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ORIGINAL ARTICLE

Parametric identification of nonlinear systems using multiple trials

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Abstract It is observed that the harmonic balance (HB) method of parametric identification of nonlinear system may not give right identification results for a single test data. A multiple-trial HB scheme is suggested to obtain improved results in the identification, compared with a single sample test. Several independent tests are conducted by subjecting the system to a range of harmonic excitations. The individual data sets are combined to obtain the matrix for inversion. This leads to the mean square error minimization of the entire set of periodic orbits. It is shown that the combination of independent test data gives correct results even in the case where the individual data sets give wrong results.

Keywords Harmonic balance · Method of least squares · Multiple trials · Nonlinear system identification

Abbreviations

HB	Harmonic balance
MDOF	Multidegree of freedom
DFT	Discrete Fourier transform
FFT	Fast Fourier transform

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Nomenclature

m	Mass
c	Coefficient of damping
k	Linear spring stiffness
α	Coefficient of cubic stiffness
F	Amplitude of harmonic excitation
Ω	Frequency of excitation
t	Time
$x(t)$	Displacement response
$\Delta x(t)$	Periodic perturbation in $x(t)$
T	Period of excitation/response
M	Number of harmonics in the response
$[a_0\ a_1\ b_1\ \cdot\ \cdot\ a_M\ b_M]$	Fourier coefficients for the response
$p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$	Time series of $\ddot{x}(t)$, $\dot{x}(t)$, $x(t)$ and $x^3(t)$
$p_5(t)$	Time series of excitation
N	Number of discrete time samples in a period
Δt	Sampling time interval
$[G]$	Matrix containing the discrete time values of $p_1(t)$, $p_2(t)$, etc. as columns

$[G]^+$	Psuedo-inverse of $[G]$ matrix
$[G]_j$	The G matrix for the jth trial, also used for the jth subset selected in the total data set
$[D]$	$[G]^T[G]$
$[\Delta G]$	Noise perturbed $[G]$
$[\Delta G]_{\mathrm{p}}$	Periodic perturbation in $[G]$
$\{r\}$	Actual parameter set
$\{r\}_i$	Identified parameter set
$\{r\}_{\mathrm{t}}$	The $\{r\}_i$ using the total data set
$\{r\}_{\mathrm{p}}$	The $\{r\}_i$ when there is periodic perturbation
$\{r\}_{pt}$	The $\{r\}_{\mathrm{p}}$ with the total set
$\{f\}$	Discrete time series of external force
n_{p}	Number of parameters
$m_i,\ c_i,\ k_i,\ \alpha_i$	Identified parameters
$m_{\mathrm{e}},\ c_{\mathrm{e}},\ k_{\mathrm{e}},\ \alpha_{\mathrm{e}}$	Normalized error in the parameters
n	Number of independent trials
$[H]$	Assembled version of $[G]$
$[H]_{\mathrm{p}}$	Periodic perturbed $[H]$
$[\Delta H]_{\mathrm{p}}$	Periodic perturbation in $[H]$
$\{g\}$	Assembled version of $\{f\}$
$[h]$	The random subset of $[H]$
$\{q\}$	The random subset of $\{f\}$
ω_n	Undamped natural frequency
η	Frequency ratio, Ω/ω_n
E_{p}	Parametric error for the total set
E_{t}	Parametric error for the total set
$\{r\}_{\mathrm{t}}$	Identified parameter with the total data set
l_c	logarithm of condition number
l_{t}	logarithm of condition number for total set
I	Inertia force
I_i	Inertia force based on m_i
ε_n	Noise to signal ratio
ε_{p}	Periodic perturbation to signal ratio
$v(k)$	Histogram count for the kth parameter
σ	Standard deviation
μ	Mean
s_i	Number of terms in the polynomial type stiffness in the ith branch of MDOF system
k_i	ith stiffness coefficient in the polynomial-type stiffness nonlinearity
$[k_i]$	Coefficients (set) of polynomial-type stiffness in the ith branch of MDOF system
x_{ij}	Relative displacement between stations i and j
v_{ij}	Relative velocity between stations i and j
$[Q]_{ij}$	The $[G]$ like submatrices used in MDOF systems

Subscripts

c	Subsets
e	Error
i	Identified system
min	Minimum
max	Maximum
p	Periodic perturbation
pt	Periodic perturbation, total set
s	Selected
t	Total

Superscripts

$+$	Pseudo-inverse
n	Noise
T	Transpose

Overhead

$\sim$	Average
$\hat{}$	Direct mean value

1 Introduction

Although the harmonic balance (HB) method is widely used for the analysis of nonlinear systems [1], it is not as popular for parametric identification of nonlinear systems. In the conventional HB method of nonlinear system identification, a periodic force is used to excite the system and a periodic response is induced. The steady state response is resolved into its Fourier components and is substituted in the original equation to obtain an algebraic form of the differential equation. This set of algebraic equations is solved by a psuedo inversion to obtain the unknown parameters of the system. This inversion can lead to erroneous results, if an appropriate choice of the excitation parameters is not made. In other words, the method based on a single set of excitation parameters is not robust.

In one of the early classical papers on nonlinear system identification, Masri and Caughey [2] used the state variables of nonlinear systems to express the system characteristics in terms of orthogonal functions. Various works [3–7] have been done in the area of nonlinear vibratory system identification. Yasuda et al. [8] applied the principle of HB for the identification of nonlinear multidegree of freedom (MDOF) systems. They approximated the nonlinearity in the system with polynomials and the response of the system was approximated with a truncated Fourier series. However, most of these studies were limited to a single set of excitation parameters and the robustness of the technique for various excitation parameter sets were not examined.

The basic idea of the scheme is mentioned in a work of Yuan and Feeny [9]. However, any detailed study of the present type has not been reported. To overcome the limitations of single-trial experiments, a scheme for obtaining the system parameters by conducting a limited number of independent trials is proposed in this paper. These tests are conducted with different levels of harmonic excitations. The obtained data sets are assembled for the analysis, instead of being averaged to get the result. It is seen that this scheme overcomes the limitations of the conventional HB scheme. Also, this method will enable the experimenter to employ a relatively arbitrary set of excitation test signals for identification.

2 Parametric identification using harmonic balance method

A review of the parametric identification using HB method [10] is given in this section. Let us consider vibratory systems governed by nonlinear ordinary differential equations and subjected to harmonic force excitation. Further, one can assume that the response of the system is periodic. In the HB method, the periodic response of the system is expressed in terms of a truncated Fourier series with terms having frequencies, which are integer multiples/submultiples of the excitation frequency. The coefficients of the harmonic terms are obtained such that the resulting time series matches with the original one to the required degree of accuracy. The original response of the system, in general, cannot be represented in an analytical form, as the system is nonlinear. The HB solution thus obtained can be used for system analysis as well as system identification.

In the identification scheme, the HB solution is substituted into the equation of motion, which gives an algebraic equation in terms of system parameters. Imposing the condition that this equation should be satisfied at all sample points yields a system of linear algebraic equations. This is solved using a psuedo-inversion technique to obtain the unknown system parameters.

2.1 Identification using harmonic balance method

To illustrate the procedure, the harmonically forced Duffing oscillator is considered.

$$m\ddot{x} + c\dot{x} + kx + \alpha x^3 = F\cos\Omega t \tag{1}$$

where m, c, k, and α are the parameters of the system to be determined from the known input-response data. For identification purpose, the oscillator is assumed to be excited with known harmonic excitation parameters, F and Ω . Assume that the response $x(t)$ of a vibrating system is known and it is with a fundamental period $T{=}2\pi/\Omega$. Expressing the response in a truncated M-term harmonic Fourier series one has

$$x(t) = a_0 + \sum_{j=1}^{M} (a_j \cos\, j\Omega t + b_j \sin\, j\Omega t) \tag{2}$$

Substituting this periodic solution in Equation (1), one obtains

$$\begin{aligned} &-m\sum_{j=1}^{M} (a_j j^2\Omega^2 \cos\, j\Omega t + b_j j^2\Omega^2 \sin\, j\Omega t) \\ &+c\sum_{j=1}^{M} (-a_j j\Omega \sin\, j\Omega t + b_j j\Omega \cos\, j\Omega t) \\ &+k\left(a_0 + \sum_{j=1}^{M} (a_j \cos\, j\Omega t + b_j \sin\, j\Omega t)\right) \\ &+\alpha\left(a_0 + \sum_{j=1}^{M} (a_j \cos\, j\Omega t + b_j \sin\, j\Omega t)\right)^3 \\ &= F\cos\Omega t \end{aligned} \tag{3}$$

Equation (3) can be written compactly as

$$mp_1(t) + cp_2(t) + kp_3(t) + \alpha p_4(t) = p_5(t) \tag{4}$$

where,

$$p_1(t) = -\Omega^2 \sum_{j=1}^{M} j^2(a_j \cos\ j\Omega t + b_j \sin\ j\Omega t);$$

$$p_2(t) = \Omega \sum_{j=1}^{M} j(-a_j \sin\ j\Omega t + b_j \cos\ j\Omega t)$$

$$p_3(t) = a_0 + \sum_{j=1}^{M} (a_j \cos\ j\Omega t + b_j \sin\ j\Omega t);$$

$$p_4(t) = \left(a_0 + \sum_{j=1}^{M} (a_j \cos\ j\Omega t + b_j \sin\ j\Omega t)\right)^3 \tag{5}$$

$$p_5(t) = F \cos\ \Omega t$$

For a set of N discrete time samples in one excitation time period T, the matrix form of the above equation is

$$\begin{bmatrix} p_1(0) & p_2(0) & p_3(0) & p_4(0) \\ p_1(\Delta t) & p_2(\Delta T) & p_3(\Delta T) & p_4(\Delta T) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p_1((N-1)\Delta t) & p_2((N-1)\Delta t) & p_3((N-1)\Delta t) & p_4((N-1)\Delta t) \end{bmatrix} \times \begin{Bmatrix} m \\ c \\ k \\ \alpha \end{Bmatrix} = \begin{Bmatrix} p_5(0) \\ p_5(\Delta t) \\ \cdot \\ \cdot \\ p_5((N-1)\Delta t) \end{Bmatrix} \tag{6}$$

