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Analysis of periodic-quasiperiodic nonlinear systems via Lyapunov-Floquet transformation and normal forms

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Abstract In this paper a general technique for the analysis of nonlinear dynamical systems with periodic-quasiperiodic coefficients is developed. For such systems the coefficients of the linear terms are periodic with frequency ω while the coefficients of the nonlinear terms contain frequencies that are incommensurate with ω. No restrictions are placed on the size of the periodic terms appearing in the linear part of system equation. Application of Lyapunov-Floquet transformation produces a dynamically equivalent system in which the linear part is time-invariant and the time varying coefficients of the nonlinear terms are quasiperiodic. Then a series of quasiperiodic near-identity transformations are applied to reduce the system equation to a normal form. In the process a quasiperiodic homological equation and the corresponding 'solvability condition' are obtained. Various resonance conditions are discussed and examples are included to show practical significance of the method. Results obtained from the *quasiperiodic time-dependent normal form theory* are compared with the numerical solutions. A close agreement is found.

Keywords Nonlinear systems · Periodic · Quasiperiodic · Lyapunov-Floquet transformation · Normal forms

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1. Introduction

An important class of dynamical systems may be represented by a set of linear/nonlinear differential equations with periodic/quasiperiodic coefficients. Bogoljubov et al. [1], Jorba and Simó [2], and Jorba et al. [3], among others, have all considered the reducibility of such systems to approximate time-invariant forms using a small parameter approach. Normal forms of quasiperiodic nonlinear systems with time-invariant linear part have been studied by E. G. Belaga as reported by Arnold [4]. The more recent techniques have, in general, been limited to systems with constant nonlinear coefficients, and are restricted by small parameters multiplying the nonlinear and/or time-varying terms. Belhaq et al. [5] and Guennoun et al. [6] consider a homogeneous Mathieu equation with quasiperiodic linear coefficients and a constant nonlinear coefficient. The small parameter technique of multiple scales is applied twice to the system to obtain an approximate time-invariant system. In another study (see Belhaq and Houssni, [7]) the system under investigation contains quadratic and cubic nonlinearities as well as parametric (linear terms) and external excitations of incommensurate frequencies. The small parameter techniques of generalized averaging and multiple-scale perturbation are employed to obtain a solutions. Rand and his associates (see Mason and Rand [8], Zounes and Rand [9], Zounes and Rand [10]) analyze a linear homogeneous quasiperiodic Mathieu equation via several methods, viz., numerical integration, Lyapunov exponents, regular perturbation, Lie transform perturbation and harmonic balance. Most of these methods require small parameter restrictions.

In this paper, we propose a technique for solving much wider class of problems where the nonlinear terms contain quasiperiodically time-varying coefficients and the linear terms have periodic coefficients. This type of systems generally arises in the analysis of parametrically excited coupled systems where under certain conditions one of the equations decouples and an explicit solution can be expressed as a periodic function of time. To illustrate this point consider the following coupled system.

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1 x_1 + \varepsilon x_1 x_2^2 = f_0 \sin\omega_1 t$$
$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2 x_2 + \varepsilon x_1^2 x_2 + (\delta \sin\omega_2 t)x_2^3 = 0 \tag{a}$$

where ε and δ are positive constants. Under the assumptions that $m_1 \gg m_2$ and $m_1 \gg \varepsilon$, the solution to the first equation of (a) is

$$x_1 = A\sin(\omega_1 t + \phi) \tag{b}$$

Substituting this solution into the second equation of (a), we obtain

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2 + \varepsilon(A\sin(\omega_1 t + \phi))^2)x_2 + \delta(\sin\omega_2 t + \phi)x_2^3 = 0 \tag{c}$$

where, of course, the linear term in x_2 has both constant and periodically time-varying coefficients (with frequency ω_1) and the cubic term has a periodic coefficient of an incommensurate frequency, ω_2.

The periodic coefficient of the linear term is not required to be small. In fact, other than numerical integration methods, no known techniques for analyzing a system of this type exist. By applying the (L–F) transformation to the system, the linear terms in the transformed domain have constant coefficients. Next, we develop a technique of quasiperiodic *time-dependent normal forms* (TDNF) as a generalization of the periodic TDNF [4, 17] where successive near-identity transformations are made in attempts to reduce the nonlinear terms, beginning with the lowest order. Without any loss of generality, we assume that the quasiperiodic terms contain only two incommensurate frequencies and thus the coefficients of nonlinear terms are expressed as a double Fourier series. Following a procedure similar to the periodic case, a quasiperiodic homological equation is obtained which yields the ‘solvability condition’. If the ‘solvability condition’ is not satisfied, then the so called ‘resonant terms’ can not be removed and remain in the simplified equation. Both *time-independent* and *time-dependent resonances* are discussed. If there are no resonance terms, the equation is reduced to a linear from with constant coefficients and the solution is readily obtained in the transformed domain. If the system dynamics in the original coordinates are desired, one must simply reverse the sequence of transformations that have been applied.

