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A Novel Method for Finding Minimum-norm Solutions to Pseudomonotone Variational Inequalities

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Abstract

In this paper, we introduce a novel iterative method for finding the minimum-norm solution to a *pseudomonotone* variational inequality problem in Hilbert spaces. We establish strong convergence of the proposed method and its linear convergence under some suitable assumptions. Some numerical experiments are given to illustrate the performance of our method. Our result improves and extends some existing results in the literature.

Keywords Subgradient extragradient method · Variational inequality problem · Pseudomonotone operator · Strong convergence · Convergence rate

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1 Introduction

Throughout this paper, assume that C is a nonempty, convex and closed subset of the real Hilbert space H with the inner product $\langle , \rangle$ and the norm $\|.\|$. Let $F : H \to H$ be a Lipschitz continuous operator. The object of our investigation is the following variational inequality problem (shortly, $VI(C, F)$):

Find $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0 \ \ \forall x \in C. \tag{1}$$

We denote the solution set of $VI(C, F)$ by $Sol(C, F)$.

Many problems in various fields such as physic, economics, engineering, optimization theory can be led to variational inequalities. Iterative methods for solving these problems have been proposed and analyzed (see, for example, Facchinei and Pang (2003), Gibali et al. (2017), Kassay et al. (2011), Kinderlehrer and Stampacchia (1980), Konnov (2001) and references therein). One of the most famous methods for solving $VI(C, F)$ is the extragradient method introduced by Korpelevich (1976). In this method, one needs to calculate two projections onto C at each iteration. This may affect the efficiency of the method when finding a projection onto a closed and convex set C is not an easy problem.

In recent years, many authors are interested in the extragradient method and improved it in various ways, see, e.g. Anh et al. (2020), Censor et al. (2011a, b, 2012b), Reich, Thong and Dong et al. (2021), Reich, Thong and Cholamjiak et al. (2021), Thong and Hieu (2018), Thong and Vuong (2021), Yang et al. (2018), Yang (2021) and references therein. The subgradient extragradient method, proposed by Censor et al. (2012a) for solving $VI(C, F)$ in real Hilbert spaces is one of these modifications.

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda F x_n), \\ T_n = \{x \in H : \langle x_n - \lambda F x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda F y_n), \end{cases} \tag{2}$$

where $\lambda \in (0, \frac{1}{L})$, and L is a Lipschitz constant of F. This method replaces two projections onto C by one projection onto C and one onto a half-space. The sequence $\{x_n\}$ generated by (2) converges weakly to an element of $Sol(C, F)$ provided that $Sol(C, F)$ is nonempty.

Kraikaew and Seajung (2014) used the subgradient extragradient method and Halpern method to introduce an algorithm for solving $VI(C, F)$ as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda F x_n), \\ T_n = \{x \in H : \langle x_n - \lambda F x_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}(x_n - \lambda F y_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) z_n, \end{cases} \tag{3}$$

where $\lambda \in (0, \frac{1}{L})$, $\{\alpha_n\} \subset (0,1)$, $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$. They proved that the sequence $\{x_n\}$ generated by (3) converges strongly to $P_{Sol(C,F)}x_0$ if F is monotone and L-Lipschitz continuous. The main disadvantage of Algorithms (2), (3) is a requirement to know the Lipschitz constant of F or at least to know some its estimation.

Very recently, Yang (2021) proposed a modification of subgradient extragradient method with step size rule using the inertial-type method as follows: Given $\lambda_0 > 0, \mu < \mu_0 \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary

$$
\begin{aligned}
w_n &= x_n + \alpha_n(x_n - x_{n-1}),\\
y_n &= P_C(w_n - \lambda_n F w_n),\\
T_n &:= \{x \in H : \langle w_n - \lambda_n F w_n - y_n, x - y_n\rangle \le 0\},\\
x_{n+1} &= P_{T_n}(w_n - \lambda_n F y_n),\\
\lambda_{n+1} &= \begin{cases} \min\{\mu \dfrac{\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2}{2\langle F w_n - F y_n, x_{n+1} - y_n\rangle}, \lambda_n\} & \text{if } \langle F w_n - F y_n, z_n - y_n\rangle > 0,\\ \lambda_n & \text{otherwise.}\end{cases}
\end{aligned}
$$

Under the pseudomonotonicity and sequential weak continuity of the mapping, the convergence of the algorithm was established without the knowledge of the Lipschitz constant of the mapping.

Motivated and inspired by the above mentioned works, and by the ongoing research in these directions, in this paper, we suggest a new iterative scheme for finding the minimum-norm solution to $VI(C, F)$ (1). It is worth pointing out that the proposed algorithm does not require the prior knowledge of the Lipschitz-type constant of the variational inequality mapping and only requires to compute one projection onto a feasible set per iteration as well as without the assumption on the weakly sequential continuity of the mapping. Moreover, the convergence rate is obtained under strong pseudomonotonicity and Lipschitz continuity assumptions of the variational inequality mapping.

The paper is organized as follows. In Sect. 2, we recall some basic definitions and results. In Sect. 3, we present and analyze the convergence of the proposed algorithms. Finally in Sect. 4, we present some numerical experiments to illustrate the performance of the proposed method.

2 Preliminaries

Lemma 2.1 ([Cottle and Yao (1992), *Lemma* 2.1]) *Consider the problem $VI(C, F)$ with C being a nonempty, closed, convex subset of a real Hilbert space H and $F : C \to H$ being pseudo-monotone and continuous. Then, x^* is a solution of $VI(C, F)$ if and only if*

$$\langle Fx, x - x^*\rangle \ge 0 \quad \forall x \in C.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following results hold*:

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \ \forall x, y \in H$;
(ii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle \ \forall x, y \in H$.

Definition 2.1 Let $T : C \to H$ be an operator, where C is a closed and convex subset of a real Hilbert space H. Then

– T is called L-Lipschitz continuous with $L > 0$ if

$$\|Tx - Ty\| \le L\|x - y\| \quad \forall x, y \in C.$$

– T is called monotone if

$$\langle Tx - Ty, x - y\rangle \ge 0 \quad \forall x, y \in C.$$

– T is said to be *pseudo-monotone* if

$$\langle Tx, y - x\rangle \ge 0 \Longrightarrow \langle Ty, y - x\rangle \ge 0.$$

It is called δ-strongly pseudo-monotone if there is $\delta > 0$ such that

$$\langle Tx, y - x\rangle \ge 0 \Longrightarrow \langle Ty, y - x\rangle \ge \delta\|y - x\|^2.$$

– T is said to be *weakly sequentially continuous* if, for each sequence $\{x_n\}$ in C, $\{x_n\}$ converges weakly to a point $x \in C$, then $\{Tx_n\}$ converges weakly to Tx.
– T is called weakly closed on C if for any $\{x_n\} \subset C$, $x_n \rightharpoonup x$, and $T(x_n) \rightharpoonup y$, then $T(x) = y$.
– T is said to have $*$-property on C, if the function $\|T(x)\|$ is weakly lower-semicontinuous (w.l.s.c.) on C, i.e., for any $\{x_n\} \subset C$, $x_n \rightharpoonup x$,

$$\|T(x)\| \le \liminf_{n\to\infty} \|T(x_n)\|.$$

A relation between the weakly sequential continuity, weak closedness and $*$-property are revealed in the following simple statement.

Lemma 2.3

(i) *Any weakly sequentially continuous operator is weakly closed and have the $*$-property.*
(ii) *A weakly closed operator, mapping bounded subsets into bounded subsets, is weakly sequentially continuous.*
(iii) *An operator having the $*$-property and mapping bounded subsets into bounded subsets is not necessarily weakly sequentially continuous, and hence is not necessarily weakly closed.*

Proof *i.* Suppose T is weakly sequentially continuous on C. Then it is weakly closed by definition. Further, let $C \ni x_n \rightharpoonup x$, then $T(x_n) \rightharpoonup T(x)$, and due to the weak lower continuity of the norm, one gets $\|T(x)\| \le \liminf_{n\to\infty} \|T(x_n)\|$, which means the $*$-property of T.

ii. Assume that T is weakly closed and maps bounded subsets into bounded subsets. Let $x_n \rightharpoonup x$, than the sequence $\{x_n\}$ is bounded, hence, the set $\{T(x_n)\}$ is also bounded. Let ζ be a weak cluster point of $\{T(x_n)\}$. There exists a weakly convergent subsequence $T(x_{n_k}) \rightharpoonup \zeta$. Since $x_{n_k} \rightharpoonup x$, by the weak closedness of T, one gets $\zeta = T(x)$. Thus, $T(x_n) \rightharpoonup T(x)$.

iii. Let H be a real Hilbert space with an orthonormal basis $\{e_n\}$ and C be a closed ball centered at 0 with radius $r := \sqrt{2}$. Define the operator $T : C \to H$ by $T(x) := \|x\|x$. Obviously, T maps bounded subsets into bounded subsets. Further, T has the $*$-property. Indeed, let $x_n \rightharpoonup x$, then $\|T(x)\| = \|x\|^2 \le \left(\liminf_{n\to\infty} \|x_n\|\right)^2 \le \liminf_{n\to\infty} \|x_n\|^2 = \liminf_{n\to\infty} \|T(x_n)\|$. On the other hand, T is not weakly sequentially continuous. Indeed, let $x_n = e_n + e_1$. Then $x_n \rightharpoonup e_1$, and for $n \ge 2$, $T(x_n) = \sqrt{2}(e_n + e_1) \rightharpoonup \sqrt{2}e_1 \ne T(e_1) = 2e_1$.