Equation (6) can be written compactly as

$$[G]\{r\} = \{f\} \tag{7}$$

where $\{r\}$ is the actual system parameter set, $\{r\} = \{m, c, k, \alpha\}^{\mathrm{T}}$ where the superscript T represents transpose. The estimated value of $\{r\}$, denoted as $\{r\}_i$ is obtained as

$$\{r\}_i = [G]^{+}\{f\} = [D]^{-1}[G]^{\mathrm{T}}\{f\} \tag{8}$$

where $[G]^{+}$ is the pseudo-inverse of $[G]$ and

$$[D] = [G]^T[G] = \begin{bmatrix} p_1(0) & . & . & p_1((N-1)\Delta t) \\ p_2(0) & . & . & p_2((N-1)\Delta t) \\ p_3(0) & . & . & p_3((N-1)\Delta t) \\ p_4(0) & . & . & p_4((N-1)\Delta t) \end{bmatrix}$$

$$\begin{bmatrix} p_1(0) & p_2(0) & p_3(0) & p_4(0) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p_1((N-1)\Delta t) & p_2((N-1)\Delta t) & p_3((N-1)\Delta t) & p_4((N-1)\Delta t) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=0}^{N-1} (p_1(i\Delta t))^2 & \sum p_1(i\Delta t)p_2(i\Delta t) & \sum p_1(i\Delta t)p_3(i\Delta t) & \sum p_1(i\Delta t)p_4(i\Delta t) \\ \cdot & \sum (p_2(i\Delta t))^2 & \sum p_2(i\Delta t)p_3(i\Delta t) & \sum p_2(i\Delta t)p_4(i\Delta t) \\ \cdot & \cdot & \sum (p_3(i\Delta t))^2 & \sum p_3(i\Delta t)p_4(i\Delta t) \\ \cdot & \cdot & \cdot & \sum (p_4(i\Delta t))^2 \end{bmatrix} \tag{9}$$

Pseudo-inverse $[G]^{+}$ is the unique minimal two-norm solution [11] to the problem, $\min_{G \in N \times n_p} ||Gr - f||_2$ where n_{p} is the number of parameters.

The formulation similar to the above can be applied to other types of systems having smooth nonlinearities. Invertibility of the D matrix plays a crucial role in successful identification. While it is clear that the excitation parameters will influence the relative values of the elements of $[D]$, the appropriate choice of these parameters *a priori* is difficult.

2.2 Implementation

If the periodic response of system is given, an approximate solution for the system can be accomplished by performing a discrete Fourier transform (DFT) on the response data. The data points should be equally spaced in time. In practice, a fast Fourier transform (FFT) is used. The FFT coefficients, which are complex, may be converted into equivalent real-valued Fourier coefficients. Following this, the system identification by pseudo-inverse as mentioned above can be done.

Parametric identification of the Duffing oscillator illustrated in the previous section is carried out using the HB method. To generate data for the study, the periodic response of a known system to a harmonic excitation is obtained by numerical integration in MATLAB. The above data is transformed to the frequency domain, to obtain the Fourier series solution to the problem. Using the above input–output data again, the inverse form of the HB method is made use of to identify the parameters of the system. The result obtained for a case with single harmonic excitation is shown in Table 1 and Fig. 1. The number of sampling points N considered in this case is 128. Thus, the equally spaced sampling

Table 1 Comparison of original and identified parameters

Excitation parameters: $F = 2$, $\Omega = 0.4$
Original system parameters, $\{r\}^{\mathrm{T}} = [m, c, k, \alpha] = [1.0, 0.2, 1, 1]$
Corresponding identified parameters, $\{r\}_{\mathrm{i}}^{\mathrm{T}} = [1.0000, 0.1994, 0.9988, 1.0010]$

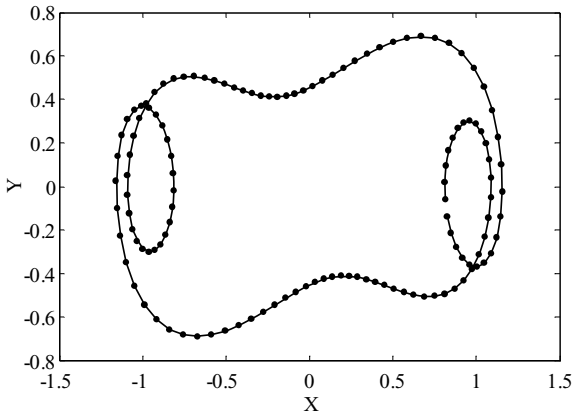


Fig. 1 Phase plane plot of the original and the identified system. Solid line is the one-period steady state response of the original system. The dot symbols stand for the corresponding identified system.

time interval is $\Delta t = T/N$ where T is the steady state period of the response. Relatively insignificant Fourier coefficient terms are not included in the analysis.

2.3 Direct determination of error in parameters

If the identification is carried out using the simulated data as mentioned earlier, the parameters of the original and the identified system are available for error estimation. Error measure based on difference in parameter values for a Duffing oscillator can be defined as

$$m_{\mathrm{e}} = (m - m_{\mathrm{i}})/m; \qquad c_{\mathrm{e}} = (c - c_{\mathrm{i}})/c;$$
$$k_{\mathrm{e}} = (k - k_{\mathrm{i}})/k; \qquad \alpha_{\mathrm{e}} = (\alpha - \alpha_{\mathrm{i}})/\alpha$$

where suffix i represents the identified parameters with m, c, k, and α being the mass, damping coefficient, linear stiffness, and nonlinear stiffness, respectively, and m_e, c_e, k_e, and α_e are the normalized errors in the identification of mass, damping, linear stiffness, and nonlinear stiffness, respectively. The total parameter error E_{p} is computed as

$$E_{\mathrm{p}} = \sqrt{\left(m_{\mathrm{e}}^2 + c_{\mathrm{e}}^2 + k_{\mathrm{e}}^2 + \alpha_{\mathrm{e}}^2\right)/n_{\mathrm{p}}} \tag{10}$$

where n_{p} is equal to four in this case. The aim of parametric identification is to find the parameters $\{r\}_{\mathrm{i}}$ such that $||\{r\} - \{r\}_{\mathrm{i}}||$ is a minimum. Equation (8) gives a least square error in the algebraic system of equations, which is the same as minimizing the mean square error in the net forces at all instants of time. Let $x_i(t)$ be the periodic orbit obtained by solving Equation (1) using $\{r\}_i$. The corresponding induced inertia force, damping force, linear spring force, and nonlinear spring force are $m_{\mathrm{i}}\ddot{x}_i$, $c_{\mathrm{i}}\dot{x}_i$, $k_{\mathrm{i}}x_i$, and $\alpha_{\mathrm{i}}x_i^3$, respectively. Using these quantities, another possible estimate of error can obtained based on the normalized forces such as $\frac{1}{T}\int_T \left(\frac{m\ddot{x}(t) - m_{\mathrm{i}}\ddot{x}_i(t)}{m\ddot{x}(t)}\right)^2 dt$. However, this error norm does not guarantee the minimization of E_{p}. Thus, in order to assess the quality of identification, the most direct estimate as in Equation (10) is suggested.

An interpretation can be given for Equation (10) in terms of errors in forces. Let the system with the identified parameters be forced through the original periodic trajectory. The corresponding induced force $\{f\}_{\mathrm{i}}$ will be different from the actual excitation force. However, the pseudo-inversion gives a least square error in $||\{f\} - \{f\}_{\mathrm{i}}||_2$. Let '$I$' denote actual inertia force and I_i the inertia force based on m_i computed on the original trajectory. It can be seen that the normalized error in the inertia force assuming the response is the same, is given by

$$\frac{1}{N}\sum_{j=1}^{N}\left(\frac{I(j) - I_i(j)}{I(j)}\right)^2$$
$$= \frac{1}{N}\sum_{j=1}^{N}\left(\frac{m\ddot{x}(j) - m_i\ddot{x}(j)}{m\ddot{x}(j)}\right)^2 = \left(\frac{m - m_i}{m}\right)^2 = m_{\mathrm{e}}^2 \tag{11}$$

which leads to the same definition as parametric error. The parametric error is a valid measure only if the assumed model is correct.

3 The multiple-trial scheme of identification

Consider a single degree of freedom (SDOF) system excited by a harmonic force where the steady

state response is periodic. The input–output data is collected and the experiment is repeated with different excitations to obtain force–displacement data sets $\{(f)_j, (x)_j\}, \quad j = 1, 2, \ldots, n$, generated from n independent trials of identification tests. Let N be the number of sample points in a set. Perform HB on each data set and the resulting algebraic equations can be written in matrix form as $[G]_j\{r\} = \{f\}_j$ for each of the n trials as in Equation (7). Each one of the data sets can be used to generate system parameters by the method of least squares/pseudo-inversion. It is possible that there are no consistent identification results from the above. This is due to improper excitation used in the identification. However, *a priori* knowledge of the right excitation to be given is often not available for an experimenter.

With an aim of getting the system parameters from the multiple data sets, one can assemble the entire data set by appending one by one and the total set is denoted by the equation

$$[H]\{r\} = \{g\} \tag{12}$$

where

$$[H] = \begin{bmatrix} [G]_1 \\ [G]_2 \\ . \\ . \\ [G]_n \end{bmatrix} \quad \text{and} \quad \{g\} = \begin{Bmatrix} \{f\}_1 \\ \{f\}_2 \\ . \\ . \\ \{f\}_n \end{Bmatrix}$$

This set has $T_\mathrm{p} = n\,N$ data points. Now one can carry out the identification with this total data set to obtain

$$\{r\}_\mathrm{t} = [H]^+\{g\} \tag{13}$$

where $\{r\}_\mathrm{t}$ is the identified parameter set using total data set. It is seen that in many cases there is remarkable improvement in the identification results, especially when the original data sets are not good enough. The above idea is illustrated next.

