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Stability and Boundedness Results for Solutions of Certain Third Order Nonlinear Vector Differential Equations

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Abstract. In this paper, we investigate the asymptotic stability of the zero solution and boundedness of all solutions of a certain third order nonlinear ordinary vector differential equation. The results are proved using Lyapunov's second (or direct method). Our results include and improve some well known results existing in the literature.

Key words: boundedness, differential equation of third order, Lyapunov function, stability

1. Introduction

From the relevant literature, it is well known that in a sequence of results the third order ordinary scalar differential equations of the form

$$\dddot{x} + a_1\ddot{x} + a_2\dot{x} + a_3x = p(t, x, \dot{x}, \ddot{x}) \tag{1}$$

in which a_1, a_2 and a_3 are not necessarily constants, have been the object of much study by several authors till now. In these works, the qualitative behaviors of solutions of third order nonlinear ordinary scalar differential equations of the form (1), namely, stability of solutions, instability of solutions, boundedness of solutions, existence of periodic solutions and boundary value problems have been studied extensively by the authors. One may refer to [1], for a survey, as well as [2–15] and the references quoted therein for some publications on these topics. However, according to our observations in the relevant literature, only a few researches (see [16–25]), have been carried out about the stability, boundedness of solutions and the existence of periodic solutions of third order nonlinear vector differential equations of the form

$$\dddot{X} + A_1\ddot{X} + A_2\dot{X} + A_3X = P(t, X, \dot{X}, \ddot{X}) \tag{2}$$

in which $X \in \Re^n$, $t \in [0, \infty)$, A_1, A_2 and A_3 are not necessarily $n \times n$-constant matrices and $P\colon \Re^+ \times \Re^n \times \Re^n \times \Re^n \to \Re^n$. Now, some researches related to equations of the form (2) can be summarized as follows: In 1975, Edziwy Spolhk [19] obtained sufficient conditions for global asymptotic stability of the trivial solution of the third order linear vector differential equation

$$\dddot{Y} + A\ddot{Y} + B\dot{Y} + CY = 0,$$

where A, B and C are $n \times n$-constant symmetric matrices. In 1966, 1983 and 1993, respectively, Ezeilo and Tejumola [20], Afuwape [17] and Meng [23] studied the ultimately boundedness and existence of periodic solutions of the nonlinear vector differential equation

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}).$$

Later, in 1985, Afuwape [18] also considered the vector differential equation

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

and for the above equation the author gives ultimate boundedness results that are generalizations of some earlier works exiting in the literature. Besides these works, in 1985, Abou-El-Ela [16] gave sufficient conditions that ensure that all solutions of real vector differential equations of the form

$$\dddot{X} + F(X, \dot{X})\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

are ultimately bounded. Afterward, in 1995, Feng [21] established sufficient conditions under which the nonlinear vector differential

$$\dddot{X} + A(t)\ddot{X} + B(t)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X})$$

equation has at least unique periodic solution. Further, in 1999, Tiryaki [24] constituted similar results for solutions of vector differential equations of the form

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}),$$

and Tunç [25] also proved some results on the ultimate boundedness of solutions and the existence of periodic solutions of vector differential equation

$$\dddot{X} + F(X, \dot{X})\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}).$$

It is worth mentioning that the first author of this paper (see [26, 27]), more recently, established some similar results on the same topic for the third order nonlinear scalar differential equations as follows:

$$\dddot{x} + a(t)f(x, \dot{x}, \ddot{x})\ddot{x} + b(t)g(x, \dot{x}) + c(t)h(x) = p(t)$$

and

$$\dddot{x} + a(t)f(x, \dot{x}, \ddot{x})\ddot{x} + b(t)g(x, \dot{x}) + c(t)h(x) = p(t, x, \dot{x}, \ddot{x}),$$

and

$$\dddot{x} + \psi(x, \dot{x}, \ddot{x})\ddot{x} + f(x, \dot{x}) = p(t, x, \dot{x}, \ddot{x}),$$

respectively.

