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A Closed-Loop Supply Chain Equilibrium Model with Random and Price-Sensitive Demand and Return

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Abstract This paper proposes a decentralized closed-loop supply chain network model consisting of raw material suppliers, manufacturers, retailers, and recovery centers. We assume that the demands for the product and the corresponding returns are random and price-sensitive. Retailers and recovery centers face penalties associated with shortage demand and supply, respectively. We derive the optimality conditions of the various decision-makers, and establish that the governing equilibrium conditions can be formulated as a finite-dimensional variational inequality problem. The qualitative properties of the solution to the variational inequality are discussed. Numerical examples are provided to illustrate the effects of demand and return uncertainties on quantity shipments and prices.

Keywords Closed-loop supply chain · Network equilibrium · Variational inequality · Random demand · Random return

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1 Introduction

The topic of closed-loop supply chain (CLSC) modeling and analysis has been of great interest, both from practical and research perspectives due to the importance of managing reverse logistics flows (Fleischmann et al. 2000) and to the increasing consumer awareness of environmental issues (Bloomberg et al. 2002). Many researchers have addressed the management of reverse flows and CLSCs (see Souza (2013), for a recent critical review on CLSC models). Sheu et al. (2005) presented a mathematical approach to model logistics operational problems of green-supply chain management. Nagurney and Toyasaki (2005) proposed a variational inequality formulation to model the management of reverse supply chain flows including electronic waste and recycling. Hammond and Beullens (2007) expanded the model of Nagurney and Toyasaki (2005) to consider a CLSC network consisting of manufacturers and consumers at various demand markets. Yang et al. (2009) used the theory of variational inequality to develop a CLSC network consisting of raw material suppliers, manufacturers, retailers, consumers and recovery centers. The authors discussed several examples to show the effects of CLSC parameters on equilibrium flows and net revenues. Feng et al. (2014) proposed a CLSC network model with demand for the underlying product being both price and time dependent. From the design point of view, Ramezani et al. (2014) studied CLSC by incorporating a set of fuzzy constraints to account for the lack of knowledge and uncertain goal of the decision makers. However, as pointed out by Guide and Van Wassenhove (2009), there is limited contribution in the literature that addresses the complexity that arises from the large number of actors in a decentralized closed-loop supply chain system. Furthermore, Thierry et al. (1995) found that intensity of the competition is increased when combined with the product end-of-life issues. Therefore, a holistic approach to study the competition and interaction among the decision-makers within a CLSC network is needed in order to study the system behavior and obtain useful managerial insights.

The above studies do not consider the effects of the demand and return uncertainties on quantity shipments and prices. In fact, demands and returns for a product are not known with certainty but we may obtain some information such as the cumulative distribution functions based on historical data. Demand uncertainty is a known problem faced by firms to determine suitable levels of output before demand is known, which is classically known as the "newsboy" problem in operational research literature (e.g., Atkinson 1979; Chen and Chen 2010; Huang et al. 2011). Shi et al. (2011) studied the production planning problem for a multi-product closed loop system, in which the demands and their returns are uncertain and price-sensitive, but developed the model to include only the manufacturer's decision-making problem. Qiang et al. (2013) proposed a CLSC network model using the variational inequality approach. They discussed the effects of competition, distribution channel investment, yield and conversion rate, combined with uncertainties in demand, on equilibrium quantity transactions and prices. However, the authors did not consider the return uncertainties in the above work. In particular, we note that Dong et al. (2004) studied the demand uncertainties in the decentralized supply chain network; but their model only considers the forward supply chain network. To our best knowledge, there are very few papers discussing both random return and demand in the CLSC network. The only work that we are aware of is the paper by Sun et al. (2013). The authors proposed a multi-period supply chain network model and assumed the end-of-life product return is random, depending on the collecting price. The authors studied the characteristics of the optimal price and designed a monotonic pricing policy.

In this paper, we extend the work of Yang et al. (2009) to consider demand and return uncertainties. In contrast to the aforementioned models, we propose a new CLSC model that captures demand and return uncertainties through price-dependent functions. Note that the proposed model fits in the framework of stochastic variational inequality/stochastic Nash equilibrium studied by Sobel et al. (1971), Ravat and Shanbhag (2011) and Jiang and Xu (2008). As argued in Jiang and Xu (2008), if the distribution functions of the random elements involved are assumed to be known then the problem reduces to a classical deterministic optimization problem. Throughout this paper, we assume that the distribution functions of the random demand and random returns are known. The case of unknown distribution functions is left for future work.