3.1 Illustration–Duffing oscillator

First, a Duffing oscillator with the following parameter values is considered as an example. The parameters chosen are $[m, c, k, \alpha\,] = [1, 0.02, 1, 0.02]$. The excitation is selected randomly from the range specified as follows.

Force F: $F_\mathrm{min} = 0.1$, $F_\mathrm{max} = 1$; $F = F_\mathrm{min} + \mathrm{rand}\,(1)\,(F_\mathrm{max} - F_\mathrm{min})$ where rand(.) is a random number with a uniform distribution over the interval [0, 1] and frequency ratio $\eta = (\Omega/\omega_n)$: $\eta_\mathrm{min} = 0.1$, $\eta_\mathrm{max} = 0.5$; $\eta = \eta_\mathrm{min} + \mathrm{rand}\,(1)\,(\eta_\mathrm{max} - \eta_\mathrm{min})$ where ω_n is the undamped natural frequency of the corresponding linear system. A total of five simulation trials are conducted. The result of the identification is given in Table 2. The overall results are extremely inaccurate. This is due to the very weak nonlinearity chosen for these cases as well as the low excitation levels. The damping estimate is correct in these cases as the corresponding column vector for $\dot{x}$ is nearly orthogonal to other columns of the $[G]$ matrix. Since the parametric error is unknown to the experimenter, the correctness of the results is to be inferred from the consistency of the identified parameter values. Nearly identical parameter values obtained in say, three independent trials may be taken as the correct result. Such a case does not occur here. An orbital error check can be made based on these parameters to get a confidence in the results. This is explained at the end of this section. A question is whether the above data set is to be discarded to obtain a new set of data. A simple solution would be to continue with extra trials until the nearly repeated results are achieved. Another question, which needs to be posed, is that, given such a data set is there a solution hidden in the combination

Table 2 Duffing oscillator: Result of identification based on individual tests

				Identified parameters					
Trial	F	Ω	η	m	c	k	α	E_p	l_c
1	0.9850	0.3946	0.3946	0.8596	0.0200	0.9872	0.0110	0.2361	13.0192
2	0.5755	0.4583	0.4583	0.8747	0.0201	0.9756	0.0155	0.1284	14.3657
3	0.9533	0.2471	0.2471	1.4778	0.0200	1.0367	0.0099	0.3477	17.3513
4	0.5919	0.3558	0.3558	0.9640	0.0200	0.9973	0.0148	0.1315	11.2770
5	0.7105	0.4449	0.4449	1.2139	0.0199	1.0371	0.0294	0.2598	15.7134

of these data? The answer to this question is attempted using the examples as discussed later.

With reference to the Duffing oscillator, the identification is carried out with the full set and the parametric error E_t based on $\{r\}_t$ is obtained. The identified parameters using the total data set is, $\{r\}_t^T = [m \;\; c \;\; k\,\alpha]_t = [0.9987, \;\; 0.0200, \;\; 0.9999, \;\; 0.0201]$ and $E_t = 0.0019$. Accurate identification is obtained in this example even though the excitation levels are low and the nonlinearity weak; the original data sets separately gave extremely wrong results.

Examining the possible reason for the improvement of results, consider the condition number of the two cases namely for the original data sets and the total set. The condition number is the ratio of the largest singular value to the smallest singular value of a matrix. The last column of Table 2 gives the logarithm of the condition number for the D matrix denoted as l_c. The corresponding condition number for the total data set is 6.2341.There is a substantial reduction in the condition number for the total set compared with the original ones. The mixing of data from different sets improves the linear independence of the total collection of data. Since the trials are independent, the total data set shows a better quality of inversion apart from containing maximum information. Kundert et al. [12] have developed a time-point selection algorithm using near orthogonal selection of the data set for the analysis of almost periodic circuits. However, such a procedure is not considered in the case of the system identification.

In the previous case, the excitations were chosen to be poor to demonstrate the method. If the original data sets are better individually, then also this scheme works, and gives better results. Thus, effort should be made to conduct trials with an aim to collect best data sets possible.

The algorithm for multiple-trial nonlinear system identification can be summarized as

1. Conduct identification tests to collect input–output data for a number of independent tests say, five, by changing the excitation over a wide range of force amplitude and frequency. Only periodic response is to be considered. Care should be taken to collect the best possible data.
2. Carry out HB on the above data sets and obtain the necessary matrices for identification.
3. Use the total assembled data set in HB form to identify the parameters.
4. Conduct an orbital check as explained later.

One can do an orbital error check on each of the test orbits to ensure that no test orbits deviate much from the original ones. For this, a plot of all the original test trajectories and the trajectories generated from the identified parameters, $\{r\}_t$ are given for visual comparison in Fig. 2. The trajectories of the identified system are obtained by solving the equations of motion with an initial condition of the original trajectory. An rms criterion may also be used to quantify this. A force-based check can also be done as shown in Fig. 3.

If the system is known to operate within the force excitation ranges conducted in the trials, this closeness in the orbits will suffice and in such cases, the orbit sketch can be used to check the results. The success of the method depends on the quality of the data sets and there may be cases when the scheme does not yield a good result. However, in such cases, the checks based on orbital error as mentioned above guards against picking up a wrong answer. The number of tests in this study is arbitrarily fixed as five and can be lower or higher. Even a combination of two independent trials may give excellent results.

The excitation parameters for trial tests may be generated from a uniform grid of points in the F–η plane. A randomly generated data within the same range used in this study is found to give better results. Due to random generation of data, the excitation frequencies of each trial are different and thus the fundamental period of the response is different. Thus, the Fourier expansion of the respective responses is in terms of different frequency sets for each trial. This creates more linear independency among the total test data.

3.2 Identification with noise

Now the identification is done when there is noise in the signal. For this purpose, a uniformly distributed noise with a specified noise-to-signal ratio, ε_n and a zero mean value is added to the simulated response signal. Since the addition of noise in the displacement amplifies the velocity and acceleration signals, the noise is added to the acceleration signal and the resulting velocity and displacement signals are obtained by numerical integration. The corresponding noise levels will decrease progressively from acceleration to displacement. The procedure used for generating the noise is described in Appendix.

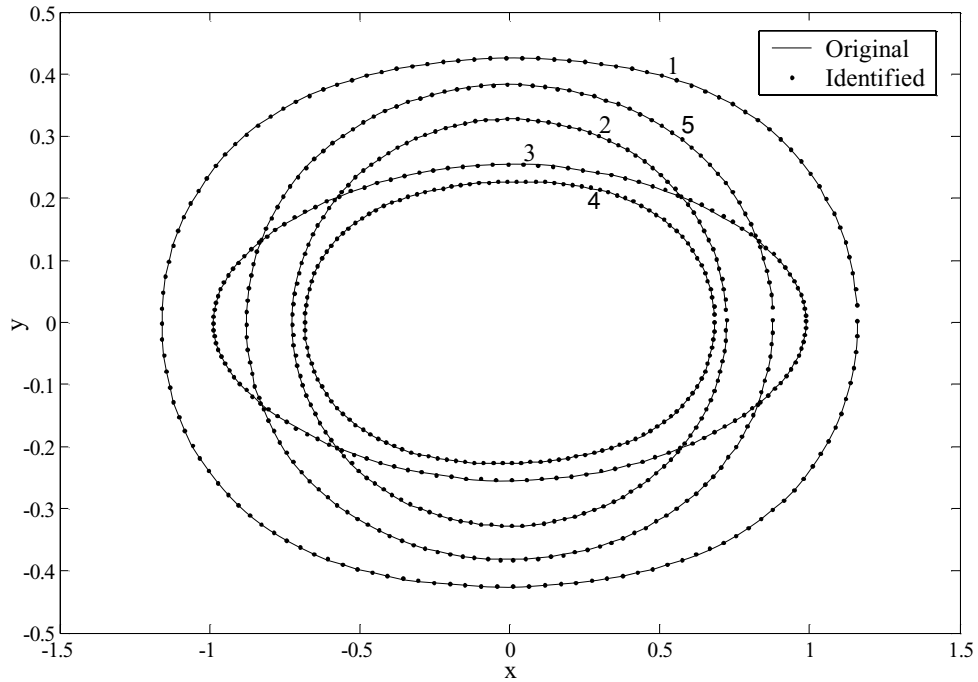


Fig. 2 State space orbits obtained from the original and the identified parameter from the total set. The trajectory number indicated corresponds to row entries in Table 2

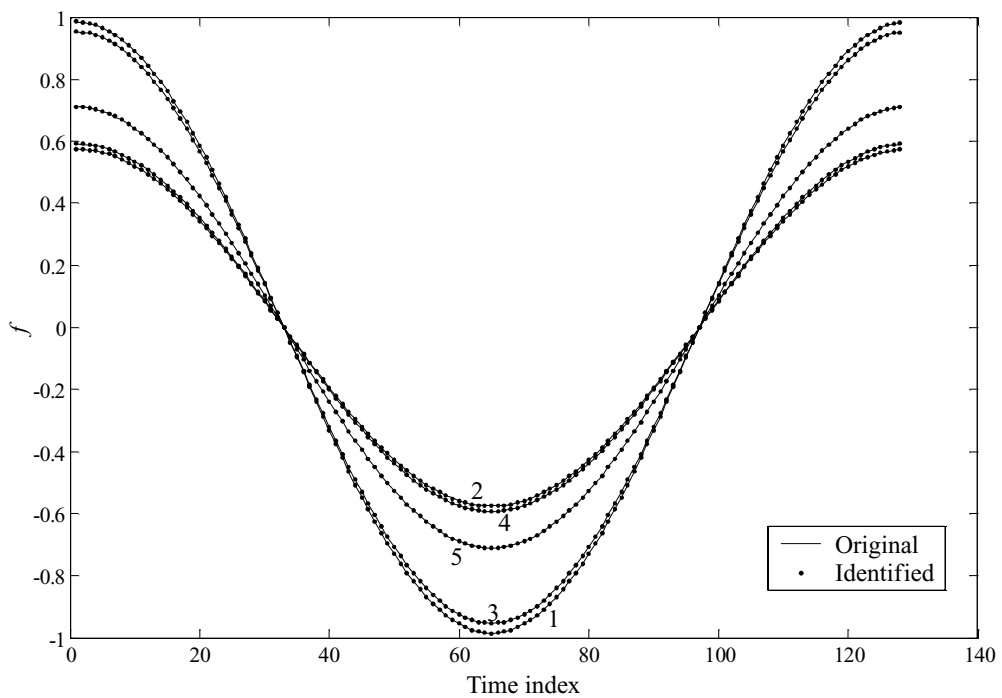


Fig. 3 Original excitation force and the estimated induced force based on the identified parameters from total set, $\{f\}_{it} = [G]\{r\}_{\mathrm{t}}$