This research is concerned with nonlinear vector differential equations of the form

$$\dddot{X} + F(X, \dot{X}, \ddot{X})\ddot{X} + B(t)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}), \quad (3)$$

where $X \in \Re^n$ and $t \in [0, \infty)$; F and B are $n \times n$-symmetric continuous matrix functions; $H\colon \Re^n \to \Re^n$ and $P\colon \Re^+ \times \Re^n \times \Re^n \times \Re^n \to \Re^n$, and H and P are continuous. Let

$$J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right), \quad \dot{B}(t) = \frac{d}{dt}(b_{ij}(t)), \quad (i, j = 1, 2, \ldots, n),$$

where $(x_1, x_2, \dots, x_n)$, $(h_1, h_2, \dots, h_n)$ and $(b_{ij}(t))$ are the components of X, H and B, respectively. It is also assumed, as basic throughout what follows, that the Jacobian matrices $J_H(X)$ and $\dot{B}(t)$ exist and are symmetric and continuous.

In what follows it will be convenient to use the equivalent differential system:

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= -F(X, Y, Z)Z - B(t)Y - H(X) + P(t, X, Y, Z), \end{aligned} \tag{4}$$

obtained from (3) by setting $\dot{X} = Y$, $\ddot{X} = Z$. The symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, and $\lambda_i(A)(i = 1, 2, \dots, n)$ are the eigenvalues of $n \times n$-matrix.

The motivation for the present investigation has come from the papers mentioned above. It should be also noted that the equation studied here is different than that considered in the earlier papers [7, 16–21, 24, 25].

2. Stability and Boundedness of Solutions

In the case $P \equiv 0$ the following result is established.

Theorem 1. *In addition to the fundamental assumptions on F, B and H, suppose that:*

(i) *There exists an $n \times n$-real continuous operator $A(X, Y)$ for any vectors X, Y in $\Re^n$ such that*

$$H(X) = H(Y) + A(X, Y)(X - Y), \quad (H(0) = 0),$$

whose eigenvalues $\lambda_i(A(X, Y))$, $(i = 1, 2, \dots, n)$, satisfy

$$0 < \delta_h \le \lambda_i(A(X, Y)) \le \Delta_h$$

for fixed constants δ_h and Δ_h;

(ii) *There exists a real $n \times n$-constant symmetric matrix A such that the matrices A, $B(t)$, $\dot{B}(t)$, $(F(X, Y, Z) - A)$ have positive eigenvalues and pair wise commute with themselves as well as with the operator $A(X, Y)$ for any X, Y in $\Re^n$, and that*

$$\delta_a = \min_{1\le i\le n} \{\lambda_i(A), \lambda_i(F(X, Y, Z))\}, \quad \Delta_a = \max_{1\le i\le n} \{\lambda_i(A), \lambda_i(F(X, Y, Z))\},$$

$$\delta_b = \min_{1\le i\le n, t\in[0,\omega]} (\lambda_i(B(t))), \quad \Delta_b = \max_{1\le i\le n, t\in[0,\omega]} (\lambda_i(B(t))),$$

$\Delta_h \le k\delta_a\delta_b$*(where k is positive constant to be determined later in the proof),*

$$0 \le \lambda_i(F(X, Y, Z) - A) \le \frac{\sqrt{\varepsilon}}{2} \quad \text{and} \quad \varepsilon = \max |\lambda_i(\dot{B}(t))| \quad (i = 1, 2, \dots, n),$$

where

$$\varepsilon \le \frac{1}{2} \min \left\{ \left(\frac{\delta_b\delta_h}{4\Delta_b + 4}\right)^2, \left(\frac{\delta_a\delta_b}{6\Delta_a + 7}\right)^2, \frac{\delta_a^2}{4}, 1 \right\}.$$

Then, the zero solution of system (4) is asymptotically stable.

In the case $P \neq 0$ we have the following result.

Theorem 2. *Let all the conditions of Theorem 1 be satisfied, and in addition we assume that there exist a finite constant $K > 0$ and a non-negative and continuous function $\theta = \theta(t)$ such that the vector*

P satisfies

$$\|P(t, X, Y, Z)\| \le \theta(t) + \theta(t)(\|X\| + \|Y\| + \|Z\|),$$

where $\int_0^t \theta(s)ds \le K < \infty$ *for all* $t \ge 0$. *Then there exists a constant* $D > 0$ *such that any solution* $(X(t), Y(t), Z(t))$ *of (4) determined by*

$$X(0) = X_0, \qquad Y(0) = Y_0, \qquad Z(0) = Z_0$$

satisfies

$$\|X(t)\| \le D, \quad \|Y(t)\| \le D, \quad \|Z(t)\| \le D$$

for all $t \ge 0$.

