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Continuous-Time Bilinear System Identification

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Abstract. The objective of this paper is to describe a new method for identification of a continuous-time multi-input and multi-output bilinear system. The approach is to make judicious use of the linear-model properties of the bilinear system when subjected to a constant input. Two steps are required in the identification process. The first step is to use a set of pulse responses resulting from a constant input of one sample period to identify the state matrix, the output matrix, and the direct transmission matrix. The second step is to use another set of pulse responses with the same constant input over multiple sample periods to identify the input matrix and the coefficient matrices associated with the coupling terms between the state and the inputs. Numerical examples are given to illustrate the concept and the computational algorithm for the identification method.

Key words: bilinear system, Markov parameters, nonlinear system, system identification, system realization

1. Introduction

System identification is a methodology used to characterize a dynamical or other engineering system with measurements of the input–output signals. Mathematicians and engineers have developed a number of approaches to address the identification problem. The identification of a linear time-invariant system is relatively well understood and theoretically well developed [1, 2]. This is not true for the identification of a nonlinear system, although some progress has been made in the identification of nonlinear systems over the past few decades [3–19].

There is a class of nonlinear systems called bilinear systems whose dynamics are jointly linear in the state and the force variables. It is a simple nonlinear extension of a linear system. The concept of bilinear systems was introduced in the 1960's (see the surveys of Bruni et al. [5] and Mohler et al. [6]). Mohler [7] and Elliott [8]) provide a survey of bilinear-related system-theory methods and their contributions to problems such as stabilization, controllability, and observability. Bilinear systems have been studied extensively and applied successfully to several problems [15]. Recently, research activities in identification of bilinear systems have been focused on the so-called "discrete-time" model identification [19]. The discrete-time model is an approximation obtained by linearizing the continuous one with a method such as the finite difference. In contrast, we focus on the identification of a continuous-time bilinear system without any approximation.

A new method is introduced in this paper for identification of a continuous-time multi-input and multi-output bilinear system. When the input of a bilinear system is a constant, the bilinear system becomes a linear system. This special characteristic is the basis for the identification method. Two steps are required for the identification process. The first step begins with generating a set of pulse responses with a constant input applied one at a time over one sample period. The pulse responses are then used to form a Hankel matrix consisting of system Markov parameters to identify the state matrix, the output matrix, and the direct transmission matrix. The identification step is quite similar, if not identical, to the identification of a linear system [1, 2]. This step establishes a specific set of coordinates for the whole identification process. This set of coordinates is not unique, depending mainly on the size of Hankel matrix and the resulting choice of matrix that represents the observability matrix. The second step starts by generating another set of pulse responses with the same constant input as the first step but for multiple sample periods. These multiple-pulse responses are used to define another set of Markov parameters to form a Hankel-like matrix for each input. The observability matrix obtained in the first step is then applied to the Hankel-like matrix to compute the corresponding controllability matrix of the input to identify the input vector and the coefficient matrix associated with the coupling terms between the state and the input.

Simple examples are given to demonstrate how to apply the method to identify a continuous-time bilinear system and how to transfer the identified model from one set of coordinates to the other set of coordinates. The coordinate transformation also serves as a way of verifying the identified system.

2. Basic Formulation

Let x and A_c be the state vector of dimension $n \times 1$ and its corresponding state matrix of $n \times n$, and u and B_c be the input vector of $r \times 1$ and its corresponding input matrix of $n \times r$. Subscript c signifies the associated quantity in the continuous-time domain. The bilinear state equation in the continuous-time domain is commonly expressed by

$$\dot{x} = A_c x + B_c u + \sum_{i=1}^{r} N_{ci} x u_i \tag{1}$$

where the coupling term xu_i between the state vector x and each individual u_i $(i = 1, \ldots, r)$ in the input vector u is weighted by the matrix N_{ci} of $n \times n$. The measurement equation is identical to the one for a linear system that is commonly described by

$$y = Cx + Du \tag{2}$$

where y is the output measurement vector of $m \times 1$, C is the output matrix of $m \times n$ and D is the direct transmission matrix of $m \times r$.

For simplicity, consider only one input at a time. Equation (1) reduces to

$$\dot{x} = A_c x + b_{ci} u_i + N_{ci} x u_i \tag{3}$$

where b_{ci} is the ith column of B_c associated with the input u_i. Assuming $u_i = \upsilon_i$ where υ_i is a pre-specified constant, the continuous-time state equation (3) further reduces to

$$\dot{x} = (A_c + \upsilon_i N_{ci})x + b_{ci} \upsilon_i \tag{4}$$

The discrete-time model of this system is

$$x(k+1) = \bar{A}_i x(k) + \bar{b}_i; \; i = 1, 2, \ldots, r \tag{5}$$

with the measurement equation

$$y_i(k) = Cx(k) + \bar{d}_i \tag{6}$$

where the quantities $\bar{A}_i$, $\bar{b}_i$, and $\bar{d}_i$ are determined by

$$\bar{A}_i = e^{(A_c+\upsilon_i N_{ci})\Delta t} \tag{7}$$

$$\bar{b}_i = \int_0^{\Delta t} e^{(A_c+\upsilon_i N_{ci})\tau} d\tau \; b_{ci}\upsilon_i \tag{8}$$

