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# Homotopy Solutions for a Generalized Second-Grade Fluid Past a Porous Plate

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**Abstract.** The flow of a second-grade fluid past a porous plate subject to either suction or blowing at the plate has been studied. A modified model of second-grade fluid that has shear-dependent viscosity and can predict the normal stress difference is used. The differential equations governing the flow are solved using homotopy analysis method (HAM). Expressions for the velocity have been constructed and discussed with the help of graphs. Analysis of the obtained results showed that the flow is appreciably influenced by the material and normal stress coefficient. Several results of interest are deduced as the particular cases of the presented analysis.

**Key words:** generalized second-grade fluid, HAM solutions, hydrodynamic fluid, porous plate, shear-dependent viscosity

## 1. Introduction

The governing equations that describe the flow of a Newtonian fluid are the Navier–Stokes equations. There are many fluids (such as polymeric liquids, biological fluids, liquid crystals) that show viscoelastic behaviour and cannot be described simply as Newtonian fluids. Interest in flows of non-Newtonian fluids has increased substantially over the past decades because of wide use of these fluids in chemical process industries, food and construction engineering, in petroleum production, in power engineering and commercial applications. Moreover, the boundary layer concept of non-Newtonian fluids is of special importance owing to its application to many engineering problems, among which we cite the possibility of reducing frictional drag on the hulls of ships and submarines.

Due to the complexity of fluids, many constitutive equations have been proposed. Amongst the many models that have been used to describe the non-Newtonian behaviour exhibited by certain fluids, the fluids of differential type have acquired special status. The non-linear response of such fluids constitute an important area of mathematical modelling. The equation of motion are highly non-linear and finding analytic solutions is not easy. Although the power-law model [1, 2] is mostly used in engineering but it cannot capture the normal stress differences or yield stresses. One particular class of differential type fluids for which one can reasonably hope to derive analytic solutions is class of second-grade fluids. Also, second-grade fluid model can predict the normal stress effects (which could lead to phenomena such as 'die swell' and 'rod-climbing'). Dunn and Fosdick [3], Dunn and Rajagopal [4], and Fosdick and Rajagopal [5] have performed complete thermodynamic analysis of second-grade fluids. Important theoretical studies of such fluids have been made by Rajagopal [6–8], Hayat et al. [9–12], Benharbit and Siddiqui [13], Rajagopal and Gupta [14], Bandelli and Rajagopal [15], Bendelli [16], Fetecau and Zierep [17], and Fetecau et al. [18].

It is worth mentioning that the departure of viscoelastic behaviour of non-Newtonain fluids from the Navier–Stokes equations manifests itself in a variety of ways: non-Newtonian viscosity (shear thinning or shear thickening), stress relaxation, non-linear creeping, development of normal stress differences and yield stress. The interesting case of non-linearity is the generalized second-grade fluid which results from a modification of constitutive equation incorporating a shear–rate-dependent viscosity. The generalized second-grade fluid model is not only capable of predicting normal stress differences but it can also be used for shear thinning and shear thickening also. Valuable contributions in this direction have been made by Man [19], Massoudi and Phuoc [20], and Straughan [21, 22]. Gupta and Massoudi [23], and Franchi and Straughan [24] also examined the second-grade models when viscosity is a function of temperature.

In this work, we analyse the flow of a generalized second-grade fluid past a porous plate. In the constitutive equation, the viscosity depends on the shear rate. The layout of the paper is as follows. In Section 2, the formulation of the problem is given and basic notation is introduced. In Section 3, the solutions for velocity are given using homotopy analysis method (HAM) [25–35]. Section 4 deals with the discussion of the results. Finally, conclusions are given in Section 5.

## 2. Development of the Flow

Let us consider the flow of a generalized second-grade fluid (in the region $0 \leq y < \infty$) past a porous plate at $y = 0$. The flow far away from the plate is uniform. A coordinate system is chosen in which $x$-axis is parallel to the plate and the $y$-axis perpendicular to the plate. $u$ and $v$ denote the velocity components in the $x$- and $y$-directions, respectively. The incompressible flow is governed by the continuity, linear momentum:

$$\operatorname{div} \mathbf{V} = 0. \tag{1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}. \tag{2}$$

In previous equations $\mathbf{V}$ is the velocity vector, $t$ the time, $\rho$ the density, $\mathbf{b}$ the body force and $d/dt$ the material time derivative.