Remark. This study has indicated that Feng's result could also be achieved without his assumption

$$\varepsilon = \max|\lambda_i(\dot{A}(t))| \quad \text{and} \quad \varepsilon \le \frac{1}{2}\min\left\{\left(\frac{\delta_b\delta_h}{4\Delta_b + 4}\right)^2, \left(\frac{\delta_a\delta_b}{6\Delta_a + 7}\right)^2, \frac{\delta_a^2}{4}, 1\right\}$$

established in [21]. Thus, our results improve and include the result in [21].

Our main tool, in the proofs of the theorems, is the function Lapunov's $V = V(t, X, Y, Z)$ defined by

$$2V = \frac{1}{4}\langle BX, BX\rangle + \frac{3}{2}\langle BY, Y\rangle + \langle Z, Z\rangle + \left\langle Z + AY + \frac{1}{2}BX, Z + AY + \frac{1}{2}BX\right\rangle, \tag{5}$$

where A is an $n \times n$-constant matrix and $B = B(t)$ is an $n \times n$-matrix function.

Now, we dispose of some well known algebraic results which will be required in the proofs. The first of these is quite standard one:

Lemma 1. *Let D be a real symmetric* $n \times n$ *matrix. Then for any* $X \in \Re^n$

$$\delta_d\|X\|^2 \le \langle DX, X\rangle \le \Delta_d\|X\|^2,$$

where δ_d *and* Δ_d *are, respectively, the least and greatest eigenvalues of the matrix D.*

Proof. See [17]. □

Next, we require the following lemma.

Lemma 2. *Let Q, D be any two real* $n \times n$ *commuting symmetric matrices. Then*

(i) *The eigenvalues* $\lambda_i(QD)(i = 1, 2, \ldots,)$ *of the product matrix QD are real and satisfy*

$$\max_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D) \ge \lambda_i(QD) \ge \min_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D).$$

(ii) *The eigenvalues* $\lambda_i(Q + D)(i = 1, 2, \dots,)$ *of the sum of matrices* Q *and* D *are real and satisfy*

$$\left\{\max_{1\le j\le n} \lambda_j(Q) + \max_{1\le k\le n} \lambda_k(D)\right\} \ge \lambda_i(Q + D) \ge \left\{\min_{1\le j\le n} \lambda_j(Q) + \min_{1\le k\le n} \lambda_k(D)\right\},$$

where $\lambda_j(Q)$ *and* $\lambda_k(D)$ *are, respectively, the eigenvalues of* Q *and* D.

Proof. See [17]. □

Now, the properties of the function $V(t, X, Y, Z)$ are summarized with Lemmas 3 and 4.

Lemma 3. *If the conditions of Theorem 1 hold, then there exists a positive constant* δ_1 *such that*

$$V(t, X, Y, Z) \ge \delta_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2)$$

is valid for every solution of (4).

Proof. Now, V here is the same as the function V defined in [21], except the case A is an $n \times n$-constant matrix. If we realize similar estimates as shown for V there, we easily obtain that

$$V(t, X, Y, Z) \ge \delta_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2). \tag{6}$$

This completes the proof of the lemma. □

Let $(X(t), Y(t), Z(t))$ be an arbitrary solution of (4). Define $v(t) = V(t, X(t), Y(t), Z(t))$. We can easily prove the following lemma.

Lemma 4. *Assume that all the conditions of Theorem 1 are satisfied. Then*

$$\dot{v}(t) \le 0 \quad \text{for all } t \ge 0 \tag{7}$$

and especially

$$\dot{v}(t) = \frac{d}{dt}V(t, X, Y, Z) < 0 \textit{ provided } \|X\|^2 + \|Y\|^2 + \|Z\|^2 > 0. \tag{8}$$

Proof. A straightforward calculation from (5) and (4) yields that

$$\dot{v} = \frac{d}{dt}V(t, X(t), Y(t), Z(t)) = -V_1 - V_2 - V_3,$$

where

$$\begin{aligned}
V_1 = {} & \frac{1}{4}\langle BX, H(X)\rangle + \frac{1}{4}\langle AY, BY\rangle + \frac{1}{2}\langle F(X, Y, Z)Z, Z\rangle - \frac{1}{2}\langle BX, \dot{B}X\rangle \\
& - \frac{1}{2}\langle \dot{B}X, Z\rangle - \frac{3}{4}\langle \dot{B}Y, Y\rangle - \frac{1}{2}\langle AY, \dot{B}X\rangle + \frac{1}{2}\langle BX, (F(X, Y, Z) - A)Z\rangle \\
& + \langle AY, (F(X, Y, Z) - A)Z\rangle + \langle (F(X, Y, Z) - A)Z, Z\rangle, \\
V_2 = {} & \frac{1}{8}\langle BX, H(X)\rangle + 2\langle H(X), Z\rangle + \frac{1}{2}\langle F(X, Y, Z)Z, Z\rangle, \\
V_3 = {} & \frac{1}{8}\langle BX, H(X)\rangle + \langle H(X), AY\rangle + \frac{1}{4}\langle AY, BY\rangle.
\end{aligned}$$