$$\bar{d}_i = d_i \upsilon_i \tag{9}$$

and d_i is the ith column of D associated with the input u_i. The quantity Δt is the time interval for data sampling. Assuming that the initial state $x(0)$ is a zero vector of n by 1, i.e., $x(0) = 0_{n\times 1}$, the state response for the discrete-time model described by Equation (5) can be computed by:

$$\begin{aligned}
x(0) &= 0_{n\times 1} \\
x(1) &= \bar{A}_i x(0) + \bar{b}_i = \bar{b}_i \\
x(2) &= \bar{A}_i x(1) + \bar{b}_i = \bar{A}_i\bar{b}_i + \bar{b}_i \\
x(3) &= \bar{A}_i x(2) + \bar{b}_i = \bar{A}_i^2\bar{b}_i + \bar{A}_i\bar{b}_i + \bar{b}_i \\
&\vdots \\
x(N) &= \bar{A}_i x(N-1) + \bar{b}_i = \bar{A}_i^{N-1}\bar{b}_i + \bar{A}_i^{N-2}\bar{b}_i + \cdots + \bar{A}_i\bar{b}_i + \bar{b}_i
\end{aligned} \tag{10}$$

After the time $t \geq N\Delta t$, let $u_i(t) = 0$. Note that in the discrete-time domain, the input force $u_i(k)$ at time index k implies that $u_i(t)$ is constant over the time period $(k+1)\Delta t > t \geq k\Delta t$. The state Equation (3) reduces to the simple form

$$\dot{x} = A_c x \tag{11}$$

Its discrete-time model is

$$x(k+1) = Ax(k) \tag{12}$$

where

$$A = e^{A_c\Delta t} \tag{13}$$

The free decay response after $t > N\Delta t$becomes

$$\begin{aligned}
x(N+1) &= Ax(N) = A\left(\bar{A}_i^{N-1}\bar{b}_i + \bar{A}_i^{N-2}\bar{b}_i + \cdots + \bar{A}_i\bar{b}_i + \bar{b}_i\right) \\
x(N+2) &= A^2x(N) = A^2\left(\bar{A}_i^{N-1}\bar{b}_i + \bar{A}_i^{N-2}\bar{b}_i + \cdots + \bar{A}_i\bar{b}_i + \bar{b}_i\right) \\
&\vdots \\
x(N+\ell) &= A^\ell x(N) = A^\ell\left(\bar{A}_i^{N-1}\bar{b}_i + \bar{A}_i^{N-2}\bar{b}_i + \cdots + \bar{A}_i\bar{b}_i + \bar{b}_i\right)
\end{aligned} \tag{14}$$

where ℓ is an integer indicating the data length of the free-decay response.

From Equations (2), (10), and (14), the measurement quantities $y_i(k)$for $k = 0, 1, \cdots, N + \ell$ due to the force excitation of $u_i = \upsilon_i$ (constant force) for $k < N$ can thus be computed as

$$\begin{aligned}
y_i(0) &= Cx(0) + \bar{d}_i = \bar{d}_i \\
y_i(1) &= Cx(1) + \bar{d}_i = C\bar{b}_i + \bar{d}_i \\
y_i(2) &= Cx(2) + \bar{d}_i = C\left(\bar{A}_i\bar{b}_i + \bar{b}_i\right) + \bar{d}_i \\
&\vdots \\
y_i(N) &= Cx(N) = C\left(\bar{A}_i^{N-1}\bar{b}_i + \bar{A}_i^{N-2}\bar{b}_i + \cdots + \bar{A}_i\bar{b}_i + \bar{b}_i\right) \\
y_i(N+1) &= Cx(N+1) = CA\left(\bar{A}_i^{N-1}\bar{b}_i + \bar{A}_i^{N-2}\bar{b}_i + \cdots + \bar{A}_i\bar{b}_i + \bar{b}_i\right) \\
&\vdots \\
y_i(N+\ell) &= Cx(N+\ell) = CA^{\ell}\left(\bar{A}_i^{N-1}\bar{b}_i + \bar{A}_i^{N-2}\bar{b}_i + \cdots + \bar{A}_i\bar{b}_i + \bar{b}_i\right)
\end{aligned} \tag{15}$$

The upper portion, $y_i(0), y_i(1), \cdots, y_i(N-1)$, of Equation (15), corresponds to the multiple-pulse response resulting from a constant force over multiple sample periods. But the lower portion, $y_i(N), y_i(N+1), \cdots, y_i(N+\ell)$, corresponds to the free-decay response.

It is clear that the continuous-time system matrices/vectors, $A_c, b_{ci}, N_{ci}, C, d_i$, are embedded in the output quantities shown in Equation (15). First of all, we need to use the output measurements to extract the discrete-time matrices/vectors, $A, C, \bar{A}_i, \bar{b}_i, d_i$. It is worth to stress that the multiple-pulse response and the free-decay response result from two different discrete models.

The free-decay response, $y_i(N), y_i(N+1), \ldots, y_i(N+\ell)$, after $t > N\Delta t$ is quite similar, if not identical, to the pulse response for a linear system. Any linear system identification technique may be applied to compute the state matrix A and the output matrix C. The key idea is to make judicious use of this linear portion of the bilinear system. The identification problems for linear systems have been extensively studied and many good techniques have been developed and implemented [1, 2].

3. System Identification Method

The identification method requires two steps. The first step is to identify the state matrix A_c, the output matrix C, and the data transmission matrix D. The second step is to determine the input matrices B_c, and N_i for the coupling term between the state vector x and the ith input u_i.