Using additive and multiplicative functions for both demand and return, we demonstrate the joint concavity of the retailers and recovery centers' profits as functions of both shipment quantities and prices. This important result allows us to formulate the optimality conditions of all closed-loop supply chain members as a finite-dimensional variational inequality in which both retailers and recovery centers should account for penalties and salvage values associated with shortage and excess supply respectively. The variational inequality approach is commonly used to solve equilibrium problems, see for instance Dafermos (1980) and Shao et al. (2006) for traffic equilibrium, Dafermos (1990), He et al. (2011), and Jofré et al. (2007) for economic equilibrium and Friesz et al. (1984), Nagurney and Zhang (1996) and Takayama and Judge (1971) for spatial price equilibrium. To solve the variational inequality, we adopt the extra-gradient method of Khobotov (1987). Numerical examples are performed to illustrate the effect of demand and return uncertainties on quantity shipments and prices.

The contributions of this paper include the following:

i) This paper proposes a closed-loop supply chain equilibrium model with the consideration of both demand and return uncertainties.
ii) We consider that the demands and returns for a product are price-sensitive. Both demand and return have a deterministic and an uncertain components. The deterministic component is modeled using traditional linear, logarithmic, and power functions whereas the uncertain component is modeled using additive or multiplicative functions.
iii) We consider that the distribution functions of the random demand and return are assumed to be known which allows us to formulate the closed-loop supply chain problem as a deterministic optimization problem.
iv) We develop a new variational inequality formulation in which both retailers and recovery centers should account for penalties and salvage values associated with shortage and excess supply respectively.
v) The model and the solution method allow us to evaluate the performance of closed-loop supply chains under demand and return uncertainties, and to assess

the effectiveness of closed-loop supply chain decisions under the effects of uncertainty, reverse logistics and net revenues.

This paper is organized as follows: In Section 2, we present the CLSC network model with random demands and returns consisting of multiple tiers of supply chain members. In Section 3, the optimality conditions of the various decision makers are derived using the theory of variational inequality. Section 4 presents the equilibrium pattern and its qualitative properties. This section also outlines the solution algorithm and discusses convergence results. Section 5 presents several examples to describe the effects of random demands and returns on the equilibrium shipments and expected profits. Section 6 concludes the paper. Finally, all proofs are given in the appendices.

2 The Closed-Loop Supply Chain Network Model with Random Demands and Returns

In this section, we present the closed-loop supply chain network model with random demands and returns. As shown in Fig. 1 (provided in Yang et al. (2009)), the network consists of two groups of supply chain members: (1) Forward supply chain members illustrated at the left side of Fig. 1, including raw material suppliers, manufacturers and retailers; (2) Reverse supply chain members displayed at the right side of Fig. 1, including retailers, recovery centers and manufacturers. Note that manufacturers and

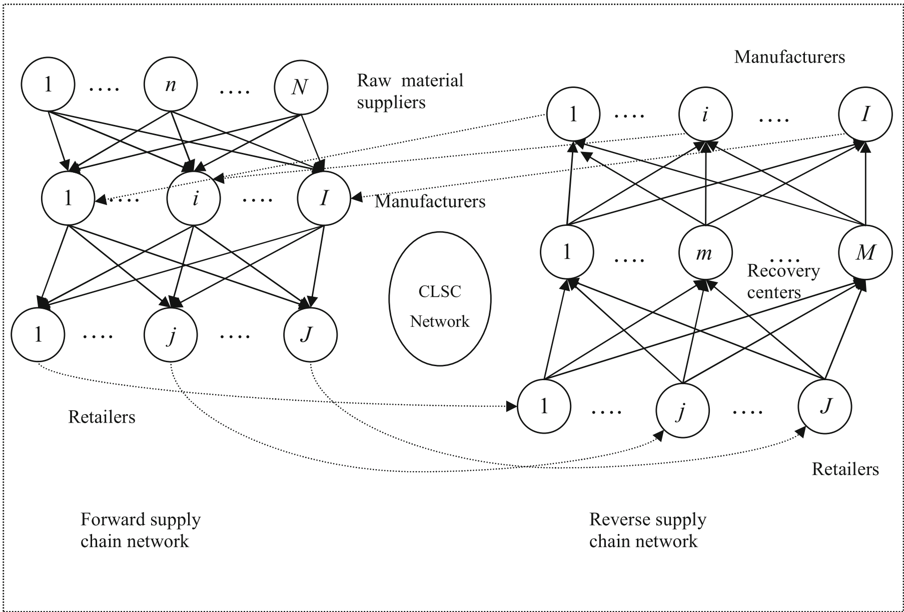

Fig. 1 The closed-loop supply chain network

retailers are the nodes to connect the forward supply chain network and the reverse supply chain network to establish the closed-loop supply chain network.