In view of the assumptions of Theorem 1, it follows that

$$
\begin{aligned}
H(X) &= H(0) + A(X,0)X = A(X,0)X,\\
\langle BX, H(X)\rangle &= \langle BX, A(X,0)X\rangle \geq \delta_b\delta_h\|X\|^2,\\
\langle AY, BY)\rangle &\geq \delta_a\delta_b\|Y\|^2,\ \langle F(X,Y,Z)Z, Z\rangle \geq \delta_a\|Z\|^2, \quad -\langle BX, \dot{B}X\rangle\\
&\geq -\Delta_b\sqrt{\varepsilon}\|X\|^2, \quad -\langle \dot{B}Y, Y\rangle \geq -\sqrt{\varepsilon}\|X\|^2,\\
\langle (F(X,Y,Z) - A)Z, Z\rangle &\geq 0.
\end{aligned}
$$

Obviously, for some constants $k_1 > 0$, $k_2 > 0$ (which will be chosen later), we obtain

$$
\begin{aligned}
-\frac{1}{2}\langle AY, \dot{B}X\rangle &= \frac{1}{4}\left\|k_1^{-1}\sqrt{A\dot{B}}X - k_1\sqrt{A\dot{B}}Y\right\|^2 - \frac{1}{4}k_1^{-2}\langle AX, \dot{B}X\rangle - \frac{1}{4}k_1^2\langle AY, \dot{B}Y\rangle\\
&\geq -\frac{1}{4}k_1^{-2}\langle AX, \dot{B}X\rangle - \frac{1}{4}k_1^2\langle AY, \dot{B}Y\rangle \geq -\frac{1}{4}k_1^{-2}\sqrt{\varepsilon}\Delta_a\|X\|^2 - \frac{1}{4}k_1^2\sqrt{\varepsilon}\Delta_a\|Y\|^2.
\end{aligned}
$$

Taking

$$
k_1^2 = \min\left\{2^{-1}\Delta_a\Delta_b^{-1}, 4^{-1}\right\},
$$

we get

$$
-\frac{1}{2}\langle AY, \dot{B}X\rangle \geq -\frac{1}{2}\sqrt{\varepsilon}\Delta_b\|X\|^2 - \frac{1}{16}\sqrt{\varepsilon}\Delta_a\|Y\|^2.
$$

Next, we have

$$
\begin{aligned}
\frac{1}{2}\langle BX, (F(X,Y,Z) - A)Z\rangle &= \frac{1}{4}\left\|k_2^{-1}\sqrt{B}\sqrt{F(X,Y,Z) - A}X - k_2\sqrt{B}\sqrt{F(X,Y,Z) - A}Z\right\|^2\\
&\quad -\frac{1}{4}k_2^{-2}\langle BX, (F(X,Y,Z) - A)X\rangle - \frac{1}{4}k_2^2\langle BZ, (F(X,Y,Z) - A)Z\rangle\\
&\geq -\frac{1}{4}k_2^{-2}\langle BX, (F(X,Y,Z) - A)X\rangle - \frac{1}{4}k_2^2\langle BZ, (F(X,Y,Z) - A)Z\rangle\\
&\geq -\frac{1}{8}k_2^{-2}\sqrt{\varepsilon}\Delta_b\|X\|^2 - \frac{1}{8}k_2^2\sqrt{\varepsilon}\Delta_b\|Z\|^2.
\end{aligned}
$$

If we choose $k_2^2 = \min\{4^{-1}\Delta_b, \Delta_b^{-1}\}$, then

$$
\frac{1}{2}\langle BX, (F(X,Y,Z) - A)Z\rangle \geq -\frac{1}{2}\sqrt{\varepsilon}\|X\|^2 - \frac{1}{8}\sqrt{\varepsilon}\|Z\|^2.
$$

Finally, in similar way, we can easily obtain

$$
\langle AY, (F(X,Y,Z) - A)Z\rangle \geq -\frac{1}{4}\sqrt{\varepsilon}\Delta_a\|Y\|^2 - \frac{5}{8}\sqrt{\varepsilon}\|Z\|^2.
$$