Table 3 Duffing oscillator: comparison of results with and without noise

	(a) Individual sets				(b) Total set[a]		
	E_p	E_p^n	l_c	l_c^n	$\{r\}$	$\{r\}_\mathrm{t}$	v
1	0.2361	0.7905	13.0192	12.5991	1.0000	0.9987	1.0087
2	0.1284	1.0332	14.3657	12.4591	0.0200	0.0200	0.0208
3	0.3477	1.2146	17.3513	15.0934	1.0000	0.9999	1.0029
4	0.1315	0.6810	11.2770	11.2283	0.0200	0.0201	0.0190
5	0.2598	1.2216	15.7134	12.7870			

[a] $E_\mathrm{t} = 0.0019$, $E_\mathrm{t}^\mathrm{n} = 0.0313$; $l_\mathrm{t} = 6.2341$, $l_\mathrm{t}^\mathrm{n} = 6.2340$

Consider the data set as given in Table 2. A uniform random distributed noise with a noise-to-signal ratio, $\varepsilon_n = 0.01$ (1%) is added to the acceleration signal. The corresponding ε_n values for $\dot{x}$, x, and x^3 is found to be 0.0024, 0.0027, and 0.0059, respectively for the first set of data. Now the identification is done based on the noisy $[G]$ matrix. Table 3a shows the results without and with noise. The results with the noise are denoted with a superscript 'n.' This is repeated by adding noise with the same ε_n to other trial sets and the order of magnitude of noise-to-signal ratio obtained in other terms are the same as for the first set. The parametric error increases considerably with noise for individual sets. This is because the individual $[G]$ matrices are ill conditioned. The identification based on the combined noisy signal is found and the result is shown in Table 3b. The parametric error with the total set E_t^n is about 0.03, which is small, but acceptable.

3.2.1 Parameter averaging

In the previous section, only a single noise signal was used. One can consider an ensemble of such signals and keeping the ε_n the same, obtaining the E_t^n for each case. A plot of E_t^n with respect to the set number is shown in Fig. 4. Let J be the ensemble size. The mean value of E_t^n is obtained as $\hat{E}_\mathrm{t}^\mathrm{n} = \frac{1}{J}\sum_{j=1}^{J} E_{\mathrm{t}j}^\mathrm{n}$ and is equal to 0.0279 and the standard deviation of the set of E_t^n is 0.0138. An estimate of the parameters is done based on the average values of the parameters in these tests. Defining the average values of parameters, where the overhead $\sim$ represents the average values, we have

$$\tilde{m} = \frac{1}{J}\sum_{j=1}^{J} m_j^\mathrm{n}, \quad \tilde{c} = \frac{1}{J}\sum_{j=1}^{J} c_j^\mathrm{n},$$

$$\tilde{k} = \frac{1}{J}\sum_{j=1}^{J} k_j^\mathrm{n}, \quad \tilde{\alpha} = \frac{1}{J}\sum_{j=1}^{J} \alpha_j^\mathrm{n}$$

$$\tilde{m}_\mathrm{e} = \left(\frac{\tilde{m}-m}{m}\right), \quad \tilde{c}_\mathrm{e} = \left(\frac{\tilde{c}-c}{c}\right),$$

$$\tilde{k}_\mathrm{e} = \left(\frac{\tilde{k}-k}{k}\right), \quad \tilde{\alpha}_\mathrm{e} = \left(\frac{\tilde{\alpha}-\alpha}{\alpha}\right)$$

and

$$\tilde{E}^\mathrm{n} = \sqrt{\left(\tilde{m}_\mathrm{e}^2 + \tilde{c}_\mathrm{e}^2 + \tilde{k}_\mathrm{e}^2 + \tilde{\alpha}_\mathrm{e}^2\right)/n_\mathrm{p}} \tag{16}$$

For $J = 25$ tests, parameters values of $\tilde{m}$, $\tilde{c}$, $\tilde{k}$, and $\tilde{\alpha}$ are, respectively, equal to 0.9972, 0.0203, 1.0001, and 0.0201. The corresponding parametric error, based on the average values of these parameters is, $\tilde{E}^\mathrm{n} = 0.0074$. Thus, it can be observed that $\tilde{E}^\mathrm{n} < \hat{E}_\mathrm{t}^\mathrm{n}$. Moreover the $\tilde{E}^\mathrm{n}$is close to E_t. Thus, a better estimate of the parameters can be made by this parameter averaging process. In an experimental situation, data from a sequence of several periodic orbit sets may be collected for each trial to obtain the data for parameter averaging. Instead of this, one could have averaged the noisy signal and then carried out the parameter identification.

4 System with quadratic damping

Consider a system with quadratic damping

$$m\ddot{x} + c_1\dot{x} + c_2|\dot{x}|\dot{x} + kx = f(t) \tag{17}$$

The actual system parameters are $m = 1$, $c_1 = 0.02$, $c_2 = 0.005$, and $k = 1$. The excitation parameters are chosen in the range $F_\mathrm{min} = 5$, $F_\mathrm{max} = 10$; $\eta_\mathrm{min} = 0.1$, $\eta_\mathrm{max} = 0.5$. The system parameters identified using individual and total data sets are given in Tables 4 and 5, respectively. For the quadratic damping case also, there is significant improvement by considering the total set for identification than by identification using the individual set.

Table 4 Quadratic damping: Identification results based on individual tests

				Identified parameters					
Trial	F	Ω	η	m	c_1	c_2	k	E_p	l_c
1	8.9597	0.3462	0.3462	0.9865	0.0231	0.0039	0.9984	0.1320	10.9945
2	8.6910	0.4687	0.4687	0.4909	0.0403	0.0004	0.8882	0.7330	16.5512
3	7.0285	0.1705	0.1705	9.9123	0.0252	0.0000	1.2591	4.4879	27.2577
4	9.5845	0.4742	0.4742	0.4907	0.0424	0.0005	0.8855	0.7631	16.5660
5	9.4682	0.2641	0.2641	1.5366	0.0293	0.0009	1.0375	0.5393	17.5227

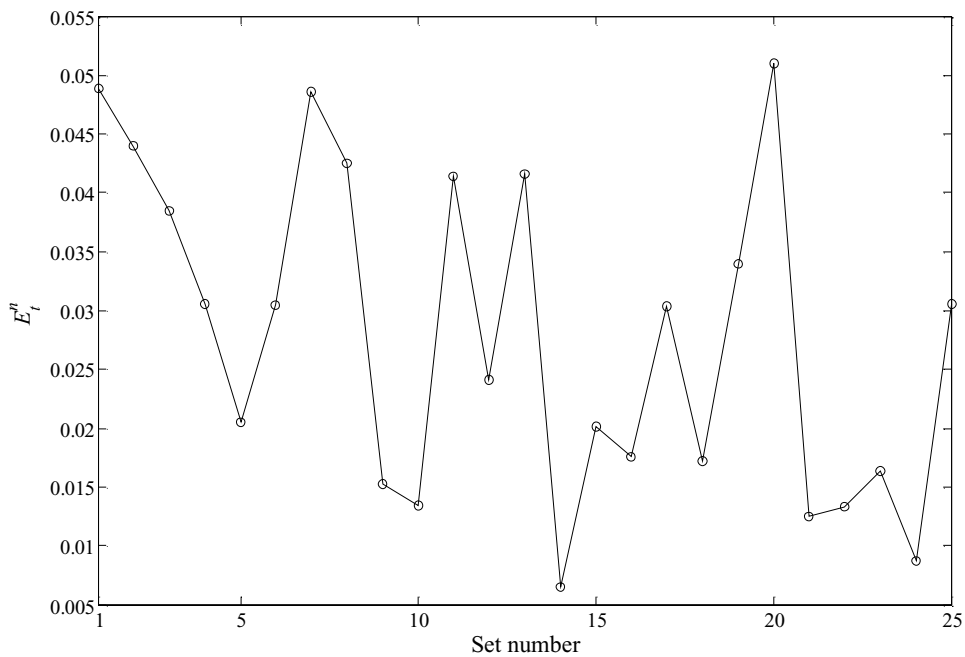


Fig. 4 Duffing oscillator. Variation of the parametric error for different trials of random noise.

Table 5 Quadratic damping: results with total data

$\{r\}_\mathrm{t}\ m = 0.9999,\ c_1 = 0.0205,\ c_2 = 0.0049\ ,\ k = 1.0000$
$E_\mathrm{t}\ 0.0177 \quad l_\mathrm{t} = 6.3506$

4.1 Identification with noise

The identification with noise is done with the same procedure as explained in Section 3.1. A uniform random distributed noise with a noise-to-signal ratio, $\varepsilon_n = 0.0099$ is added to the acceleration signal. The corresponding ε_n values for $\dot{x}$, $x|\dot{x}|$, and x are found to be 0.0019, 0.0032, and 0.0017, respectively for the first set of data for trial 1 in Table 4. Table 6a shows the results with and without noise. The identification based on the combined noisy signal is found and the result is shown in Table 6b. The parametric error with the total set E_t^n is about 0.044, which is small.

