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ORIGINAL ARTICLE

Linearization criteria for a system of second-order quadratically semi-linear ordinary differential equations

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Abstract Conditions are derived for the linearizability via invertible maps of a system of n second-order quadratically semi-linear differential equations that have no lower degree lower order terms in them, i.e., for the symmetry Lie algebra of the system to be $sl(n+2, \mathbb{R})$. These conditions are stated in terms of the coefficients of the equations and hence provide simple invariant criteria for such systems to admit the maximal symmetry algebra. We provide the explicit procedure for the construction of the linearizing transformation. In the simplest case of a system of two second-order quadratically semi-linear equations without the linear terms in the derivatives, we also provide the construction of the linearizing point transformation using complex variables. Examples are given to illustrate our approach for two- and three-dimensional systems.

Keywords Lie symmetry algebra · Linearization · System of second-order ordinary differential equations

1 Introduction

It is of interest to provide general criteria for the linearizability of systems of nonlinear ordinary differential equations, as they can then be reduced to easily solvable systems. Linearization criteria via invertible transformations for ordinary differential equations (ODEs) have been of great interest and have been dealt with by many authors over the years (see, e.g., [3, 7, 8, 11]). These are also of current interest both for scalar and systems of ODEs (see, e.g., [5, 6, 12]). The Lie algebraic criteria for linearization via point transformations for systems of second-order ODEs have been studied in [12] and the decoupling problem in [10].

A recently proved theorem [4], providing the relation between symmetries of systems of geodesic equations and the underlying manifold, leads to a simple procedure to check global linearizability for a large class of systems of second-order quadratically semi-linear ODEs. Related prior works can be seen at [1–3]. The class of equations for which the criteria can be simply stated consists of those that are formally like the system of geodesic equations. These equations have no linear terms in the derivatives or terms involving derivatives of the dependent variables in a linear fashion.

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It is natural to expect that there is a connection between the symmetries of differential equations and the isometries of manifolds. A connection was found in the theorem proved in [4], which states that the symmetries of the system of geodesic equations in spaces of constant positive or negative curvature (being maximally symmetric spaces) are simply the direct product of the two-dimensional dilatation group and the isometry group of the manifold. Classification of symmetries of systems of geodesic equations using projective methods was earlier given in [1]. For zero curvature, the isometry group is $SO(p,q) \otimes_s \mathbb{R}^n$, where $p+q=n$ and $\otimes_s$ stands for the semi-direct product, while the symmetry group of the geodesic equations is $SL(n+2,\mathbb{R})$. It can be expected that for less symmetric spaces the geodesics would continue to inherit the isometry group apart from continuing to carry the symmetry of re-parametrization of the geodetic parameter. This was conjectured in [4].

Using the Einstein summation convention, the system of geodesic equations can be written as

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \quad i,j,k = 1,\ldots,n, \tag{1}$$

where the Γ^i_{jk} are the Christoffel symbols and depend on x^i. Any system of ODEs of this form can be dealt with as if it was a system of geodesic equations. Consequently, the results obtained here are more generally applicable and the Γ^i_{jk} need no longer be thought of as Christoffel symbols but simply as coefficients of the relevant terms in the system of ODEs.

In Section 2, we state a theorem that yields simple criteria for the general system (1) to be linearizable via invertible transformations, i.e., have the symmetry algebra $sl(n+2,\mathbb{R})$, and write down explicitly the criteria for linearization of a system of three second-order ODEs. In Section 3, we solve the problem for a system of two equations in terms of the coefficients of the system explicitly and then construct the transformations that do the reduction to a linear system with $sl(4,\mathbb{R})$ symmetry algebra. An example that illustrates the construction procedure is included in Section 4. Herein other examples, including three-dimensional systems, are also given which illustrate various other points. Section 5 gives a summary and discussion of the results.

2 Linearization criteria for geodesic equations

Our criteria use some geometrical notation that we provide first for completeness. For a metric tensor, g_{ij}, the Christoffel symbols are defined by

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}), \tag{2}$$

where the "," stands for the partial derivative, i.e., $f_{,i} = \partial f/\partial x^i$. For the geometrically relevant, covariant derivative, we use ";", which is the same as the "," for a scalar but defined for vectors by

$$A^i_{;j} = A^i_{,j} + \Gamma^i_{jk}A^k, \quad B_{i;j} = B_{i,j} - \Gamma^k_{ij}B_k. \tag{3}$$

The Christoffel symbols are symmetric in the lower pair of indices, i.e., $\Gamma^i_{jk} = \Gamma^i_{kj}$. The Riemann tensor, which gives a measure of the curvature of the space, is defined by

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}, \tag{4}$$

and has the properties that

$$R^i{}_{jkl} = -R^i{}_{jlk}, \tag{5}$$

$$R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0, \tag{6}$$

and

$$R^i{}_{jkl;m} + R^i{}_{jlm;k} + R^i{}_{jmk;l} = 0. \tag{7}$$

The tensor, written in fully covariant form by "lowering" the first index

$$R_{ijkl} = g_{im}R^m{}_{jlk}, \tag{8}$$

satisfies the additional property that

$$R_{ijkl} = -R_{jikl}. \tag{9}$$

The following theorem provides simple criteria for linearization of a system of ODEs of the form (1) by means of invertible transformations.

