Nonlinear Dyn (2007) :367–375
DOI 10.1007/s11071-006-9036-x

ORIGINAL ARTICLE

Hopf bifurcation of the generalized Lorenz canonical form

Tiecheng Li · Guanrong Chen · Yun Tang · Lijun Yang

Received: 25 April 2005 / Accepted: 16 February 2006 / Published online: 13 October 2006

Abstract Hopf bifurcation of a unified chaotic system – the generalized Lorenz canonical form (GLCF) – is investigated. Based on rigorous mathematical analysis and symbolic computations, some conditions for stability and direction of the periodic obits from the Hopf bifurcation are derived.

Keywords Chaos · Lorenz system · Canonical form · Hopf bifurcation

1. Introduction

Bifurcation properties of the classic Lorenz system is very well known (see, e.g., [1–4]).

Recently, Chen and Ueta [5, 6] constructed a three-dimensional autonomous differential equation with only two quadratic terms, and found a new chaotic attractor [7], referred to as Chen's attractor thereafter by others. This system displays very complex dynamics like the Lorenz system, but they are topologically not equivalent. The stability and various bifurcations especially the Hopf bifurcation of Chen's system were studied in [6, 8, 9], showing various bifurcations in this system such as tangent bifurcation, pitchfork bifurcation, period-doubling bifurcation, homoclinic bifurcation, coexistence of two stable limit cycles and one chaotic attractor, as well as some periodic solutions emerging from Hopf bifurcation but ending in homoclinic bifurcation, etc. Moreover, the travelling of nontrivial trajectories is discussed in [10], and the unstable periodic orbits of the system are detected in [11].

More recently, it was discovered that there is a very large and general class of relevant chaotic systems – the family of generalized Lorenz systems, named *Generalized Lorenz Canonical Form* (GLCF) [13, 20] – defined according to the system structures, which covers all the aforementioned chaotic systems as special cases. This family of chaotic systems has only one parameter, satisfying $-1 < \tau < +\infty$, and there is another large family of chaotic systems, named *Hyperbolic Generalized Lorenz Canonical Forms*, satisfying $-\infty < \tau \leq -1$ [12]. Notably, the Lorenz system satisfies $0 < \tau < +\infty$; the Lü system, $\tau = 0$; [22] the Chen system, $-1 < \tau < 0$; the Shimizu-Morioka system, $\tau = -1$ [23]. It has been mathematically proved that Smale horseshoe and horseshoe chaos, hence Shil'nikov chaos, exist in the GLCF [14].

Given the above background, it is clearly important to study the bifurcation of the GLCF, which is the scope of the present paper. In Section 2, some stability analysis on the GLCF will be derived based on the center manifold theory and symbolic computations (for general references concerning symbolic computation

T. Li (✉) · Y. Tang · L. Yang
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China
e-mail: tli@math.tsinghua.edu.cn

G. Chen
Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong SAR, P. R. China

of Hopf bifurcations, see [8, 15–19]). Section 3 analyzes the bifurcation of periodic orbits of the GLCF, and gives some conditions for stability and directions of the periodic obits from the Hopf bifurcation. Section 4 concludes the investigation.

2. Stability analysis

GLCF is described by the following three-dimensional smooth quadratic autonomous system [13, 20]:

$$\begin{cases} \dot{x} = ax - (x-y)z \\ \dot{y} = -by - (x-y)z \\ \dot{z} = -cz + (x-y)(x+\tau y), \end{cases} \tag{1}$$

where $(a, b, c, \tau) \in R_+^3 \times R$. This system contains the classical Lorenz system and the newly discovered Chen system as two special and extreme cases, along with infinitely many chaotic systems in between [6, 12, 13, 20].

In the following, some stability properties of system (1) are studied first.

Proposition 1. *The equilibria of system (1) are all isolated; moreover, when* $\tau \geq b/a$ *system (1) has only one equilibrium* $S_0 = (0, 0, 0)$*, and when* $\tau < b/a$ *system (1) has three equilibria:* $S_0 = (0, 0, 0)$

$$S_+ = \left(\frac{b\sqrt{abc}}{(a+b)\sqrt{b-a\tau}}, -\frac{a\sqrt{abc}}{(a+b)\sqrt{b-a\tau}}, \frac{ab}{a+b} \right),$$

$$S_- = \left(-\frac{b\sqrt{abc}}{(a+b)\sqrt{b-a\tau}}, \frac{a\sqrt{abc}}{(a+b)\sqrt{b-a\tau}}, \frac{ab}{a+b} \right).$$

Proof: The equilibria of system (1) are the solutions of the following equations:

$$\begin{cases} ax - (x-y)z = 0 \\ -by - (x-y)z = 0 \\ -cz + (x-y)(x+\tau y) = 0. \end{cases} \tag{2}$$

Condition (2) leads to

$$\begin{cases} ax - (x-y)z = 0 \\ y = -\frac{a}{b}x \\ z = \frac{1}{c}(x-y)(x+\tau y), \end{cases}$$

which gives, by eliminating y and z from the above equations, the following:

$$ax - \frac{1}{c}\left(1 + \frac{a}{b}\right)^2 \left(1 - \frac{a\tau}{b}\right) x^3 = 0;$$

that is,

$$x(ab^3c - (a+b)^2(b - a\tau)x^2) = 0.$$

Solving this equation for x yields $x = 0$ when $\tau \geq b/a$, and yields $x = 0$ and $x = \pm b\sqrt{abc}/((a+b)\sqrt{b-a\tau})$ when $\tau < b/a$. Therefore, when $\tau \geq b/a$ system (1) has only one equilibrium $S_0 = (0, 0, 0)$, but when $\tau < b/a$ it has three equilibria: $S_0 = (0, 0, 0)$ and

$$S_+ = \left(\frac{b\sqrt{abc}}{(a+b)\sqrt{b-a\tau}}, -\frac{a\sqrt{abc}}{(a+b)\sqrt{b-a\tau}}, \frac{ab}{a+b} \right),$$

$$S_- = \left(-\frac{b\sqrt{abc}}{(a+b)\sqrt{b-a\tau}}, \frac{a\sqrt{abc}}{(a+b)\sqrt{b-a\tau}}, \frac{ab}{a+b} \right).$$