4.1.1 Parameter averaging

Now the parameter averaging is once again done with noise added. A plot of E_t^n with respect to the set number is shown in Fig. 5. Mean value $\hat{E}_\mathrm{t}^\mathrm{n}$ is 0.0467 and the standard deviation of E_t^n is 0.0362. Mean values of the parameters based on these 25 tests for $\tilde{m}$, $\tilde{c}_1$, $\tilde{c}_2$, and $\tilde{k}$ are, respectively, equal to 0.9984, 0.0201, 0.0049, and 1.0002. The corresponding parametric error, based on parameter average is, $\tilde{E}^\mathrm{n} = 0.0103$. Thus, it can be observed that $\tilde{E}^\mathrm{n} < \hat{E}_\mathrm{t}^\mathrm{n}$. Moreover, in this case, $\tilde{E}^\mathrm{n}$ is even less than E_t , which appears to be paradoxical. Thus, in this case also a better

Table 6 Qudratic damping: Comparison of results with and without noise

	(a) Individual sets				(b) Total set[a]		
Trial	E_{p}	$E_{\mathrm{p}}^{\mathrm{n}}$	l_c	$l_{\mathrm{c}}^{\mathrm{n}}$	$\{r\}$	$\{r\}_{\mathrm{t}}$	$\{r\}_{\mathrm{t}}^{\mathrm{n}}$
1	0.1320	0.4464	10.9945	10.9411	1.0000	0.9999	1.0012
2	0.7330	1.4354	16.5512	13.7537	0.0200	0.0205	0.0185
3	4.4879	1.2166	27.2577	16.2942	0.0050	0.0049	0.0052
4	0.7631	1.4170	16.5660	14.0147	1.0000	1.0000	1.0007
5	0.5393	0.6871	17.5227	14.4778			

$E^{\mathrm{a}}{}_{\mathrm{t}} = 0.0177$, $E_{\mathrm{t}}^{\mathrm{n}} = 0.0440$; $l_{\mathrm{t}} = 6.3506$, $l_{\mathrm{t}}^{\mathrm{n}} = 6.3499$

estimate of the parameters is obtained by the parameter averaging method.

5 Multidegree of freedom (MDOF) system

A 3-DOF system shown in Fig. 6 is identified using the multiple-trial method. The basic formulation as given in Section 2 is extended to this case and the details are given below. Referring to the figure $[k_1] = [k_{11}\ k_{12}\ \ldots\ k_{1s_1}]$, $[k_2] = [k_{21}\ k_{22}\ \ldots\ k_{2s_2}]$, and $[k_3] = [k_{31}\ k_{32}\ \ldots\ k_{3s_3}]$ are the coefficients of the polynomial type stiffness, and s_1, s_2, and s_3 are the respective number of polynomial terms in the nonlinear stiffness.

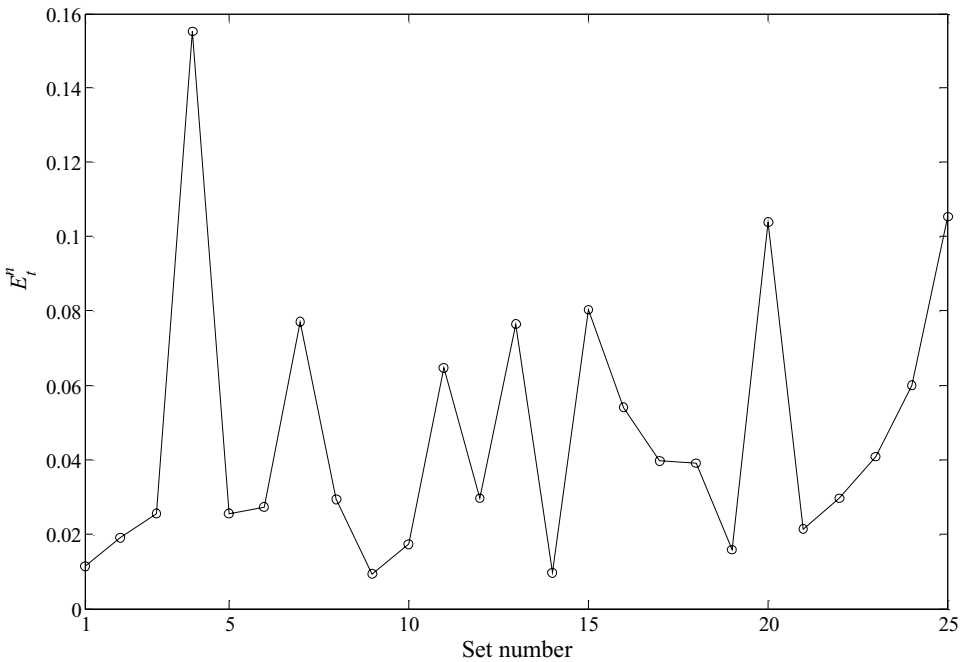
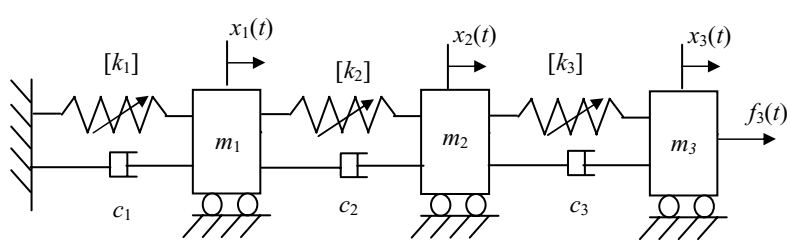


Fig. 5 Oscillator with quadratic damping . Variation of the parametric error for different trials of random noise

Fig. 6 MDOF system



The equations of motion for the case where c_3 is quadratic damping are given by

$$m_1\ddot{x}_1 + \sum_{j=1}^{s_1} k_{1j}x_1^j + c_1\dot{x}_1 - \sum_{j=1}^{s_2} k_{2j}(x_2 - x_1)^j - c_2(\dot{x}_2 - \dot{x}_1) = 0 \tag{18a}$$

$$m_2\ddot{x}_2 + \sum_{j=1}^{s_2} k_{2j}(x_2 - x_1)^j + c_2(\dot{x}_2 - \dot{x}_1) - \sum_{j=1}^{s_3} k_{3j}(x_3 - x_2)^j - c_3(\dot{x}_3 - \dot{x}_2)|(\dot{x}_3 - \dot{x}_2)| = 0 \tag{18b}$$

$$m_3\ddot{x}_3 + \sum_{j=1}^{s_3} k_{3j}(x_3 - x_2)^j + c_3(\dot{x}_3 - \dot{x}_2)|(\dot{x}_3 - \dot{x}_2)| = f_3(t) = F_3\cos(\Omega t) \tag{18c}$$

where $f_3(t)$ is the external excitation acting on mass 3 which is assumed harmonic. The system of equations is numerically solved. The steady state response, which is periodic, is taken for analysis. The FFT of the time series x_1, x_2, and x_3 is carried out. Let Ω be the fundamental frequency of the total six-dimensional orbit. The displacement and velocities in terms of real Fourier series can be expressed as

$$\begin{aligned}
x_1(t) &= a_{01} + \sum_{j_1=1}^{M_1}\left(a_{j_1}\cos j_1\Omega t + b_{j_1}\sin j_1\Omega t\right)\\
x_2(t) &= a_{02} + \sum_{j_2=1}^{M_2}\left(a_{j_2}\cos j_2\Omega t + b_{j_2}\sin j_2\Omega t\right)\\
x_3(t) &= a_{03} + \sum_{j_3=1}^{M_3}\left(a_{j_3}\cos j_3\Omega t + b_{j_3}\sin j_3\Omega t\right)
\end{aligned} \tag{19a}$$

$$\begin{aligned}
v_1(t) &= \dot{x}_1(t) = \Omega\sum_{j_1=1}^{M_1}\left(-a_{j_1}j_1\sin j_1\Omega t + b_{j_1}j_1\cos j_1\Omega t\right)\\
v_2(t) &= \dot{x}_2(t) = \Omega\sum_{j_2=1}^{M_2}\left(-a_{j_2}j_2\sin j_2\Omega t + b_{j_2}j_2\cos j_2\Omega t\right)\\
v_3(t) &= \dot{x}_3(t) = \Omega\sum_{j_3=1}^{M_3}\left(-a_{j_3}j_3\sin j_3\Omega t + b_{j_3}j_3\cos j_3\Omega t\right)
\end{aligned} \tag{19b}$$

The accelerations are given by,

$$\begin{aligned}
a_1(t) &= \ddot{x}_1(t) = -\Omega^2\sum_{j_1=1}^{M_1}\left(a_{j_1}j_1^2\cos j_1\Omega t + b_{j_1}j_1^2\sin j_1\Omega t\right)\\
a_2(t) &= \ddot{x}_2(t) = -\Omega^2\sum_{j_2=1}^{M_2}\left(a_{j_2}j_2^2\cos j_2\Omega t + b_{j_2}j_2^2\sin j_2\Omega t\right)\\
a_3(t) &= \ddot{x}_3(t) = -\Omega^2\sum_{j_3=1}^{M_3}\left(a_{j_3}j_3^2\cos j_3\Omega t + b_{j_3}j_3^2\sin j_3\Omega t\right)
\end{aligned} \tag{19c}$$

Let

$$\begin{aligned}
x_{21}(t) &= x_2(t) - x_1(t); \quad x_{32}(t) = x_3(t) - x_2(t)\\
v_{21}(t) &= v_2(t) - v_1(t); \quad v_{32}(t) = v_3(t) - v_2(t)
\end{aligned} \tag{20}$$

The equations of motion take the form

$$m_1a_1(t) + c_1v_1(t) + \sum_{i=1}^{s_1} k_{1i}(x_1(t))^{\mathrm{i}} - \sum_{i=1}^{s_2} k_{2i}(x_{21}(t))^{\mathrm{i}} - c_2v_{21}(t) = 0 \tag{21a}$$

$$m_2a_2(t) + \sum_{i=1}^{s_2} k_{2i}(x_{21}(t))^{\mathrm{i}} + c_2v_{21}(t) - \sum_{i=1}^{s_3} k_{3i}(x_{32}(t))^{\mathrm{i}} - c_3v_{32}(t)|v_{32}(t)| = 0 \tag{21b}$$

$$m_3a_3(t) + \sum_{i=1}^{s_3} k_{3i}(x_{32}(t))^{i} + c_3v_{32}(t)|v_{32}(t)| = f_3(t) \tag{21c}$$