The Cauchy stress tensor $\mathbf{T}$ for generalized second-grade fluid is given by [19]

$$\mathbf{T} = -p\mathbf{I} + \mu \Pi^{m/2} \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{3}$$

where $\mathbf{I}$ is the identity tensor, $p$ the pressure, $\mu$ the viscosity coefficient, $\alpha_1$ and $\alpha_2$ the normal stress coefficients, $m$ the material parameter and

$$\Pi = \frac{1}{2} tr \mathbf{A}_1^2 \tag{4}$$

is the second invariant of the symmetric part of the velocity gradient. The kinematical tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined through [36]

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^\top, \tag{5}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1 \mathbf{L} + \mathbf{L}^\top \mathbf{A}_1, \tag{6}$$

where

$\mathbf{L} = \operatorname{grad} \mathbf{V}$.

It is worth mentioning here that power-law fluid has a shear-dependent viscosity but which can exhibit no normal stress differences. The generalized second-grade fluid, which has been used successfully in modelling the flow of icy mesh [37], exhibits both shear thinning and normal stress differences.

It should be noted that for $m = 0$, the model defined by Equation (3) reduces to that considered by Rajagopal [6–8], Rajagopal and Gupta [14], Hayat et al. [9–12], and Siddiqui et al. [38]. If $\alpha_1 = \alpha_2 = 0$, on the other hand, the power-law model [1] is recovered from Equation (3). Moreover, if $\alpha_1 = \alpha_2 = 0$ and $m = 0$ we are left with the classical model of Navier and Stokes.

For the problem in question, let us seek velocity field of the following form

$$\mathbf{V} = u(y)\hat{i} + v(y)\hat{j}, \tag{7}$$

where $\hat{i}$ and $\hat{j}$ are the unit vectors in the $x$- and $y$-directions. It follows from Equation (1) that

$$v(y) = -V_0 = \text{constant} \tag{8}$$

in which $V_0 > 0$ is a scale of suction velocity and $V_0 < 0$ is blowing at the plate.

Using Equations (7) and (8) and employing the same arguments as in Rajagopal and Gupta [14], Equation (2) in absence of body forces gives

$$\frac{d}{dy}\left\{\mu\left[\left|\frac{du}{dy}\right|^2\right]^{m/2}\frac{du}{dy}\right\} + \rho V_0\frac{du}{dy} - \alpha_1 V_0\frac{d^3u}{dy^3} = 0. \tag{9}$$

The boundary conditions are

$$\begin{aligned} &u(0) = 0, \\ &\quad u \to U_\infty \quad \text{as} \quad y \to \infty. \end{aligned} \tag{10}$$

Since Equation (9) is a third order ordinary differential equation, we thus have one boundary condition less than that necessary to solve Equation (9). While it is possible to augment the boundary conditions based on the asymptotic structures for the velocity field or the stress. We thus have

$$\frac{du}{dy} \to 0 \quad \text{as} \quad y \to \infty \tag{11}$$

as there is no shear in the free stream.

Let us introduce the following non-dimensional parameters

$$y^* = \frac{yU_\infty}{\nu}, \quad u^* = \frac{u}{U_\infty}, \quad V_0^* = \frac{V_0}{U_\infty}, \quad \mu^* = \frac{\mu}{\mu_0}. \tag{12}$$

Equation (9) and boundary conditions (10) and (11) after dropping asterisks take the following form

$$\alpha V_0\frac{d^3u}{dy^3} - V_0\frac{du}{dy} - \mu\Gamma\frac{d}{dy}\left\{\left[\left|\frac{du}{dy}\right|^2\right]^{m/2}\frac{du}{dy}\right\} = 0, \tag{13}$$

$$\begin{aligned} &u(0) = 0, \\ &\quad u \to 1 \quad \text{as} \quad y \to \infty, \\ &\frac{du}{dy} \to 0 \quad \text{as} \quad y \to \infty, \end{aligned} \tag{14}$$

where

$$\alpha = \frac{\alpha_1 U_\infty^2}{\rho \nu^2}, \quad \Gamma_m = \frac{\mu_0}{\rho U_\infty^2}\left(\frac{U_\infty^2}{\nu}\right)^{m+1}. \tag{15}$$