Lemma 2.4 (*Saejung and Yotkaew* (2012)) *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in* $(0, 1)$ *with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n b_n \quad \forall n \ge 1.$$

If $\limsup_{k\to\infty} b_{n_k} \le 0$ *for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying* $\liminf_{k\to\infty} (a_{n_k+1} - a_{n_k}) \ge 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Definition 2.2 (Ortega and Rheinboldt (1970)) Let $\{x_n\}$ be a sequence in H.

(i) $\{x_n\}$ is said to converge R-linearly to x^* with rate $\rho \in [0, 1)$ if there is a constant $c > 0$ such that

$$\|x_n - x^*\| \le c\rho^n \quad \forall n \in \mathbb{N}.$$

(ii) $\{x_n\}$ is said to converge Q-linearly to x^* with rate $\rho \in [0, 1)$ if

$$\|x_{n+1} - x^*\| \le \rho\|x_n - x^*\| \quad \forall n \in \mathbb{N}.$$

3 Main results

In this work, we assume the following conditions:

Condition 1 The feasible set C is nonempty, closed, and convex.

Condition 2 The mapping $F : H \to H$ is L-Lipschitz continuous, pseudomonotone on H. However, the information of L is not necessary to be known.

Condition 3 The solution set $Sol(C, F)$ is nonempty.

The proposed algorithm is of the form:

Algorithm 3.1

Initialization: Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying $\sum_{n=1}^{\infty} \alpha_n < +\infty$. Let $\theta > 0$, $\tau_1 > 0$, $\mu \in (0,1)$ and $x_0, x_1 \in H$ be arbitrary. We assume that $\{\theta_n\}$, $\{\epsilon_n\}$ and $\{\gamma_n\}$ are three positive sequences such that $\{\theta_n\} \subset [0, \theta)$ and $\epsilon_n = o(\gamma_n)$, i.e., $\lim_{n\to\infty} \dfrac{\epsilon_n}{\gamma_n} = 0$, where $\{\gamma_n\} \subset (0,1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \gamma_n = 0, \quad \sum_{n=1}^{\infty} \gamma_n = \infty.$$

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{4}$$

Step 2. Set $u_n = (1-\gamma_n)(x_n + \theta_n(x_n - x_{n-1}))$ and compute

$$q_n = P_C(u_n - \tau_n F u_n).$$

Step 3. Compute

$$x_{n+1} = P_{T_n}(u_n - \tau_n F q_n),$$

where $T_n := \{x \in H | \langle u_n - \tau_n F u_n - q_n, x - q_n \rangle \leq 0\}$.

Update

$$\tau_{n+1} := \begin{cases} \min\{\mu \frac{\|u_n - q_n\|^2 + \|x_{n+1} - q_n\|^2}{2\langle Fu_n - Fq_n, x_{n+1} - q_n \rangle}, \tau_n + \alpha_n\} & \text{if } \langle Fu_n - Fq_n, x_{n+1} - q_n \rangle > 0, \\ \tau_n + \alpha_n & \text{otherwise.} \end{cases} \tag{5}$$

Set $n := n + 1$ and go to **Step 1**.

Remark 3.1 As noted in Liu and Yang (2020), the sequence generated by (5) is allowed to increase from iteration to iteration. Hence, our results in this work are different from those in Yang et al. (2018), Yang (2021).

Lemma 3.5 *(Liu and Yang (2020)) Assume that Condition 2 holds. Let $\{\tau_n\}$ be the sequence generated by (5). Then*

$$\lim_{n\to\infty} \tau_n = \tau \text{ with } \tau \in \left[\min\left\{\tau_1, \frac{\mu}{L}\right\}, \tau_1 + \alpha\right],$$

where $\alpha = \sum_{n=1}^{\infty} \alpha_n$. *Moreover*

$$2\langle Fu_n - Fq_n, x_{n+1} - q_n\rangle \le \frac{\mu}{\tau_{n+1}}(\|u_n - q_n\|^2 + \|x_{n+1} - q_n\|^2). \tag{6}$$

Theorem 3.1 *Assume that Conditions 1–3 hold. If the mapping* $F : H \to H$ *satisfies the* $*$*-property then the sequence* $\{x_n\}$, *generated by Algorithm* 3.1, *converges strongly to an element* $z \in Sol(C, F)$, *where* $z = P_{Sol(C,F)}(0)$.

Proof To improve readability, we split the proof of our main theorem into some parts.

Claim 1.

$$\|x_{n+1} - z\|^2 \le \|u_n - z\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1} - q_n\|^2.$$

Since $z \in C \subset T_n$ and P_{T_n} is firmly nonexpansive, see, for example Reich and Shafrir (1987), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 = \|P_{T_n}(u_n - \tau_n Fq_n) - P_{T_n} z\|^2 &\le \langle x_{n+1} - z, u_n - \tau_n Fq_n - z\rangle \\
&= \frac{1}{2}\|x_{n+1} - z\|^2 + \frac{1}{2}\|u_n - \tau_n Fq_n - z\|^2 - \frac{1}{2}\|x_{n+1} - u_n + \tau_n Fq_n\|^2 \\
&= \frac{1}{2}\|x_{n+1} - z\|^2 + \frac{1}{2}\|u_n - z\|^2 + \frac{1}{2}\tau_n^2\|Fq_n\|^2 - \langle u_n - z, \tau_n Fq_n\rangle \\
&\quad - \frac{1}{2}\|x_{n+1} - u_n\|^2 - \frac{1}{2}\tau_n^2\|Fq_n\|^2 - \langle x_{n+1} - u_n, \tau_n Fq_n\rangle \\
&= \frac{1}{2}\|x_{n+1} - z\|^2 + \frac{1}{2}\|u_n - z\|^2 - \frac{1}{2}\|x_{n+1} - u_n\|^2 - \langle x_{n+1} - z, \tau_n Fq_n\rangle.
\end{aligned}$$

This implies that

$$\|x_{n+1} - z\|^2 \le \|u_n - z\|^2 - \|x_{n+1} - u_n\|^2 - 2\langle x_{n+1} - z, \tau_n Fq_n\rangle. \tag{7}$$

Since z is the solution of VI, we have $\langle Fz, x - z\rangle \ge 0$ for all $x \in C$. By the pseudomontonicity of F on C we have $\langle Fx, x - z\rangle \ge 0$ for all $x \in C$. Taking $x := q_n \in C$ we get

$$\langle Fq_n, z - q_n\rangle \le 0.$$

Thus,

$$\langle Fq_n, z - x_{n+1}\rangle = \langle Fq_n, z - q_n\rangle + \langle Fq_n, q_n - x_{n+1}\rangle \le \langle Fq_n, q_n - x_{n+1}\rangle. \tag{8}$$

From (7) and (8) we obtain

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \|u_n-z\|^2-\|x_{n+1}-u_n\|^2+2\tau_n\langle Fq_n,q_n-x_{n+1}\rangle\\ &= \|u_n-z\|^2-\|x_{n+1}-q_n\|^2-\|q_n-u_n\|^2-2\langle x_{n+1}-q_n,q_n-u_n\rangle\\ &\quad +2\tau_n\langle Fq_n,q_n-x_{n+1}\rangle\\ &= \|u_n-z\|^2-\|x_{n+1}-q_n\|^2-\|q_n-u_n\|^2+2\langle u_n-\tau_nFq_n-q_n,x_{n+1}-q_n\rangle.\end{aligned} \tag{9}$$

Since $q_n = P_{T_n}(u_n - \tau_n F u_n)$ and $x_{n+1} \in T_n$ we have

$$\begin{aligned}&2\langle u_n-\tau_nFq_n-q_n,x_{n+1}-q_n\rangle\\ &\qquad = 2\langle u_n-\tau_nFu_n-q_n,x_{n+1}-q_n\rangle+2\tau_n\langle Fu_n-Fq_n,x_{n+1}-q_n\rangle\\ &\qquad \le 2\tau_n\langle Fu_n-Fq_n,x_{n+1}-q_n\rangle.\end{aligned} \tag{10}$$