3.1. Identification of A_c, C, and D

First, let us apply a pulse of magnitude υ_1 to the system for one time step Δt to generate the pulse response for the first input u_1. From Equation (15) for $N = 1$, the pulse response has the following expression.

$$\begin{aligned}
y_1(0) &= \bar{d}_1 \\
y_1(1) &= C\bar{b}_1 \\
y_1(2) &= CA\bar{b}_1
\end{aligned} \tag{16}$$

$$\vdots$$
$$y_1(\ell+1) = CA^{\ell}\bar{b}_1$$

All other input pulse responses can be similarly generated to yield

$$\begin{array}{ccc} y_2(0) = \bar{d}_2 & \cdots & y_r(0) = \bar{d}_r \\ y_2(1) = C\bar{b}_2 & \cdots & y_r(1) = C\bar{b}_r \\ y_2(2) = CA\bar{b}_2 & \cdots & y_r(2) = CA\bar{b}_r \\ \vdots & \vdots & \vdots \\ y_2(\ell+1) = CA^{\ell}\bar{b}_2 & \cdots & y_r(\ell+1) = CA^{\ell}\bar{b}_r \end{array} \tag{17}$$

Equation (17) is obtained by replacing the subscript 1 representing the first input with the other input integers 2 through r. Let us define the system Markov parameters to be

$$\begin{aligned} Y_1(0) &= [\, y_1(0)\ y_2(0) \cdots y_r(0)\,] = [\,\bar{d}_1\ \bar{d}_2 \cdots \bar{d}_r\,] = \bar{D} \\ Y_1(1) &= [\, y_1(1)\ y_2(1) \cdots y_r(1)\,] = [\,C\bar{b}_1\ C\bar{b}_2 \cdots C\bar{b}_r\,] \\ Y_1(2) &= [\, y_1(2)\ y_2(2) \cdots y_r(2)\,] = [\,CA\bar{b}_1\ CA\bar{b}_2 \cdots CA\bar{b}_r\,] \\ &\vdots \\ Y_1(\ell+1) &= [\, y_1(\ell+1)\ y_2(\ell+1) \cdots y_r(\ell+1)\,] = [\,CA^{\ell}\bar{b}_1\ CA^{\ell}\bar{b}_2 \cdots CA^{\ell}\bar{b}_r\,] \end{aligned} \tag{18}$$

The use of the subscript 1 for $Y_1(k)$ ($k = 1, 2, \ldots, \ell + 1$) is intended to signify one-time-step pulse response. Equation (18) provides the basic parameters for system identification. Indeed, let us form a Hankel matrix as follows.

$$H_1 = \begin{bmatrix} Y_1(1) & Y_1(2) & \cdots & Y_1(\beta) \\ Y_1(2) & Y_1(3) & \cdots & Y_1(\beta+1) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1(\alpha) & Y_1(\alpha+1) & \cdots & Y_1(\alpha+\beta-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix} [\,\bar{B}_1\ A\bar{B}_1 \cdots A^{\beta-1}\bar{B}_1\,] \tag{19}$$

where

$$\bar{B}_1 = [\,\bar{b}_1\ \bar{b}_2 \cdots \bar{b}_r\,] \tag{20}$$

The matrix product on the right-hand side of Equation (19) shows the relationship between the system Markov parameters and the discrete-time system matrices. Obviously the Hankel matrix H_1 has the rank n that is the order of the state matrix A if we choose α and β such that αm and βr are larger than or equal to n where m is the number of outputs and r is the number of inputs. Using the singular value decomposition (SVD) to decompose the Hankel matrix H_1 yields

$$H_1 = U_1 \Sigma_1 V_1^T \tag{21}$$

where Σ_1 of $n \times n$ is a square matrix containing n non-zero singular values. The matrix U_1 is of dimension $\alpha m \times n$ and the matrix V_1 is of dimension $\beta r \times n$.

From Equation (19), one may choose

$$U_1 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix} \tag{22}$$

and

$$\Sigma_1 V_1^T = [\, \bar{B}_1 \;\; A\bar{B}_1 \; \cdots \; A^{\beta-1}\bar{B}_1 \,] \tag{23}$$

This choice is not unique. Many other choices are also valid. The other common choice is

$$U_1 \Sigma_1^{1/2} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix} \tag{24}$$

and

$$\Sigma_1^{1/2} V_1^T = [\, \bar{B}_1 \;\; A\bar{B}_1 \; \cdots \; A^{\beta-1}\bar{B}_1 \,] \tag{25}$$

Note that the choice of Equation (22) has the advantage that

$$U_1^T U_1 = I_{n\times n} \Rightarrow U_1^{\dagger} = U_1^T \tag{26}$$

because U_1 is a unitary matrix resulting from the property of the SVD. Nevertheless, the choice of Equations (24) and (25) has a nice property of balanced coordinates. Equation (22) or (24) is commonly called observability matrix whereas Equation (23) or (25) is referred to as the controllability matrix.