The proposition is thus proved. □

Since the Jacobian of system (1) at S_0 is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix},$$

the equilibrium S_0 of system (1) is always a saddle point, which possesses a 1-dimensional local unstable manifold and a 2-dimensional local stable manifold.

It is interesting to consider the stability of system (1) at S_+ and S_- with $\tau < b/a$. Notice the invariance of system (1) under the transformation $(x, y, z) \mapsto (-x, -y, z)$. Thus, one only needs to consider the stability of system (1) at S_+.

Let $x = X + b\sqrt{abc}/((a+b)\sqrt{b-a\tau})$, $y = Y - a\sqrt{abc}/((a+b)\sqrt{b-a\tau})$, and $z = Z + ab/(a+b)$. Then, system (1) becomes

$$\begin{cases} \dot{X} = \dfrac{a^2}{a+b}X + \dfrac{ab}{a+b}Y - \dfrac{\sqrt{abc}}{\sqrt{b-a\tau}}Z - (X-Y)Z \\ \dot{Y} = -\dfrac{ab}{a+b}X - \dfrac{b^2}{a+b}Y - \dfrac{\sqrt{abc}}{\sqrt{b-a\tau}}Z - (X-Y)Z \\ \dot{Z} = \dfrac{(a+2b-a\tau)\sqrt{abc}}{(a+b)\sqrt{b-a\tau}}X + \dfrac{(-b+2a\tau+b\tau)\sqrt{abc}}{(a+b)\sqrt{b-a\tau}} \\ \quad \times Y - cZ + (X-Y)(X+\tau Y). \end{cases} \tag{3}$$

Hence, one has to consider the stability of system (3) at (0, 0, 0) with $\tau < b/a$.

Denote

$$A_+ = \begin{pmatrix} \frac{a^2}{a+b} & \frac{ab}{a+b} & -\frac{\sqrt{abc}}{\sqrt{b-a\tau}} \\ -\frac{ab}{a+b} & -\frac{b^2}{a+b} & -\frac{\sqrt{abc}}{\sqrt{b-a\tau}} \\ \frac{(a+2b-a\tau)\sqrt{abc}}{(a+b)\sqrt{b-a\tau}} & \frac{(-b+2a\tau+b\tau)\sqrt{abc}}{(a+b)\sqrt{b-a\tau}} & -c \end{pmatrix}.$$

Then, the characteristic equation of A_+ is

$$s^3 + (-a+b+c)s^2 + \frac{c(b^2+a^2\tau)}{b-a\tau}s + 2abc = 0. \tag{4}$$

Proposition 2. *The condition $\tau < b/a$ and the real parts of the roots of Equation (4) being negative together are equivalent to the conditions of $a < b + c$ and $b^2(3a-b-c)/(a^2(-a+3b+c)) < \tau < b/a$. In this case, the equilibrium* (0, 0, 0) *of system (3) is asymptotically stable.*

Proof: By the Routh-Hurwitz Theorem, the real parts of the roots of (4) are negative if and only if

$$\frac{c(b^2+a^2\tau)}{b-a\tau} > 0, \quad 2abc > 0,$$
$$(-a+b+c)\frac{c(b^2+a^2\tau)}{b-a\tau} - 2abc > 0,$$

so the conditions of $\tau < b/a$ and the real parts of the roots of Equation (4) being negative together are equivalent to

$$\tau < \frac{b}{a}, \ (b^2+a^2\tau) > 0,$$
$$(-a+b+c)\frac{(b^2+a^2\tau)}{b-a\tau} > 2ab,$$

which, in turn, are equivalent to the following:

$$\tau < \frac{b}{a}, \ -a+b+c > 0,$$
$$(-a+b+c)(b^2+a^2\tau) > 2ab(b-a\tau);$$

namely,

$$\tau < \frac{b}{a}, \ -a+b+c > 0,$$
$$a^2\tau(-a+3b+c) > b^2(3a-b-c).$$

The proposition is thus proved. □

Next, denote

$$\sum = \left\{ (a,b,c,\tau) \in R_+^3 \times R \mid a < b+c, \right.$$
$$\left. \tau = \frac{b^2(3a-b-c)}{a^2(-a+3b+c)} \right\}.$$

Proposition 3. *The conditions of $\tau < b/a$ and Equation (4) having roots with zero real parts together are equivalent to the condition $(a,b,c,\tau) \in \sum$. In this case, Equation (4) has one pair of purely imaginary conjugate roots, $s = \pm i\sqrt{2abc}/\sqrt{-a+b+c}$, together with a negative root, $s = a - b - c$.*

