Nonlinear Dynamics (2005) 45: 45–53
DOI: 10.1007/s11071-006-1890-z

Pseudo Almost Periodic Solutions to a Class of Semilinear Differential Equations

TOKA DIAGANA
Department of Mathematics, Howard University, 2441, 6th Street N.W., Washington, D.C. 20059, USA;
(e-mail: tdiagana@howard.edu)

(Received: 22 March 2005; accepted: 29 July 2005)

Abstract. This paper is concerned with the existence and uniqueness of pseudo almost periodic solutions to a class of semilinear differential equations involving the algebraic sum of two (possibly noncommuting) densely defined closed linear operators acting on a Hilbert space. Sufficient conditions for the existence and uniqueness of pseudo almost periodic solutions to those semilinear equations are obtained.

Key words: almost periodic function, Banach fixed-point principle, existence and uniqueness of a pseudo almost periodic solution, infinitesimal generator of a c_0-group, pseudo almost periodic function, the method of the invariant subspaces for unbounded linear operators

1. Introduction

Let $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a Hilbert space. In [1] the author had shown through the so-called method of the *invariant subspaces* for unbounded linear operators [1–4] that under some suitable assumptions, every bounded solution to the abstract differential equation

$$u'(t) = Au(t) + Bu(t) + g(t), \quad t \in \mathbb{R}, \tag{1}$$

where A, B are densely defined closed linear operators on $\mathbb{H}$, and $g : \mathbb{R} \mapsto \mathbb{H}$ is a continuous function, is *pseudo almost periodic*.

In this paper, we combine the above-mentioned method and the classical Banach fixed-point principle to obtain some sufficient conditions, which do guarantee the existence and uniqueness of a pseudo almost periodic solution to the class of the semilinear differential equations

$$u'(t) = Au(t) + Bu(t) + f(t, Cu(t)), \quad t \in \mathbb{R}, \tag{2}$$

where A, B are densely defined closed (possibly noncommuting) linear operators on $\mathbb{H}$, $C\colon \mathbb{H} \mapsto \mathbb{H}$ is a nonzero bounded linear operator, and $f\colon \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$ is a jointly continuous function.

The concept of the pseudo almost periodicity, which is the central question in this paper was first initiated by Zhang in [5–7] and is a natural generalization of the of the classical (Bochner) almost periodicity. Thus, this new concept is welcome to implement another existing generalization of the (Bochner) almost periodicity, the so-called notion of asymptotically almost periodicity due to Fréchet, see e.g., [4, 8–10]. More details on the concepts of almost periodicity and pseudo almost periodicity and related applications can be found in [1, 4–7, 9, 11–17] and the references therein.

The existence and uniqueness of pseudo almost periodic solutions to some semilinear differential equations has been studied in [5–7, 11–13, 16] and the references therein. Here, it goes back to examine some sufficient conditions which do guarantee the existence and uniqueness of a pseudo almost periodic solution to Equation (2), by using the above-mentioned techniques.

Let us recall some definitions and notations that we shall use in the sequel.

2. Preliminaries

2.1. Invariant and Reducing Subspaces for Linear Operators

Let $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $\mathcal{M} \subset \mathbb{H}$ be a closed subspace. Let A be a densely defined closed unbounded linear operator on $\mathbb{H}$ and let $P_{\mathcal{M}}$ denote the orthogonal projection onto the closed subspace $\mathcal{M}$.

Definition 2.1. $\mathcal{M} \subset \mathbb{H}$ is called an invariant subspace for A if the linear operator A maps $D(A) \cap \mathcal{M}$ into $\mathcal{M}$.

Example 2.2. Let $\mathbb{H}$ be a Hilbert space and let $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ be a densely defined closed linear operator on $\mathbb{H}$.
(i) $\mathcal{M} = N(A) = \{u \in D(A) : Au = 0\}$ is an invariant subspace for A.
(ii) If A is self-adjoint, then each eigenspace $\mathcal{M}_\lambda = N(\lambda I - A)$ is invariant for the linear operator A.

Example 2.3 Let $\mathbb{H} = L^2([\alpha, \beta])$. If $V \in L^2([\alpha, \beta] \times [\alpha, \beta])$, let A be the integral operator defined by

$$(A\phi)(s) := \int_\alpha^s V(s,t)\phi(t)\,dt, \quad s \in [\alpha, \beta].$$

Setting $\mathcal{M}_\gamma = \{\phi \in L^2([\alpha, \beta]) : \phi = 0 \text{ a.e. on } [\alpha, \gamma]\}$, it is not hard to see that $(\mathcal{M}_\gamma)_{\gamma \in [\alpha,\beta]}$ is invariant for A.

