation of velocity profiles u and v with the change in parameter Ω for 10th-order approximation at $\hbar = -0.1$

and the coefficients $b_{m,n}$, $c_{m,n}$ and $d_{m,n}$ are

$$b_{m,n} = (n+1)a_{m,n+1}, \tag{43}$$

$$c_{m,n} = (n+1)b_{m,n+1}, \tag{44}$$

$$d_{m,n} = (n+1)c_{m,n+1}. \tag{45}$$

For the detailed procedure of deriving the aforementioned relations, the reader is referred to [19]. With the aforementioned recurrence formulae, we can calculate all coefficients $a_{m,n}$ using only the first three

$$a_{0,0} = 0, \qquad a_{0,1} = 1, \qquad a_{0,2} = 0 \tag{46}$$

given by the initial guess approximation for the function $F(\eta)$ in Eq. (26). The corresponding M-order approximation of Eqs. (24) and (25) is then given by

$$\sum_{m=0}^{M} F_m(\eta) = \sum_{n=1}^{4M+2} \sum_{m=n-1}^{4M+1} a_{m,n}\eta^n. \tag{47}$$

Therefore, explicit, totally analytic solution of the present flow is

$$F(\eta) = \sum_{m=0}^{\propto} F_m(\eta) = \lim_{M\to\infty}\left[\sum_{n=1}^{4M+2} \sum_{m=n-1}^{4M+1} a_{m,n}\eta^n\right]. \tag{48}$$

4 Convergence of the analytic solution

The expression given in Eq. (48) contains the auxiliary parameter $\hbar$ that gives the convergence region and rate of approximation for the HAM. In Fig. 1(a) and (b), the $\hbar$-curves are plotted for different order of approximations for the nondimensional velocity fields u and v. From Fig. 1(a) and (b), it is clear that the range for the admissible values for $\hbar$ is $-0.2 \le h < 0$. Our calculations indicate that the real and imaginary parts of the series given by Eq. (48) converge in the whole region of z when $\hbar = -0.1$.

5 Results and discussion

Figures 2(a) and (b) shows the variation of the velocity profiles u and v for different values of β. This figure shows that the velocity profile u decreases with increase in β, whereas the velocity profile v increases and then decreases with increase in β when other parameters $M = K = m_0 = \Omega = 1$ are fixed. From these figures, it is observed that the increase in the velocity profile v is smaller in magnitude when compared with the velocity profile u. Figures 3(a) and (b) depict the variation of the velocity profiles u and v for various values of M when $\beta = K = m_0 = \Omega = 1$. This figure indicates that the velocity profile u increases with increase in M, whereas the velocity profile v decreases with increase in M, showing the reverse behavior as observed in the previous case. However, the decrease in the magnitude of the velocity profile v is smaller than that in the increase of the velocity profile u. Figures 4(a) and (b) shows the variation of K on the velocity profiles u and v for $M = \beta = m_0 = \Omega = 1$. The velocity profile u decreases with increase in K, whereas the velocity profile v increases with increase in K, but with a small change when compared with the velocity profile u. In Figs. 5(a) and (b), it is observed that the velocity profiles u and v increase for large values of m_0 when $M = K = \beta = \Omega = 1.5$. The magnitude of decrease in the velocity profile u is smaller than that of decrease of u. Figures 6(a) and (b) depict the variation in the velocity profiles u and v for different values of Ω. From this figure, it is clearly seen that the velocity profile u decreases with increase in Ω. However, the velocity profile v increases by increasing Ω. The increase is smaller for the velocity profile v when compared with that of the velocity profile u when $M = K = m_0 = \beta = 1$.

6 Concluding remarks

In this work, the flow of a third-grade fluid over a moved plate is investigated in the presence of Hall current. The governing equation is modeled on the basis of the modified Darcy's law for a third-grade fluid. The resulting nonlinear problem has been solved analytically using HAM. It is noted that the velocity profiles u and v have reversed behavior for physical parameters β, M, K, and Ω, but the velocity profiles u and v show the same behavior for the physical parameter m_0. It is further found that the boundary layer thickness for v is large when compared to u.

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