It follows from (6) that

$$2\tau_n\langle Fu_n-Fq_n,x_{n+1}-q_n\rangle \le \mu\frac{\tau_n}{\tau_{n+1}}\|u_n-q_n\|^2+\mu\frac{\tau_n}{\tau_{n+1}}\|q_n-x_{n+1}\|^2. \tag{11}$$

Combining (10) and (11), we obtain

$$2\langle u_n-\tau_nFq_n-q_n,x_{n+1}-q_n\rangle \le \mu\frac{\tau_n}{\tau_{n+1}}\|u_n-q_n\|^2+\mu\frac{\tau_n}{\tau_{n+1}}\|q_n-x_{n+1}\|^2. \tag{12}$$

Substituting (12) into (9) we obtain

$$\|x_{n+1}-z\|^2 \le \|u_n-z\|^2-(1-\mu\frac{\tau_n}{\tau_{n+1}})\|q_n-u_n\|^2-(1-\mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1}-q_n\|^2.$$

Claim 2. The sequence $\{x_n\}$ is bounded. Indeed, we have

$$\begin{aligned}\|u_n-z\| &= \|(1-\gamma_n)(x_n+\theta_n(x_n-x_{n-1}))-z\|\\ &= \|(1-\gamma_n)(x_n-z)+(1-\gamma_n)\theta_n(x_n-x_{n-1})-\gamma_nz\|\\ &\le (1-\gamma_n)\|x_n-z\|+(1-\gamma_n)\theta_n\|x_n-x_{n-1}\|+\gamma_n\|z\|\\ &= (1-\gamma_n)\|x_n-z\|+\gamma_n[(1-\gamma_n)\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|+\|z\|].\end{aligned} \tag{13}$$

On the other hand, since (4) we have

$$\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\| \le \frac{\epsilon_n}{\gamma_n} \to 0,$$

which implies that $\lim_{n\to\infty}\left[(1-\gamma_n)\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|+\|z\|\right] = \|z\|$, hence there exists $M > 0$ such that

$$(1-\gamma_n)\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|+\|z\| \le M. \tag{14}$$

Combining (13) and (14) we obtain

$$\|u_n - z\| \le (1-\gamma_n)\|x_n - z\| + \gamma_n M.$$

Moreover, we have $\lim_{n\to\infty}(1-\mu\frac{\tau_n}{\tau_{n+1}}) = 1-\mu > \frac{1-\mu}{2}$, hence there exists $n_0 \in \mathbb{N}$ such that $1-\mu\frac{\tau_n}{\tau_{n+1}} > 0 \ \ \forall n \ge n_0$. By Claim 1 we obtain

$$\|x_{n+1} - z\| \le \|u_n - z\| \ \ \forall n \ge n_0. \tag{15}$$

Thus

$$\begin{aligned}\|x_{n+1} - z\| &\le (1-\gamma_n)\|x_n - z\| + \gamma_n M\\ &\le \max\{\|x_n - z\|, M\} \le \ldots \le \max\{\|x_{n_0} - z\|, M\}.\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded.

Claim 3.

$$\begin{aligned}(1-\mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 + (1-\mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1} - q_n\|^2\\ \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \gamma_n M_1.\end{aligned}$$

Indeed, we have $\|u_n - z\| \le (1-\gamma_n)\|x_n - z\| + \gamma_n M$, this implies that

$$\begin{aligned}\|u_n - z\|^2 &\le (1-\gamma_n)^2\|x_n - z\|^2 + 2\gamma_n(1-\gamma_n)M\|x_n - z\| + \gamma_n^2 M^2\\ &\le \|x_n - z\|^2 + \gamma_n[2(1-\gamma_n)M\|x_n - z\| + \gamma_n M^2]\\ &\le \|x_n - z\|^2 + \gamma_n M_1,\end{aligned} \tag{16}$$

where $M_1 := \max\{2(1-\gamma_n)M\|x_n - z\| + \gamma_n M^2 : \ n \in \mathbb{N}\}$. Substituting (16) into Claim 1 we get

$$\|x_{n+1} - z\|^2 \le \|x_n - z\|^2 + \gamma_n M_1 - (1-\mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 - (1-\mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1} - q_n\|^2.$$

Or equivalently

$$\begin{aligned}(1-\mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 + (1-\mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1} - q_n\|^2\\ \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \gamma_n M_1.\end{aligned}$$

Claim 4.

$$\begin{aligned}\|x_{n+1} - z\|^2 \le (1-\gamma_n)\|x_n - z\|^2 + \gamma_n\Big[2(1-\gamma_n)\|x_n - z\|\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|\\ + \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| + 2\|z\|\|u_n - x_{n+1}\| + 2\langle -z, x_{n+1} - z\rangle\Big],\end{aligned}$$

$\forall n \ge n_0$. Indeed, using Lemma 2.2 ii) and (15) we get

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \|u_n-z\|^2 \ \ \forall n\ge n_0 = \|(1-\gamma_n)(x_n-z)\\ &\quad +(1-\gamma_n)\theta_n(x_n-x_{n-1})-\gamma_n z\|^2 \ \ \forall n\ge n_0\\ &\le \|(1-\gamma_n)(x_n-z)+(1-\gamma_n)\theta_n(x_n-x_{n-1})\|^2\\ &\quad +2\gamma_n\langle -z,u_n-z\rangle \ \ \forall n\ge n_0\\ &\le (1-\gamma_n)^2\|x_n-z\|^2+2(1-\gamma_n)\theta_n\|x_n-z\|\|x_n-x_{n-1}\|\\ &\quad +\theta_n^2\|x_n-x_{n-1}\|^2+2\gamma_n\langle -z,u_n-x_{n+1}\rangle\\ &\quad +2\gamma_n\langle -z,x_{n+1}-z\rangle \ \ \forall n\ge n_0\\ &\le (1-\gamma_n)\|x_n-z\|^2+\gamma_n\Bigg[2(1-\gamma_n)\|x_n-z\|\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|\\ &\quad +\theta_n\|x_n-x_{n-1}\|\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|\\ &\quad +2\|z\|\|u_n-x_{n+1}\|+2\langle -z,x_{n+1}-z\rangle\Bigg] \ \ \forall n\ge n_0.\end{aligned}$$

Claim 5. $\{\|x_n-z\|^2\}$ converges to zero.

Indeed, by Lemma 2.4 it suffices to show that $\limsup_{k\to\infty}\langle -z,x_{n_k+1}-z\rangle\le 0$ and $\limsup_{k\to\infty}\|u_{n_k}-x_{n_k+1}\|\le 0$ for every subsequence $\{\|x_{n_k}-z\|\}$ of $\{\|x_n-z\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|)\ge 0.$$

For this purpose, suppose that $\{\|x_{n_k}-z\|\}$ is a subsequence of $\{\|x_n-z\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|)\ge 0$. Then

$$\begin{aligned}\liminf_{k\to\infty}(\|x_{n_k+1}-z\|^2-\|x_{n_k}-z\|^2) &= \liminf_{k\to\infty}[(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|)\\ &\qquad (\|x_{n_k+1}-z\|+\|x_{n_k}-z\|)]\ge 0.\end{aligned}$$

By Claim 3 we obtain

$$\begin{aligned}&\limsup_{k\to\infty}\left[(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}})\|u_{n_k}-q_{n_k}\|^2+(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}})\|x_{n_k+1}-q_{n_k}\|^2\right]\\ &\le \limsup_{k\to\infty}\left[\|x_{n_k}-z\|^2-\|x_{n_k+1}-z\|^2+\gamma_{n_k}M_1\right]\\ &\le \limsup_{k\to\infty}\left[\|x_{n_k}-z\|^2-\|x_{n_k+1}-z\|^2\right]+\limsup_{k\to\infty}\gamma_{n_k}M_1\\ &= -\liminf_{k\to\infty}\left[\|x_{n_k+1}-z\|^2-\|x_{n_k}-z\|^2\right]\\ &\le 0.\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\|q_{n_k}-u_{n_k}\|=0 \text{ and } \lim_{k\to\infty}\|x_{n_k+1}-q_{n_k}\|=0.$$

Thus

$$\lim_{k\to\infty}\|x_{n_k+1}-u_{n_k}\|=0. \tag{17}$$

Now, we show that

$$\|x_{n_k+1}-x_{n_k}\|\to 0 \quad \text{as} \quad k\to\infty. \tag{18}$$

Indeed, using $\lim_{n\to\infty}\gamma_n=0$ we have

$$\begin{aligned}\|x_{n_k}-u_{n_k}\| &= \|(1-\gamma_{n_k})(x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1}))-x_{n_k}\|\\ &= \|\theta_{n_k}(x_{n_k}-x_{n_k-1})-\gamma_{n_k}(x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1}))\|\\ &\le \theta_{n_k}\|x_{n_k}-x_{n_k-1}\|+\gamma_{n_k}\|x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1})\|\\ &= \gamma_{n_k}\frac{\theta_{n_k}}{\gamma_{n_k}}\|x_{n_k}-x_{n_k-1}\|+\gamma_{n_k}\|x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1})\|\to 0.\end{aligned} \tag{19}$$