## 3. Analytic Solutions for Integer Material Parameter (*m*)

Here, we seek solutions with $du/dy > 0$, $\alpha_1 > 0$ and consider the case that $m$ is a positive integer. When $m = 1$, Equation (13) becomes

$$\alpha V_0 \frac{d^3u}{dy^3} - V_0 \frac{du}{dy} - 2\mu\Gamma_1 \frac{du}{dy}\frac{d^2u}{dy^2} = 0, \tag{16}$$

When $m = 2$, Equation (13) is

$$\alpha V_0 \frac{d^3u}{dy^3} - V_0 \frac{du}{dy} - 3\mu\Gamma_2 \left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} = 0, \tag{17}$$

and when $m = 3$,

$$\alpha V_0 \frac{d^3u}{dy^3} - V_0 \frac{du}{dy} - 4\mu\Gamma_3 \left(\frac{du}{dy}\right)^3 \frac{d^2u}{dy^2} = 0, \tag{18}$$

in which $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ can be taken from Equation (15).

Now to solve the non-linear differential equations (16)–(18) subject to boundary conditions (14), we apply HAM to give an explicit, uniformly valid and totally analytic solutions.

### 3.1. The Zero-Order Deformation Equation

Due to the boundary conditions (14) and governing Equation (13), it is straightforward to choose

$$u_0(y) = \left(1 - e^{-\frac{y}{\sqrt{\alpha}}}\right), \tag{19}$$

as the initial guess of $u(y)$ and

$$\mathcal{L}[\bar{u}(y;p)] = \left(\alpha V_0 \frac{\partial^3}{\partial y^3} - V_0 \frac{\partial}{\partial y}\right)\bar{u}(y;p) \tag{20}$$

as the auxiliary linear operator, respectively.

Furthermore, we define using Equation (13) the non-linear operator

$$\mathcal{N}[\bar{u}(y;p)] = \alpha V_0 \frac{\partial^3 \bar{u}(y;p)}{\partial y^3} - V_0 \frac{\partial \bar{u}(y;p)}{\partial y} - \mu\Gamma_m \frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial \bar{u}(y;p)}{\partial y}\right)^2\right]^{m/2} \frac{\partial \bar{u}(y;p)}{\partial y}\right\}. \tag{21}$$

Then, we construct the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\bar{u}(y;p) - u_0(y)] = p\hbar\, \mathcal{N}[\bar{u}(y;p)], \tag{22}$$

subject to the boundary conditions

$$\begin{aligned} \bar{u}(0; p) &= 0, \\ \bar{u}(y; p) &\to 1 \quad \text{as} \quad y \to \infty, \\ \frac{\partial \bar{u}(y; p)}{\partial y} &\to 0 \quad \text{as} \quad y \to \infty, \end{aligned} \tag{23}$$

where $p$ is an embedding parameter and $\hbar$ is a non-zero auxiliary parameter.

When $p = 0$, the solution of Equations (22) and (23) is

$$\bar{u}(y; 0) = u_0(y). \tag{24}$$

When $p = 1$, Equations (22) and (23) are equivalent to Equations (13) and (14), provided

$$\bar{u}(y; 1) = u(y). \tag{25}$$

Thus, as $p$ increases from 0 to 1, $\bar{u}(y; p)$ varies from the initial approximation $u_0(y)$ to the exact solution $u(y)$ governed by Equations (13) and (14). Note that we have great freedom to choose the auxiliary parameter $\hbar$ . Assume that $\hbar$ is properly chosen so that the zero-order deformation equations (22) and (23) have solutions for all $p \in [0, 1]$ and thus the term

$$u_k(y) = \frac{1}{k!} \frac{\partial^k \bar{u}(y; p)}{\partial p^k}\bigg|_{p=0}, \tag{26}$$