The Equations (21) can be written in matrix form as

$$\begin{bmatrix} [Q]_{11} & [Q]_{12} & [0] \\ [0] & [Q]_{22} & [Q]_{23} \\ [0] & [0] & [Q]_{33} \end{bmatrix} \begin{Bmatrix} \{r\}_1 \\ \{r\}_2 \\ \{r\}_3 \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{F_{e3}\} \end{Bmatrix} \tag{22}$$

where

$$[Q]_{11} = \begin{bmatrix} a_1(1) & v_1(1) & x_1(1) & (x_1(1))^2 & (x_1(1))^3 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_1(N) & v_1(N) & x_1(N) & (x_1(N))^2 & (x_1(N))^3 \end{bmatrix}; \quad (23a)$$

$$[Q]_{12} = \begin{bmatrix} 0 & -v_{21}(1) & -x_{21}(1) & -(x_{21}(1))^2 & -(x_{21}(1))^3 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & -v_{21}(N) & -x_{21}(N) & -(x_{21}(N))^2 & -(x_{21}(N))^3 \end{bmatrix} \quad (23b)$$

$$[Q]_{22} = \begin{bmatrix} a_2(1) & v_{21}(1) & x_{21}(1) & (x_{21}(1))^2 & (x_{21}(1))^3 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_2(N) & v_{21}(N) & x_{21}(N) & (x_{21}(N))^2 & (x_{21}(N))^3 \end{bmatrix} \quad (23c)$$

$$[Q]_{23} = \begin{bmatrix} 0 & -v_{32}(1)|v_{32}(1)| & -x_{32}(1) & -(x_{32}(1))^2 & -(x_{32}(1))^3 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & -v_{32}(N)|v_{32}(N)| & -x_{32}(N) & -(x_{32}(N))^2 & -(x_{32}(N))^3 \end{bmatrix} \quad (23d)$$

$$[Q]_{33} = \begin{bmatrix} a_3(1) & v_{32}(1)|v_{32}(1))| & x_{32}(1) & (x_{32}(1))^2 & (x_{32}(1))^3 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_3(N) & v_{32}(N)|v_{32}(N)| & x_{32}(N) & (x_{32}(N))^2 & (x_{32}(N))^3 \end{bmatrix} \quad (23e)$$

The external force on mass 3 is $\{f_3\} = [f_3(1) \cdots f_3(N)]^{\mathrm{T}}$. The three sets of parameters are

$$\{r\}_1 = \begin{bmatrix} m_1 & c_1 & k_{11} & . & . & k_{1s_1} \end{bmatrix}^{\mathrm{T}},$$

$$\{r\}_2 = \begin{bmatrix} m_2 & c_2 & k_{21} & . & . & k_{2s_2} \end{bmatrix}^{\mathrm{T}},$$

and $$\{r\}_3 = \begin{bmatrix} m_3 & c_3 & k_{31} & . & . & k_{3s_3} \end{bmatrix}^{\mathrm{T}}$$

The parameters are identified in the following sequence as

$$[Q]_{33}\{r\}_3 = \{f_3\} \Rightarrow \{r\}_3 = [Q]_{33}^{+}\{f_3\} \quad (24a)$$

$$[Q]_{22}\{r\}_2 + [Q]_{23}\{r\}_3 = \{0\} \Rightarrow \{r\}_2 = -[Q]_{22}^{+}[Q]_{23}\{r\}_3 \quad (24b)$$

$$[Q]_{11}\{r\}_1 + [Q]_{12}\{r\}_2 = \{0\} \Rightarrow \{r\}_1 = -[Q]_{11}^{+}[Q]_{12}\{r\}_2 \quad (24c)$$

The system parameters are given in Table 7. The harmonic excitation is given on the third mass. The undamped natural frequencies of the system are $\omega_{n1} = 26.2$ rad/s, $\omega_{n2} = 118.1$ rad/s, and $\omega_{n3} = 144.4$ rad/s, respectively. The excitation parameter range $F_{\min} = 1000$, $F_{\max} = 2000$, $\eta_{\min} = 0.2$, $\eta_{\max} = 0.5$ where $\eta = \Omega/\omega_{n1}$. Thus, the excitation is below the first natural frequency. The results of the individual trials are given in Tables 8 where l_{c1}, l_{c2}, and l_{c3} are the logarithm of condition number of inversion of $([Q]_{11}^{\mathrm{T}}[Q]_{11})$, etc.

The identification results are poor in all the trials as indicated by the parametric error. Now the identification is made with the total data set. For this, the assembled version of the $[Q]_{33}$, $[Q]_{22}$, and $[Q]_{11}$ is used as indicated in Equation (22). The result is shown in Table 9. This result has an acceptable accuracy.

The condition numbers are compared with the two cases (i) original sets and (ii) the set corresponding to Table 9. There is decrease in the condition number for the total set. Trajectory plots based on the actual and identified parameters, $\{r\}_{\mathrm{t}}$ are shown in Fig. 7.

6 Identification with additional noncontributing terms in the model

In the examples given so far, a correct model structure was assumed for the system. A more general case is where the actual model structure is not exactly known or cannot be assumed. In such cases, a model with potentially more number of terms is assumed and the identification is carried out. An illustration is given next. Let the actual system be a Duffing oscillator, governed by Equation (1). A multiple-trial test is conducted and the results are shown in Table 10.

6.1 The individual test results with larger model

The following model with fourth-order polynomial terms in stiffness is assumed.

$$m\ddot{x} + c\dot{x} + k_1 x + k_2 x^2 + k_3 x^3 + k_4 x^4 = F\cos\Omega t \quad (25)$$

Table 7 MDOF system: System parameters

Mass (kg)	Stiffness (N/m $\times 10^4$)	Damping (N s/m $\times 10^3$)
$m_1 = 1$	$k_1 = 1$ (Linear)	$c_1 = 0.04$ (Viscous)
$m_2 = 2$	$[k_2] = [0.5\ 0.1]$ (Duffing)	$c_2 = 0.03$ (Viscous)
$m_3 = 2.5$	$k_3 = 2$ (Linear)	$c_3 = 0.02$ (Quadratic)

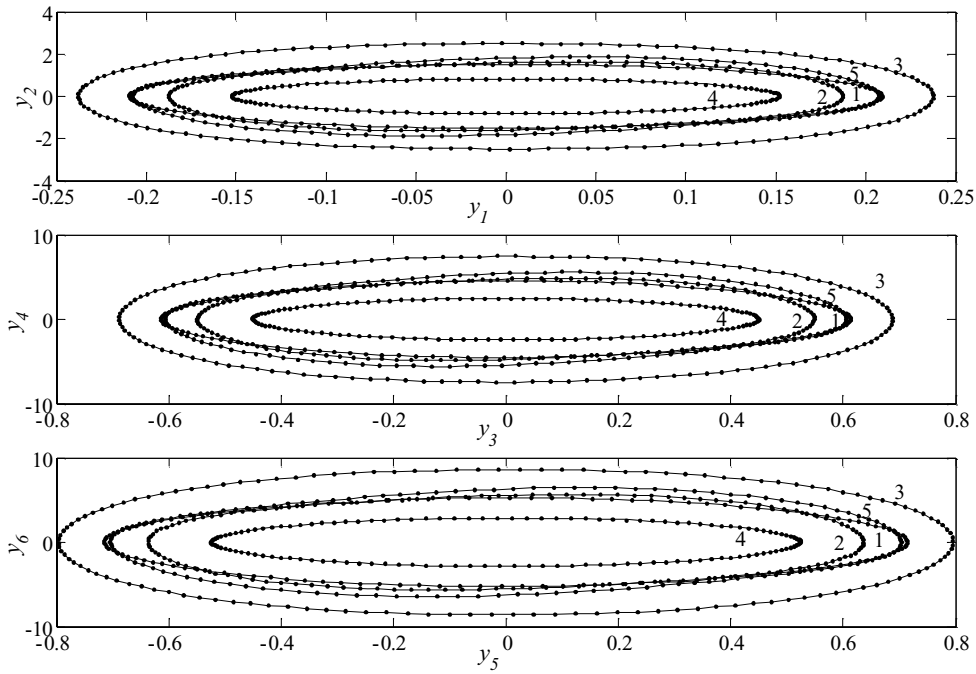


Fig. 7 MDOF system. Trajectories based on the original (*solid line*) and identified (*dotted line*) system (i) State variables of mass 1 (ii) mass 2 and (iii) mass 3. All the sets of trajectories are matching quite well

The identification is carried out with the same $\{f(t), x(t)\}$data as used in Table 10. The G matrix has six columns corresponding to $\ddot{x}$, $\dot{x}$, x, x^2, x^3, and x^4. The results are given in Table 11. It can be seen that the results with quadratic and quartic extra terms are nearly matching with the results of the correct model.

Now a model with still more terms, a quadratic damping term and polynomial up to fifth degree having eight parameters is considered. It is given by the equation

$$m\ddot{x} + c_1\dot{x} + c_2\dot{x}|\dot{x}| + k_1x + k_2x^2 + k_3x^3 + k_4x^4 + k_5x^5 = F\cos\Omega t \tag{26}$$

The result based on this model using total set is given in Table 12. Though the identification is nearly accurate, it is not as good as before. The k_5 value is not negligible. There is a small error in k_3. The introduction of extra terms in the model may affect the quality of results in some cases. A scheme is now proposed by which the presence of spurious parameters can be detected.