Theorem 1. *If, for a system of n second-order quadratically semi-linear ODEs for n dependent variables of the form (1), the Riemann tensor constructed from the* *coefficients of the quadratic terms treated as Christoffel symbols, is zero, then the resulting system of ODEs is linearizable via a point transformation of the dependent variables, and the admitted symmetry algebra is* $sl(n+2,\mathbb{R})$.

Proof: Treat the system of ODEs (1) as geodesic equations. From Theorem 1 of [4], since the space is flat it will have the corresponding isometries and hence the symmetry algebra of the system of geodesic equations is $sl(n+2,\mathbb{R})$. Note that the Γ^i_{jk} need not necessarily be consistently read as Christoffel symbols. However, the compatibility condition for them to be Christoffel symbols is simply that the Riemann tensor constructed from them be well defined and satisfy the conditions (4)–(9). When $R^i_{jkl}=0$ this is automatically guaranteed. Hence, the system is linearizable via a point transformation of the dependent variables. Note that geometrically the space is locally flat and the geodesics are locally straight lines, regardless of the choice of coordinates that may be curvilinear. Hence the system of ODEs is linearizable. Note, further, that if the manifold is simply connected the space is globally flat and the geodesics are globally straight in the manifold. □

Corollary . *If the coefficients are constant, then the linearizability condition reduces to*

$$\Gamma^i_{mk}\Gamma^m_{jl}=\Gamma^i_{ml}\Gamma^m_{jk}. \tag{10}$$

Note that these conditions are trivially satisfied when $k=l$ *but are nontrivial otherwise.*

To make the criterion more explicit, we write out the conditions in the theorem for a system of three geodesic equations in which $i, j, k = 1, 2, 3$ in (1). For this case the linearizabilty conditions are

$$\Gamma^i_{jl,k}-\Gamma^i_{jk,l}+\Gamma^i_{mk}\Gamma^m_{jl}-\Gamma^i_{ml}\Gamma^m_{jk}=0. \tag{11}$$

If we take $k=1, l=2$, then

$$\left(\Gamma^i_{j2}\right)_x-\left(\Gamma^i_{j1}\right)_y+\Gamma^i_{m1}\Gamma^m_{j2}-\Gamma^i_{m2}\Gamma^m_{j1}=0. \tag{12}$$

These give nine conditions. If we take $k=1, l=3$, then

$$\left(\Gamma^i_{j3}\right)_x-\left(\Gamma^i_{j1}\right)_z+\Gamma^i_{m1}\Gamma^m_{j3}-\Gamma^i_{m3}\Gamma^m_{j1}=0. \tag{13}$$

Here too nine conditions arise. If we take $k=2, l=3$, then

$$\left(\Gamma^i_{j3}\right)_y-\left(\Gamma^i_{j2}\right)_z+\Gamma^i_{m2}\Gamma^m_{j3}-\Gamma^i_{m3}\Gamma^m_{j2}=0 \tag{14}$$

which give nine conditions. Not all the conditions in (11), (12) and (13) are linearly independent. In fact, only nine of them are in this case. In general, there will be n^2 independent equations. A general (not limited to a system of three equations) computer algorithm to check these conditions is being developed [9].

3 System of two equations and flat space

The essential principle may be seen by considering a system of two geodesic equations for two functions of one variable

$$x''=a(x,y)x'^2+2b(x,y)x'y'+c(x,y)y'^2, \tag{15}$$

$$y''=d(x,y)x'^2+2e(x,y)x'y'+f(x,y)y'^2, \tag{16}$$

where we have used the prime instead of the dot usually used in geodesic equations to fit with the notation familiar in differential equations. We read off the Christoffel symbols as the negative of the coefficients of the quadratic terms. Thus,

$$\Gamma^1_{11}=-a,\quad \Gamma^1_{12}=-b,\quad \Gamma^1_{22}=-c,$$
$$\Gamma^2_{11}=-d,\quad \Gamma^2_{12}=-e,\quad \Gamma^2_{22}=-f. \tag{17}$$

For a known metric tensor, the Christoffel symbols are given by (2). If $R^i_{jkl}=0$, the equation is linearizable as the space is flat.