Theorem 2.4. *The equality $P_{\mathcal{M}} A P_{\mathcal{M}} = A P_{\mathcal{M}}$ is a necessary and sufficient condition for a subspace $\mathcal{M}$ to be invariant for a linear operator A.*

Proof. Suppose $P_{\mathcal{M}} A P_{\mathcal{M}} = A P_{\mathcal{M}}$. If $x \in D(A) \cap \mathcal{M}$, then $x = P_{\mathcal{M}} x \in D(A)$, and so, $Ax = AP_{\mathcal{M}}x = P_{\mathcal{M}} A P_{\mathcal{M}} x \in \mathcal{M}$.

Conversely, if $\mathcal{M}$ is invariant for A; let $x \in \mathbb{H}$ such that $P_{\mathcal{M}} x \in D(A)$. Then $AP_{\mathcal{M}} x \in \mathcal{M}$ and so $P_{\mathcal{M}} A P_{\mathcal{M}} x = A P_{\mathcal{M}} x$, hence $AP_{\mathcal{M}} \subset P_{\mathcal{M}} A P_{\mathcal{M}}$. Since $D(AP_{\mathcal{M}}) = D(P_{\mathcal{M}} A P_{\mathcal{M}})$ it follows that $AP_{\mathcal{M}} = P_{\mathcal{M}} A P_{\mathcal{M}}$. □

Definition 2.5. A closed proper subspace $\mathcal{M}$ of the Hilbert space $\mathbb{H}$ is said to reduce an operator A if $P_{\mathcal{M}} D(A) \subset D(A)$ and both $\mathcal{M}$ and $H \ominus \mathcal{M}$, the orthogonal complement of $\mathcal{M}$, are invariant for A.

Theorem 2.6. *A closed subspace $\mathcal{M}$ of $\mathbb{H}$ reduces an operator A if and only if $P_{\mathcal{M}} A \subset A P_{\mathcal{M}}$.*

Proof. See the proof of [18, Theorem 4.11., p. 29].

□

Remark 2.7. Notice that the inclusion $P_{\mathcal{M}}A \subset AP_{\mathcal{M}}$ yields if $x \in D(A)$, then $P_{\mathcal{M}}x \in D(A)$ and $P_{\mathcal{M}}Ax = AP_{\mathcal{M}}x$.

From now on, $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$, $D(T)$, $R(T)$ and $N(T)$ stand for a Hilbert space, the domain, the range, and the kernel of a given (possibly unbounded) linear operator T, respectively. If S, T are densely defined closed (unbounded) linear operators on $\mathbb{H}$, then their algebraic sum is the linear operator defined by $D(S+T) = D(S) \cap D(T)$ and $(S+T)u := Su + Tu$, for each $u \in D(S) \cap D(T)$.

Throughout the rest of the paper, we suppose that the algebraic sum $A + B$ of A and B appearing in Equation (2) is densely defined.

2.2. Pseudo Almost Periodic Functions

Let $(BC(\mathbb{H}), \|\cdot\|_\infty)$ be the Banach space of bounded continuous functions $g : \mathbb{R} \mapsto \mathbb{H}$ endowed with the sup norm defined by $\|g\|_\infty := \sup_{t\in\mathbb{R}} \|g(t)\|$. Similarly, $BC(\mathbb{R}\times\Omega)$ where $\Omega \subset \mathbb{H}$ is an open subset denotes the vector space of bounded continuous functions $F : \mathbb{R}\times\Omega \mapsto \mathbb{H}$.

Definition 2.8 [8] A function $f \in BC(\mathbb{H})$ is called almost periodic if for each $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the following property

$$\|f(t+\tau) - f(t)\| < \varepsilon \quad (t \in \mathbb{R}).$$

The number τ above is called an *ε-translation number* of f, and the collection of such functions will be denoted $AP(\mathbb{H})$.