From (17) and (19), we get

$$\|x_{n_k+1}-x_{n_k}\|\le\|x_{n_k+1}-u_{n_k}\|+\|x_{n_k}-u_{n_k}\|\to 0.$$

Since the sequence $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $z^*\in H$, such that

$$\limsup_{k\to\infty}\langle -z, x_{n_k}-z\rangle=\lim_{j\to\infty}\langle -z, x_{n_{k_j}}-z\rangle=\langle -z, z^*-z\rangle. \tag{20}$$

Using (19), we get

$$u_{n_k}\rightharpoonup z^* \text{ as } k\to\infty,$$

Using (17), we obtain

$$x_{n_k}\rightharpoonup z^* \text{ as } k\to\infty.$$

Now, we show that $z^*\in Sol(C,F)$. Indeed, since $q_{n_k}=P_C(u_{n_k}-\tau_{n_k}Fu_{n_k})$, we have

$$\langle u_{n_k}-\tau_{n_k}Fu_{n_k}-q_{n_k}, x-q_{n_k}\rangle\le 0 \ \ \forall x\in C,$$

or equivalently

$$\frac{1}{\tau_{n_k}}\langle u_{n_k}-q_{n_k}, x-q_{n_k}\rangle\le\langle Fu_{n_k}, x-q_{n_k}\rangle \ \ \forall x\in C.$$

Consequently

$$\frac{1}{\tau_{n_k}}\langle u_{n_k} - q_{n_k}, x - q_{n_k}\rangle + \langle Fu_{n_k}, q_{n_k} - u_{n_k}\rangle \leq \langle Fu_{n_k}, x - u_{n_k}\rangle \ \ \forall x \in C. \tag{21}$$

Being weakly convergent, $\{u_{n_k}\}$ is bounded. Then, by the Lipschitz continuity of F, $\{Fu_{n_k}\}$ is bounded. As $\|u_{n_k} - q_{n_k}\| \to 0$, $\{q_{n_k}\}$ is also bounded and $\tau_{n_k} \geq \min\{\tau_1, \frac{\mu}{L}\}$. Passing (21) to limit as $k \to \infty$, we get

$$\liminf_{k\to\infty}\langle Fu_{n_k}, x - u_{n_k}\rangle \geq 0 \ \ \forall x \in C. \tag{22}$$

Moreover, we have

$$\langle Fq_{n_k}, x - q_{n_k}\rangle = \langle Fq_{n_k} - Fu_{n_k}, x - u_{n_k}\rangle + \langle Fu_{n_k}, x - u_{n_k}\rangle + \langle Fq_{n_k}, u_{n_k} - q_{n_k}\rangle. \tag{23}$$

Since $\lim_{k\to\infty} \|u_{n_k} - q_{n_k}\| = 0$ and F is L-Lipschitz continuous on H, we get

$$\lim_{k\to\infty} \|Fu_{n_k} - Fq_{n_k}\| = 0$$

which, together with (22) and (23) implies that

$$\liminf_{k\to\infty}\langle Fq_{n_k}, x - q_{n_k}\rangle \geq 0.$$

Next, we choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by N_k the smallest positive integer such that

$$\langle Fq_{n_j}, x - q_{n_j}\rangle + \epsilon_k \geq 0 \ \ \forall j \geq N_k. \tag{24}$$

Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, since $\{q_{N_k}\} \subset C$ we can suppose $Fq_{N_k} \neq 0$ (otherwise, q_{N_k} is a solution) and, setting

$$v_{N_k} = \frac{Fq_{N_k}}{\|Fq_{N_k}\|^2},$$

we have $\langle Fq_{N_k}, v_{N_k}\rangle = 1$ for each k. Now, we can deduce from (24) that for each k

$$\langle Fq_{N_k}, x + \epsilon_k v_{N_k} - q_{N_k}\rangle \geq 0.$$

From F is pseudomonotone on H, we get

$$\langle F(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - q_{N_k}\rangle \geq 0.$$

This implies that

$$\langle Fx, x - q_{N_k}\rangle \geq \langle Fx - F(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - q_{N_k}\rangle - \epsilon_k\langle Fx, v_{N_k}\rangle. \tag{25}$$

Now, we show that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Indeed, since $u_{n_k} \rightharpoonup z^*$ and $\lim_{k\to\infty} \|u_{n_k} - q_{n_k}\| = 0$, we obtain $q_{N_k} \rightharpoonup z^*$ as $k \to \infty$. By $\{q_n\} \subset C$, we obtain $z^* \in C$. Since F has $*$-property, we have

$$0 < \|Fz^*\| \le \liminf_{k\to\infty} \|Fq_{n_k}\|.$$

Since $\{q_{N_k}\} \subset \{q_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \le \limsup_{k\to\infty} \|\epsilon_k v_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Fq_{n_k}\|} \right) \le \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Fq_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$.

Now, letting $k \to \infty$, then the right-hand side of (25) tends to zero by F is uniformly continuous, $\{u_{N_k}\}, \{v_{N_k}\}$ are bounded and $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Thus, we get

$$\liminf_{k\to\infty} \langle Fx, x - q_{N_k} \rangle \ge 0.$$

Hence, for all $x \in C$ we have

$$\langle Fx, x - z^* \rangle = \lim_{k\to\infty} \langle Fx, x - q_{N_k} \rangle = \liminf_{k\to\infty} \langle Fx, x - q_{N_k} \rangle \ge 0.$$

By Lemma 2.1, we get

$$z^* \in \mathrm{Sol(C, F)}.$$

Since (20) and the definition of $z = P_{Sol(C,F)}(0)$, we have

$$\limsup_{k\to\infty} \langle -z, x_{n_k} - z \rangle = \langle -z, z^* - z \rangle \le 0. \tag{26}$$

Combining (18) and (26), we have

$$\begin{aligned} \limsup_{k\to\infty} \langle -z, x_{n_k+1} - z \rangle &\le \limsup_{k\to\infty} \langle -z, x_{n_k} - z \rangle \\ &= \langle -z, z^* - z \rangle \\ &\le 0. \end{aligned} \tag{27}$$

Hence, by (27), $\lim_{n\to\infty} \frac{\theta_n}{\gamma_n} \|x_n - x_{n-1}\| = 0$, $\lim_{k\to\infty} \|x_{n_k+1} - u_{n_k}\| = 0$, Claim 5 and Lemma 2.4, we have $\lim_{n\to\infty} \|x_n - z\| = 0$, which was to be proved.

Remark 3.2 It should be noted that if the operator F is monotone, the $*$ property is redundant, see Denisov et al. (2015), Vuong (2018).

4 Convergence Rate

In this section we establish a convergence rate for the so-called relaxed inertial subgradient extragradient method. Actually, we consider the following modification of Algorithm 3.1:

Algorithm 4.2

Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers which satisfies $\sum_{n=1}^{\infty}\alpha_n < +\infty$. Given $\theta \in [0,1), \gamma \in (0,\frac{1}{2}),\ \mu \in (0,1)$ and $\tau_1 > 0$, Let $x_0, x_1 \in H$ be arbitrary. Let

$$\begin{aligned} u_n &= x_n + \theta(x_n - x_{n-1}),\\ q_n &= P_C(u_n - \tau_n F u_n),\\ z_n &= P_{T_n}(u_n - \tau_n F q_n), \end{aligned}$$

where $T_n := \{x \in H | \langle u_n - \tau_n F u_n - q_n, x - q_n\rangle \le 0\}$,

$$x_{n+1} = (1-\gamma)x_n + \gamma z_n.$$

Update

$$\tau_{n+1} := \begin{cases} \min\{\mu \dfrac{\|u_n - q_n\|^2 + \|z_n - q_n\|^2}{2\langle Fu_n - Fq_n, z_n - q_n\rangle}, \tau_n + \alpha_n\} \text{ if } \langle Fu_n - Fq_n, z_n - q_n\rangle > 0,\\ \tau_n + \alpha_n \text{otherwise.} \end{cases}$$

Throughout this section, the operator F is assumed to be L-Lipschitz continuous on H and δ-strongly pseudo-monotone on C. We now prove that the iterative sequence generated by Algorithm 4.2 converges strongly to the unique solution of problem (VI) with an R-linear rate.