exists for $k \geq 1$. Then, by Taylor's theorem and using Equation (24), we can expand $\bar{u}(y; p)$ in power series of $p$ as follows

$$\bar{u}(y; p) = u_0(y) + \sum_{k=1}^{+\infty} u_k(y) p^k. \tag{27}$$

Furthermore, assuming that $\hbar$ is so properly chosen that the power series (27) is convergent at $p = 1$, we have from Equation (25) the solution series

$$u(y) = u_0(y) + \sum_{k=1}^{+\infty} u_k(y). \tag{28}$$

### 3.2. The High-Order Deformation Equation

For brevity, define the vectors

$$\mathbf{u}_k = \{u_0(y), u_1(y), u_2(y), \ldots, u_k(y)\}. \tag{29}$$

Differentiating the zero-order deformation equations (22) and (23) $k$ times with respect to $p$ and then dividing them by $k!$ and finally setting $p = 0$, we have the high-order deformation equation

$$\mathcal{L}[u_k(y) - \chi_k u_{k-1}(y)] = \hbar\, \mathcal{R}_k(\mathbf{u}_{k-1}), \quad k \geq 1 \tag{30}$$

subject to the boundary conditions

$$
\begin{aligned}
u_k(0) &= 0,\\
u_k &\to 0 \quad \text{as} \quad y \to \infty,\\
\frac{du_k}{dy} &\to 0 \quad \text{as} \quad y \to \infty,
\end{aligned} \tag{31}
$$

in which

$$
\begin{aligned}
\mathcal{R}_k(\mathbf{u}_{k-1}) &= \frac{1}{(k-1)!}\frac{\partial^{k-1}\mathcal{N}[\bar{u}(y;p)]}{\partial p^{k-1}}\bigg|_{p=0},\\
\chi_k &= \begin{cases} 1, & k > 1,\\ 0, & k = 1. \end{cases}
\end{aligned} \tag{32}
$$

Note that $\mathcal{R}_k(\mathbf{u}_{k-1})$ is dependent on the integer material parameter $m$. When $m = 1$, we have

$$
\mathcal{R}_k(\mathbf{u}_{k-1}) = \alpha V_0 u'''_{k-1} - V_0 u'_{k-1} - 2\mu\Gamma_1 \sum_{n=0}^{k-1} u'_{k-1-n} u''_n, \tag{33}
$$

when $m = 2$

$$
\mathcal{R}_k(\mathbf{u}_{k-1}) = \alpha V_0 u'''_{k-1} - V_0 u'_{k-1} - 3\mu\Gamma_2 \sum_{n=0}^{k-1} u'_{k-1-n} \sum_{i=0}^{n} u'_{n-i} u''_i, \tag{34}
$$

when $m = 3$

$$
\mathcal{R}_k(\mathbf{u}_{k-1}) = \alpha V_0 u'''_{k-1} - V_0 u'_{k-1} - 4\mu\Gamma_3 \sum_{n=0}^{k-1} u'_{k-1-n} \sum_{i=0}^{n} u'_{n-i} \sum_{j=0}^{i} u'_{i-j} u''_j, \tag{35}
$$

where primes denote the derivative with respect to $y$.

### 3.3. Solution Expressions When $m$ is Positive Integer

Now, it is easy to solve the linear $k$th-order deformation equations (30) subject to boundary conditions (31). Solving Equation (30) subject to boundary conditions (31) upto third-order of approximations, we obtain the four-term solution consisting of equation (13) and boundary conditions (14). Hence, the four-term solution of Equations (13) and (14) can be expressed as

$$
u(y) = u_0(y) + u_1(y) + u_2(y) + u_3(y), \tag{36}
$$

where $u_0(y)$ is given by Equation (19) and when $m = 1$

$$
u_1(y) = \frac{\hbar\,\mu\Gamma_1}{3\alpha V_0}\left(1 - e^{-\frac{y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{37}
$$

$$
u_2(y) = \frac{\hbar\,\mu\Gamma_1}{18\alpha^2 V_0^2}\left[6\alpha V_0(1+\hbar) - \hbar\,\mu\Gamma_1\left(1 - 3e^{-\frac{y}{\sqrt{\alpha}}}\right)\right]\left(1 - e^{-\frac{y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}. \tag{38}
$$