6.2 Identification using the subsets: Coefficient of variation

In the previous sections, the total data set was used in the identification and the effectiveness of the scheme was demonstrated. Now an identification study using the subsets of the total data sets is considered. Consider the equation with the full data set $[H]\{r\} = \{g\}$. This set has a total of $N_t = nN$ data points. One can pick up any combination out of N_t nonrepeated data points for identification. Thus, any subset of n_s ($n_s < N_t$) nonrepeated points can be randomly selected from the total set and the resulting relation containing n_s number of equations is denoted as $[h]\{r\} = \{q\}$. Let n_c be the

Table 8 MDOF system: Individual trial results

	Identified parameters										Result summary					
Trial	m_1	c_1	$k_1 \times 10^4$	m_2	c_2	$k_{21} \times 10^3$	$k_{22} \times 10^3$	m_3	c_3	$k_3 \times 10^4$	Force (N)	Frequency (Ω rad/s)	η	E_p	l_{c1}	l_{c2}	l_{c3}
1	−2.0574	40.1700	0.9935	2.2356	29.9195	5.0613	0.8849	2.8208	18.9771	2.0114	1950.1	7.0601	0.2693	0.9692	13.2581	17.6159	16.3496
2	1.1802	39.9223	0.9999	1.9796	30.1095	5.0038	0.8841	2.4645	19.0176	1.9971	1606.8	9.0641	0.3458	0.0697	11.8524	16.7772	16.1011
3	−0.4547	39.7019	1.0170	2.6907	30.0271	5.1438	1.2792	2.7378	18.9302	2.0231	1891.3	11.2354	0.4286	0.4823	13.8258	20.1914	18.9826
4	−87.358	39.0209	0.8186	13.8265	30.0156	5.4227	0.4205	−0.1457	27.5196	1.9442	1456.5	5.3880	0.2056	28.0067	15.6311	19.3082	17.3841
5	1.0367	40.1614	1.0000	2.1047	29.8614	4.9972	1.0164	2.4043	22.6896	1.9957	1821.4	8.7395	0.3334	0.0489	11.3358	16.0809	15.4130

number of such independent selections from the total set. Thus, one gets matrix equations as $[h]_j\{r\} = \{q\}_j$ where suffix $j = 1, 2, \ldots, n_c$. Since the elements of these sets are formed from different periodic orbits, these sets cannot be pictorially represented as orbits. However, these subcollections are valid sets for the purpose of identification.

Now we carry out the identification with these sets,$\{[h]_j, \{q\}_j\}$. It is seen that the identification results are nearly the same, though not as good as that done with the total set provided the number of elements in each subsets are sufficiently large. The identification results with these subsets will be used to obtain a level of confidence in the correctness of the assumed model. As mentioned earlier, a set of n_s indices from the N_t numbers is selected randomly. n_s is taken as 128. The pseudo inversion of this set $\{r\}_j = [h]_j^+\{q\}_j$ gives the identification parameters and the corresponding parametric error E_p for the jth set is found. The process is repeated for $n_\mathrm{c} = 10\,000$ sets. The computation mainly involves a pseudo inversion and is executed very fast. The multiple-trial scheme facilitates an almost unlimited number of choices for the selection of data points for identification, each one of them giving a result, which is a candidate parameter set. The identified parameter values across 100 samples are shown in Figure 8 for each of the eight parameters. The histograms of the corresponding full data are shown in Figure 9. Using the mean value and the standard deviation of these respective data, a Gaussian density function is determined. It is scaled (multiplied) by the area of the histogram and is drawn in the same figure in order to see whether the distribution is normal or not. It can be seen that the histogram closely matches with the Gaussian distribution. A dimensionless term called the coefficient of variation C_v, which is defined as the ratio of the standard deviation σ to the mean value $\mu(C_\mathrm{v} = \sigma/\mu)$ of the parameter is also determined. Table 13 gives the values of μ, σ, and C_v for all the eight parameters. C_v is a measure of fluctuation in the data among the samples. A higher value of C_v indicates that the corresponding data is inaccurate. The C_v for the parameters c_2 , k_2, k_4, and k_5 is an order of magnitude higher when compared to other parameters. It can be inferred that the error in the parameters is linked to the high value of C_v. This observation is useful because C_v for each parameter can be obtained in a real test and it is a measure of confidence in the result. The result in Table 13, however, has to be properly interpreted. A relatively high value of C_v

Table 9 MDOF system: Identified parameters using total data set

Mass (kg)	Stiffness (N/m $\times 10^4$)	Damping (N s/m)
$m_1 = 0.9737$	$k_1 = 0.9998$ (Linear)	$c_1 = 40.0396$ (Viscous)
$m_2 = 2.0370$	$[k_2] = [0.5000\ 0.1009]$ (Duffing)	$c_2 = 29.9033$(Viscous)
$m_3 = 2.4746$	$k_3 = 1.9986$ (Linear)	$c_3 = 20.1792$ (Quadratic)
$E_t = 0.0115$	$l_{c1} = 10.7013\ l_{c2} = 16.0757\ l_{c3} = 14.6022$	

Table 10 Results for Duffing oscillator model, $\{r\} = [1\ 0.02\ 1\ 0.02]^T$

				Identified parameters				
Trial	F	Ω	η	m	c	k	α	E_p
1	0.3080	0.0480	0.0480	144.5197	0.0200	1.3322	0.0000	71.7618
2	0.5374	0.0343	0.0343	109.9303	0.0199	1.1324	−0.0006	54.4676
3	0.7859	0.0457	0.0457	63.9448	0.0199	1.1409	−0.0012	31.4769
4	0.1167	0.0283	0.0283	328.7730	0.0221	1.2620	−0.0000	163.8873
5	0.5002	0.0429	0.0429	115.4889	0.0201	1.2166	−0.0138	57.2508
Total set				0.9847	0.0200	1.0000	0.0200	0.0078

Table 11 Results for a model having up to fourth-order polynomial terms in stiffness

	Identified parameters						
Trial	m	c	k_1	k_2	k_3	k_4	E_p
1	144.3894	0.0200	1.3319	−0.0000	0.0000	0.0000	71.7618
2	109.1251	0.0199	1.1314	−0.0000	−0.0004	−0.0000	54.4676
3	61.2064	0.0199	1.1347	−0.0000	−0.0003	0.0000	31.4769
4	359.7605	0.0221	1.2867	−0.0009	−0.0000	0.0483	163.8873
5	71.7278	0.0201	1.1338	−0.0000	−0.0010	−0.0000	57.2508
Total set	0.9847	0.0200	1.0000	−0.0000	0.0200	0.0001	

Table 12 Results for a model containing eight parameters

	Identified parameters							
	m	c_1	c_2	k_1	k_2	k_3	k_4	k_5
Total set	1.0117	0.0202	−0.0119	1.0001	−0.0000	0.0191	0.0001	0.0014

together with a nearly zero mean value for the parameters confirms that the parameter is negligible. If such parameters are detected, they may be excluded from the model and the parametric estimation can be redone with the reduced model. The sensitivity of a parameter to a given excitation is analogous to the reciprocal of C_v. In the case of a correct model, also there is scope for using C_v to detect poor identification, however, that is not done here.

Larger-sized subsets give smaller C_v and vice versa. However, the mean values of parameters and their relative C_v values remain nearly unchanged. In the next

Table 13 The coefficient of variation

	m	c_1	c_2	k_1	k_2	k_3	k_4	k_5
$\{r\}^T$	1	0.02	0	1	0	0.02	0	0
$\{r\}_t^T$	1.0117	0.0202	−0.0119	1.0001	−0.0000	0.0191	0.0001	0.0014
μ	1.0119	0.0202	−0.0113	1.0001	−0.0000	0.0191	0.0001	0.0015
σ	0.0271	0.0007	0.0240	0.0000	0.0000	0.0002	0.0001	0.0003
c_v	0.0268	0.0358	−2.1284	0.0000	−0.9554	0.0129	0.8005	0.2049