Theorem 4.2 *Assume that $F : H \to H$ is L-Lipschitz continuous on H and δ-strongly pseudo-monotone on C. Let $\theta \in \left[0, \frac{\delta}{L+\delta}\right)$, $\mu \in \left(\frac{\theta}{1+\theta}\frac{L}{\delta}, \frac{1-\theta}{1+\theta}\right)$ and $\tau_1 > \frac{\mu}{L}$. Then the sequence $\{x_n\}$ generated by Algorithm* 4.2 *converges in norm with an R-linear convergence rate to the unique element z in Sol(C,F).*

Proof Since $\langle Fz, q_n - z\rangle \ge 0$, the δ-strong pseudo-monotonicity of F on C yields the inequality

$$\langle Fq_n, q_n - z\rangle \ge \delta\|q_n - z\|^2.$$

This implies that

$$\langle Fq_n, z - z_n\rangle = \langle Fq_n, z - q_n\rangle + \langle Fq_n, q_n - z_n\rangle \le -\delta\|q_n - z\|^2 + \langle Fq_n, q_n - z_n\rangle. \tag{28}$$

Now, using (28) and a similar argument as in Claim 1 of Theorem 3.1, we get

$$\begin{aligned} \|z_n - z\|^2 &\le \|u_n - z\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|z_n - q_n\|^2 - 2\delta\tau_n\|q_n - z\|^2\\ &\le \|u_n - z\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 - 2\delta\tau_n\|q_n - z\|^2. \end{aligned}$$

Since $\theta < \dfrac{\delta}{L+\delta}$, it follows that

$$\frac{\theta}{1+\theta}\frac{L}{\delta} < \frac{1-\theta}{1+\theta}$$

therefore there always exists

$$\mu \in \left(\frac{\theta}{1+\theta}\frac{L}{\delta}, \frac{1-\theta}{1+\theta}\right).$$

From $\mu < \dfrac{1-\theta}{1+\theta}$, one finds $\dfrac{1-\mu}{2} > \dfrac{\theta}{1+\theta}$ and $\mu > \dfrac{\theta}{1+\theta}\dfrac{L}{\delta}$ implies that $\delta\dfrac{\mu}{L} > \dfrac{\theta}{1+\theta}$. Fix $\epsilon \in \left(\dfrac{\theta}{1+\theta}, \min\left\{\dfrac{1-\mu}{2}, \delta\dfrac{\mu}{L}\right\}\right)$. We have

$$\lim_{n\to\infty}(1-\mu\frac{\tau_n}{\tau_{n+1}}) = 1-\mu > 2\epsilon$$

and

$$\lim_{n\to\infty}\delta\tau_n = \delta\tau \geq \delta\min\left\{\tau_1, \frac{\mu}{L}\right\} = \delta\frac{\mu}{L} > \epsilon.$$

Therefore, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we get

$$\begin{aligned}\|z_n - z\|^2 &\leq \|u_n - z\|^2 - 2\epsilon\|q_n - u_n\|^2 - 2\epsilon\|q_n - z\|^2\\ &\leq \|u_n - z\|^2 - \epsilon\|u_n - z\|^2\\ &= (1-\epsilon)\|u_n - z\|^2.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|(1-\gamma)x_n + \gamma z_n - z\|^2\\ &= \|(1-\gamma)(x_n - z) + \gamma(z_n - z)\|^2\\ &= (1-\gamma)\|x_n - z\|^2 + \gamma\|z_n - z\|^2 - (1-\gamma)\gamma\|x_n - z_n\|^2\\ &= (1-\gamma)\|x_n - z\|^2 + \gamma\|z_n - z\|^2 - \frac{1-\gamma}{\gamma}\|x_{n+1} - x_n\|^2\\ &\leq (1-\gamma)\|x_n - z\|^2 + \gamma(1-\epsilon)\|u_n - z\|^2 - \frac{1-\gamma}{\gamma}\|x_{n+1} - x_n\|^2 \ \forall n \geq N.\end{aligned}$$

We also have

$$\begin{aligned}\|u_n - z\|^2 &= \|(1+\theta)(x_n - z) - \theta(x_{n-1} - z)\|^2\\ &= (1+\theta)\|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 + \theta(1+\theta)\|x_n - x_{n-1}\|^2.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\gamma)\|x_n-z\|^2+\gamma(1-\epsilon)[(1+\theta)\|x_n-z\|^2-\theta\|x_{n-1}-z\|^2\\ &\quad+\theta(1+\theta)\|x_n-x_{n-1}\|^2]-\frac{1-\gamma}{\gamma}\|x_{n+1}-x_n\|^2\ \forall n\ge N\\ &\le (1-\gamma(1-(1-\epsilon)(1+\theta)))\|x_n-z\|^2-\gamma(1-\epsilon)\theta\|x_{n-1}-z\|^2\\ &\quad+\gamma(1-\epsilon)\theta(1+\theta)\|x_n-x_{n-1}\|^2-\frac{1-\gamma}{\gamma}\|x_{n+1}-x_n\|^2\ \forall n\ge N\\ &\le (1-\gamma(1-(1-\epsilon)(1+\theta)))\|x_n-z\|^2+\gamma(1-\epsilon)\theta(1+\theta)\|x_n-x_{n-1}\|^2\\ &\quad-\frac{1-\gamma}{\gamma}\|x_{n+1}-x_n\|^2\ \forall n\ge N.\end{aligned}$$

Since $\gamma\in(0,\frac{1}{2})$, it implies $\frac{1-\gamma}{\gamma}>1$. Hence, we obtain

$$\begin{aligned}\|x_{n+1}-z\|^2+\|x_{n+1}-x_n\|^2 &\le \|x_{n+1}-z\|^2+\frac{1-\gamma}{\gamma}\|x_{n+1}-x_n\|^2\\ &\le (1-\gamma(1-(1-\epsilon)(1+\theta)))\|x_n-z\|^2\\ &\quad+\gamma(1-\epsilon)\theta(1+\theta)\|x_n-x_{n-1}\|^2\ \forall n\ge N.\end{aligned}$$

This follows that

$$\begin{aligned}\|x_{n+1}-z\|^2+\|x_{n+1}-x_n\|^2 &\le (1-\gamma(1-(1-\epsilon)(1+\theta)))\Big[\|x_n-z\|^2\\ &\quad+\frac{1}{(1-\gamma(1-(1-\epsilon)(1+\theta)))}\gamma(1-\epsilon)\theta(1+\theta)\|x_n-x_{n-1}\|^2\Big]\ \forall n\ge N.\end{aligned}$$

We now show that

$$1-\gamma(1-(1-\epsilon)(1+\theta))\in(0,1)$$

and

$$\frac{1}{1-\gamma(1-(1-\epsilon)(1+\theta))}\gamma(1-\epsilon)\theta(1+\theta)\in(0,1).$$

Indeed, since $\epsilon\in(\frac{\theta}{1+\theta},\min\left\{\frac{1-\mu}{2},\delta\frac{\mu}{L}\right\})$, this implies that $\epsilon>\frac{\theta}{1+\theta}$, or, $1-\epsilon<\frac{1}{1+\theta}$ that is $(1-\epsilon)(1+\theta)<1$, hence $1-\gamma(1-(1-\epsilon)(1+\theta))\in(0,1)$. It is easy to see that

$$\frac{1}{1-\gamma(1-(1-\epsilon)(1+\theta))}\gamma(1-\epsilon)\theta(1+\theta)\in(0,1).$$

Therefore, we deduce

$$\|x_{n+1}-z\|^2+\|x_{n+1}-x_n\|^2\le(1-\gamma(1-(1-\epsilon)(1+\theta)))[\|x_n-z\|^2+\|x_n-x_{n-1}\|^2]\ \forall n\ge N.$$

Letting $a_n:=\|x_n-z\|^2+\|x_n-x_{n-1}\|^2$ and $\xi:=(1-\gamma(1-(1-\epsilon)(1+\theta)))$, we get

$$\|x_{n+1} - z\|^2 \leq a_{n+1} \leq \xi a_n \leq \xi^{n-N+1} a_N = \frac{\xi}{\xi^N} a^N \xi^n.$$

Thus, the sequence $\{x_n\}$ converges R-linearly to z, as desired.

Remark 4.3 It should be emphasized that we obtain the linear convergence rate of Algorithm 4.2 instead of the strong convergence as in Shehu et al. (2021).

Remark 4.4 In Theorem 4.2, the $*$- property of F is not assumed.