$$u_3(y) = \frac{\hbar\mu\Gamma_1 e^{-\frac{y}{\sqrt{\alpha}}}}{270\alpha^3 V_0^3}\left[\begin{array}{l} 26\hbar^2\Gamma_1^2\mu^2\left(1-e^{-\frac{3y}{\sqrt{\alpha}}}\right) - 45\hbar\mu\Gamma_1\{\hbar\mu\Gamma_1 - 2(1+\hbar)V_0\alpha\}\left(1-e^{-\frac{2y}{\sqrt{\alpha}}}\right) \\ +10\left\{9(1+\hbar)^2V_0^2\alpha^2 - 12\hbar(1+\hbar)V_0\alpha\mu\Gamma_1 + 2\hbar^2\Gamma_1^2\mu^2\right\}\left(1-e^{-\frac{y}{\sqrt{\alpha}}}\right)\end{array}\right], \tag{39}$$

For $m = 2$

$$u_1(y) = \frac{\hbar\mu\Gamma_2}{8\alpha^{3/2}V_0}\left(1-e^{-\frac{2y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{40}$$

$$u_2(y) = \frac{\hbar\mu\Gamma_2}{64\alpha^3V_0^2}\left[8\alpha^{3/2}V_0(1+\hbar) + 3\hbar\mu e^{-\frac{2y}{\sqrt{\alpha}}}\right]\left(1-e^{-\frac{2y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{41}$$

$$u_3(y) = \frac{\hbar\mu\Gamma_2 e^{-\frac{y}{\sqrt{\alpha}}}}{35840\alpha^{9/2}V_0^3}\left[\begin{array}{l} 351\hbar^2\Gamma_2^2\mu^2\left(1-e^{-\frac{6y}{\sqrt{\alpha}}}\right) + 576\hbar^2\Gamma_2\mu^2\left(1-e^{-\frac{5y}{\sqrt{\alpha}}}\right) \\ \quad -210\hbar^2\Gamma_2(3+2\Gamma_2)\mu^2\left(1-e^{-\frac{4y}{\sqrt{\alpha}}}\right) \\ \quad +112\hbar(1+\hbar)(15+16\Gamma_2)\mu\alpha^{3/2}V_0\left(1-e^{-\frac{3y}{\sqrt{\alpha}}}\right) \\ \quad -210\hbar\mu\{8(1+\hbar)(1+\Gamma_2)\alpha^{3/2}V_0 - \hbar\Gamma_2^2\mu\}\left(1-e^{-\frac{2y}{\sqrt{\alpha}}}\right) \\ \quad +4480(1+\hbar)^2\alpha^3V_0^2\left(1-e^{-\frac{y}{\sqrt{\alpha}}}\right)\end{array}\right], \tag{42}$$

and when $m = 3$

$$u_1(y) = \frac{\hbar\mu\Gamma_3}{15\alpha^2V_0}\left(1-e^{-\frac{3y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{43}$$

$$u_2(y) = \frac{\hbar\mu\Gamma_3}{225\alpha^4V_0^2}\left[15\alpha^2V_0(1+\hbar) + \hbar\mu\Gamma_3\left(1+5e^{-\frac{3y}{\sqrt{\alpha}}}\right)\right]\left(1-e^{-\frac{3y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{44}$$

$$u_3(y) = \frac{\hbar\mu\Gamma_3 e^{-\frac{y}{\sqrt{\alpha}}}}{111375\alpha^6V_0^3}\left[\begin{array}{l} 1180\hbar^2\Gamma_3^2\mu^2\left(1-e^{-\frac{9y}{\sqrt{\alpha}}}\right) + 165\hbar\mu\Gamma_3\{30(1+\hbar)\alpha^2V_0 \\ \quad -7\hbar\mu\Gamma_3\}\left(1-e^{-\frac{6y}{\sqrt{\alpha}}}\right) + 33\left\{225(1+\hbar)^2\alpha^4V_0^2\right. \\ \quad \left. -120\hbar(1+\hbar)\alpha^2V_0\mu\Gamma_3 + 2\hbar^2\Gamma_3^2\mu^2\right\}\left(1-e^{-\frac{3y}{\sqrt{\alpha}}}\right)\end{array}\right]. \tag{45}$$

## 4. Discussion of Results

In this section, we present various results obtained from the flow analysed in this investigation. The velocity profiles are plotted in Figures 1–6 for various values of the normal stress coefficient $\alpha$ and porosity parameter $V_0$ for the three cases when $m = 1, 2$ and 3. Figures 1 and 2 are prepared when $m = 1$, Figures 3 and 4 for $m = 2$ and Figures 5 and 6 for $m = 3$.