Proof: The conditions $\tau < b/a$ and $s = i\omega (\omega \in R)$ satisfying Equation (4) together imply that

$$\tau < \frac{b}{a}, \quad 2abc + (a-b-c)\omega^2 = 0,$$
$$\omega^3 = \frac{c(b^2+a^2\tau)}{b-a\tau}\omega,$$

which are equivalent to

$$\tau < \frac{b}{a}, \quad a-b-c < 0, \quad \omega^2 = \frac{2abc}{-a+b+c},$$

$$\frac{2ab}{-a+b+c} = \frac{(b^2+a^2\tau)}{b-a\tau};$$

namely,

$$\tau < \frac{b}{a}, \quad a-b-c<0, \quad \omega^2 = -\frac{2abc}{a+b+c},$$
$$\tau = \frac{b^2(3a-b-c)}{a^2(-a+3b+c)};$$

that is,

$$\frac{b^2(3a-b-c)}{a^2(-a+3b+c)} < \frac{b}{a}, \quad a-b-c<0,$$
$$\omega^2 = \frac{2abc}{-a+b+c}, \quad \tau = \frac{b^2(3a-b-c)}{a^2(-a+3b+c)},$$

which, in turn, are equivalent to

$$a-b-c<0, \quad \omega^2 = \frac{2abc}{-a+b+c},$$
$$\tau = \frac{b^2(3a-b-c)}{a^2(-a+3b+c)}.$$

It is then easy to verify that when

$$a-b-c<0, \quad \tau = \frac{b^2(3a-b-c)}{a^2(-a+3b+c)},$$

the three roots of Equation (4) are $s = \pm i\sqrt{2abc}/\sqrt{-a+b+c}$ and $s = a-b-c$.

The proposition is thus proved. □

Now, denote

$$\mu(a,b,c) = (b-a)(3a^4 - 16a^3b + 18a^2b^2 - 16ab^3 + 3b^4 - 10a^3c + 18a^2bc - 18ab^2c + 10b^3c + 12a^2c^2 - 8abc^2 + 12b^2c^2 - 6ac^3 + 6bc^3 + c^4). \tag{5}$$

Theorem 1. *If* $(a,b,c,\tau) \in \sum$, *then the equilibrium* $(0,0,0)$ *of system (3) possesses a 1-dimensional local stable manifold and a 2-dimensional local center manifold. Moreover, for the flow restricted to the 2-dimensional local center manifold,* $(0,0,0)$ *is an unstable focus when* $\mu(a,b,c) < 0$ *and a stable focus when* $\mu(a,b,c) > 0$.

Proof: Under the condition $(a,b,c,\tau) \in \Sigma$, it follows from Proposition 3 that the eigenvalues of A_+ are $s = \pm i\sqrt{2abc}/\sqrt{-a+b+c}$ and $s = a-b-c$; so $(0,0,0)$ is a degenerate critical point of system (3), possessing a 1-dimensional local stable manifold and a 2-dimensional local center manifold.

Next, it is to prove that the point $(0,0,0)$ is a focus for the flow restricted to the 2-dimensional local center manifold.

Denote

$$\lambda = a-b-c, \quad \omega = \frac{\sqrt{2abc}}{\sqrt{-a+b+c}},$$
$$\sigma = \frac{\sqrt{(a+b)(-a+b+c)}}{a\sqrt{c(-a+3b+c)}},$$

and let

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ 0 & \sigma & \sigma \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix},$$

where

$$u_{11} = -\frac{\omega^2 - ab + b^2}{\omega(\omega^2 + (a-b)^2)}, \quad u_{12} = \frac{a}{\omega^2 + (a-b)^2},$$
$$u_{13} = \frac{a-c}{c\lambda}, \quad u_{21} = -\frac{\omega^2 + a^2 - ab}{\omega(\omega^2 + (a-b)^2)},$$
$$u_{22} = -\frac{b}{\omega^2 + (a-b)^2}, \quad u_{23} = -\frac{b+c}{c\lambda}.$$

Then

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

where

$$v_{11} = -\frac{\omega((b+c)\omega^2 + ac(a-b) + b\lambda^2)}{(a+b)(\omega^2+\lambda^2)},$$
$$v_{12} = -\frac{\omega((a-c)\omega^2 + bc(a-b) + a\lambda^2)}{(a+b)(\omega^2+\lambda^2)},$$
$$v_{13} = \frac{c\omega}{(\omega^2+\lambda^2)\sigma}, \quad v_{21} = -\frac{c\lambda(\omega^2+a^2-ab)}{(a+b)(\omega^2+\lambda^2)},$$
$$v_{22} = \frac{c\lambda(\omega^2-ab+b^2)}{(a+b)(\omega^2+\lambda^2)}, \quad v_{23} = \frac{\omega^2+(a-b)\lambda}{(\omega^2+\lambda^2)\sigma},$$
$$v_{31} = \frac{c\lambda(\omega^2+a^2-ab)}{(a+b)(\omega^2+\lambda^2)},$$
$$v_{32} = -\frac{c\lambda(\omega^2-ab+b^2)}{(a+b)(\omega^2+\lambda^2)},$$
$$v_{33} = -\frac{c\lambda}{(\omega^2+\lambda^2)\sigma}.$$