5 Numerical Illustrations

In this section, we present some numerical experiments in solving variational inequality problems. In the first example, we compare the proposed algorithm with three well-known algorithms including Algorithm 1 of Yang et al. (2018), Algorithm 3.1 of Yang (2021), and the subgradient extragradient algorithm (SEGM) of Censor et al. (2012a) and illustrate the convergence of the proposed algorithm. In the second example, we compare the proposed algorithm with Algorithm 1 of Yang et al. (2018), Algorithm 3.1 of Yang (2021) and [Shehu et al. (2021), Algorithm 1]. In the last example, we compare the proposed algorithm with Algorithm 3.1 of Yang (2021) and [Shehu et al. (2021), Algorithm 1]. All the numerical experiments are performed on an HP laptop with Intel(R) Core(TM)i5-6200U CPU 2.3GHz with 4 GB RAM. The programs are written in Matlab 2015a.

Remark 5.5 We usually choose $\alpha_n = \theta_0$ for Algorithm 3.1 of Yang (2021) because α_n and θ_n have similar roles in their algorithm as well as in our proposed algorithm. Similarly, we take $\lambda = \tau_0$ for the subgradient extragradient algorithm of Censor et al. (2012a).

In the numerical experiments, we choose parameters as follows:

$$\text{Proposed algorithm: } \theta_0 = 0.5, \theta_n = \begin{cases} \min\{\theta_0, \frac{\gamma_n^2}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \theta_0 & \text{otherwise.} \end{cases}$$

Algorithm 3.1: $\alpha_n = \theta_0 = 0.5$.

Example 1 Assume that $F : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $F(x) = Mx + q$ with $M = NN^T + S + D$, N is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix, whose diagonal entries are positive (so M is positive definite), q is a vector in $\mathbb{R}^m$, and

$$C := \{x \in \mathbb{R}^m : x_i \geq -1, i = 1, \cdots, m\}.$$

Table 1 Numerical results obtained by other algorithms

Methods	$m = 50$			$m = 100$		
	Sec.	Iter.	Error.	Sec.	Iter.	Error.
Proposed Alg	0.11	31	9.9773e-07	0.75	43	9.4795e-07
Algorithm 1	5.8	2000	0.0029	34	2000	0.0384
Algorithm 3.1	5.9	2000	4.9922e-05	35	2000	0.0043
SEGM	5.6	2000	0.0012	30	2000	0.0272

It is clear that F is monotone and Lipschitz continuous with the Lipschitz constant $L = \|M\|$. For $q = 0$, the unique solution of the corresponding variational inequality is $x^* = 0$.

For the experiment, all entries of B, S and D are generated randomly from a normal distribution with mean zero and unit variance. The process is started with the initial $x_0 = (1, ..., 1)^T \in \mathbb{R}^m$ and $x_1 = 0.9x_0$. To terminate algorithms, we use the condition $D_n = \|x_n - x^*\|^2 \leq \epsilon$ with $\epsilon = 10^{-6}$ or the number of iterations ≥ 2000 for all algorithms. We choose $\mu = 0.5$ for the proposed algorithm, Algorithms 1 of Yang et al. (2018), Algorithm 3.1 of Yang (2021) and $\gamma_n = \frac{1}{n+1}$ for the proposed algorithm.
Case 1: We take $\lambda = \frac{0.7}{\|M\|}$ for the subgradient extragradient algorithm of Censor et al. (2012a) and $\tau_0 = \frac{0.7}{\|M\|}$ for Algorithm 1 of Yang et al. (2018), Algorithm 3.1 of Yang (2021) and the proposed algorithm. The numerical results are described in Table 1 and Figs. 1 and 2.

Case 2: We take $\lambda = \frac{0.9}{\|M\|}$ for the subgradient extragradient algorithm of Censor et al. (2012a) and $\tau_0 = \frac{0.9}{\|M\|}$ for Algorithm 1 of Yang et al. (2018), Algorithm 3.1 of Yang (2021) and the proposed algorithm. The numerical results are described in Table 2 and Figs. 3 and 4.

Tables 1 and 2 and Figs. 1–4 give the errors of the proposed algorithm, algorithm of Censor et al. (2012a), Algorithm 1 of Yang et al. (2018), Algorithm 3.1 of Yang

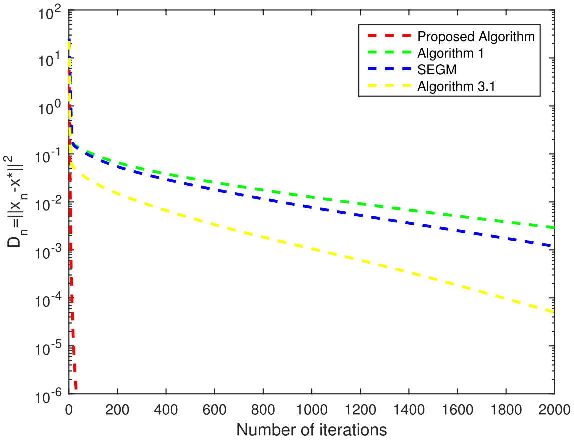

Fig. 1 Comparison of all algorithms with $m = 50$

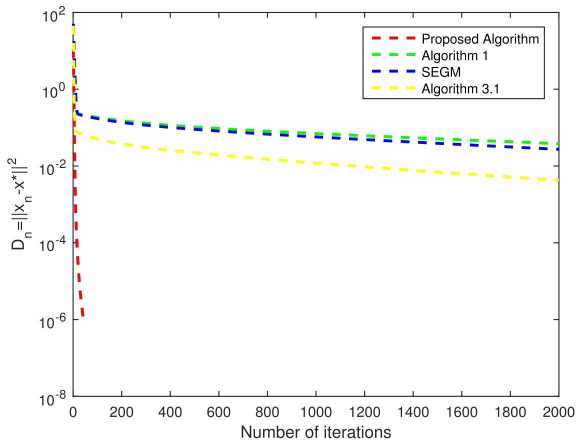

Fig. 2 Comparison of all algorithms with $m = 100$

Table 2 Numerical results obtained by other algorithms

Methods	$m = 50$			$m = 100$		
	Sec.	Iter.	Error.	Sec.	Iter.	Error.
Proposed Alg	0.12	34	9.3168e-07	0.76	45	9.8653e-07
Algorithm 1	5.62	2000	0.0034	34	2000	0.0409
Algorithm 3.1	5.7	2000	8.2121e-05	35	2000	0.0054
SEGM	5.59	2000	3.6547e-04	30	2000	0.0184

(2021) as well as their execution times. They show that the proposed algorithm is less time consuming and more accurate than those of Yang et al. (2018), Yang (2021), Censor et al. (2012a).

In Fig. 5 we illustrate the convergence rate of the proposed algorithm for different choices of the θ with $\lambda = \frac{0.7}{\|M\|}$, $\mu = 0.5, m = 50$ and $\gamma_n = \frac{1}{n+1}$.

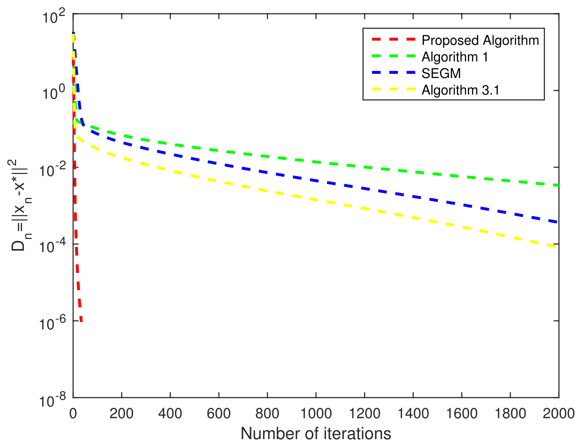

Fig. 3 Comparison of all algorithms with $m = 50$

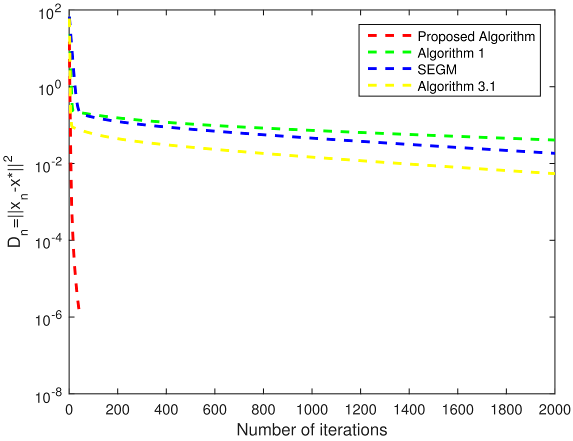

Fig. 4 Comparison of all algorithms with $m = 100$

In Fig. 6 we illustrate the convergence rate of the proposed algorithm for different choices of the γ_n with $\lambda = \frac{0.7}{\|M\|}$, $\mu = 0.5, m = 50$ and $\theta = 0.5$.

Figures 5 and 6 show that the rate of convergence of the proposed algorithm in general depends strictly on the convergent rate of sequence of γ_n and θ.