2. Problem formulation

Consider the nonlinear systems represented by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \varepsilon\mathbf{f_2}(\mathbf{x}(t), t) + \varepsilon^2\mathbf{f_3}(\mathbf{x}(t), t) + \cdots + \varepsilon^{k-1}\mathbf{f_k}(\mathbf{x}(t), t) + \varepsilon^k O(|\mathbf{x}(t)|^{k+1}, t) \tag{1}$$

where ε is a book keeping (and generally small positive) parameter, and $\mathbf{A}(t)$ is an $n \times n$, T_1 periodic matrix such that $\mathbf{A}(t) = \mathbf{A}(t + T_1)$. The $n \times 1$ vectors $\mathbf{f}_i(\mathbf{x}(t), t)$, $i = 2, 3, \ldots k$ are T_2 periodic homogeneous monomials in $\mathbf{x}$ of order i such that T_1 and T_2 are incommensurate (i.e., $T_1 \neq kT_2$, where k is any integer). Following Sinha et al. [14], the state transition matrix (STM) for the linear part of the system can be factored as

$$\Phi(t) = \mathbf{Q}(t)e^{\mathbf{R}t} \tag{2}$$

where $\mathbf{Q}(t)$ is typically $2T_1$ periodic such that $\mathbf{Q}(t) = \mathbf{Q}(t + 2T_1)$ and $\mathbf{R}$ is a real-valued $n \times n$ constant matrix. Applying the Lyapunov-Floquet (L–F) transformation $\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{y}(t)$ to Equation (1) yields

$$\dot{\mathbf{y}}(t) = \mathbf{R}\mathbf{y}(t) + \varepsilon\mathbf{Q^{-1}}(t)\mathbf{f}_2(\mathbf{Q}(t)\mathbf{y}(t), t) + \cdots + \varepsilon^{k-1}\mathbf{Q^{-1}}(t)\mathbf{f_k}(\mathbf{Q}(t)\mathbf{y}(t), t) + \varepsilon^k O(|\mathbf{Q}(t)\mathbf{y}(t)|^{k+1}, t) \tag{3}$$

Notice that the linear terms in $y(t)$ now have constant coefficients, and the nonlinear terms are quasiperiodic.

Further, application of the modal transformation $\mathbf{y}(t) = \mathbf{M}\mathbf{z}(t)$ puts the linear part of the system in the

Jordan canonical form and Equation (3) takes the form

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{J}\mathbf{z}(t) + \varepsilon \mathbf{M}^{-1}\mathbf{Q}^{-1}(t)\mathbf{f}_2(\mathbf{M}\mathbf{Q}(t)\mathbf{z}(t), t) + \cdots \\ &\quad + \varepsilon^{k-1}\mathbf{M}^{-1}\mathbf{Q}^{-1}(t)\mathbf{f}_k(\mathbf{M}\mathbf{Q}(t)\mathbf{z}(t), t) \\ &\quad + \varepsilon^k O(|\mathbf{M}\mathbf{Q}(t)\mathbf{z}(t)|^{k+1}, t) \qquad (4)\end{aligned}$$

where $\mathbf{J}$ is the Jordan canonical form of $\mathbf{R}$. Rewriting $\mathbf{M}^{-1}\mathbf{Q}^{-1}(t)\mathbf{f}_r(\cdot) = \mathbf{w}_r(\cdot)$, Equation (4) becomes

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{J}\mathbf{z}(t) + \varepsilon\, \mathbf{w}_2(\mathbf{z}(t), t) + \varepsilon^2\, \mathbf{w}_3(\mathbf{z}(t), t) + \cdots \\ &\quad + \varepsilon^{k-1}\mathbf{w}_k(\mathbf{z}(t), t) + \varepsilon^k O(|\mathbf{z}(t)|^{k+1}, t) \qquad (5)\end{aligned}$$

It is important to point out that the linear terms in $\mathbf{z}$ have constant complex coefficients and are in Jordan form, and that the nonlinear terms have quasiperiodic coefficients. In the following, a *quasiperiodic time-dependent normal form* (TDNF) theory is developed for systems represented by Equation (5).

2.1. Quasiperiodic time dependent normal form (TDNF) theory

As in the periodic case, we construct a sequence of transformations, beginning with the lowest order of nonlinearity, to successively remove the nonlinear terms of Equation (5). In order to remove the nonlinear terms of order r, $\mathbf{w}_r(\mathbf{z}, t)$, the following near-identity transformation is applied:

$$\mathbf{z} = \mathbf{v} + \varepsilon \mathbf{h}_r(\mathbf{v}, t) \qquad (6)$$

where the unknown nonlinear function $\mathbf{h}_r(\mathbf{v}, t)$ contains terms of similar forms as $\mathbf{w}_r(\mathbf{v}, t)$ and is quasiperiodic. We propose to choose the coefficients of $\mathbf{h}_r(\mathbf{v}, t)$ such that when transformation (6) is applied to system (5) all rth order nonlinearities are canceled out, if possible.

Applying transformation (6) to Equation (5) and following a procedure similar to that of periodic case (*c.f.*, reference [11, 17]), we obtain

$$\begin{aligned}\dot{\mathbf{v}} &= \mathbf{J}\mathbf{v} + \varepsilon\left(\mathbf{J}\mathbf{h}_r - \frac{\partial \mathbf{h}_r}{\partial t} - \frac{\partial \mathbf{h}_r}{\partial \mathbf{v}}\mathbf{J}\mathbf{v} + \mathbf{w}_r(\mathbf{v} + \varepsilon \mathbf{h}_r, t)\right) \\ &\quad - \varepsilon^2 \frac{\partial \mathbf{h}_r}{\partial \mathbf{v}}\left(\mathbf{J}\mathbf{h}_r - \frac{\partial \mathbf{h}_r}{\partial t} - \frac{\partial \mathbf{h}_r}{\partial \mathbf{v}}\mathbf{J}\mathbf{v} + \mathbf{w}_r(\mathbf{v} + \varepsilon \mathbf{h}_r, t)\right) \\ &\quad + \cdots + \varepsilon^2 \mathbf{w}_{r+1}(\mathbf{v} + \varepsilon \mathbf{h}_r, t) + \cdots \\ &\quad + \varepsilon^{k-1}\mathbf{w}_k(\mathbf{v} + \varepsilon \mathbf{h}_r, t) \qquad (7)\end{aligned}$$