2.1 Model Assumptions

Similar to the CLSC literature in the deterministic case, we make the following assumptions:

i) The raw material suppliers only supply raw materials to the manufacturers in the forward supply chain.
ii) The manufacturers in the closed-loop supply chain network produce a homogeneous product, from both raw materials and recycled materials. This assumption can be relaxed by taking into account the quality depreciation of products made from recycled materials.
iii) Each retailer is responsible for dealing with its own demand market. Such an assumption has been used in CLSC literature (see Qiang et al. (2013)).
iv) The recovery centers only collect the recyclable product shipments from the demand markets in the original supply chain. They only supply reusable materials to the manufacturers in this supply chain.
v) All the supply chain members compete in a noncooperative manner under the Cournot-Nash equilibrium framework, that is, each decision maker will determine his optimal decision variables, given the optimal ones of the competitors. An other approach would be to consider a Stackelberg-Nash setting where each decision maker anticipate the reaction of other competitors to his decision (e.g., Sherali et al. (1983)). Such model would then fit the framework of EPECs (equilibrium programs with equilibrium constraints). While the latter are much more difficult to solve than variational inequalities, such approaches have been proposed in the literature, especially in the fields of energy and revenue management (e.g., Hu and Ralph (2007), Jiang et al. (2004), Metzler et al. (2003) and Oggioni et al. (2012)).

To consider uncertainties in CLSC, two more assumptions are made within the model:

i) The demand for the product at each retailer outlet is random and depends on retailer prices.
ii) The return for the recyclable product at each recovery center is random and depends on buy-back prices.

2.2 Model Definitions

Definitions of indices, parameters and variables in the closed-loop supply chain network are described below:

2.2.1 Indices

i index of manufacturers in the CLSC network, $i = \{1, \dots, I\}$.

j index of retailers in the CLSC network, $j = \{1, \ldots, J\}$.
m index of recovery centers in the CLSC network, $m = \{1, \ldots, M\}$.
n index of raw material suppliers in the CLSC network, $n = \{1, \ldots, N\}$.

2.3 Parameters

β_i^r fraction of usable material that can be transformed from raw materials for manufacturer i. $\beta_i^r \in [0, 1]$.
$\bar{\beta}_i^r$ fraction of useless raw material for manufacturer i. These useless materials are sent to the landfill $\bar{\beta}_i^r = 1 - \beta_i^r$.
β_i^u fraction of usable material that can be transformed from recycled materials for manufacturer i. $\beta_i^u \in [0, 1]$.
$\bar{\beta}_i^u$ fraction of useless recycled material for manufacturer i. These useless materials are sent to the landfill. $\bar{\beta}_i^u = 1 - \beta_i^u$.
χ_m fraction of usable recycled product that can be transformed to reusable material for recovery center m. $\chi_m \in [0, 1]$.
$\bar{\chi}_m$ fraction of useless recycled material for recovery center m. These useless materials are sent to the landfill. $\bar{\chi}_m = 1 - \chi_m$.
r return ratio of used products at all demand markets.
f^{rec} recycling fee (per product unit) charged by the corresponding Environment Protection Agency for manufacturing a given units of products.
s^{rec} unit of subsidy of environment protection to recovery centers offered by the EPA.
c^r cost (per unit of disposed raw materials) to the landfill.
c^u cost (per unit of disposed reusable materials) to the landfill.
λ_j^+ per-unit salvage value of having excess supply at retailer j
λ_j^- per-unit cost of having excess demand at retailer j.
λ_m^+ per-unit salvage value of having excess supply at recovery center m.
λ_m^- per-unit cost of having shortage at recovery center m.