System (3) becomes

$$\begin{cases} \dot\xi = \omega\eta + \alpha_{11}\xi^2 + \alpha_{12}\xi\eta + \alpha_{22}\eta^2 + \alpha_{13}\xi\zeta \\ \quad +\alpha_{23}\eta\zeta + \alpha_{33}\zeta^2 \\ \dot\eta = -\omega\xi + \beta_{11}\xi^2 + \beta_{12}\xi\eta + \beta_{22}\eta^2 + \beta_{13}\xi\zeta \\ \quad +\beta_{23}\eta\zeta + \beta_{33}\zeta^2 \\ \dot\zeta = \lambda\zeta + \gamma_{11}\xi^2 + \gamma_{12}\xi\eta + \gamma_{22}\eta^2 + \gamma_{13}\xi\zeta \\ \quad +\gamma_{23}\eta\zeta + \gamma_{33}\zeta^2, \end{cases} \tag{6}$$

where

$$\begin{aligned}
\alpha_{11} &= (u_{11}-u_{21})(u_{11}+\tau u_{21})v_{13},\\
\alpha_{12} &= (u_{12}-u_{22})(u_{11}+\tau u_{21})v_{13} + (u_{11}-u_{21})\\
&\quad \times(-\sigma(v_{11}+v_{12}) + (u_{12}+\tau u_{22})v_{13}),\\
\alpha_{22} &= (u_{12}-u_{22})(-\sigma(v_{11}+v_{12}) + (u_{12}+\tau u_{22})v_{13}),\\
\alpha_{13} &= (u_{13}-u_{23})(u_{11}+\tau u_{21})v_{13} + (u_{11}-u_{21})\\
&\quad \times(-\sigma(v_{11}+v_{12}) + (u_{13}+\tau u_{23})v_{13}),\\
\alpha_{23} &= (u_{13}-u_{23})(-\sigma(v_{11}+v_{12}) + (u_{12}+\tau u_{22})v_{13})\\
&\quad +(u_{12}-u_{22})(-\sigma(v_{11}+v_{12}) + (u_{13}+\tau u_{23})v_{13}),\\
\alpha_{33} &= (u_{13}-u_{23})(-\sigma(v_{11}+v_{12}) + (u_{13}+\tau u_{23})v_{13}),\\
\beta_{11} &= (u_{11}-u_{21})(u_{11}+\tau u_{21})v_{23},\\
\beta_{12} &= (u_{12}-u_{22})(u_{11}+\tau u_{21})v_{23} + (u_{11}-u_{21})\\
&\quad \times(-\sigma(v_{21}+v_{22}) + (u_{12}+\tau u_{22})v_{23}),\\
\beta_{22} &= (u_{12}-u_{22})(-\sigma(v_{21}+v_{22}) + (u_{12}+\tau u_{22})v_{23}),\\
\beta_{13} &= (u_{13}-u_{23})(u_{11}+\tau u_{21})v_{23} + (u_{11}-u_{21})\\
&\quad \times(-\sigma(v_{21}+v_{22}) + (u_{13}+\tau u_{23})v_{23}),\\
\beta_{23} &= (u_{13}-u_{23})(-\sigma(v_{21}+v_{22}) + (u_{12}+\tau u_{22})v_{23})\\
&\quad +(u_{12}-u_{22})(-\sigma(v_{21}+v_{22}) + (u_{13}+\tau u_{23})v_{23}),\\
\beta_{33} &= (u_{13}-u_{23})(-\sigma(v_{21}+v_{22}) + (u_{13}+\tau u_{23})v_{23}),\\
\gamma_{11} &= (u_{11}-u_{21})(u_{11}+\tau u_{21})v_{33},\\
\gamma_{12} &= (u_{12}-u_{22})(u_{11}+\tau u_{21})v_{33} + (u_{11}-u_{21})\\
&\quad \times(-\sigma(v_{31}+v_{32}) + (u_{12}+\tau u_{22})v_{33}),\\
\gamma_{22} &= (u_{12}-u_{22})(-\sigma(v_{31}+v_{32}) + (u_{12}+\tau u_{22})v_{33}),\\
\gamma_{13} &= (u_{13}-u_{23})(u_{11}+\tau u_{21})v_{33} + (u_{11}-u_{21})\\
&\quad \times(-\sigma(v_{31}+v_{32}) + (u_{13}+\tau u_{23})v_{33}),\\
\gamma_{23} &= (u_{13}-u_{23})(-\sigma(v_{31}+v_{32}) + (u_{12}+\tau u_{22})v_{33})\\
&\quad +(u_{12}-u_{22})(-\sigma(v_{31}+v_{32}) + (u_{13}+\tau u_{23})v_{33}),\\
\gamma_{33} &= (u_{13}-u_{23})(-\sigma(v_{31}+v_{32}) + (u_{13}+\tau u_{23})v_{33}).
\end{aligned}$$

Denote by $W^c_{\text{loc}}(0_+)$ the local center manifold of system (6) near the origin:

$$\begin{aligned} W^c_{\text{loc}}(0_+) &= \{(\xi,\eta,\zeta) \mid \zeta = h(\xi,\eta),\\ &|\xi| + |\eta| \ll 1,\ h(0,0) = \partial_\xi h(0,0)\\ &= \partial_\eta h(0,0) = 0\}. \end{aligned}$$

Substituting $\zeta = h(\xi, \eta)$ into (6) yields the vector field restricted to the center manifold, as

$$\begin{cases} \dot\xi = \omega\eta + \varphi(\xi,\eta) \\ \dot\eta = -\omega\xi + \psi(\xi,\eta), \end{cases} \tag{7}$$

where

$$\begin{aligned} \varphi(\xi,\eta) &= \alpha_{11}\xi^2 + \alpha_{12}\xi\eta + \alpha_{22}\eta^2 + \alpha_{13}\xi h\\ &\quad + \alpha_{23}\eta h + \alpha_{33}h^2,\\ \psi(\xi,\eta) &= \beta_{11}\xi^2 + \beta_{12}\xi\eta + \beta_{22}\eta^2 + \beta_{13}\xi h + \beta_{23}\eta h\\ &\quad + \beta_{33}h^2, \end{aligned}$$