Similarly,

Definition 2.9. A function $F \in BC(\mathbb{R}\times\Omega)$ is called almost periodic in $t \in \mathbb{R}$ uniformly in any $K \subset \Omega$ a bounded subset if for each $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length $l_\varepsilon > 0$ contains a number τ with the following property

$$\|F(t+\tau, x) - F(t, x)\| < \varepsilon, \quad (t \in \mathbb{R},\ x \in K).$$

Here again, the number τ above is called an ε-translation number of F, and the class of such functions will be denoted $AP(\mathbb{R}\times\Omega)$.

More details on properties of almost periodic functions $f : \mathbb{R} \mapsto \mathbb{H}$ and as well as those of the form $F : \mathbb{R}\times\mathbb{H} \mapsto \mathbb{H}$ can be found in the literature, especially in [4, 8, 9, 17] and the references therein.

From now on, one supposes that $\Omega = \mathbb{H}$ and set

$$AP_0(\mathbb{H}) := \left\{ f \in BC(\mathbb{H}) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(s)\| ds = 0 \right\}, \quad \text{and}$$

$$AP_0(\mathbb{R}\times\mathbb{H}) = \left\{ F \in BC(\mathbb{R}\times\mathbb{H}) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|F(t,u)\| dt = 0,\ \forall u \in \mathbb{H} \right\}.$$

Definition 2.10. A function $f \in BC(\mathbb{H})$ is called pseudo almost periodic if it can be expressed as

$$f = g + \phi,$$

where $g \in AP(\mathbb{H})$ and $\phi \in AP_0(\mathbb{H})$.

The collection of such functions will be denoted by $PAP(\mathbb{H})$.

Let us mention that the functions g and ϕ appearing in Definition 2.10 are respectively called the almost periodic and the ergodic perturbation components of f. Furthermore, the decomposition in Definition 2.10 is unique [5].

We now equip $PAP(\mathbb{H})$ the collection of pseudo almost periodic functions from $\mathbb{R}$ into $\mathbb{H}$ with the sup norm. It is well-known that $(PAP(\mathbb{H}), \|\cdot\|_\infty)$ is a Banach space, see e.g., [16].

Definition 2.11. A function $f \in BC(\mathbb{R} \times \mathbb{H})$ is called pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{H}$ if it can be expressed as

$$f = g + \phi,$$

where $g \in AP(\mathbb{R} \times \mathbb{H})$ and $\phi \in AP_0(\mathbb{R} \times \mathbb{H})$.

The collection of such functions will be denoted by $PAP(\mathbb{R} \times \mathbb{H})$.

The following assumptions will be made:

(H.1) The function $f\colon \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$, $(t, u) \mapsto f(t, u)$ is pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{H}$, i.e. $f = g + \phi$, where $g \in AP(\mathbb{R} \times \mathbb{H})$ and $\phi \in AP_0(\mathbb{R} \times \mathbb{H})$; and that f satisfies Lipschitz condition in $u \in \mathbb{H}$ for each $t \in \mathbb{R}$, i.e., there exists $L > 0$,

$$\|f(t, u) - f(t, v)\| \leq L \, . \, \|u - v\|,$$

for all $u, v \in \mathbb{H}$ and $t \in \mathbb{R}$;

(H.2) there exists $\mathcal{M} \subset \mathbb{H}$ a closed subspace which reduces both A and B. In this event, we denote by $P_{\mathcal{M}}$, $Q_{\mathcal{M}} = (I - P_{\mathcal{M}}) = P_{\mathbb{H} \ominus \mathcal{M}}$, the orthogonal projections onto $\mathcal{M}$ and $\mathbb{H} \ominus \mathcal{M}$, respectively;

(H.3) A, B are the infinitesimal generators of c_0-groups of bounded operators $(T(s))_{s \in \mathbb{R}}$, $(R(s))_{s \in \mathbb{R}}$, respectively, such that, there exist $M, K, c, d > 0$ with

$$\|T(s-\sigma)P_{\mathcal{M}}\| \leq M e^{-c(s-\sigma)} \quad \text{for each } s \geq \sigma, \quad \text{and}$$
$$\|T(s-\sigma)Q_{\mathcal{M}}\| \leq M e^{-c(\sigma-s)} \quad \text{for each } s \leq \sigma, \quad \text{and}$$
$$\|R(s-\sigma)P_{\mathcal{M}}\| \leq K e^{-d(s-\sigma)} \quad \text{for each } s \geq \sigma, \quad \text{and}$$
$$\|R(s-\sigma)Q_{\mathcal{M}}\| \leq K e^{-d(\sigma-s)} \quad \text{for each } s \leq \sigma;$$

(H.4) $R(A) \subset R(P_{\mathcal{M}}) = N(Q_{\mathcal{M}})$;

(H.5) $R(B) \subset R(Q_{\mathcal{M}}) = N(P_{\mathcal{M}})$.