2.4 Variables

q_{ni} nonnegative raw material shipment from supplier n to manufacturer i. Group the shipments of all the raw materials into the column vector $Q_1 \in \mathbb{R}_+^{NI}$.
q_{mi} nonnegative reusable material shipment from recovery center m to manufacturer i. Group the shipments of all the reusable materials into the column vector $Q_2 \in \mathbb{R}_+^{MI}$.
q_{ij} nonnegative product shipment from manufacturer i to retailer j. Group the shipments of all the manufacturers into the column vector $Q_3 \in \mathbb{R}_+^{IJ}$.
p_{ni} selling price from supplier n to manufacturer i.
p_{mi} selling price from recovery center m to manufacturer i.
p_{ij} selling price from manufacturer i to retailer j.
p_j selling price at retailer outlet j. Group the prices of all the retailers into the column vector $P_1 \in \mathbb{R}_+^J$.
p_m buy-back price from recovery center m. Group the prices of all the recovery centers into the column vector $P_2 \in \mathbb{R}_+^M$.

3 Equilibrium Conditions of the Closed-Loop Supply Chain Network Members

In this section, we derive the optimality conditions of the various decision-makers in the closed-loop supply chain.

3.1 Raw Material Suppliers and their Equilibrium Conditions

We assume that each raw material supplier n is faced with a procurement cost, $f_n^r\left(\sum_{i=1}^{I} q_{ni}\right)$, which, in general, depends upon the entire material shipment, $\sum_{i=1}^{I} q_{ni}$. We associate with each raw material supplier and manufacturer pair (n, i) a transaction cost denoted by $c_{ni}(q_{ni})$.

Given these two costs, we can express the criterion of profit maximization for each raw material supplier n as:

$$\max_{q_{ni}} \Pi_n = \sum_{i=1}^{I} p_{ni}^* q_{ni} - f_n^r\left(\sum_{i=1}^{I} q_{ni}\right) - \sum_{i=1}^{I} c_{ni}(q_{ni}). \tag{1}$$

Equation 1 states that a raw material supplier's profit is equal to sales revenue minus costs associated with procurement and transaction. Note that p_{ni}^* denote the optimal prices from each raw material supplier n to each manufacturer i. We will discuss later how these optimal prices are recovered after solving the complete closed-loop supply chain equilibrium model.

We assume that the raw material suppliers compete in noncooperative manner. Also, we assume that procurement and transaction cost functions for each supplier are continuous and convex. Therefore, the optimality conditions for all raw material suppliers simultaneously can be expressed as the following variational inequality (e.g. Nagurney et al. (2002)): Determine $Q_1^* \in \mathbb{R}_+^{NI}$ satisfying:

$$\sum_{n=1}^{N}\sum_{i=1}^{I}\left[\frac{\partial f_n^r\left(\sum_{i=1}^{I} q_{ni}^*\right)}{\partial q_{ni}} + \frac{\partial c_{ni}(q_{ni}^*)}{\partial q_{ni}} - p_{ni}^*\right] \times [q_{ni} - q_{ni}^*] \geq 0,\ \forall Q_1 \in \mathbb{R}_+^{NI}. \tag{2}$$

3.2 Manufacturers and their Equilibrium Conditions

Each manufacturer i must decide on: (a) the amount of product to ship to retailers; (b) the amount of raw materials to get from suppliers; (c) the amount of reusable materials to input from recovery centers (see Fig. 2 adopted from Yang et al. (2009)). Manufacturer i, incurs a production cost from raw materials, $f_i^r\left(\beta_i^r, \sum_{n=1}^{N} q_{ni}\right)$, and a remanufacturing cost of reusable materials, $f_i^u\left(\beta_i^u, \sum_{m=1}^{M} q_{mi}\right)$. These costs depend on the recovery level (β_i^r or β_i^u) designed into the product.