By setting the coefficient of ε to zero, we obtain the well known homological equation

$$\mathbf{J}\mathbf{h}_r - \frac{\partial \mathbf{h}_r}{\partial t} - \frac{\partial \mathbf{h}_r}{\partial \mathbf{v}}\mathbf{J}\mathbf{v} + \mathbf{w}_r(\mathbf{v}, t) = 0 \qquad (8)$$

However, in this case the coefficients of $\mathbf{w}_r(\mathbf{v}, t)$ are quasiperiodic.

In order to find the solution of the homological equation given (8), we expand $\mathbf{h}_r(\mathbf{v}, t)$ and $\mathbf{w}_r(\mathbf{v}, t)$ in double finite Fourier series as

$$w_r(\mathbf{v}, t) = \sum_{\sum m_l = r} \sum_{j=1}^{n} \sum_{p_1=-q_1}^{q_1} \sum_{p_2=-q_2}^{q_2} a_{r,p_1,p_2,m_l} e^{i(\mathbf{p}\cdot\omega)t} \mathbf{v}^{\mathbf{m}} \mathbf{e}_j \qquad (9)$$

$$h_r(\mathbf{v}, t) = \sum_{\sum m_l = r} \sum_{j=1}^{n} \sum_{p_1=-q_1}^{q_1} \sum_{p_2=-q_2}^{q_2} h_{r,p_1,p_2,m_l} e^{i(\mathbf{p}\cdot\omega)t} \mathbf{v}^{\mathbf{m}} \mathbf{e}_j \qquad (10)$$

assuming two principal frequencies $\boldsymbol{\omega} = \{\omega_1\ \omega_2\}$, for simplicity. For a more general situation, we can assume $f_r()$ in Equation (1) will have frequency contents of $\omega_2, \omega_3, \ldots \omega_p$ and then $w_r()$ and $h_r()$ have to be expanded in p-tuple Fourier series. a_{r,p_1,p_2,m_l} are the known Fourier coefficients of the quasiperiodic functions from Equation (5); h_{r,p_1,p_2,m_l} are the unknown Fourier coefficients of the rth order normal form relation; $\mathbf{e}_j$ is the jth member of the natural basis; $\mathbf{v}^{\mathbf{m}} = v_1^{m_1} \ldots v_n^{m_n}$.

Upon substitution of these expressions into the homological Equation (8) we obtain a set of linear algebraic equations to be solved for the unknown Fourier coefficients of the near-identity transformation coefficients. A term-by-term comparison of the double Fourier coefficients provides the solution for the coefficients of the near-identity transformation as

$$h_{r,p_1,p_2,m_l} = \frac{a_{r,p_1,p_2,m_l}}{i\,(\mathbf{p}\cdot\boldsymbol{\omega}) + \mathbf{m}\cdot\boldsymbol{\lambda} - \lambda_j}; \qquad i = \sqrt{-1} \qquad (11)$$

where $\boldsymbol{\lambda} = \{\lambda_1\, \lambda_2\, \ldots\, \lambda_n\}$ are the eigenvalues of the Jordan matrix $\mathbf{J}$ and $\mathbf{p} = \{p_1\ p_2\}$. The difference between Equation (11) for the quasiperiodic case and the periodic case is the addition of multiple frequencies in the denominator. It is obvious that when the *solvability condition*

$$i(\mathbf{p}\cdot\boldsymbol{\omega})+\mathbf{m}\cdot\boldsymbol{\lambda}-\lambda_j\neq 0 \tag{12}$$

is satisfied, the corresponding nonlinear term can be eliminated. Otherwise, the corresponding resonant terms will remain in the reduced equation so that Equation (7) takes the form

$$\begin{aligned}\dot{\boldsymbol{v}} &= \mathbf{J}\mathbf{v}+\varepsilon\mathbf{w}^*_{\mathbf{r}}(\mathbf{v},t)+\varepsilon^2\mathbf{w}_{\mathbf{r+1}}(\mathbf{v},t)+\cdots\\ &\quad+\varepsilon^{k-1}\mathbf{w}_k(\mathbf{v},t)+O(\varepsilon^2)\end{aligned} \tag{13}$$

where $\mathbf{w}^*_{\mathbf{r}}$ contains only the "resonating" terms. Next, the $(r+1)$th order terms are removed, and so on, to obtain

$$\begin{aligned}\dot{\bar{\boldsymbol{v}}} &= \mathbf{J}\bar{\mathbf{v}}+\varepsilon\mathbf{w}^*_r(\bar{\mathbf{v}},t)+\varepsilon^2\mathbf{w}^*_{r+1}(\bar{\mathbf{v}},t)+\cdots\\ &\quad+\varepsilon^{k-1}\mathbf{w}^*_k(\bar{\mathbf{v}},t)\end{aligned} \tag{14}$$

where the $(\cdot)^*$ denotes the resonant terms that could not be removed.