Equations (22) and (23) produce the following solutions

$$C = \text{the first } m \text{ rows of } U_1 \tag{27}$$

$$\bar{B}_1 = \text{the first } r \text{ columns of } \Sigma_1 V_1^T \tag{28}$$

Since the choices of controllability and observability matrices are not unique, the identified matrices C and $\bar{B}_1$ are not unique. To determine the state matrix A, let us first define and observe the following matrices.

$$U_{1\uparrow} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-2} \end{bmatrix} \tag{29}$$

and

$$U_{1\downarrow} = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{\alpha-1} \end{bmatrix} = U_{1\uparrow}A \tag{30}$$

Deleting the last m rows of U_1 forms the matrix $U_{1\uparrow}$ whereas deleting the first m rows of U_1 yields the matrix $U_{1\downarrow}$. It is then clear that the state matrix A can be determined by

$$A = U_{1\uparrow}^{\dagger} U_{1\downarrow} \tag{31}$$

For the identified state matrix to have the rank n, both $(\alpha - 1)m \times n$ matrices $U_{1\uparrow}$ and $U_{1\downarrow}$ must also have the rank n. This implies that α must be chosen such that $(\alpha - 1)m > n$, i.e., $\alpha m > n$. Of course, we have assumed that the pulse force υ_i for $i = 1, 2, \ldots, r$ are chosen so that all system modes are excitable and observable.

With the aid of Equation (13), Equation (31) produces the continuous-time state matrix as

$$A_c = \frac{1}{\Delta t}\log(A) = \frac{1}{\Delta t}\log(U_{1\uparrow}^{\dagger} U_{1\downarrow}) \tag{32}$$

Note that the conversion from a discrete-time state matrix to a continuous-time state matrix may not be unique.

To this end, we have determined A_c from Equation (32), C from Equation (27), $\bar{B}_1$ from Equation (28), and $\bar{D}$ from Equation (18). The original transmission matrix D can be recovered using Equation (9) to have

$$D = \bar{D}\ \mathrm{diag}[\ 1/\upsilon_1\ \ 1/\upsilon_2\ \cdots\ 1/\upsilon_r\ ] \tag{33}$$

Let us stress that the identified matrices A_c, $\bar{B}_1$ and C are not uniquely determined but D is coordinate invariant and so is uniquely determined.

3.2. Identification of B$_c$ and N$_{ci}$

The second step begins with generating the two-sample-period pulse response for all inputs with one input at a time, i.e., a force is applied with the same magnitude as above to the system for two time steps $2\Delta t$. From Equation (15) for $N = 2$, we obtain

$$\begin{array}{lcl}
y_1(0) = \bar{d}_1 & \cdots & y_r(0) = \bar{d}_r \\
y_1(1) = C\bar{b}_1 + \bar{d}_1 & \cdots & y_r(1) = C\bar{b}_r + \bar{d}_r \\
y_1(2) = C[\bar{A}_1\bar{b}_1 + \bar{b}_1] & \cdots & y_r(2) = C[\bar{A}_r\bar{b}_r + \bar{b}_r] \\
\vdots & \vdots & \vdots \\
y_1(\ell+2) = CA^{\ell}[\bar{A}_1\bar{b}_1 + \bar{b}_1] & \cdots & y_r(\ell+2) = CA^{\ell}[\bar{A}_r\bar{b}_r + \bar{b}_r]
\end{array} \tag{34}$$

Now define the system Markov parameters from the two-sample-period pulse response as

$$
\begin{aligned}
Y_2(2) &= [\, y_1(2)\ y_2(2)\ \cdots\ y_r(2)\,] \\
&= [\, C(\bar{A}_1\bar{b}_1 + \bar{b}_1)\ C(\bar{A}_2\bar{b}_2 + \bar{b}_2)\ \cdots\ C(\bar{A}_r\bar{b}_r + \bar{b}_r)\,] \\
Y_2(3) &= [\, y_1(3)\ y_2(3)\ \cdots\ y_r(3)\,] \\
&= [\, CA(\bar{A}_1\bar{b}_1 + \bar{b}_1)\ CA(\bar{A}_2\bar{b}_2 + \bar{b}_2)\ \cdots\ CA(\bar{A}_r\bar{b}_r + \bar{b}_r)\,] \\
&\ \vdots \\
Y_2(\ell+2) &= [\, y_1(\ell+2)\ y_2(\ell+2)\ \cdots\ y_r(\ell+2)\,] \\
&= [\, CA^{\ell}(\bar{A}_1\bar{b}_1 + \bar{b}_1)\ CA^{\ell}(\bar{A}_2\bar{b}_2 + \bar{b}_2)\ \cdots\ CA^{\ell}(\bar{A}_r\bar{b}_r + \bar{b}_r)\,]
\end{aligned}
\tag{35}
$$

Subscript 2 for $Y_2(k)$ ($k = 2, 3, \ldots, \ell + 2$) signifies two-sample-period pulse response. Let us form a $\alpha m \times r$ matrix as follows.

$$
H_2 = \begin{bmatrix} Y_2(2) \\ Y_2(3) \\ \vdots \\ Y_2(\alpha+1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix} \bar{B}_2 \tag{36}
$$

where

$$
\bar{B}_2 = [\,(\bar{A}_1\bar{b}_1 + \bar{b}_1)\ (\bar{A}_2\bar{b}_2 + \bar{b}_2)\ \cdots\ (\bar{A}_r\bar{b}_r + \bar{b}_r)\,] \tag{37}
$$

With the help of Equation (22), the $n \times r$ matrix $\bar{B}_2$ in Equation (36) can be solved by

$$
\bar{B}_2 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix}^{\dagger} H_2 = U_1^{\dagger} H_2 \tag{38}
$$