Remark 2.12

(i) If A, B are infinitesimal generators of c_0-groups of bounded operators, then their algebraic sum $A + B$ need not be the infinitesimal generator of a c_0-group of bounded operators.

(ii) Since $A + B$ is assumed to be densely defined, then from the assumption (H.2) it follows that both $\mathcal{M}$ and $[\mathbb{H} \ominus \mathcal{M}]$ are invariant for $A + B$.

(iii) The method of the invariant subspaces consists of imposing the assumptions (H.2), (H.4), and (H.5) on A and B.

3. Existence and Uniqueness of Pseudo Almost Periodic Solutions

Throughout the rest of the paper, $C:\mathbb{H} \mapsto \mathbb{H}$ denotes a nonzero bounded linear operator.

Theorem 3.1 *Under assumptions (H.1), (H.2), (H.3), (H.4), and (H.5), Equation (2) has a unique pseudo almost periodic solution whenever*

$$\|C\| < \frac{1}{L}\left[\left(\frac{M}{c}\right) + \left(\frac{K}{d}\right)\right]^{-1}.$$

The proof of our main result (Theorem 3.1) requires the following technical lemmas:

Lemma 3.2. *Under assumptions (H.1), (H.2), (H.3), (H.4), and (H.5), every bounded solution to Equation (2) can be expressed as: $u = \zeta(u) + \xi(u)$, where*

$$\zeta(u)(t) := \int_{-\infty}^{t} T(t-s)P_{\mathcal{M}}g(s, Cu(s))\,ds + \int_{-\infty}^{t} R(t-s)Q_{\mathcal{M}}g(s, Cu(s))\,ds, \quad \textit{and}$$
$$\xi(u)(t) := \int_{-\infty}^{t} T(t-s)P_{\mathcal{M}}\phi(s, Cu(s))\,ds + \int_{-\infty}^{t} R(t-s)Q_{\mathcal{M}}\phi(s, Cu(s))\,ds.$$

Proof. (Lemma 3.2). Let u be a bounded solution to Equation (2). In view of (H.2), u can be decomposed as

$$u(t) = P_{\mathcal{M}}u(t) + (I - P_{\mathcal{M}})u(t), \quad \forall t \in \mathbb{R},$$

where $P_{\mathcal{M}}u(t) \in R(P_{\mathcal{M}}) = N(Q_{\mathcal{M}})$ and $Q_{\mathcal{M}}u(t) \in N(P_{\mathcal{M}}) = R(Q_{\mathcal{M}})$.

We have

$$\begin{aligned}
\frac{d}{dt}(P_{\mathcal{M}}u(t)) &= P_{\mathcal{M}}\frac{d}{dt}u(t) \\
&= P_{\mathcal{M}}Au(t) + P_{\mathcal{M}}Bu(t) + P_{\mathcal{M}}f(t, Cu(t)) \\
&= AP_{\mathcal{M}}u(t) + P_{\mathcal{M}}Bu(t) + P_{\mathcal{M}}f(t, Cu(t)), \quad \text{by (H.2),} \\
&= AP_{\mathcal{M}}u(t) + P_{\mathcal{M}}f(t, Cu(t)), \quad \text{by (H.5).}
\end{aligned}$$

From the previous equation and the fact that $P_{\mathcal{M}}$ is a bounded linear operator on $\mathbb{H}$ it is clear that $P_{\mathcal{M}}u(t)$ is a bounded solution to the differential equation

$$\frac{d}{dt}(z(t)) = Az(t) + P_{\mathcal{M}}f(t, Cu(t)).$$

It follows that (see [13]):

$$P_{\mathcal{M}}u(t) = \int_{-\infty}^{t} T(t-s)P_{\mathcal{M}}f(s, Cu(s))\,ds.$$

And hence,

$$P_{\mathcal{M}}u(t) = \int_{-\infty}^{t} T(t-s)P_{\mathcal{M}}g(s, Cu(s))\,ds + \int_{-\infty}^{t} T(t-s)P_{\mathcal{M}}\phi(s, Cu(s))\,ds,$$

by (H.1).