Because of the quasiperiodicity of the functions, the *time-independent resonance* may occur only if the double Fourier series expansion in Equation (12) contains constant terms, i.e., terms corresponding to $p_1=p_2=0$. *Time-dependent resonances* may also occur in certain cases. Such possibilities are discussed as in the following.

2.2. The resonant cases

2.2.1. Time-independent resonance

Resonance may occur in quasiperiodic systems when the double Fourier series contains a constant term in the expansion. A close examination of Equation (11) reveals that resonance may only occur when the eigenvalues are purely imaginary, i.e., if there is no dissipation in the system. For example, in the case of cubic nonlinearity, Equation (14) takes the form (see Pandiyan and Sinha [12])

$$\begin{Bmatrix}\dot{v}_1\\ \dot{v}_2\end{Bmatrix}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{Bmatrix}v_1\\ v_2\end{Bmatrix}+\varepsilon\begin{Bmatrix}\alpha_1 v_1^2 v_2\\ \alpha_2 v_1 v_2^2\end{Bmatrix} \tag{15}$$

where α_1 and α_2 are the coefficients of the resonant terms, resulting from the double Fourier series expansion. We multiply the first equation of system (15) by v_2, the second equation by v_1, and add to obtain

$$\frac{d}{dt}(\boldsymbol{v}_1\boldsymbol{v}_2)=\varepsilon(\alpha_1+\alpha_2)v_1^2v_2^2 \tag{16}$$

which can be integrated to yield v_1v_2 as an explicit function of t, say $c(t)$. Then from Equation (15), $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ may be obtained as

$$\begin{aligned}v_1 &= e^{(\lambda_1+\varepsilon\alpha_1 c(t))\,t}\,v_{10}\\ v_2 &= e^{(\lambda_2+\varepsilon\alpha_2 c(t))\,t}\,v_{20}\end{aligned} \tag{17}$$

It is to be noted that such resonances always occur if the eigenvalues $\lambda_1\&\lambda_2$ are purely imaginary, and the stability of the system entirely depends on α_1 and α_2.

2.2.2. Time-dependent resonance

It is clear that terms of the form $w_{r,l,\mathbf{m},n}e^{i(p_1\omega_1+p_2\omega_2)t}v_1^{m_1}v_2^{m_2}\mathbf{e_l}$ remain when the corresponding resonance condition

$$i(p_1\omega_1+p_2\omega_2)+m_1\lambda_1+m_2\lambda_2-\lambda_1=0 \tag{18}$$

is satisfied, where $m_1+m_2=r$, $\omega_1={}^{2\pi}/_{T_1}$, and $\mathbf{e_l}$ is the lth member of the natural basis. Butcher and Sinha [13] showed that *time-dependent resonances* $(p\neq 0)$ occur for the periodic case (where there is only one frequency ω_1) for purely imaginary eigenvalues with specific absolute values. For the quasiperiodic case the results can be extended as follows.

It is obvious that Equation (18) has no $p\neq 0$ solutions for real or complex $\lambda_{1,2}$. Therefore, the two eigenvalues are restricted to be purely imaginary pairs of the form $\lambda_{1,2}=\pm i\beta$ so that only the stable Hamiltonian case with no damping is relevant. Furthermore, the case of purely imaginary characteristic exponents at the fold stability boundary (multipliers, $\mu_{1,2}=+1$) is discounted since the corresponding zero eigenvalues of $\mathbf{R}$ imply that only time-independent resonances are present. Equation (18) thus becomes

$$\mathbf{p}\cdot\boldsymbol{\omega}=\beta s;\quad \text{where}\quad s=m_2-m_1\pm 1 \tag{19}$$

where $(+1)$ corresponds to $l=1$ and (-1) corresponds to $l=2$. The values of s for all $\mathbf{m}$ and l were tabulated in Butcher and Sinha [13] (Table 1). It was shown that there are exactly eight different combinations of l, m_1, and m_2 which yield five different values of s from -4 to 4. The two combinations which result in $s = 0$ correspond to the time-independent resonances ($p_1 = p_2 = 0$) which occur in the entire stable part of the parameter plane. For the quasiperiodic case there are no time-dependent resonances when $s = 0$; however, if the two frequencies were rationally related, then time-dependent resonances would occur when $p_1\omega_1 + p_2\omega_2 = 0$. The following shows when time-dependent resonances occur for $s \neq 0$.

It was shown in Butcher and Sinha [13] that the parametric period T_1 satisfies $T_1\beta < \pi$ which can also be expressed as $\beta > \omega_1/2$. Multiplying this inequality by $|s|$ and using Equation (19) results in

$$|\mathbf{p}\cdot\boldsymbol{\omega}| < \frac{|s|\omega_1}{2}; \quad \operatorname{sgn}(\mathbf{p}\cdot\boldsymbol{\omega}) = \operatorname{sgn}(s); \quad p_{1,2} = \pm 1,\ \pm 2, \ldots \tag{20}$$

Equation (20) may be solved for combinations of p_1 and p_2 which result in time-dependent resonances and the corresponding value of s (which determines m_1, m_2, and l). Equation (19) may then be solved for $\beta = (\mathbf{p}\cdot\boldsymbol{\omega})/s$ which yields a valid set of time-dependent resonant eigenvalues as $\lambda_{1,2} = \pm i\beta$. It is important to note that, although there may be many values of p_1 and p_2 which solve Equation (20), the corresponding resonances will not occur simultaneously since each individual resonance requires the above values of the eigenvalues of the constant matrix $\mathbf{R}$ (which in turn require the Floquet multipliers to be $\mu = e^{(\pm i2\pi\beta/\omega_1)}$). However, for a particular given imaginary eigenvalue pair, some resonance may be found which corresponds to a pair that is arbitrarily close to the given pair. Hence, the stable region of the parameter plane is foliated with such resonances. Also, unlike the periodic case in which time-dependent resonances occur when the linear solution has a component that is *MT*-periodic ($M = 1, 2, 3, \ldots$), the resonances in the quasiperiodic case require solution components with periods irrationally related to the period T_1. Hence, the symbolic computational technique used in (Reference [13]) to compute the parameter regions where *MT*-periodic solutions occur cannot be applied here.