Example 2 In the second example, we study an important Nash–Cournot oligopolistic market equilibrium model, which was proposed originally by Murphy et al. (1982) as a convex optimization problem. Later, Harker reformulated it as a monotone variational inequality in Harker (1984). We provide only a short description of the problem, for more details we refer to Facchinei and Pang (2003), Harker (1984), Murphy et al. (1982). There are N firms, each of them supplies a homogeneous product in a non-cooperative fashion. Let $q_i \geq 0$ be the ith firm's supply at cost $f_i(q_i)$ and $p(Q)$ be the inverse demand curve, where $Q \geq 0$ is the total supply in the market, i.e., $Q = \sum_{i=1}^{N} q_i$. A Nash equilibrium solution for the market defined above is a set of nonnegative output levels $(q_1^*, q_2^*, ...q_N^*)$ such that q_i^* is an optimal solution to the following problem for all $i = 1, 2.....N$:

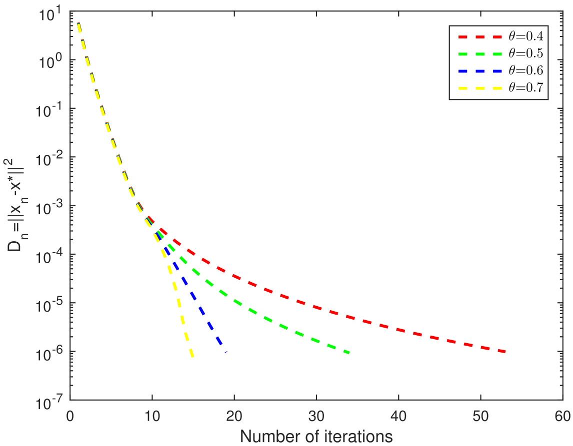

Fig. 5 Convergence rate of the proposed algorithms for different choice of the θ

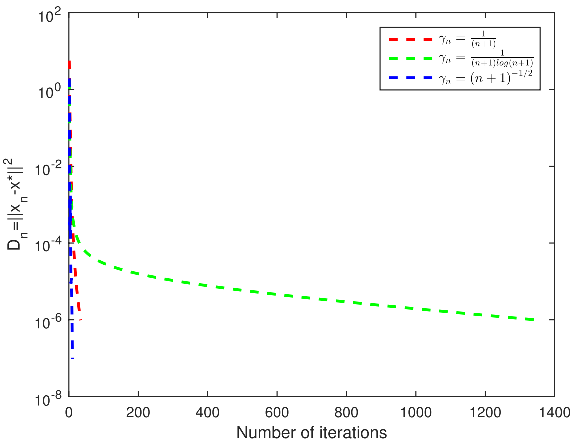

Fig. 6 Convergence rate of the proposed algorithm for different choice of the γ_n

$$\max_{q_i \geq 0} q_i p(q_i + Q_i^*) - f_i(q_i) \tag{29}$$

where

$$Q_i^* = \sum_{j=1, j\neq i}^{N} q_j^*.$$

A variational inequality equivalent to (29) is (see Harker (1984))

$$\text{find} \quad (q_1^*, q_2^*, ...q_N^*) \in \mathbb{R}_+^N \quad \text{such that} \quad \langle F(q^*), q - q^* \rangle \geq 0 \quad \forall q \in \mathbb{R}_+^N,$$

where $F(q^*) = (F_1(q^*), F_2(q^*), ...F_N(q^*))$ and

$$F_i(q^*) = f_i'(q_i^*) - p\left(\sum_{j=1}^{N} q_j^*\right) - q_i^* p'\left(\sum_{j=1}^{N} q_j^*\right).$$

As in the classical example of the Nash-Cournot equilibrium Harker (1984), Murphy et al. (1982), we consider an oligopoly with N firms, each with the inverse demand function p and the cost function f_i take the form

$$p(Q) = 5000^{1/1.1} Q^{-1/1.1} \quad \text{and} \quad f_i(q_i) = c_i q_i + \frac{\beta_i}{\beta_i + 1} L_i^{\frac{-1}{\beta_i}} q_i^{\frac{\beta_i+1}{\beta_i}}.$$

Each entry of c_i, L_i and β_i are drawn independently from the uniform distributions with the following parameters

$$c_i \sim \mathscr{U}(1, 100), \quad L_i \sim \mathscr{U}(0.5, 5), \quad \beta_i \sim \mathcal{U}(0.5, 2).$$

We choose $\mu = 0.5, \tau = \tau_0 = 0.01$ for the proposed algorithm, Algorithm 1 of Yang et al. (2018), Algorithm 3.1 of Yang (2021), $\gamma_n = \frac{1}{n+1}$ for the proposed algorithm and

Table 3 Numerical results are obtained by other algorithms

Methods	$N = 500$			$N = 1000$		
	Sec.	Iter.	Error.	Sec.	Iter.	Error.
Proposed Alg	0.038	35	5.8688e-11	0.042	38	5.6226e-11
Algorithm 1	0.054	98	9.9443e-11	0.089	103	9.9463e-11
Algorithm 3.1	0.1278	200	7.4879e-08	0.15	200	3.6203e-07
Algorithm 1 of Shehu et al.	0.045	79	9.2591e-11	0.060	81	8.7509e-11

$\theta = 0.9$, $\rho_n = \rho = 0.4$ for Shehu et al. [Shehu et al. (2021), Algorithm 1]. The process is started with the initial $x_0 = 10 * (1, ..., 1)^T \in \mathbb{R}^N$ and $x_1 = 0.9 * x_0$ and stopping conditions is Residual $:= \|u_n - q_n\| \leq 10^{-10}$ or the number of iterations ≥ 200 for all algorithms. The numerical results are described in Table 3 and Figs. 7 and 8.

Example 3 Consider the following fractional programming problem:

$$\min f(x) = \frac{x^T Qx + a^T x + a_0}{b^T x + b_0}$$

subject to $x \in X := \{x \in \mathbb{R}^m : b^T x + b_0 > 0\}$, where Q is an $m \times m$ symmetric matrix, $a, b \in \mathbb{R}^m$, and $a_0, b_0 \in \mathbb{R}$. It is well known that f is pseudoconvex on X when Q is positive-semidefinite. We consider the following cases:

Case 1:

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, a_0 = -2, b_0 = 4.$$

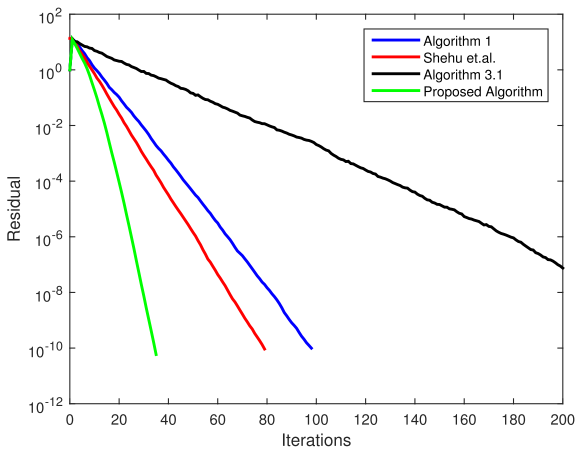

Fig. 7 Comparison of all algorithms with $N = 500$

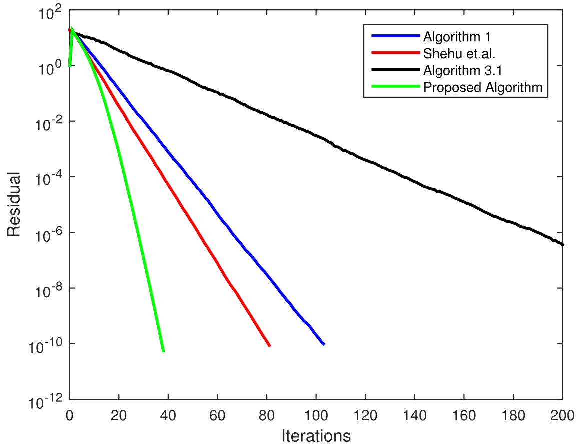

Fig. 8 Comparison of all algorithms with $N = 1000$

We minimize f over $C := \{x \in \mathbb{R}^4 : 1 \le x_i \le 10, i = 1, \cdots, 4\} \subset X$. It is easy to verify that Q is symmetric and positive definite in $\mathbb{R}^4$ and consequently, f is pseudo-convex on X.

We choose $\mu = 0.5, \tau = \tau_0 = 0.5$ for the proposed algorithm and Algorithm 3.1 of Yang (2021), $\theta = 0.9, \rho_n = \rho = 0.4, \alpha_n = \frac{1}{n\sqrt{n}}$ for [Shehu et al. (2021), Algorithm 1] and $\alpha_n = \frac{1}{n\sqrt{n}}, \gamma_n = \frac{1}{\gamma'(n+\gamma'')}$, where $\gamma' = 10^6$, $\gamma'' = 10^5$ for the proposed algorithm.