in which h satisfies

$$\begin{aligned} \partial_\xi h\ \dot\xi + \partial_\eta h\ \dot\eta &= \lambda h + \gamma_{11}\xi^2 + \gamma_{12}\xi\eta + \gamma_{22}\eta^2\\ &\quad + \gamma_{13}\xi h + \gamma_{23}\eta h + \gamma_{33}h^2; \end{aligned}$$

that is,

$$\begin{aligned} &\partial_\xi\ h\ (\omega\eta + \alpha_{11}\xi^2 + \alpha_{12}\xi\eta + \alpha_{22}\eta^2 + \alpha_{13}\xi h\\ &\quad + \alpha_{23}\eta h + \alpha_{33}h^2) + \partial_\eta h\ (-\omega\xi + \beta_{11}\xi^2\\ &\quad + \beta_{12}\xi\eta + \beta_{22}\eta^2 + \beta_{13}\xi h + \beta_{23}\eta h + \beta_{33}h^2)\\ &= \lambda\ h + \gamma_{11}\xi^2 + \gamma_{12}\xi\eta + \gamma_{22}\eta^2 + \gamma_{13}\xi h\\ &\quad + \gamma_{23}\eta h + \gamma_{33}h^2. \end{aligned} \tag{8}$$

To this end, assume that h has the form

$$h(\xi,\eta) = h_{11}\xi^2 + h_{12}\xi\eta + h_{22}\eta^2 + \cdots.$$

Then, substituting it into (8) and then equating coefficients of $\xi^m\eta^n$, $m+n\geq 2$, on both sides, one obtains

$$h_{11} = -\frac{(\lambda^2+2\omega^2)\gamma_{11} - \lambda\omega\gamma_{12} + 2\omega^2\gamma_{22}}{\lambda(\lambda^2+4\omega^2)},$$

$$h_{12} = -\frac{2\omega\gamma_{11} + \lambda\gamma_{12} - 2\omega\gamma_{22}}{\lambda^2+4\omega^2},$$

$$h_{22} = -\frac{2\omega^2\gamma_{11} + \lambda\omega\gamma_{12} + (\lambda^2+2\omega^2)\gamma_{22}}{\lambda(\lambda^2+4\omega^2)},$$

...

The polar coordinates $\xi = r\cos\theta$, $\eta = r\sin\theta$ transform system (7) into

$$\begin{cases} \dot{r} = R(r,\theta) \\ \dot{\theta} = -\omega + Q(r,\theta), \end{cases} \tag{9}$$

where

$$R(r,\theta) = \varphi(r\cos\theta, r\sin\theta)\cos\theta + \psi(r\cos\theta, r\sin\theta)\sin\theta,$$
$$Q(r,\theta) = \frac{1}{r}(\psi(r\cos\theta, r\sin\theta)\cos\theta - \varphi(r\cos\theta, r\sin\theta)\sin\theta).$$

Denote by $P(r)$ the Poincaré map of the following system:

$$\frac{dr}{d\theta} = \frac{R(r,\theta)}{-\omega + Q(r,\theta)} \tag{10}$$

along the ray $\theta = 0$, and denote by $d(r)$ the succession function

$$d(r) = P(r) - r\,.$$

Then, some tedious calculations yield

$$d(r) = -\frac{b\ (a+b)^4\ c\ \pi\ \mu(a,b,c)}{\lambda\ a^3\ (-a+3b+c)^2\ \sigma^2\ \omega^5\ (\omega^2+(a-b)^2)\ (\omega^2+\lambda^2)(4\omega^2+\lambda^2)} r^3 + O(r^4). \tag{11}$$

One can easily check that

$$-\frac{b\ (a+b)^4\ c\ \pi}{\lambda\ a^3\ (-a+3b+c)^2\ \sigma^2\ \omega^5\ (\omega^2+(a-b)^2)\ (\omega^2+\lambda^2)(4\omega^2+\lambda^2)} > 0. \tag{12}$$

The theorem is thus proved. □

Corollary 1. *If* $a < b+c$, $\tau = b^2(3a-b-c)/\ (a^2(-a+3b+c))$, *then the equilibrium* $(0,0,0)$ *of system (3) is unstable when* $\mu(a,b,c) < 0$ *and asymptotically stable when* $\mu(a,b,c) > 0$.

3. Bifurcation of periodic orbits

In this section, consider the periodic orbit from bifurcation of system (1) at S_+ and S_-.

Owing to the invariance of system (1) under the transformation $(x,y,z) \mapsto (-x,-y,z)$, one only needs to study the bifurcation at S_+.

Theorem 2. *If* $(a,b,c,\tau) \in \sum$, $L: \varepsilon\ (|\varepsilon| \ll 1) \mapsto (j(\varepsilon), k(\varepsilon), p(\varepsilon), q(\varepsilon)) \in R_+^3 \times R$ *is a smooth curve transversal to* $\sum$ *at* (a,b,c,τ) *when* $\varepsilon = 0$, *which means that* $(j(0), k(0), p(0), q(0)) = (a,b,c,\tau)$ *and*

$$\left[\frac{k^2(\varepsilon)(3j(\varepsilon)-k(\varepsilon)-p(\varepsilon))}{j^2(\varepsilon)(-j(\varepsilon)+3k(\varepsilon)+p(\varepsilon))} - q(\varepsilon)\right]'\Bigg|_{\varepsilon=0} \neq 0,$$

then system (1) with the parameters $(j(\varepsilon), k(\varepsilon), p(\varepsilon), q(\varepsilon))$:

$$\begin{cases} \dot{x} = j(\varepsilon)x - (x-y)z \\ \dot{y} = -k(\varepsilon)y - (x-y)z \\ \dot{z} = -p(\varepsilon)z + (x-y)(x+q(\varepsilon)y), \end{cases} \tag{13}$$

has a Hopf bifurcation at S_+ *as* ε *passes* 0. *Moreover,*

(i) the bifurcating periodic solutions are unstable when $\mu(a,b,c) < 0$ *and asymptotically stable when* $\mu(a,b,c) > 0$;
(ii) the direction of bifurcation is $\varepsilon < 0$ *when*

$$\mu(a,b,c)\left[\frac{k^2(\varepsilon)(3j(\varepsilon)-k(\varepsilon)-p(\varepsilon))}{j^2(\varepsilon)(-j(\varepsilon)+3k(\varepsilon)+p(\varepsilon))} - q(\varepsilon)\right]'\Bigg|_{\varepsilon=0} < 0,$$

and $\varepsilon > 0$ *when*

$$\mu(a,b,c)\left[\frac{k^2(\varepsilon)(3j(\varepsilon)-k(\varepsilon)-p(\varepsilon))}{j^2(\varepsilon)(-j(\varepsilon)+3k(\varepsilon)+p(\varepsilon))} - q(\varepsilon))\right]'\Bigg|_{\varepsilon=0} > 0.$$

Proof: Let

$$x = X + \frac{k(\varepsilon)\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{(j(\varepsilon)+k(\varepsilon))\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}},$$

$$y = Y - \frac{j(\varepsilon)\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{(j(\varepsilon)+k(\varepsilon))\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}},$$

$$z = Z + \frac{j(\varepsilon)k(\varepsilon)}{j(\varepsilon)+k(\varepsilon)},$$

and

$$A_+(\varepsilon) = \begin{pmatrix} \frac{j^2(\varepsilon)}{j(\varepsilon)+k(\varepsilon)} & \frac{j(\varepsilon)k(\varepsilon)}{j(\varepsilon)+k(\varepsilon)} & -\frac{\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}} \\ -\frac{j(\varepsilon)k(\varepsilon)}{j(\varepsilon)+k(\varepsilon)} & -\frac{k^2(\varepsilon)}{j(\varepsilon)+k(\varepsilon)} & -\frac{\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}} \\ \frac{(j(\varepsilon)+2k(\varepsilon)-j(\varepsilon)q(\varepsilon))\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{(j(\varepsilon)+k(\varepsilon))\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}} & \frac{(-k(\varepsilon)+2j(\varepsilon)q(\varepsilon)+k(\varepsilon)q(\varepsilon))\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{(j(\varepsilon)+k(\varepsilon))\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}} & -p(\varepsilon) \end{pmatrix}.$$

Then, system (13) becomes

$$\begin{cases} \dot{X} = \frac{j^2(\varepsilon)}{j(\varepsilon)+k(\varepsilon)}X + \frac{j(\varepsilon)k(\varepsilon)}{j(\varepsilon)+k(\varepsilon)}Y \\ \quad - \frac{\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}}Z - (X-Y)Z \\ \dot{Y} = -\frac{j(\varepsilon)k(\varepsilon)}{j(\varepsilon)+k(\varepsilon)}X - \frac{k^2(\varepsilon)}{j(\varepsilon)+k(\varepsilon)}Y \\ \quad - \frac{\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}}Z + (Y-X)Z \\ \dot{Z} = \frac{(j(\varepsilon)+2k(\varepsilon)-j(\varepsilon)q(\varepsilon))\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{(j(\varepsilon)+k(\varepsilon))\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}}X \\ \quad + \frac{(-k(\varepsilon)+2j(\varepsilon)q(\varepsilon)+k(\varepsilon)q(\varepsilon))\sqrt{j(\varepsilon)k(\varepsilon)p(\varepsilon)}}{(j(\varepsilon)+k(\varepsilon))\sqrt{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}}Y \\ \quad - p(\varepsilon)Z + (X-Y)(X+q(\varepsilon)Y), \end{cases} \tag{14}$$

and the characteristic equation of $A_+(\varepsilon)$ is

$$s^3 + (-j(\varepsilon)+k(\varepsilon)+p(\varepsilon))s^2 + \frac{p(\varepsilon)(k^2(\varepsilon)+j^2(\varepsilon)q(\varepsilon))}{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}s + 2j(\varepsilon)k(\varepsilon)p(\varepsilon) = 0\,. \tag{15}$$

It follows from Proposition 3 that when $\varepsilon = 0$, (15) has one pair of purely imaginary conjugate roots, $\pm i\sqrt{2abc}/\sqrt{-a+b+c}$, and a negative root, $a-b-c$.

Denote by $s(\varepsilon)$ the branch of the solution of (15) with $s(0) = i\sqrt{2abc}/\sqrt{-a+b+c}$. Then

$$s'(\varepsilon) = -\frac{(-j(\varepsilon)+k(\varepsilon)+p(\varepsilon))'s^2(\varepsilon) + \left(\frac{p(\varepsilon)(k^2(\varepsilon)+j^2(\varepsilon)q(\varepsilon))}{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}\right)'s(\varepsilon) + 2(j(\varepsilon)k(\varepsilon)p(\varepsilon))'}{3s^2(\varepsilon) + 2(-j(\varepsilon)+k(\varepsilon)+p(\varepsilon))s(\varepsilon) + \frac{p(\varepsilon)(k^2(\varepsilon)+j^2(\varepsilon)q(\varepsilon))}{k(\varepsilon)-j(\varepsilon)q(\varepsilon)}}, \tag{16}$$