The general discussion of the system of two equations when $R^i_{jkl}\neq 0$ is given in [9]. Here we restrict our attention to the case when the space is flat. In Cartesian coordinates, we can take $g_{11}=g_{22}=1$, $g_{12}=0$. However, if we want the coordinate transformations from the coordinates that were used in the equation to the Cartesian coordinates, we need to solve the set of six equations which result from Theorem 1. These reduce to the four (2^2) equations

$$a_y-b_x+be-cd=0, \tag{18}$$

$$b_y-c_x+(ac-b^2)+(bf-ce)=0, \tag{19}$$

$$d_y-e_x-(ae-bd)-(df-e^2)=0, \tag{20}$$

$$(b+f)_x=(a+e)_y, \tag{21}$$

which together with (2) and (17) yield

$$p_x = -2(ap + dq), \tag{22}$$

$$q_x = -bp - (a+e)q - dr, \tag{23}$$

$$r_x = -2(bq + er), \tag{24}$$

$$p_y = -2(bp + eq), \tag{25}$$

$$q_y = -cp - (b+f)q - er, \tag{26}$$

$$r_y = -2(cq + fr). \tag{27}$$

Note that the compatibility of this set of six equations gives the above four linearization conditions (18)–(21). Also note that in the case of a system of three equations, we would get eighteen such equations out. In general, these would be $n^2(n+1)/2$, $n \geq 2$.

So far, we have not obtained the required coordinate transformation. We only know that the metric could be chosen to be the identity matrix in Cartesian coordinates, and in the given coordinates we have

$$g_{11} = p, \tag{28}$$

$$g_{12} = q = g_{21}, \tag{29}$$

$$g_{22} = r. \tag{30}$$

To obtain the Cartesian coordinates $u(x, y)$, $v(x, y)$ in which the metric is the identity matrix, use the tensor transformation laws for the covariant second rank tensor, $g_{ab}(\mathbf{x})$,

$$g_{ab}(\mathbf{x}) = \frac{\partial u^i}{\partial x^a}\frac{\partial u^j}{\partial x^b} g_{ij}(\mathbf{u}), \tag{31}$$

where $\mathbf{x} = (x^1, x^2) = (x, y)$, $\mathbf{u} = (u^1, u^2) = (u, v)$ and require that $g_{ij}(\mathbf{u})$ be the identity matrix. We must then solve the equations

$$u_x^2 + v_x^2 = p, \tag{32}$$

$$u_x u_y + v_x v_y = q, \tag{33}$$

$$u_y^2 + v_y^2 = r. \tag{34}$$

One can use the three equations to write u_x and u_y in terms of v_x and v_y. Finally, one can use the compatibility condition obtained by setting $u_{xy} = u_{yx}$ and (say) Equation (34) to evaluate v_x and v_y and hence u_x and u_y. This procedure leads to a highly nonlinear first-order ODE.

Another approach is as follows. Let

$$z = u + iv, \quad i = \sqrt{-1}. \tag{35}$$

Then, the complex conjugate is

$$\bar{z} = u - iv. \tag{36}$$

Consequently, the above set of Equations (32)–(34) reduces to

$$z_x \bar{z}_x = p, \tag{37}$$

$$z_x \bar{z}_y + \bar{z}_x z_y = 2q, \tag{38}$$

$$z_y \bar{z}_y = r. \tag{39}$$

Since $p \neq 0$ we can divide the last equation by the first and the second by the first to obtain an equivalent system, viz.

$$z_y - \alpha z_x = 0, \tag{40}$$

$$\bar{z}_y - \beta \bar{z}_x = 0, \tag{41}$$

$$z_x \bar{z}_x = p, \tag{42}$$

where

$$\alpha = \frac{q \pm \sqrt{q^2 - pr}}{p}, \tag{43}$$

$$\beta = \frac{r}{q \pm \sqrt{q^2 - pr}}. \tag{44}$$

From the above procedure made explicit for a two-dimensional system, it is clear that it can be generalized to an n-dimensional system. Hence, we have the following theorem.

Theorem 2. *A system of n ODEs of the form (1) is linearizable by point transformations if and only if the curvature tensor (4) formed by treating the coefficients in (1) as Christoffel symbols is zero. The point transformation is obtainable from (31) with $a, b, i, j = 1, \ldots, n$ and $g_{ij}(\mathbf{u})$ is the identity matrix.*

4 Examples

We now consider a few examples to illustrate our approach.