For example, suppose that a 2-dimensional quasiperiodic system with cubic nonlinearities has frequencies $\Omega_\Theta = 2\pi$ (the linear parametric frequency) and $\omega_2 = 1$ (for time-dependent coefficients of nonlinear terms) and suppose that the Floquet multipliers are complex and lie on the unit circle. Thus, the eigenvalues of $\mathbf{R}$ (after performing the L-F transformation) are purely imaginary. Equation (20) gives the possible time-dependent resonances as

$$\left|\frac{p_1 + p_2}{2\pi}\right| < \frac{|s|}{2} \tag{21}$$

where $s = \pm 2\text{or} \pm 4$ as shown in Table 1 in Reference [13]. A few of the lowest order solutions are given as

$$\begin{aligned}
&(0, p_2) && p_2 = -12, \ldots, 12\\
&(1, p_2) && p_2 = -18, \ldots, 6\\
&(-1, p_2) && p_2 = -6, \ldots, 18\\
&(2, p_2) && p_2 = -25, \ldots, -1\\
&(-2, p_2) && p_2 = 1, \ldots, 25\\
& && etc.
\end{aligned} \tag{22}$$

Each pair requires separate values of the eigenvalues. For instance, the pair (1, −18), which corresponds to $s = -4$, $m_1 = 3$, $m_2 = 0$, and $l = 2$, requires the eigenvalues to be $\pm 0.4661972i$ and the Floquet multipliers to be $\mu = e^{\pm 0.4661972i}$. This would result in the resonance term $e^{-1.864790it}v_1^3\mathbf{e}_2$. However, because the pair (−1,18) (which corresponds to $s = 4$, $m_1 = 0$, $m_2 = 3$, and $l = 1$) also requires the same eigenvalues, the resonance term $e^{1.864790it}v_2^3\mathbf{e}_1$ is also present. Hence the normal form would be

$$\begin{aligned}
\dot{v}_1 &= 0.4661972iv_1 + w_{3,1,(2,1),(0,0)}v_1^2v_2\\
&\quad + w_{3,1,(0,3),(-1,18)}e^{1.864790it}v_2^3\\
\dot{v}_2 &= -0.4661972iv_2 + w_{3,2,(1,2),(0,0)}v_1v_2^2\\
&\quad + w_{3,2,(3,0),(1,-18)}e^{-1.864790it}v_1^3
\end{aligned} \tag{23}$$

in which the *time-independent resonances* shown earlier are also present.

3. Applications

In this section, application of the *quasiperiodic normal form theory* is demonstrated through two examples. In the first example, a nonlinear commutative system is analyzed. In this case, the L–F transformation is obtained in a closed form and various solutions are computed using the techniques described earlier. For this problem it is possible to obtain closed-form solutions in terms of the parameters of the system. The second example is a Mathieu-Duffing equation with cubic nonlinearities. In this case the L−F transformation matrices cannot be obtained in a closed form, so we resort to a computational algorithm that has been proven to be accurate and efficient (see Sinha, Pandiyan, and Bibb [14]).

3.1. A commutative system with quadratic nonlinerities

Consider the following system with quadratic nonlinearities:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -1+\alpha\cos^2 t & 1-\alpha\sin t\cos t \\ -1-\alpha\sin t\cos t & -1+\alpha\sin^2 t \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \varepsilon\cos(\pi\, t) \begin{Bmatrix} x_1^2 \\ x_1 x_2 \end{Bmatrix} \tag{24}$$

where α is a real-valued bifurcation parameter and ε is a positive (generally small) number. Notice the period of $\mathbf{A}(t)$ is $T_1 = \pi$ and the period of both the nonlinear term and the forcing term is $T_2 = 2$. The state transition matrix (STM) for this system and its factorization [15] are

$$\begin{aligned} \mathbf{\Phi}(t) &= \begin{bmatrix} e^{(\alpha-1)t}\cos t & e^{-t}\sin t \\ -e^{(\alpha-1)t}\sin t & e^{-t}\cos t \end{bmatrix} \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} e^{(\alpha-1)t} & 0 \\ 0 & e^{-t} \end{bmatrix} \end{aligned} \tag{25}$$

From this equation, it is obvious that

$$\mathbf{Q}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad \mathbf{Q}^{-1}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$
$$\mathbf{R} = \begin{bmatrix} \alpha-1 & 0 \\ 0 & -1 \end{bmatrix} \tag{26}$$

Applying the L–F transformation $\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{z}(t)$, system (24) becomes

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} \alpha-1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \varepsilon\cos(\pi\, t) \begin{Bmatrix} z_1^2\cos t + z_1 z_2 \sin t \\ z_1 z_2\cos t + z_2^2 \sin t \end{Bmatrix} \tag{27}$$

Notice that the matrix multiplying the linear terms is constant and in the diagonal form and the nonlinear terms are quasiperiodic. Also, the eigenvalues of the system are $(\alpha - 1)$ and (-1), so the critical value of α is 1, when one of the eigenvalues becomes zero.