Similarly, we may continue the process to generate three-sample-period pulse response, four-sample-period pulse response, etc. up to the p-sample-period pulse response for all inputs with one input at a time using a force of the same magnitude as earlier applied to the system for p time periods $p\Delta t$. From Equation (15) for $N = p$, we have

$$
\begin{aligned}
y_1(p) &= C\big(\bar{A}_1^{p-1}\bar{b}_1 + \cdots + \bar{b}_1\big) \quad \cdots \quad y_r(p) = C\big(\bar{A}_r^{p-1}\bar{b}_r + \cdots + \bar{b}_r\big) \\
y_1(p+1) &= CA\big(\bar{A}_1^{p-1}\bar{b}_1 + \cdots + \bar{b}_1\big) \ \cdots\ y_r(p+1) = CA\big(\bar{A}_r^{p-1}\bar{b}_r + \cdots + \bar{b}_r\big) \\
y_1(p+2) &= CA^2\big(\bar{A}_1^{p-1}\bar{b}_1 + \cdots + \bar{b}_1\big) \ \cdots\ y_r(p+2) = CA^2\big(\bar{A}_r^{p-1}\bar{b}_r + \cdots + \bar{b}_r\big) \\
&\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots \\
y_1(p+\ell) &= CA^{\ell}\big(\bar{A}_1^{p-1}\bar{b}_1 + \cdots + \bar{b}_1\big) \ \cdots\ y_r(p+\ell) = CA^{\ell}\big(\bar{A}_r^{p-1}\bar{b}_r + \cdots + \bar{b}_r\big)
\end{aligned}
\tag{39}
$$

Now define the system Markov parameters for the p-sample-period pulse response as

$$
\begin{aligned}
Y_p(p) &= [\, y_1(p)\ y_2(p) \cdots y_r(p)\,] \\
&= \left[\, C\left(\bar{A}_1^{p-1}\bar{b}_1 + \cdots + \bar{b}_1\right)\ C\left(\bar{A}_2^{p-1}\bar{b}_2 + \cdots + \bar{b}_2\right) \cdots C\left(\bar{A}_r^{p-1}\bar{b}_r + \cdots + \bar{b}_r\right) \right] \\
Y_p(p+1) &= [\, y_1(p+1)\ y_2(p+1) \cdots y_r(p+1)\,] \\
&= \left[\, CA\left(\bar{A}_1^{p-1}\bar{b}_1 + \cdots + \bar{b}_1\right)\ CA\left(\bar{A}_2^{p-1}\bar{b}_2 + \cdots + \bar{b}_2\right) \cdots CA\left(\bar{A}_r^{p-1}\bar{b}_r + \cdots + \bar{b}_r\right) \right] \\
&\ \ \vdots \\
Y_p(p+\ell) &= [\, y_1(p+\ell)\ y_2(p+\ell) \cdots y_r(p+\ell)\,] \\
&= \left[\, CA^{\ell}\left(\bar{A}_1^{p-1}\bar{b}_1 + \cdots + \bar{b}_1\right)\ CA^{\ell}\left(\bar{A}_2^{p-1}\bar{b}_2 + \cdots + \bar{b}_2\right) \cdots CA^{\ell}\left(\bar{A}_r^{p-1}\bar{b}_r + \cdots + \bar{b}_r\right) \right]
\end{aligned}
\tag{40}
$$

Let us form a $\alpha m \times r$ matrix as follows.

$$
H_p = \begin{bmatrix} Y_p(p) \\ Y_p(p+1) \\ \vdots \\ Y_p(p+\alpha-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix} \bar{B}_p \tag{41}
$$

where

$$
\bar{B}_p = \left[\, \left(\bar{A}_1^{p-1}\bar{b}_1 + \cdots + \bar{b}_1\right)\ \left(\bar{A}_2^{p-1}\bar{b}_2 + \cdots + \bar{b}_2\right) \cdots \left(\bar{A}_r^{p-1}\bar{b}_r + \cdots + \bar{b}_r\right) \right] \tag{42}
$$

With the help of Equation (22), the $n \times r$ matrix $\bar{B}_p$ in Equation (41) can be solved by

$$
\bar{B}_p = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix}^{\dagger} H_p = U_1^{\dagger} H_p \tag{43}
$$

To determine B_c, let us first observe the matrices $\bar{B}_1,\ \bar{B}_2, \cdots, \bar{B}_p$ defined in Equations (20), (37), and (42), and determined by Equations (28), (38), and (43), i.e.,

$$
\begin{aligned}
\bar{B}_1 &= \left[\, \bar{b}_1\ \bar{b}_2 \cdots \bar{b}_r \,\right] \\
\bar{B}_2 &= \left[\, \left(\bar{A}_1\bar{b}_1 + \bar{b}_1\right)\ \left(\bar{A}_2\bar{b}_2 + \bar{b}_2\right) \cdots \left(\bar{A}_r\bar{b}_r + \bar{b}_r\right) \right] \\
&\ \ \vdots \\
\bar{B}_p &= \left[\, \left(\bar{A}_1^{p-1}\bar{b}_1 + \cdots + \bar{b}_1\right)\ \left(\bar{A}_2^{p-1}\bar{b}_2 + \cdots + \bar{b}_2\right) \cdots \left(\bar{A}_r^{p-1}\bar{b}_r + \cdots + \bar{b}_r\right) \right]
\end{aligned}
\tag{44}
$$