Arguing similarly as above, it follows that

$$Q_{\mathcal{M}}u(t) = \int_{-\infty}^{t} R(t-s)Q_{\mathcal{M}}f(s, Cu(s))\, ds,$$

and therefore

$$Q_{\mathcal{M}}u(t) = \int_{-\infty}^{t} R(t-s)Q_{\mathcal{M}}g(s, Cu(s))\, ds + \int_{-\infty}^{t} R(t-s)Q_{\mathcal{M}}\phi(s, Cu(s))\, ds,$$

by (H.1).

One completes the proof by combining expressions of both $P_{\mathcal{M}}u$ and $Q_{\mathcal{M}}u$ above.

□

Lemma 3.3. *Under assumptions (H.1), (H.2), (H.3), (H.4), and (H.5), if* $u \in PAP(\mathbb{H})$ *is a solution to Equation (2), then* $u = \zeta(u) + \xi(u)$*, where* $\zeta(u) \in AP(\mathbb{H})$ *and* $\xi(u) \in AP_0(\mathbb{H})$ *(*ζ *and* ξ *being as in Lemma 3.2).*

Proof. (Lemma 3.3). Let $u \in PAP(\mathbb{H})$. Clearly, u is bounded. If u is a solution to Equation (2), then $u = \zeta(u) + \xi(u)$, by Lemma 3.2.

First of all, let us notice that since $f \in PAP(\mathbb{R} \times \mathbb{H})$ and satisfies Lipschitz condition, (H.1), for each $v \in PAP(\mathbb{H})$, the function $f(\cdot, v(\cdot))$ belongs to $PAP(\mathbb{H})$, see, e.g., [14, Proposition 2.2]. Furthermore, $g(\cdot, v(\cdot))$ and $\phi(\cdot, v(\cdot))$ are respectively the almost periodic and ergodic perturbation components of $f(\cdot, v(\cdot))$. In particular, $g(\cdot, u(\cdot)) \in AP(\mathbb{H})$ and $\phi(\cdot, u(\cdot)) \in AP_0(\mathbb{H})$.

We next show that $\zeta(u) \in AP(\mathbb{H})$. Since C is bounded, it follows that $t \mapsto Cu(t)$ is almost periodic, and hence $g(\cdot, Cu(\cdot)) \in AP(\mathbb{H})$. Thus for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all γ, there is $\tau \in [\gamma, \gamma + \delta]$ with

$$\|g(s+\tau, Cu(s+\tau)) - g(s, Cu(s))\| < \mu \,.\, \varepsilon, \quad \forall s \in \mathbb{R},$$

where

$$\mu = \left(\frac{M}{c} + \frac{K}{d}\right)^{-1}.$$

Considering $\zeta(u)(t+\tau) - \zeta(u)(t)$ and using the assumption (H.3) it easily follows that

$$\|\zeta(u)(t+\tau) - \zeta(u)(t)\| < \varepsilon, \quad \forall t \in \mathbb{R},$$

and hence $t \mapsto \zeta(t)$ is an almost periodic function.

It remains to show that $t \mapsto \xi(u)(t)$ is in $AP_0(\mathbb{H})$. For that, write

$$\xi(u)(t) = Y_T(t) + Y_R(t),$$

where

$$Y_T(t) := \int_{-\infty}^{t} T(t-s)P_{\mathcal{M}}\phi(s, Cu(s))\, ds,$$

and

$$Y_R(t) := \int_{-\infty}^{t} R(t-s)Q_{\mathcal{M}}\phi(s, Cu(s))\, ds.$$

We will only show that $Y_T \in AP_0(\mathbb{H})$ since the proof for Y_R follows along the same lines. Indeed, it is clear that $s \mapsto Y_T(s)$ is a bounded continuous function. Thus, the remaining task is to show that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|Y_T(t)\|\, dt = 0.$$

Using the assumption (H.3) it follows that,

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|Y_T(t)\|\, dt \le I + J,$$

where

$$I := \lim_{r\to\infty} \frac{M}{2r} \int_{-r}^{r} dt \left(\int_{-r}^{t} e^{-c(t-s)}\, \|\phi(s, Cu(s))\|\, ds\right),$$

and

$$J := \lim_{r\to\infty} \frac{M}{2r} \int_{-r}^{r} dt \int_{-\infty}^{-r} e^{-c(t-s)}\, \|\phi(s, Cu(s))\|\, ds.$$