The flow dependence of a generalized second-grade fluid on the normal stress coefficient $\alpha$ can be clearly seen from Figures 1, 3 and 5 for the three cases. From these figures, it is noted that the flow is strongly influenced by $\alpha$. It is found that with the increase in $\alpha$ the boundary layer thickness increases near the plate, while the velocity decreases for both suction and blowing.

In order to illustrate the effect of suction and blowing, for fixed values of other parameters, Figures 2, 4 and 6 are plotted. It is seen that an increase in suction results in a decrease in boundary layer thickness. This is in keeping with the expected fact that suction causes reduction in the boundary layer thickness. For the case of blowing, it is well known that in the case of Newtonian fluids, there is no solution to the Navier–Stokes equations for blowing. However, a solution to the equations of motion in the case of fluid injected into the domain is possible in the case of non-Newtonian fluids and the results

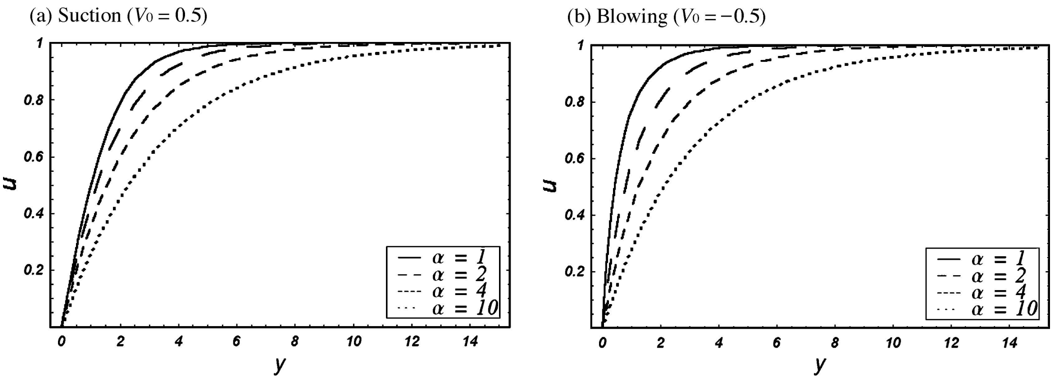


*Figure 1.* Profiles of velocity $u(y)$ for various values of normal stress coefficient $\alpha$ for fixed $\hbar = -0.5$, $\mu = 1$ and $\Gamma_1 = 1$.

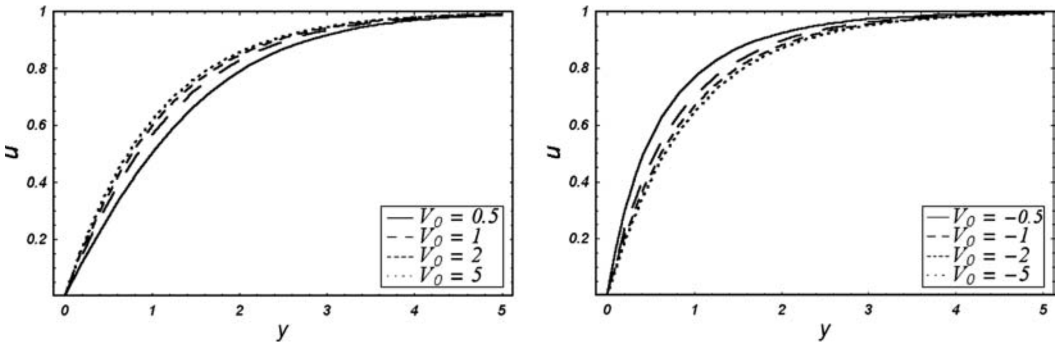


*Figure 2.* Profiles of velocity $u(y)$ for various values of suction (panel a) and blowing (panel b) for fixed $\hbar = -0.5$, $\mu = 1$, $\Gamma_1 = 1$ and $\alpha = 1$.