Applying the recursive formula

$$\bar{B}_k - \bar{B}_{k-1};\ k = 2, 3, \cdots, p \tag{45}$$

yields the controllability like matrices,

$$\mathbb{C}_i = \left[\, \bar{b}_i\ \bar{A}_i\bar{b}_i\ \cdots\ \bar{A}_i^{p-1}\bar{b}_i \,\right];\ i = 1, 2, \cdots, r \tag{46}$$

To determine the state matrix $\bar{A}_i$, let us first define the two matrices

$$\mathbb{C}_{i\leftarrow} = \left[\, \bar{b}_i\ \bar{A}_i\bar{b}_i\ \cdots\ \bar{A}_i^{p-2}\bar{b}_i \,\right] \tag{47}$$

and

$$\mathbb{C}_{i\rightarrow} = \left[\, \bar{A}_i\bar{b}_i\ \bar{A}_i^2\bar{b}_i\ \cdots\ \bar{A}_i^{p-1}\bar{b}_i \,\right] = \bar{A}_i\mathbb{C}_{i\leftarrow} \tag{48}$$

Deleting the last r columns of $\mathbb{C}_i$ forms the matrix $\mathbb{C}_{i\leftarrow}$ whereas deleting the first r columns of $\mathbb{C}_i$ yields the matrix $\mathbb{C}_{i\rightarrow}$. Equations (47) and (48) produce the solutions:

$$\bar{b}_i = \text{the first } r \text{ columns of } \mathbb{C}_i \tag{49}$$

$$\bar{A}_i = \mathbb{C}_{i\rightarrow}\mathbb{C}_{i\leftarrow}^{\dagger} \tag{50}$$

for $i = 1, 2, \ldots, r$. For the identified matrix $\bar{A}_i$ to have the rank n, both $n \times (p-1)$ matrices $\mathbb{C}_{i\leftarrow}$ and $\mathbb{C}_{i\rightarrow}$ must also have the rank n. It implies that p must be chosen such that $p - 1 \geq n$. This indicates that *the system identification method requires a total of at least* $(n+1)$ *sets of responses generated by* $(n+1)$ *different time periods of pulse input*.

Based on Equations (7) and (8) for the definitions of $\bar{A}_i$ and $\bar{b}_i$, taking the conversion from discrete-time to continuous-time produces

$$A_c + \upsilon_i N_{ci} = \frac{1}{\Delta t}\log(\bar{A}_i) = \frac{1}{\Delta t}\log(\mathbb{C}_{i\rightarrow}\mathbb{C}_{i\leftarrow}^{\dagger}) \tag{51}$$

and

$$b_{ci} = \frac{1}{\upsilon_i}\left[\mathrm{I}_{n\times n}\Delta t + \frac{1}{2!}(A_c + \upsilon_i N_{ci})(\Delta t)^2 + \frac{1}{3!}(A_c + \upsilon_i N_{ci})^2(\Delta t)^3 + \cdots\right]^{-1}\bar{b}_i \tag{52}$$

for $i = 1, 2, \ldots, r$ where $\mathrm{I}_{n\times n}$ is a $n \times n$ identity matrix, that, in turns, yields

$$B_c = \left[\, b_{c1}\ b_{c2}\ \cdots\ b_{cr} \,\right] \tag{53}$$

Again, one should be cautious to take the conversion because of its non-uniqueness problem [1]. From Equations (32) and (51), the matrices N_{ci} $(i = 1, 2, \ldots, r)$ are determined by

$$N_{ci} = \frac{1}{\upsilon_i}\left[\frac{1}{\Delta t}\log(\bar{A}_i) - A_c\right] = \frac{1}{\upsilon_i\Delta t}[\log(\mathbb{C}_{i\rightarrow}\mathbb{C}_{i\leftarrow}^{\dagger}) - \log(U_{1\uparrow}^{+}U_{1\downarrow})] \tag{54}$$

To this end, we have identified all continuous-time system matrices A_c, B_c, N_{ci}, C, and D for the bilinear system described by Equations (1) and (2) from pulse responses generated by pre-specified pulse inputs.

4. Coordinate Transformation

Let the state vector x of $n \times 1$ in Equations (1) and (2) be transformed to the new state vector $\tilde{x}$ of $n \times 1$ by the nonsingular transformation matrix Φ of $n \times n$. Equations (1) and (2) become

$$\dot{\tilde{x}} = \tilde{A}_c \tilde{x} + \tilde{B}_c u + \Sigma_{i=1}^{r} \tilde{N}_{ci} \tilde{x} u_i \tag{55}$$

and

$$y = \tilde{C} \tilde{x} + Du \tag{56}$$

where

$$\begin{aligned} \tilde{x} &= \Phi^{-1} x \\ \tilde{A}_c &= \Phi^{-1} A_c \Phi \\ \tilde{N}_{ci} &= \Phi^{-1} N_{ci} \Phi \\ \tilde{B}_c &= \Phi^{-1} B_c \\ \tilde{C} &= C \Phi \end{aligned} \tag{57}$$

It is clear that the transformed matrix $\tilde{A}_c$ is similar to the original matrix A_c in the sense that their eigenvalues are identical. The same statement is also true for the matrices $\tilde{N}_{ci}$ and N_{ci}. One question that may arise is, given two sets of matrices representing the same bilinear system, what is the transformation matrix to convert from one coordinate to the other.