which implies that

$$\begin{aligned} \Re(s'(0)) &= \frac{bc(3a^2+3b^2+4bc+c^2-6ab-3ac)}{(a+b)(2abc+(-a+b+c)^3)}j'(0) \\ &\quad - \frac{ac(3a^2-6ab+3b^2-4ac+3bc+c^2)}{(a+b)(2abc+(-a+b+c)^3)}k'(0) \\ &\quad - \frac{abc}{2abc+(-a+b+c)^3}p'(0) \\ &\quad - \frac{a^3c(-a+3b+c)^2}{2b(a+b)(2abc+(-a+b+c)^3)}q'(0) \\ &= \frac{a^3c(-a+3b+c)^2}{2b(a+b)(2abc+(-a+b+c)^3)} \\ &\quad \times \Bigg[\frac{2b^2(3a^2+3b^2+4bc+c^2-6ab-3ac))}{a^3(-a+3b+c)^2}j'(0) \\ &\quad - \frac{2b(3a^2-6ab+3b^2-4ac+3bc+c^2)}{a^2(-a+3b+c)^2}k'(0) \\ &\quad - \frac{2b^2(a+b)}{a^2(-a+3b+c)^2}p'(0) - q'(0)\Bigg] \\ &= \frac{a^3c(-a+3b+c)^2}{2b(a+b)(2abc+(-a+b+c)^3)} \\ &\quad \times \Bigg[\partial_a \frac{b^2(3a-b-c)}{a^2(-a+3b+c)}j'(0) \\ &\quad + \partial_b \frac{b^2(3a-b-c)}{a^2(-a+3b+c)}k'(0) \\ &\quad + \partial_c \frac{b^2(3a-b-c)}{a^2(-a+3b+c)}p'(0) - q'(0)\Bigg] \\ &= \frac{a^3c(-a+3b+c)^2}{2b(a+b)(2abc+(-a+b+c)^3)} \\ &\quad \left[\frac{k^2(\varepsilon)(3j(\varepsilon)-k(\varepsilon)-p(\varepsilon))}{j^2(\varepsilon)(-j(\varepsilon)+3k(\varepsilon)+p(\varepsilon))} - q(\varepsilon)\right]'\Bigg|_{\varepsilon=0}. \end{aligned}$$

Since $\frac{a^3c(-a+3b+c)^2}{2b(a+b)(2abc+(-a+b+c)^3)} > 0$ by $(a,b,c,\tau) \in \Sigma$, one has

$$\Re(s'(0) \neq 0 \tag{17}$$

from the transversality condition. The claims thus follow from the Hopf Bifurcation Theorem and Theorem 1 above, completing the proof of the theorem. □

Corollary 2. *If* $(a, b, c, \tau) \in \sum$, *then system (13) with* $(j(\varepsilon), k(\varepsilon), p(\varepsilon), q(\varepsilon)) = (a+\varepsilon, b, c, \tau)$:

$$\begin{cases} \dot{x} = (a+\varepsilon)x - (x-y)z \\ \dot{y} = -by - (x-y)z \\ \dot{z} = -cz + (x-y)(x+\tau y), \end{cases} \tag{18}$$

has a Hopf bifurcation at S_+ *as* ε *passes* 0. *Moreover,*

(i) when $\mu(a, b, c) < 0$, *the bifurcating periodic orbit is unstable and the direction of bifurcation is* $\varepsilon < 0$;

(ii) when $\mu(a, b, c) > 0$, *the bifurcating periodic orbit is asymptotically stable and the direction of bifurcation is* $\varepsilon > 0$.

Proof:

$$\left[\frac{k^2(\varepsilon)(3j(\varepsilon) - k(\varepsilon) - p(\varepsilon))}{(j^2(\varepsilon)(-j(\varepsilon) + 3k(\varepsilon) + p(\varepsilon))} - q(\varepsilon)\right]'\Bigg|_{\varepsilon=0}$$
$$= \frac{2b^2(3a^2 + 3b^2 + 4bc + c^2 - 6ab - 3ac))}{a^3(-a+3b+c)^2}$$
$$= \frac{12b^3(b+c-a)^2 + b^2c(2b+2c-3a)^2 + 3a^2b^2c}{2a^3(b+c)(-a+3b+c)^2}$$
$$> 0,$$

completing the proof of the corollary by Theorem 2. □

Observe that

$$\partial_\varepsilon \left[\frac{b^2(3a - b - (c+\varepsilon))}{a^2(-a+3b+(c+\varepsilon))} - \tau\right]\Bigg|_{\varepsilon=0} = -\frac{2b^2(a+b)}{a^2(-a+3b+c)^2},$$

$$\partial_\varepsilon \left[\frac{b^2(3a-b-c)}{a^2(-a+3b+c)} - (\tau+\varepsilon)\right]\Bigg|_{\varepsilon=0} = -1.$$

Therefore, Theorem 2 yields the following results.

Corollary 3. *If* $(a, b, c, \tau) \in \sum$, *then system (13) with* $(j(\varepsilon), k(\varepsilon), p(\varepsilon), q(\varepsilon)) = (a, b, c+\varepsilon, \tau)$:

$$\begin{cases} \dot{x} = ax - (x-y)z \\ \dot{y} = -by - (x-y)z \\ \dot{z} = -(c+\varepsilon)z + (x-y)(x+\tau y), \end{cases} \tag{19}$$

has a Hopf bifurcation at S_+ *as* ε *passes* 0. *Moreover,*

(1) when $\mu(a, b, c) < 0$, *the bifurcating periodic orbit is unstable and the direction of bifurcation is* $\varepsilon > 0$;

(2) when $\mu(a, b, c) > 0$, *the bifurcating periodic orbit is asymptotically stable and the direction of bifurcation is* $\varepsilon < 0$.