We now apply the near-identity transformation

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \varepsilon \begin{Bmatrix} h_{21(2,0)}v_1^2 + h_{21(1,1)}v_1v_2 + h_{21(0,2)}v_2^2 \\ h_{22(2,0)}v_1^2 + h_{22(1,1)}v_1v_2 + h_{22(0,2)}v_2^2 \end{Bmatrix} \tag{28}$$

in order to eliminate the quadratic terms.

Solving for the quadratic coefficients of the transformation yields

$$\begin{aligned} h_{21(2,0)} &= ae^{-(2.14159i)t} + a^* e^{(2.14159i)t} \\ &\quad + be^{-(4.14159i)t} + b^* e^{(4.14159i)t} \\ h_{21(1,1)} &= ce^{-(2.14159i)t} + c^* e^{(2.14159i)t} \\ &\quad + de^{-(4.14159i)t} + d^* e^{(4.14159i)t} \\ h_{21(0,2)} &= 0 \\ h_{22(2,0)} &= 0 \\ h_{22(1,1)} &= ae^{-(2.14159i)t} + a^* e^{(2.14159i)t} \\ &\quad + be^{-(4.14159i)t} + b^* e^{(4.14159i)t} \\ h_{22(0,2)} &= ce^{-(2.14159i)t} + c^* e^{(2.14159i)t} \\ &\quad + de^{-(4.14159i)t} + d^* e^{(4.14159i)t} \end{aligned} \tag{29}$$

where

$$a = -0.0258456 + 0.110701i$$
$$b = -0.00718276 + 0.0594961i$$

$$c = 0.0958392 + 0.0447514i$$
$$d = -0.057038 - 0.013772i \tag{30}$$

and $(\cdot)^*$ denotes the complex conjugate of $(\cdot)$. Using this transformation reduces system (27) to

$$\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} = \begin{bmatrix} \alpha - 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} \tag{31}$$

which immediately yields

$$v_1 = e^{(\alpha-1)t} v_1(0)$$
$$v_2 = e^{-t} v_2(0) \tag{32}$$

where $v_1(0)$ and $v_2(0)$ are the initial values. Since $v_1(t)$ and $v_2(t)$ are known, $z_1(t)$ and $z_2(t)$ can be obtained from Equation (28) and $\mathbf{x}(t)$ is immediately determined from $\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{z}(t)$.

The solutions of this system in the stable, center, and unstable manifolds are computed using the proposed quasiperiodic TDNF method. These results are compared with the corresponding numerical solutions in Figs. 1–3 for various values of ε. For all real values of α, Equation (27) has real eigenvalues. Therefore, the solvability condition (12) is completely satisfied and hence the quasiperiodic TDNF method reduces this system to a linear form for all values of α. The solutions in the stable manifold possess uniform convergence. It is interesting to note that in Figs. 1 and 3, the value of ε is 1.0, meaning that the nonlinearities are not small, and the accuracy is still good. Fig. 2 indicates that the solution in the unstable manifold is correctly predicted.

3.2. A Mathieu-Duffing equation

Modeling of many engineering systems with parametric excitation may be reduced to a Mathieu-Duffing type equation. For example, the forced Mathieu-Duffing equation with cubic nonlinearities has the general form

$$\ddot{\theta} + d\dot{\theta} + (a + b\cos\omega t)\,\theta + \varepsilon\, g(\theta, t) = 0 \tag{33}$$

where a, b, d, and ω are the parameters of the system; ε is a real-valued positive (generally small) multiplying factor and $g(\theta, \quad t)$ is a time-varying nonlinear function of θ. With $\{\dot{\theta}\ \ddot{\theta}\}^T = \{\dot{x}_1\ \dot{x}_2\}^T$ and assuming cubic nonlinearity, Equation (33) may be rewritten as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & +1 \\ -(a + b\cos(2\pi t)) & -d \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \varepsilon \cos t \begin{Bmatrix} 0 \\ -x_1^3 \end{Bmatrix} \tag{34}$$

This system is non-commutative; therefore we must resort to approximating the L–F transformation matrix $\mathbf{Q}(t)$ as suggested in reference [14]. A Fortran program written by Butcher [16] is utilized to compute the L–F transformation matrices for given parameter sets.

After applying the L–F and modal transformations, the system becomes

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \varepsilon(\cos t)\mathbf{M}^{-1}\mathbf{Q}^{-1}(t) \begin{Bmatrix} 0 \\ (Q_{11}(M_{11}z_1 + M_{12}z_2) \\ +Q_{12}(M_{21}z_1 + M_{22}z_2))^3 \end{Bmatrix} \tag{35}$$

Again, the coefficients of the nonlinear terms of Equation (35) may be expressed as a double Fourier series, due to the quasiperiodic nature of the terms. To this system, we apply the near-identity transformation of the form

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \varepsilon \begin{Bmatrix} h_{31(3,0)}v_1^3 + h_{31(2,1)}v_1^2 v_2 + h_{31(1,2)}v_1 v_2^2 + h_{31(0,3)}v_2^3 \\ h_{32(3,0)}v_1^3 + h_{32(2,1)}v_1^2 v_2 + h_{32(1,2)}v_1 v_2^2 + h_{32(0,3)}v_2^3 \end{Bmatrix} \tag{36}$$

where the coefficients $h_{3i}(\mathbf{v}, t)$ are unknown but quasiperiodic (with $\omega_1 = 2\pi$ and $\omega_2 = 1$), $i = 1, 2$. For the non-resonant cases, the system reduces to

$$\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} \tag{37}$$

where λ_1 and λ_2 are the eigenvalues of the transformed time-invariant linear system. The solution of this equation has been discussed earlier.