1. For $a = c = e = 1$, $b = d = f = 0$, the system of two geodesic equations are linearizable as (18)–(21) hold. Equations (22)–(27) then yield

$$p_x = -2p, \qquad p_y = -2q, \quad q_x = -2q,$$
$$q_y = -p - r, \quad r_x = -2r, \quad r_y = -2q.$$

   The solution of this system gives

$$p = r = (c_1 e^{2y-2x} + c_2 e^{-2y-2x}), \tag{45}$$
$$q = (c_2 e^{-2y-2x} - c_1 e^{2y-2x}), \tag{46}$$

   where c_1 and c_2 are positive constants. A solution of (40)–(42) then gives

$$z = \sqrt{\frac{c_1}{2}} e^{y-x} + \sqrt{\frac{c_2}{2}} e^{-y-x} + i\left(\sqrt{\frac{c_2}{2}} e^{-y-x} - \sqrt{\frac{c_1}{2}} e^{y-x}\right) \tag{47}$$

   A coordinate transformation that does the linearization is then finally given by

$$u = \sqrt{\frac{c_1}{2}} e^{y-x} + \sqrt{\frac{c_2}{2}} e^{-y-x}, \tag{48}$$
$$v = \sqrt{\frac{c_2}{2}} e^{-y-x} - \sqrt{\frac{c_1}{2}} e^{y-x}. \tag{49}$$

   The above example was also treated in [12] where the algebraic approach is utilized. Another linearizing transformation distinct from the one obtained here was given there.
2. The system of two geodesic equations which have $a = b = 0$, $c = x$, $d = f = 0$, $e = -1/x$ is linearizable since (18)–(21) are satisfied. Here the system has a singularity at $x = 0$. However, this is an apparent, or coordinate, singularity. Indeed, this is the system of geodesic equations in the Cartesian plane in polar coordinates.
3. The system which has $a = b = 0$, $c = \sin x \cos x$, $d = f = 0$, $e = -\cot x$ is not linearizable and does not admit $sl(4, \mathbb{R})$ algebra as the linearizability conditions (18) and (19) are not satisfied.
4. The system of three second-order ODEs

$$x'' + x'^2 = 0, \quad y'' + y'^2 = 0,$$
$$z'' + z'^2 = 0 \tag{50}$$

   is linearizable (the linearization conditions (12)–(14) are trivially satisfied) and a transformation that does the reduction to the simplest system is

$$u = e^x, \quad v = e^y, \quad w = e^z. \tag{51}$$

5. The following system of three second-order ODEs

$$x'' = \frac{x'^2}{x} + y'^2 \frac{xy + x}{y^2}, \tag{52}$$
$$y'' = -y'^2, \tag{53}$$
$$z'' = -z'^2 - 2y'z', \tag{54}$$

   has coordinate singularity at $x = y = 0$ but is linearizable as (12)–(14) are satisfied for the coefficient functions of the above system. A linearizing transformation is

$$u = \ln xy, \quad u = e^y, \quad w = e^{y+z}. \tag{55}$$

5 Summary and discussion

In this paper, we have demonstrated that the linearizability conditions by invertible transformations for second-order quadratically semi-linear ODEs can be very simply stated by treating the coefficients of the quadratic terms as Christoffel symbols and constructing the curvature tensor from them. As is intuitively obvious, if the corresponding space is flat the geodesics are simply straight lines and so the original system is linearizable, i.e., it has a symmetry algebra $sl(n + 2, \mathbb{R})$. As was seen in Examples 3 and 5, coordinate singularities do not interfere with the linearization criteria stated in the Theorem 1. The proof was given for a system that had no other terms than the second derivative and the quadratic terms.

The criteria presented are very simple. As an example the system of two equations was solved in full generality. The only problem that arose in the process, was the construction of the coordinate transformation that converts to the Cartesian coordinates, where the solution can be written down directly. This was achieved by using a complex transformation. The procedure was then applied to specific examples of systems of two and three equations. It is worth stressing that the procedure used is constructive and hence *yields the solution fully when the system is linearizable*!

The three dependent variable case criteria were also written down explicitly. Solving such a system in full generality does, indeed, become complicated to be done by hand. However, a code has been written [9] to achieve the implementation of checking the criteria for all cases. Here we have taken simple cases of three-dimensional systems and checked their linearizability explicitly. With the code [9] and the procedure for converting a general case to the required form, the n-dimensional system of linearizable equations can also be solved directly.

Since Aminova and Aminov [1] have discussed the case of geometric methods for symmetry analysis of geodesic equations, which is also the subject of this paper, it is necessary to point out the differences of approach and purpose of that work from ours. They are concerned with general theorems for the symmetries of the projective system of geodesic equations. Starting with (1) they use one of the variables as a local parameter to project to a system of $(n-1)$ cubically semilinear equations, with all terms in such an equation generally present. They also go on to consider the reduction of a system of two equations to a single, scalar, equation. Then they proceed to discuss the complete classification by symmetries of the two-dimensional system (1). However, they do not give invariant criteria for linearizability nor the construction procedure of the transformations that we provide. They do mention, in passing, the case of the flat space, but without discussing its use for providing the solution. We, on the other hand, are only interested in the linearizability of the system, and the explicit procedure for constructing solutions (Theorem 2) and only mention the symmetry classification in passing.

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