First, form the observability matrices for both sets of system matrices

$$Q = \begin{bmatrix} C \\ C A_c \\ \vdots \\ C A_c^{n-1} \end{bmatrix} \tag{58}$$

and

$$\tilde{Q} = \begin{bmatrix} \tilde{C} \\ \tilde{C} \tilde{A}_c \\ \vdots \\ \tilde{C} \tilde{A}_c^{n-1} \end{bmatrix} \tag{59}$$

Substituting the relationship from Equation (57) into Equation (59) yields

$$\tilde{Q} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A}_c \\ \vdots \\ \tilde{C}\tilde{A}_c^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA_c \\ \vdots \\ CA_c^{n-1} \end{bmatrix} \Phi = Q\Phi \tag{60}$$

that in turn provide the following solution for computing the transformation matrix Φ

$$\Phi = Q^{\dagger}\tilde{Q} \tag{61}$$

This transformation matrix Φ will transform the original system coordinate to another system coordinate without changing the input-to-output map.

5. Numerical Example

Consider the following example presented in Bruni et al. [4]

$$\begin{aligned} \dot{x} &= A_c x + B_c u + N_{c1} x u_1 + N_{c2} x u_2 \\ y &= Cx \end{aligned} \tag{62}$$

where

$$\begin{aligned} A_c &= \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}; \; N_{c1} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}; \; N_{c2} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ B_c &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \; C = [0 \quad 1] \end{aligned} \tag{63}$$

Assume that we do not know the order of the system. Let us generate five sets of data with the time period $\Delta t = 1$ s. The first set of data with one input at a time is generated by a unit force of period 1 s. The second set of data with one input at a time is obtained by applying a unit force of period 2 s and the fifth set of data is computed with a unit force of period 5 s. The number of time points is set to be 20 for each data record.

Figure 1 shows a total of 10 ($p \times m \times r$) responses from 2 inputs ($r = 2$), 1 output ($m = 1$), and 5 different multiple-pulse inputs ($p = 5$). Each response sampled at 1 Hz has 20 data points. These ten responses are obtained by numerically integrating the bilinear system shown in Equation (62). With $\alpha = 5$ and $\beta = 6$, the Hankel matrix H_1 shown in Equation (19) should have the size of 5×12 ($\alpha m \times \beta r$). The state matrix and the output matrix identified from this Hankel matrix are

$$\tilde{A}_c = \begin{bmatrix} -1.0629 & 3.9782 \\ 0.0148 & -1.9371 \end{bmatrix}; \; \tilde{C} = [-0.9355 \quad 0.3497] \tag{64}$$

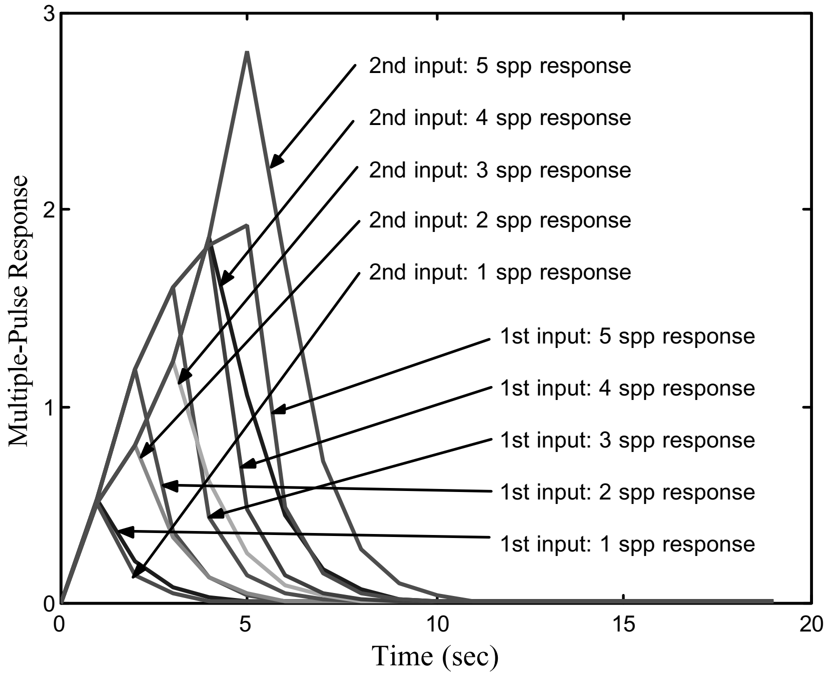

Figure 1. Five sets of pulse responses sampled at 1 Hz from two different-size inputs over five different sample periods, spp: sample-period pulse.