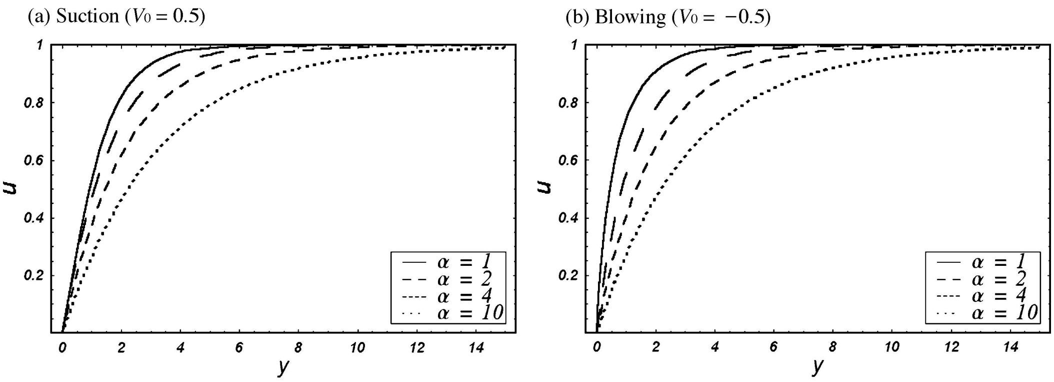


*Figure 3.* Profiles of velocity $u(y)$ for various values of normal stress coefficient $\alpha$ for fixed $\hbar = -0.5$, $\mu = 1$ and $\Gamma_2 = 1.5$.

established here are in keeping with the results of Rajagopal and Gupat [14]. As expected, the blowing causes thickening of the boundary layer and this boundary layer thickness is greater when compared to the case of suction.

It is further seen that the velocities in the three cases of $m$ have no much difference through the variation of $\alpha$. However, if we take the variation of $V_0$ then the velocities are sensitive for the different values of $m$ which shows the shear-thickening effects of the examined non-Newtonian fluid. Thus, a generalized second-grade fluid exhibits the shear-thinning and shear-thickening effects for $m < 0$ and $m > 0$, respectively.

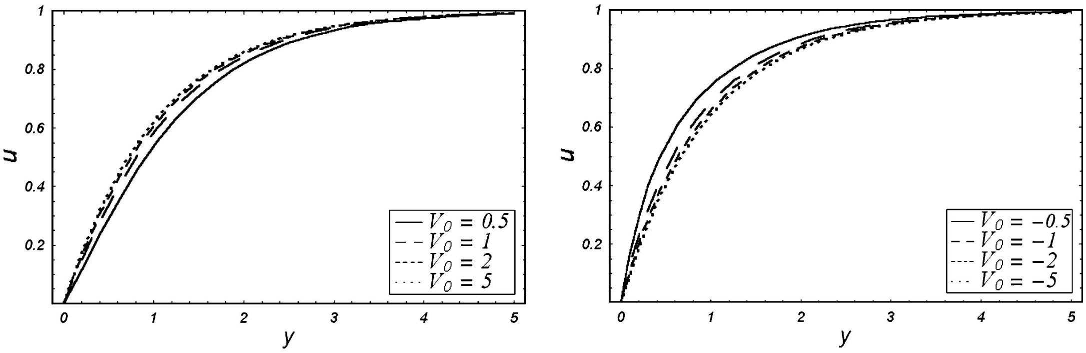


*Figure 4*. Profiles of velocity $u(y)$ for various values of suction (panel a) and blowing (panel b) for fixed $\hbar = -0.5$, $\mu = 1$, $\Gamma_2 = 1.5$ and $\alpha = 1$.