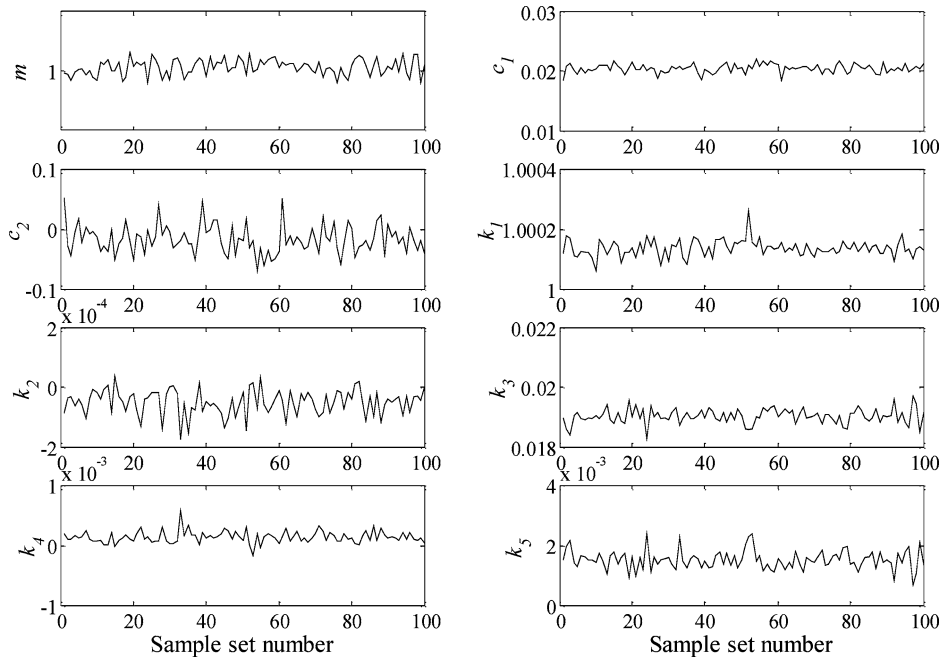


Fig. 8 Variation of identified parameter values with sample sets

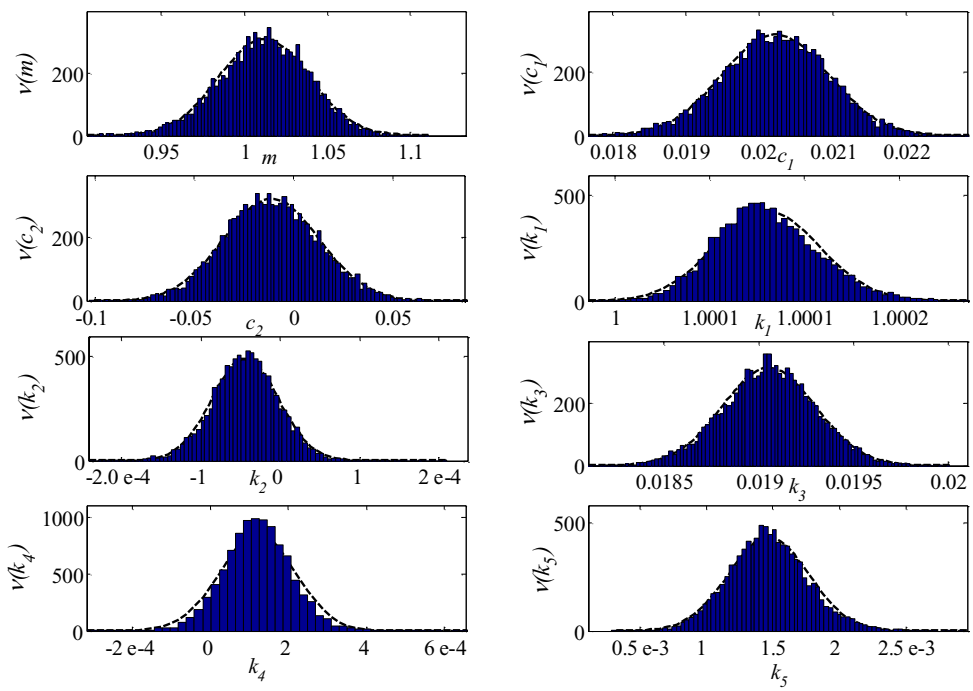


Fig. 9 Histograms for identified parameters from 10 000 samples sets. $v(i)$ denotes the number of ith parameter in the respective bin of the histogram. The *dashed line* indicates the scaled Gaussian fit

section, the problem of finding the noncontributing parameters, which give rise to errors in the model, is handled by considering a perturbation for the total data set. This is a different approach from the one discussed above.

6.3 Modeling error: Sensitivity study using periodic perturbation

Consider a small periodic perturbation $\Delta x(t)$ given to the response $x(t)$. The induced periodic perturbations, are determined for all columns of $[G]$ to get a periodic perturbation matrix $[\Delta G]_{\mathrm{p}}$. The identified parameters $\{r\}_{\mathrm{p}}$ in this case can be obtained as $\{r\}_{\mathrm{p}} = ([G] + [\Delta G]_{\mathrm{p}})^{+}\{f\}$. This process is repeated several times with randomly chosen periodic perturbations, each having a prescribed fixed rms ratio, $\varepsilon_{\mathrm{p}} = \mathrm{rms}(\Delta x)/\mathrm{rms}(x)$. The C_{v} of the parameters from the set can be determined as explained in the previous section. The same data set of the previous section is considered for illustration. Since the individual test results in Table 10 is highly erratic, it will not make sense to compute C_{v} based on a single set. However, with the combination of multiple trial data sets, the identification problem has been changed to a well-posed problem and it is meaningful to quantify C_{v} for the total data set. Each of the individual trial matrices $[G]_i$ is given nonidentical perturbations with the same value of ε_{p} to obtain the combined perturbed matrix, $[H]_{\mathrm{p}} = [H] + [\Delta H]_{\mathrm{p}}$. The identified values for the total set are given by $\{r\}_{\mathrm{pt}} = [H]_{\mathrm{p}}^{+}\{f\}$. The process is repeated 100 times and C_{v} of all the parameters are found. The calculations are done for different values of ε_{p} and the results are shown in Figure 10. The conclusion is that parameters with relatively high values of C_{v} have negligible values and can be omitted from the model. However, C_{v} cannot be used as the sole criterion for rejecting the parameters as it was mentioned in the last section. The point of cutoff cannot be ascertained based on C_{v} alone, however it is a good indicator.

7 Conclusion

A new study of nonlinear system identification incorporating a multiple-trial scheme in the conventional HB method is given in this paper. The applicability of

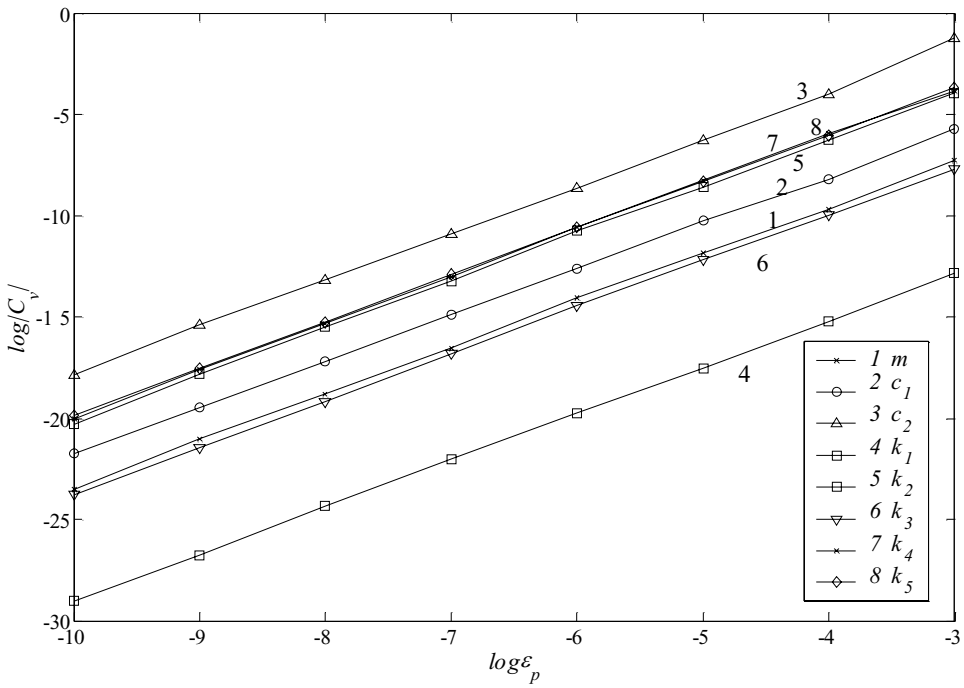


Fig. 10 Variation of C_{v} of the parameters with rms value of perturbation. ε_{p} varies uniformly from 10^{-10} to 10^{-3} in the logarithmic scale. The nonexistent parameters, which are c_2, k_2, k_4, and k_5 have higher C_{v} values compared to other parameters. They are indicated by *dotted lines* with numbers 3, 5, 7, and 8

the method is illustrated with several examples. This method works well independent of the type of nonlinear system and thus one need not be familiar with the nonlinear dynamics of a specific system to use this method. Measures to identify noncontributing terms in a model have also been suggested; this is shown to work well through a Duffing oscillator example.

Further refinements in the scheme have to be tried in which certain poor trial-set data is excluded from the total set using some criterion. This is an attempt to improve the quality of the result using the remaining data. The ranges of the excitation parameters were set somewhat arbitrarily, and it needs further studies to find out the guidelines for effective choices of these.

Appendix

Addition of a uniformly distributed random noise of prescribed noise-to-signal (*N*/*S*) ratio to a given signal. Consider a signal $x(t), 0 < t < T$. The discrete time version of $x(t)$ is denoted as $x[j], j = 1{:}N$ having uniformly spaced N points in the interval of t. The rms value of the discrete time signal,

$$R_x = \sqrt{\left(\sum_{j=1}^{N} (x[j])^2\right)/N} = \sqrt{x'x/N} \tag{A.1}$$

A uniformly distributed random noise denoted as $p(t)$ with zero mean value having the noise-to-signal rms ratio ε_{n} is to be added to the original signal. The noise-contaminated signal is $x_{\mathrm{p}}(t) = x(t) + p(t)$ and $\varepsilon_{\mathrm{n}} = R_{\mathrm{p}}/R_x$ where R_{p} is the rms value of the noise. Thus

$$R_{x\mathrm{p}}^2 = \left(1 + \varepsilon_{\mathrm{n}}^2\right) R_x^2 \tag{A.2}$$

$$R_{x\mathrm{p}} = \sqrt{x_{\mathrm{p}}^{\mathrm{T}} x_{\mathrm{p}}/N} = \sqrt{(x+p)^{\mathrm{T}}(x+p)/N} \tag{A.3}$$

$$x_{\mathrm{p}}^{\mathrm{T}} x_{\mathrm{p}} = \left(1 + \varepsilon_{\mathrm{n}}^2\right) x^{\mathrm{T}} x$$

$$x^{\mathrm{T}} x + p^{\mathrm{T}} p + 2x^{\mathrm{T}} p = \left(1 + \varepsilon_{\mathrm{n}}^2\right) x^{\mathrm{T}} x \tag{A.4}$$

$$p^{\mathrm{T}} p + 2x^{\mathrm{T}} p = \left(1 + \varepsilon_{\mathrm{n}}^2\right) x^{\mathrm{T}} x - x^{\mathrm{T}} x = N R_x^2 \varepsilon_{\mathrm{n}}^2$$

The RHS of the above equation is known and is denoted by P. The term $x^T\ p$ on the LHS is the sum of the products of x with an uncorrelated extraneous random signal p having zero mean. Hence, it will be zero. Thus $p^T p = P$ and the problem is to generate a random variable (r.v.) p such that this relation is nearly satisfied. This is the same as generating a zero mean r.v. with rms value of $\sqrt{P/N}$. The aim is to determine the range $(-a, a)$ in which p is uniformly distributed. This is determined by equating the variance of the uniform distribution $a^2/3$ to *P/N*, i.e.,

$$a = \sqrt{3} R_x \varepsilon_{\mathrm{n}} \tag{A.5}$$

Thus, the different steps to be followed in adding the noise to signal are as follows.

1. Obtain R_x for the given signal.
2. With the prescribed ε_{n}, calculate 'a' from Equation (A5).
3. Generate N uniformly distributed random variants of p in the interval $-a$ to a to be added to x, that is $x_{\mathrm{p}} = x + p$.

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