Fig. 1 Comparison of solution of commutative system in stable manifold ($\alpha = 0.5$, $\varepsilon = 1.0$)

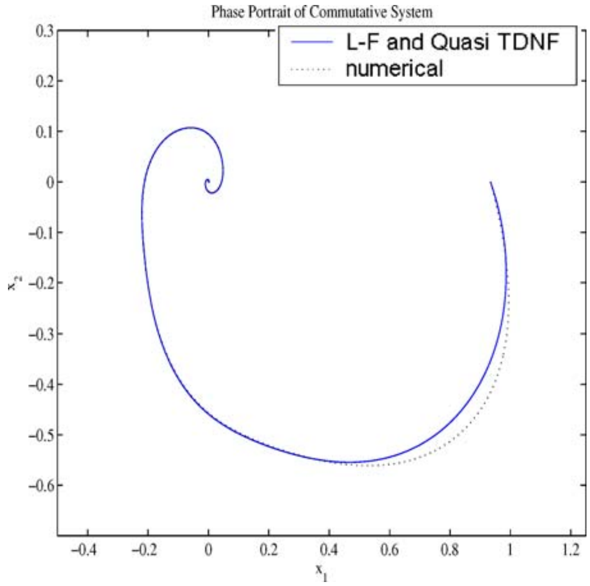


Fig. 2 Comparison of solution of commutative system in unstable manifold ($\alpha = 1.3$, $\varepsilon = 0.1$)

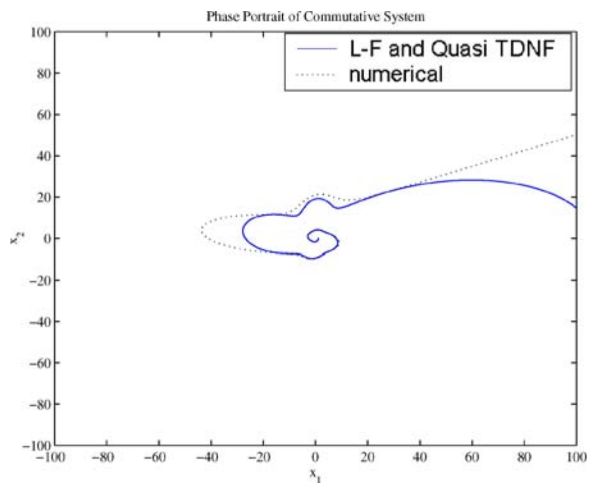
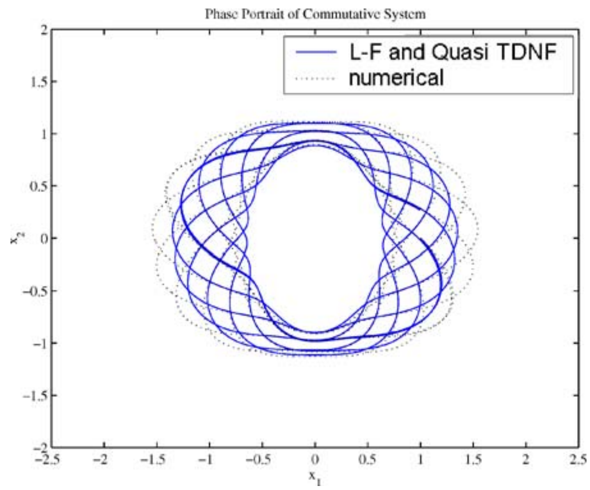


Fig. 3 Comparison of solution of commutative system in center manifold ($\alpha = 1.0$, $\varepsilon = 1.0$)



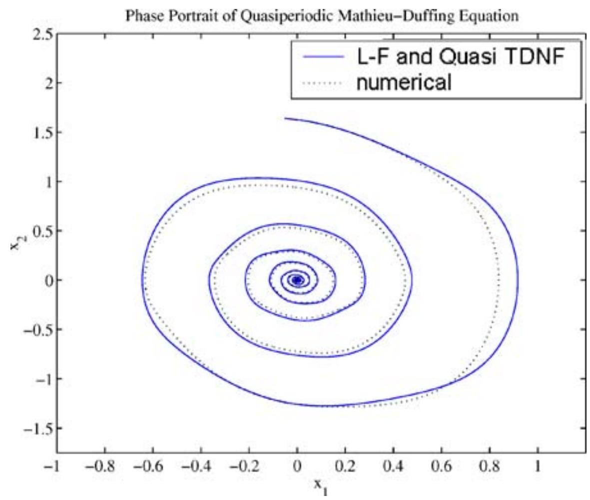


Fig. 4 Comparison of solution of Mathieu-Duffing equation in stable manifold ($a = 3$, $b = 1$, $d = 00.31623$, $\varepsilon = 1.0$)