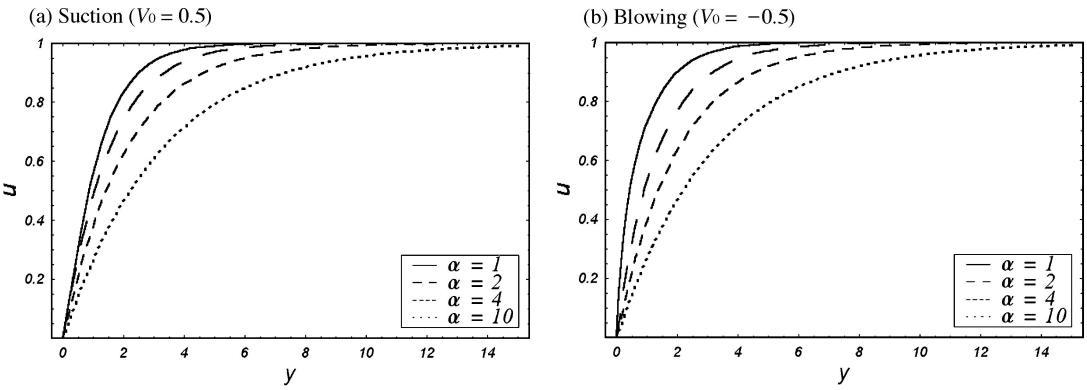


*Figure 5*. Profiles of velocity $u(y)$ for various values of normal stress coefficient $\alpha$ for fixed $\hbar = -0.5$, $\mu = 1$ and $\Gamma_3 = 1.8$.

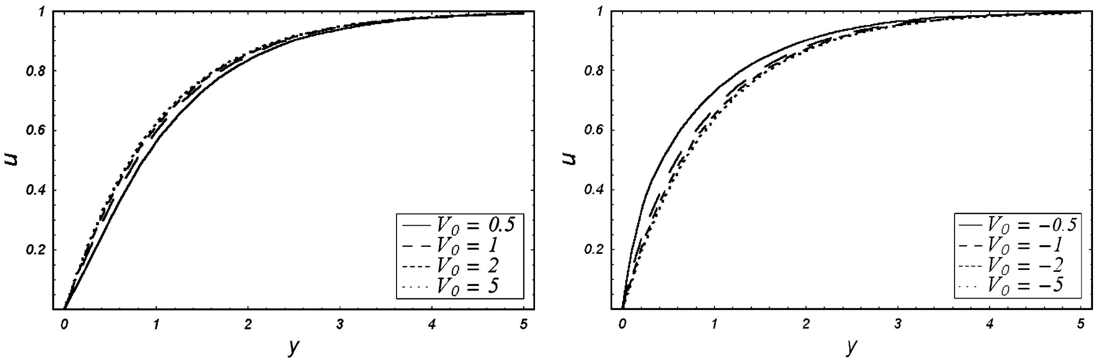


*Figure 6*. Profiles of velocity $u(y)$ for various values of suction (panel a) and blowing (panel b) for fixed $\hbar = -0.5$, $\mu = 1$, $\Gamma_3 = 1.8$ and $\alpha = 1$.

## 5. Concluding Remarks

In this paper, HAM is employed to give analytic solutions of a generalized second-grade fluid past a porous plate. For material parameter $m = 1$, 2 and 3 the explicit analytic solutions are given, which can be treated as the definition of the solution of the non-linear differential equations (16)–(18). From the presented analysis, we have the following points:

- The considered model of generalized second-grade fluids exhibits both shear-thinning and shear-thickening properties.
- Increase in the normal stress coefficient leads to an increase in the boundary layer thickness.
- Increasing the values of the suction velocity provides decrease in the boundary layer thickness.
- The increase in injection velocity increases the boundary layer thickness.
- The velocity profiles are not much sensitive to variations in $\alpha$ for various values of $m$.
- The convergence of the HAM solution series is dependent on the auxiliary (artificial) parameter $\hbar$ [32]. In most cases, one can find a proper value of $\hbar$ to ensure the convergence of the solution series, even if the corresponding physical quantity is large. It is noted from the presented analysis that the closer the value of $\hbar$ to zero from left (i.e. $-2 < \hbar < 0$), the convergence region enlarges.

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