Thus, take into consideration the above discussion, it follows that

$$
\begin{aligned}
V_1 &\geq [(1/4)\delta_b\delta_h - (\Delta_b + 1)\sqrt{\varepsilon}]\cdot\|X\|^2 + [(1/4)\delta_a\delta_b - (1/4)(6\Delta_a + 7)\sqrt{\varepsilon}]\cdot\|Y\|^2\\
&\quad + [(1/2)\delta_a - \sqrt{\varepsilon}]\cdot\|Z\|^2.
\end{aligned}
$$

Following the procedure indicated just above, we then conclude that

$$V_2 \geq 0, \quad V_3 \geq 0.$$

Hence

$$\begin{aligned}\dot{v} &\leq -[(1/4)\delta_b\delta_h - (\Delta_b + 1)\sqrt{\varepsilon}] \cdot \|X\|^2 - [(1/4)\delta_a\delta_b - (1/4)(6\Delta_a + 7)\sqrt{\varepsilon}] \cdot \|Y\|^2 \\ &\quad - [(1/2)\delta_a - \sqrt{\varepsilon}] \cdot \|Z\|^2 \\ &= -D_1\|X\|^2 - D_2\|Y\|^2 - D_3\|Z\|^2 < 0,\end{aligned}$$

where

$$D_1 = \lfloor(1/4)\delta_b\delta_h - (\Delta_b + 1)\sqrt{\varepsilon}\rfloor > 0, \quad D_2 = \lfloor(1/4)\delta_a\delta_b - (1/4)(6\Delta_a + 7)\sqrt{\varepsilon}\rfloor > 0,$$
$$D_3 = \lfloor(1/2)\delta_a - \sqrt{\varepsilon}\rfloor > 0$$

because of assumption (ii) of Theorem 1.

This completes the proof. □

Proof of Theorem 1. From Lemmas 3 and 4, we see that the function $V(t, X, Y, Z)$ is a Lyapunov function of system (4). Hence, the zero solution of system (4) is asymptotically stable [1].

This completes the proof of the theorem. □

Proof of Theorem 2. Consider the function V defined by (5). Then under the assumptions of Theorem 2 the conclusion of Lemma 3 can be obtained, that is,

$$V \geq \delta_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \tag{9}$$

and since $P(t, X, Y, Z) \neq 0$, then the conclusion of Lemma 4 can be revised as follows

$$\dot{v} = \frac{d}{dt}V \leq \langle(1/2)BX + AY + 2Z, P(t, X, Y, Z)\rangle.$$

Next, by noting the assumption of Theorem 2 on $P(t, X, Y, Z)$ and using Schwarz's inequality, we obtain

$$\begin{aligned}\dot{v} &\leq ((1/2)\|BX\| + \|AY\| + \|2Z\|) \times \|P(t, X, Y, Z)\| \\ &\leq ((1/2)\Delta_b\|X\| + \Delta_a\|Y\| + 2\|Z\|) \times (\theta(t) + \theta(t)(\|X\| + \|Y\| + \|Z\|)) \\ &\leq D_4(\|X\| + \|Y\| + \|Z\|) \times (\theta(t) + \theta(t)(\|X\| + \|Y\| + \|Z\|)),\end{aligned}$$

where $D_4 = \max\{(1/2)\Delta_b, \Delta_a, 2\}$.

Hence, by using the inequalities

$$\|X\| \leq 1 + \|X\|^2, \quad \|Y\| \leq 1 + \|Y\|^2, \quad \|Z\| \leq 1 + \|Z\|^2$$

and (9), we obtain

$$\dot{\nu} \leq D_5\theta(t) + D_6\theta(t)\nu, \tag{10}$$

where $D_5 = 3D_4$ and $D_6 = 4D_4\delta_1^{-1}$.

Integrating both sides of between (10) from 0 to $t(t \geq 0)$, leads to the inequality

$$\nu(t) - \nu(0) \leq D_5 \int_0^t \theta(s)\,ds + D_6 \int_0^t \nu(s)\theta(s)\,ds.$$

On putting $D_7 = \nu(0) + D_5 K$, it follows that

$$\nu(t) \leq D_7 + D_6 \int_0^t \nu(s)\theta(s)\,ds.$$

Gronwall–Bellman inequality yields

$$\nu(t) \leq D_7 \exp\left(D_6 \int_0^t \theta(s)\,ds\right).$$

The proof of the theorem is now complete. □

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