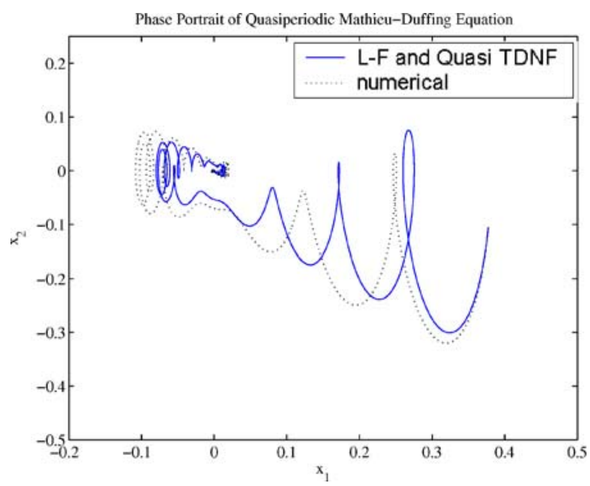


Fig. 5 Comparison of solution of Mathieu-Duffing equation in stable manifold ($a = 0$, $b = 4$, $d = 0.4243$, $\varepsilon = 1.0$)

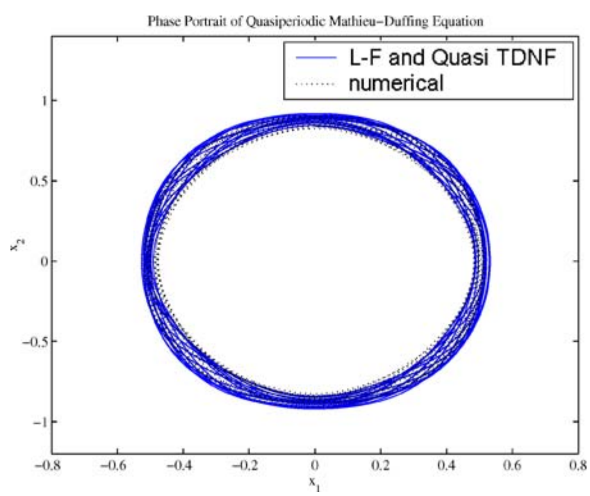


Fig. 6 Comparison of solution of Mathieu-Duffing equation in center manifold ($a = 3$, $b = 1$, $d = 0$, $\varepsilon = 1.0$)

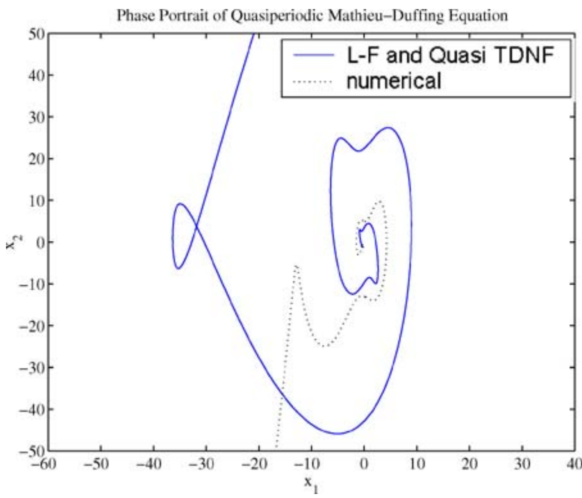


Fig. 7 Comparison of solution of Mathieu-Duffing equation in unstable manifold ($a = 1$, $b = 20$, $d = 0.5$, $\varepsilon = 0.1$)

3.2.1. Case studies

In the following, some case studies of various values of a, b, d, and ε are presented for the Mathieu-Duffing Equation (34). For the Parameter Set 1: $a = 3$, $b = 1$, $d = 0.31623$, $\varepsilon = 1.0$, the solutions are in the stable manifold. Fig. 4 shows an excellent comparison with the numerical solution. If $a = 0$, then we note that the traditional averaging (or perturbation) method has no generating solution in this situation. For this case, the TDNF and numerical solutions are shown in Fig. 5. The agreement is very close.

The dynamics in the center manifold is shown in Fig. 6. For this case $a = 3$, $b = 1$, $d = 0$, $\varepsilon = 1.0$. Since the system is underdamped the eigenvalues λ_1 & λ_2 in Equation (37) are purely imaginary. The approximate result compares well with the numerical solution.

For the parameter set: $a = 1$, $b = 20$, $d = 0.5$, $\varepsilon = 0.1$, the system is unstable and this behavior is correctly predicted as shown in Fig. 7.

4. Conclusions

In conclusion, for the first time, a *time dependent normal form technique* is developed for a class of parametrically excited dynamical systems where the linear terms have periodic coefficients while the nonlinear terms are quasiperiodic in time. For simplicity, quasiperiodicity of two frequencies is investigated in detail. The periodic coefficient of the linear term does not require a small parameter assumption, which is normally the case when one attempts to apply averaging or perturbation techniques. In addition, the question of existence of the so called 'generating solution' does not arise in this approach.

The usefulness of the method is demonstrated by applications to a commutative system and a Mathieu-Duffing type equation. Results obtained from the *quasiperiodic time dependent normal form* theory are compared with the numerical results. It is shown that the proposed analysis techniques provide accurate results for all three cases (viz., stable, unstable and neutrally stable) even if the magnitude of the parameters multiplying the nonlinear terms is moderately large. There is no limitation placed on the size of the periodic terms appearing in the linear part of the system equation. This implies that the stability and bifurcation characteristics are preserved in the entire parameter space. An extension to forced problems will be reported in the near future.

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