To complete the proof we must show that $I = J = 0$. For that, we mainly use the facts that $\phi(\cdot, Cu(\cdot)) \in AP_0(\mathbb{H})$ and $G = \sup_{t\in\mathbb{R}} \|\phi(t, Cu(t))\| < \infty$. Indeed,

$$\begin{aligned}
I &= \lim_{r\to\infty} \frac{M}{2r} \int_{-r}^{r} \|\phi(t, Cu(t))\|\, dt \left(\int_{-r}^{t} e^{-c(t-s)}\, ds\right)\\
&= \lim_{r\to\infty} \frac{M}{2r} \int_{-r}^{r} \|\phi(t, Cu(t))\|\, dt \left(\frac{1}{c}[1 - e^{-c(t+r)}]\right)\\
&\le \frac{M}{c} \cdot \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\phi(t, Cu(t))\|\, dt\\
&= 0,
\end{aligned}$$

by $\phi(\cdot, Cu(\cdot)) \in AP_0(\mathbb{H})$.

Similarly,

$$\begin{aligned}
J &= \lim_{r\to\infty} \frac{M}{2r} \int_{-\infty}^{-r} e^{cs} \|\phi(s, Cu(s))\|\, ds \int_{-r}^{r} e^{-ct} dt\\
&\le \lim_{r\to\infty} \frac{MG}{2r} \int_{-\infty}^{-r} e^{cs}\, ds \left(\frac{1}{c}[e^{cr} - e^{-cr}]\right)\\
&= \lim_{r\to\infty} \frac{MG}{2rc^2}(1 - e^{-2cr})\\
&= 0.
\end{aligned}$$

Thus $s \mapsto Y_T(s)$ belongs to $AP_0(\mathbb{H})$. In this way, $s \mapsto Y_R(s)$ is also in $AP_0(\mathbb{H})$, and therefore $\xi(u) \in AP_0(\mathbb{H})$. ☐

Proof. (Theorem 3.1). In view of Lemma 3.2, each bounded solution to Equation (2) can be written as $\Lambda(u) := \zeta(u) + \xi(u)$, where

$$\Lambda(u)(t) = \int_{-\infty}^{t} T(t-s)P_{\mathcal{M}} f(s, Cu(s))\, ds + \int_{-\infty}^{t} R(t-s)Q_{\mathcal{M}} f(s, Cu(s))\, ds.$$

Now using Lemma 3.2 it follows that Λ given above maps $PAP(\mathbb{H})$ into itself. To complete the proof we must show that Λ is a strict contraction from $(PAP(\mathbb{H}), \|\cdot\|_\infty)$ into itself.

Let $u, v \in PAP(\mathbb{H})$,

$$\begin{aligned}\left\| \int_{-\infty}^{t} T(t-s)P_{\mathcal{M}}[f(s, Cu(s)) - f(s, Cv(s))]\, ds \right\| &\leq \alpha \|u - v\|_\infty \int_{-\infty}^{t} e^{-c(t-s)}\, ds \\ &\leq \frac{\alpha}{c}\,.\, \|u - v\|_\infty\end{aligned}$$

for each $t \in \mathbb{R}$ with $\alpha = LM\|C\|$.

Consequently,

$$\sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} T(t-s)P_{\mathcal{M}}[f(s, Cu(s)) - f(s, Cv(s))]\, ds \right\| \leq \left(\frac{LM\|C\|}{c}\right).\, \|u - v\|_\infty.$$

Similarly,

$$\left\| \int_{-\infty}^{t} R(t-s)Q_{\mathcal{M}}[f(s, Cu(s)) - f(s, Cv(s))]\, ds \right\| \leq \frac{\beta}{d}\,.\, \|u - v\|_\infty$$

for each $t \in \mathbb{R}$ with $\beta = LK\|C\|$. And hence,

$$\sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} R(t-s)Q_{\mathcal{M}}[f(s, Cu(s)) - f(s, Cv(s))]\, ds \right\| \leq \left(\frac{LK\|C\|}{d}\right).\, \|u - v\|_\infty.$$