Corollary 4. *If* $(a, b, c, \tau) \in \sum$, *then system (13) with* $(j(\varepsilon), k(\varepsilon), p(\varepsilon), q(\varepsilon)) = (a, b, c, \tau+\varepsilon)$:

$$\begin{cases} \dot{x} = ax - (x-y)z \\ \dot{y} = -by - (x-y)z \\ \dot{z} = -cz + (x-y)(x+(\tau+\varepsilon)y), \end{cases} \tag{20}$$

has a Hopf bifurcation at S_+ *as* ε *passes* 0. *Moreover,*

(1) when $\mu(a, b, c) < 0$, *the bifurcating periodic orbit is unstable and the direction of bifurcation is* $\varepsilon > 0$;

(2) when $\mu(a, b, c) > 0$, *the bifurcating periodic orbit is asymptotically stable and the direction of bifurcation is* $\varepsilon < 0$.

4. Conclusions

In this paper, Hopf bifurcation of the generalized Lorenz canonical form (GLCF) has been investigated, complementing the existing bifurcation analyses on the classic Lorenz and the new Chen's systems. Based on rigorous mathematical analysis and symbolic computations, some conditions for stability and direction of the periodic obits from the Hopf bifurcation of the GLCF have been obtained. This investigation enhances our understanding of the very large family of chaotic systems, the GLCF.

Acknowledgements This research was supported by the Hong Kong Research Grants Council under the CERG grant CityU1115/03E and NNSF of China grant 10471072 NNSF of China grant 10672084.

References

1. Sparrow, C.: The Lorenz Equations: Bifurcation, Chaos, and Strange Attractor. New York, Springer-Verlag (1982)
2. Guckenheimer, J., Holmes, P.J.: Nonlinear Oscillations Dynamical Systems and Bifurcations of Vector Fields. New York, Springer-Verlag (1983)
3. Kuznetsov, Y.A.: Elements of Applied Bifurcation Theory. New York, Springer-Verlag (1995)
4. Luo, D., Wang, X., Zhu, D., Han, M.: Bifurcation Theory and Methods of Dynamical Systems. Singapore, World Scientific (1997)
5. Chen, G., Ueta, T.: Yet another chaotic attractor. Int. J. Bifurcation Chaos **9**, 1465–1466 (1999)
6. Ueta, T., Chen, G.: Bifurcation analysis of Chen's equation. Int. J. Bifurcation Chaos **10**, 1917–1931 (2000)
7. Zhou, T.S., Chen, G., Tang, Y.: Chen's attractor exists. Int. J. Bifurcation Chaos **14**, 3167–3178 (2004)
8. Li, T.C., Chen, G., Tang, Y.: On stability and bifurcation of Chen's system. Chaos Solitons Fractals **19**, 1269–1282 (2004)
9. Chang, Y., Chen, G.: Complex dynamics in Chen's system. Chaos Solitons Fractals **27**, 75–86 (2006)
10. Zhou, T.S., Tang, Y., Chen, G.: Complex dynamical behaviors of the chaotic Chen's System. Int. J. Bifurcation Chaos **13**(9), 2561–2574 (2003)
11. Yu, X., Xia, Y.: Detecting unstable periodic orbits in Chen's chaotic attractor. Int. J. Bifurcation Chaos **10**, 1987–1991 (2000)
12. Čelikovský, S., Chen, G.: Hyperbolic-type generalized Lorenz system and its canonical form, In: Proceedings of the 15th Triennial World Congress of IFAC. Barcelona, Spain, July 2002
13. Čelikovský, S., Chen, G.: On the generalized Lorenz canonical form. Chaos Solitons Fractals **26**, 1271–1276 (2005)
14. Zhou, T.S., Chen, G., Čelikovský, S.: Si'lnikov chaos in the generalized Lorenz canonical form of dynamics systems. Nonlinear Dyn. **39**, 319–334 (2005)
15. Bi, Q., Yu, P.: Symbolic computation of normal forms for semi-simple cases. J. Comput. Appl. Math. **102**(2), 195–220 (1999)
16. Chow, S.N., Li, C.Z., Wang, D.: Normal Forms and Bifurcation of Planar Vector Fields. Cambridge, Cambridge University Press (1994)
17. Lynch, S.: Dynamical Systems with Application Using Maple. Boston, Birkhauser (2001)
18. Yu, P.: Simplest normal forms of Hopf and generalized Hopf bifurcations. Int. J. Bifurcation Chaos **9**(10), 1917–1939 (1999)
19. Yu, P.: Symbolic computation of normal forms for resonant double Hopf bifurcations using a perturbation technique. J. Sound Vib. **247**(4), 615–632 (2001)
20. Čelikovský, S., Chen, G.: On a generalized Lorenz canonical form of chaotic systems. Int. J. Bifurcation Chaos **12**, 1789–1812 (2002)
21. Chen, G.: Beyond the Lorenz System,' In: Proceedings of Chinese Conference on Dynamics, Nonlinear Vibrations, and Motion Stability. Nanjing, China, Oct. 28–31, (2004)
22. Lü, J., Chen, G.: A new chaotic attractor coined. Int. J. Bifurcation Chaos **12**, 659–661 (2002)
23. Shimizu, T., Morioka, N.: On the bifurcation of a symmetric limit cycle to an asymmetric one in a simple model. Physics Letters A **76**, 201–204 (1976)