The singular values of this Hankel matrix are

$$\Sigma_1 = \mathrm{diag}[\,0.8347 \quad 0.0543 \quad 0\;0\;0\,]$$

implying that the order of the system is $n = 2$. The other matrices H_k for $k = 2, 3, \ldots, p$ shown in Equations (36) and (41) are of the size of 5×2, that produce the matrices $\bar{B}_1,\ \bar{B}_2, \cdots,\ \bar{B}_5$ of 2×2 each shown in Equation (44), and in turn yield $\mathbb{C}_1$ and $\mathbb{C}_2$ of 2×5 each shown in Equation (47). Applying Equations (45) through (54), the quantities B_c, N_{c1}, and N_{c2} can thus be identified as

$$\begin{aligned} \tilde{N}_{c1} &= \begin{bmatrix} 1.7752 & 3.2911 \\ -0.4182 & -0.7752 \end{bmatrix}; \ \tilde{N}_{c2} = \begin{bmatrix} 0.1678 & 0.3111 \\ 0.4489 & 0.8322 \end{bmatrix} \\ \tilde{B}_c &= \begin{bmatrix} -0.0929 & -0.9824 \\ -0.2484 & 0.2314 \end{bmatrix} \end{aligned} \tag{65}$$

The tilt on the top of B_c, N_{c1}, and N_{c2} signifies the identified quantities that are not uniquely determined. The quality of the identified system is evaluated next. The following transformation matrix is computed from Equation (61)

$$\Phi = \begin{bmatrix} -0.8715 & -3.6998 \\ -0.9355 & 0.3497 \end{bmatrix} \tag{66}$$

The matrices $\tilde{A}_c,\ \tilde{B}_c,\ \tilde{C},\ \tilde{N}_{c1}$ and $\tilde{N}_{c2}$would then be transformed by using Equation (57) to become A_c, B_c, C, N_{c1}, and N_{c2} shown in Equation (63).

Let us examine another case where we keep the unit force for the first input but change to 0.5 for the second input. Figure 2 shows the 10 multiple-pulse responses. Applying the same identification procedure as above, we obtain the following system matrices

$$\tilde{A}_c = \begin{bmatrix} -0.9509 & 3.9503 \\ 0.0131 & -2.094 \end{bmatrix}; \ \tilde{C} = \begin{bmatrix} -0.9253 & 0.3760 \end{bmatrix} \tag{67}$$

and

$$\begin{aligned} \tilde{N}_{c1} &= \begin{bmatrix} 1.8543 & 3.2168 \\ -0.4925 & -0.8543 \end{bmatrix}; \ \tilde{N}_{c2} = \begin{bmatrix} 0.1898 & 0.3292 \\ 0.4670 & 0.8102 \end{bmatrix} \\ \tilde{B}_c &= \begin{bmatrix} -0.0998 & -0.9755 \\ -0.2457 & 0.2591 \end{bmatrix} \end{aligned} \tag{68}$$

with the transformation matrix

$$\Phi = \begin{bmatrix} -0.9757 & -3.6737 \\ -0.9253 & 0.3760 \end{bmatrix} \tag{69}$$

Note that the Hankel singular values are

$$\Sigma_1 = \text{diag}\begin{bmatrix} 0.6753 & 0.0489\ 0\ 0\ 0 \end{bmatrix}$$

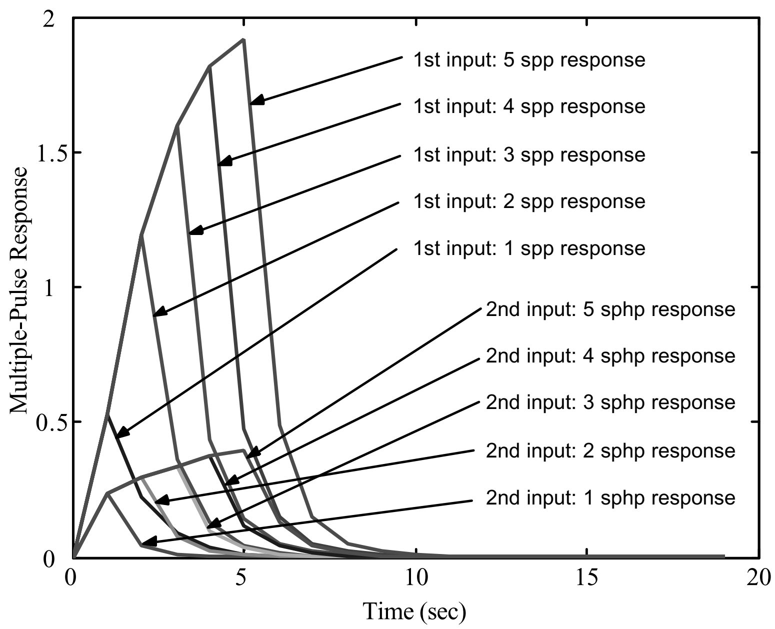

Figure 2. Five sets of pulse responses sampled at 1 Hz from two inputs of unit pulse over five different sample periods, spp: sample-period pulse, sphp: sample-period half pulse.

This set of system matrices also represents the bilinear system, because it can be transformed back to the original system exactly.

6. Concluding Remarks

A new method is introduced for identification of a continuous-time multi-input and multi-output bilinear system. The approach is to make judicious use of the linear-model properties of the bilinear system when subjected to a constant input. It has been shown in this paper that a bilinear system can be treated as a combination of two linear systems in the identification process. The first linear system is the one obtained by deleting the nonlinear terms of the bilinear system. The second linear system is given by assuming a constant input. Due to this latter property, the identification process for the bilinear system becomes a combination of two linear-system identification processes. The key is to combine these two linear-system identification processes in the same coordinate system. The resulting identified system matrices would be similar to the original ones in the sense that they represent the same bilinear system but in different coordinates. With a proper coordinate transformation, both the original model and the identified model are identical.

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