In summary,

$$\begin{aligned}\|\Lambda(u) - \Lambda(v)\|_\infty &\leq \left[\left(\frac{LM\|C\|}{c}\right) + \left(\frac{LK\|C\|}{d}\right)\right].\, \|u - v\|_\infty \\ &= \left[\left(\frac{M}{c}\right) + \left(\frac{K}{d}\right)\right]\|C\|\,.\, L\,.\, \|u - v\|_\infty.\end{aligned}$$

Thus, if

$$\|C\| < \frac{1}{L}\left[\left(\frac{M}{c}\right) + \left(\frac{K}{d}\right)\right]^{-1},$$

then the nonlinear operator $\Lambda\colon (PAP(\mathbb{H}), \|.\|_\infty) \mapsto (PAP(\mathbb{H}), \|.\|_\infty)$ is a strict contraction, and therefore by the Banach fixed point principle there exists a unique $u_0 \in PAP(\mathbb{H})$ such that $\Lambda(u_0) = u_0$.

☐

References

1. Diagana, T., 'Pseudo almost periodic solutions to some differential equations', *Nonlinear Analysis* **60**(7), 2005, 1277–1286.
2. Diagana, T., 'Some remarks on some second-order hyperbolic differential equations', *Semigroup Forum* **68**(3), 2004, 357–364.
3. Diagana, T. and N'Guérékata, G. M., 'Some extension of the Bohr-Neugebauer-N'Guerekata Theorem', *Journal of Analysis and Applications* **2**(1), 2004, 1–10.
4. N'Guérékata, G. M., *Topics in Almost Automorphy*, Springer, New York, 2005.
5. Zhang, C. Y., 'Pseudo almost periodic solutions of some differential equations', *Journal of Mathematical Analysis and Application* **151**, 1994, 62–76.
6. Zhang, C. Y., 'Pseudo almost periodic solutions of some differential equations II', *Journal of Mathematical Analysis and Application* **192**, 1995, 543–561.
7. Zhang, C. Y., 'Integration of vector-valued pseudo almost periodic functions', *Proceedings of the American Mathematical Society* **121**, 1994, 167–174.
8. Corduneanu, C., *Almost Periodic Functions*, 2nd edn., Chelsea, New York, 1989.
9. Fink, A. M., *Almost Periodic Differential Equations*, Lecture Notes in Mathematics **377**, Springer-Verlag, New York-Berlin, 1974.
10. N'Guérékata, G. M., *Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic/Plenum Publishers, New York, 2001.
11. Ait Dads, E., Ezzinbi, K., and Arino, O., 'Pseudo almost periodic solutions for some differential equations in a banach space', *Nonlinear Analysis* **28**(7), 1997, 1141–1155.
12. Ait Dads, E. and Arino, O., 'Exponential dichotomy and existence of pseudo almost periodic solutions of some differential equations', *Nonlinear Analysis* **27**(4), 1996, 369–386.
13. Amir, B. and Maniar, L., 'Composition of pseudo-almost periodic functions and Cauchy problems with operator of nondense domain', *Annales Mathematiques Blaise Pascal* **6**(1), 1999, 1–11.
14. Cuevas, C. and Pinto, M., 'Existence and uniqueness of pseudo almost periodic solutions of semilinear Cauchy problems with non-dense domain', *Nonlinear Analysis* **45**, 2001, 73–83.
15. Hino, Y., Naito, T., Minh, N. V., and Shin, J. S., *Almost Periodic Solutions of Differential Equations in Banach Spaces*, Stability and Control: Theory, Methods and Applications, vol. 15, Taylor and Francis, London, 2002.
16. Huang, F. L., Li, H. X., and Li, J. Y., 'Composition of pseudo almost-periodic functions and semilinear differential equations', *Journal of Mathematical Analysis and Application* **255**(2), 2001, 436–446.
17. Zaidman, S., *Topics in Abstract Differential Equations*, Pitman Research Notes in Mathematics Ser. II, John Wiley and Sons, New York, 1994–1995.
18. Locker, J., *Spectral Theory of Non-Self-Adjoint Two-Point Differential Operators*, AMS Mathematical Surveys and Monographs, vol. 73, 2000.
19. Diagana, T., N'Guérékata, G. M., and Minh, N. V., 'Almost Automorphic Solutions of Evolution Equations', *Proceedings of the American Mathematical Society* **134**(11), 2004, 3289–3298.
20. Diagana, T. and N'Guérékata, G. M., 'On some perturbations of some abstract differential equations', *Comment. Math. Prace Mat.* **XLIII**(2), 2003, 201–206.