The process is started with the initial $x_0 = (20, -20, 20, -20)^T$ and $x_1 = 0.9 * x_0$ and stopping conditions is Residual $:= \|u_n - q_n\| \le 10^{-10}$ or the number of iterations ≥ 1000 for all algorithms. The numerical results are described in Fig. 9 and Table 4.

Case 2: In the second experiment, we make the problem even more challenging. Let matrix $A : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$, vectors $c, d, y_0 \in \mathbb{R}^m$ and c_0, d_0 be generated from a normal distribution with mean zero and unit variance. We put $e = (1, 1, \cdots, 1)^T$

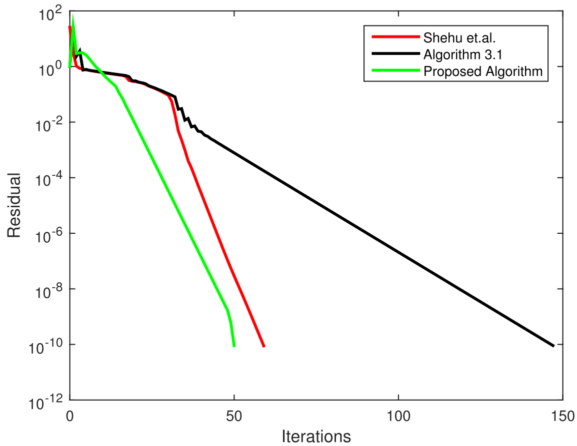

Fig. 9 Comparison of all algorithms

Table 4 Numerical results obtained by other algorithms

Method	Sec.	Iter.	Residual
Proposed Alg	0.16846	50	8.9822e-11
Algorithm 3.1	0.4882	147	9.1113e-11
Algorithm 1 of Shehu et.al.	0.19165	59	8.8626e-11

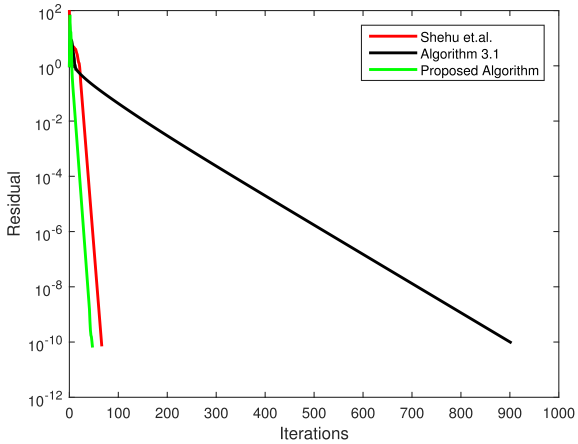

Fig. 10 Comparison of all algorithms with $m = 30$

$\in \mathbb{R}^m, Q = A^T A + I, a := e + c, b := e + d, a_0 = 1 + c_0, b_0 = 1 + d_0$. We minimize f over $C := \{x \in \mathbb{R}^m : 1 \le x_i \le 10, i = 1, \cdots, m\} \subset X$. Because Matrix Q is symmetric and positive definite in $\mathbb{R}^m$, f is pseudo-convex on X.

The process is started with the initial $x_0 := m * y_0$ and $x_1 = 0.9 * x_0$, stopping conditions and parameters as in Case 1. The numerical results are described in Figs. 10 and 11 and Table 5.

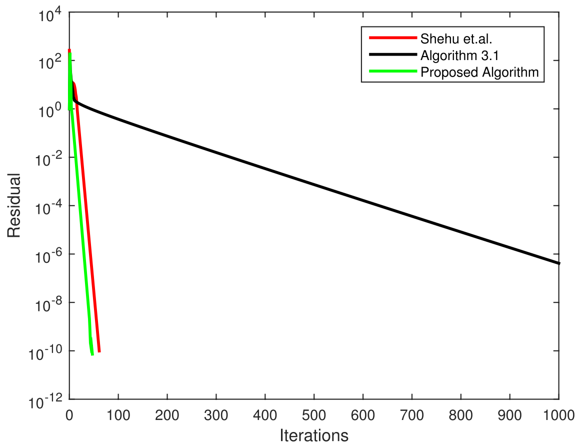

Fig. 11 Comparison of all algorithms with $m = 50$

Table 5 Numerical results obtained by other algorithms

Methods	$m = 30$			$m = 50$		
	Sec.	Iter.	Error.	Sec.	Iter.	Error.
Proposed Alg	0.174	47	7.129e-11	0.20071	47	7.301e-11
Algorithm 3.1	3.4851	901	9.8617e-11	4.2414	1000	4.1395e-07
Algorithm 1 of Shehu et al.	0.24926	66	7.874e-11	0.248	61	9.8613e-11

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Data Availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

References

Anh PK, Thong DV, Vinh NT (2020) Improved inertial extragradient methods for solving pseudo-monotone variational inequalities. Optimization. https://doi.org/10.1080/02331934.2020.1808644
Censor Y, Gibali A, Reich S (2011a) The subgradient extragradient method for solving variational inequalities in Hilbert space. J Optim Theory Appl 148:318–335
Censor Y, Gibali A, Reich S (2011b) Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim Meth Softw 26:827–845
Censor Y, Gibali A, Reich S (2012a) Algorithms for the split variational inequality problem. Numer Algor 59:301–323
Censor Y, Gibali A, Reich S (2012b) Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. Optimization 61:1119–1132
Cottle RW, Yao JC (1992) Pseudo-monotone complementarity problems in Hilbert space. J Optim Theory Appl 75:281–295
Denisov SV, Semenov VV, Chabak LM (2015) Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. Cybern Syst Anal 51:757–765
Facchinei F, Pang JS (2003) Finite-dimensional variational inequalities and complementarity problems. Springer Series in Operations Research, vol I. Springer, New York
Gibali A, Reich S, Zalas R (2017) Outer approximation methods for solving variational inequalities in Hilbert space. Optimization 66:417–437
Harker PT (1984) A variational inequality approach for the determination of oligopolistic market equilibrium. Math Program 30:105–111
Kassay G, Reich S, Sabach S (2011) Iterative methods for solving systems of variational inequalities in reflexive Banach spaces. SIAM J Optim 21:1319–1344
Kinderlehrer D, Stampacchia G (1980) An introduction to variational inequalities and their applications. Academic Press, New York
Konnov IV (2001) Combined relaxation methods for variational inequalities. Springer-Verlag, Berlin

Korpelevich GM (1976) The extragradient method for finding saddle points and other problems. Ekonomikai Matematicheskie Metody 12:747–756
Kraikaew R, Saejung S (2014) Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J Optim Theory Appl 163:399–412
Liu H, Yang J (2020) Weak convergence of iterative methods for solving quasimonotone variational inequalities. Comput Optim Appl 77:491–508
Murphy FH, Sherali HD, Soyster AL (1982) A mathematical programming approach for determining oligopolistic market equilibrium. Math Program 24:92–106
Ortega JM, Rheinboldt WC (1970) Iterative solution of nonlinear equations in several variables. Academic Press, New York
Reich S, Thong DV, Dong QL et al (2021) New algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mappings. Numer Algor 87:527–549
Reich S, Thong DV, Cholamjiak P et al (2021) Inertial projection-type methods for solving pseudomonotone variational inequality problems in Hilbert space. Numer Algor 88:813–835
Reich S, Shafrir I (1987) The asymptotic behavior of firmly nonexpansive mappings. Proc Amer Math Soc 101:246–250
Saejung S, Yotkaew P (2012) Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal 75:742–750
Shehu S, Iyiola OS, Yao JC (2021) New projection methods with inertial steps for variational inequalities. Optimization. https://doi.org/10.1080/02331934.2021.1964079
Shehu Y, Iyiola OS, Reich S (2021) A modified inertial subgradient extragradient method for solving variational inequalities. Optim Eng. https://doi.org/10.1007/s11081-020-09593-w
Thong DV, Hieu DV (2018) Weak and strong convergence theorems for variational inequality problems. Numer Algor 78:1045–1060
Thong DV, Vuong PT (2021) Improved subgradient extragradient methods for solving pseudomonotone variational inequalities in Hilbert spaces. Appl Numer Math 163:221–238
Vuong PT (2018) On the weak convergence of the extragradient method for solving pseudomonotone variational inequalities. J Optim Theory Appl 176:399–409
Yang J, Liu H, Liu Z (2018) Modified subgradient extragradient algorithms for solving monotone variational inequalities. Optimization 67:2247–2258
Yang J (2021) Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities. Appl Anal 100:1067–1078