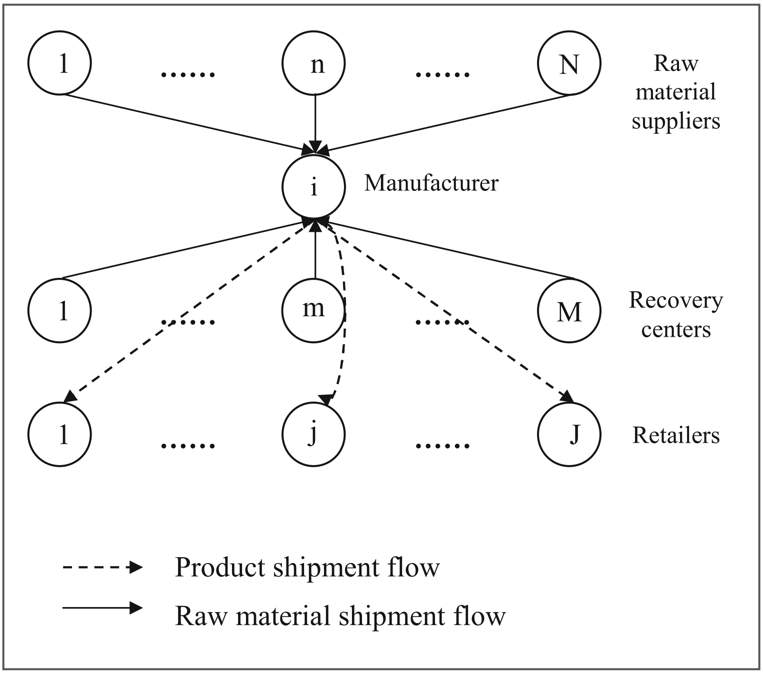

Fig. 2 Network structure of manufacturer i's transactions

We associate with each manufacturer and retailer pair (i, j) a transaction cost denoted $c_{ij}(q_{ij})$. Also let $c_{mi}(q_{mi})$ denotes the transaction cost associated with manufacturer i transacting with recovery center m.

To enforce environment legislation, recycling fees should be charged for manufacturers to make them financially responsible for the products they produced. The aggregate recycling fee for manufacturer i is equal to $f^{rec} \sum_{j=1}^{J} q_{ij}$ (Sheu et al. 2005).

Given the above costs, we can express the criterion of profit maximization for each manufacturer i as:

$$\max_{(q_{ni}, q_{mi}, q_{ij})} \Pi_i = \sum_{j=1}^{J} p_{ij}^* q_{ij} - f_i^r \left(\beta_i^r, \sum_{n=1}^{N} q_{ni} \right) - f_i^u \left(\beta_i^u, \sum_{m=1}^{M} q_{mi} \right) - \sum_{j=1}^{J} c_{ij}(q_{ij})$$
$$- \sum_{m=1}^{M} c_{mi}(q_{mi}) - \sum_{n=1}^{N} p_{ni}^* q_{ni} - \sum_{m=1}^{M} p_{mi}^* q_{mi} - f^{rec} \sum_{j=1}^{J} q_{ij}$$
$$-c^r \bar{\beta}_i^r \sum_{n=1}^{N} q_{ni} - c^u \bar{\beta}_i^u \sum_{m=1}^{M} q_{mi} \tag{3}$$

$$\text{subject to:} \quad \sum_{j=1}^{J} q_{ij} \le \beta_i^r \sum_{n=1}^{N} q_{ni} + \beta_i^u \sum_{m=1}^{M} q_{mi}. \tag{4}$$

Equation 3 states that manufacturer i's profit equals sales revenue less costs associated with production and transaction, payout to raw material suppliers and recovery centers, recycling fees and disposal cost. Constraint (4) states that the sum of all shipment quantities to retailers must be less than or equal to the sum of the quantities produced from raw materials and remanufactured from reusable materials. Once produced, the useless materials $\left(\bar{\beta}_i^r \sum_{n=1}^{N} q_{ni} + \bar{\beta}_i^u \sum_{m=1}^{M} q_{mi}\right)$ would be sent to the landfill, thus the disposal cost for manufacturer i is equal to $c^r \bar{\beta}_i^r \sum_{n=1}^{N} q_{ni} + c^u \bar{\beta}_i^u \sum_{m=1}^{M} q_{mi}$. Note that the optimal prices p_{mi}^* and p_{ij}^* become endogenous variables in the complete closed-loop supply chain equilibrium model.

We assume that the manufacturers compete in noncooperative manner. Also, we assume that production and transaction cost functions for each manufacturer are continuous and convex. Therefore, the optimality conditions for all manufacturers can be expressed simultaneously as the following variational inequality: Determine $(Q_1^*, Q_2^*, Q_3^*, \gamma_1^*) \in \mathbb{R}_+^{NI+MI+IJ+I}$ satisfying:

$$\sum_{n=1}^{N}\sum_{i=1}^{I}\left[\frac{\partial f_i^r\left(\beta_i^r, \sum_{n=1}^{N} q_{ni}^*\right)}{\partial q_{ni}} + c^r \bar{\beta}_i^r + p_{ni}^* - \beta_i^r \gamma_{1i}^*\right] \times [q_{ni} - q_{ni}^*]$$
$$+\sum_{m=1}^{M}\sum_{i=1}^{I}\left[\frac{\partial f_i^u\left(\beta_i^u, \sum_{m=1}^{M} q_{mi}^*\right)}{\partial q_{mi}} + \frac{\partial c_{mi}(q_{mi}^*)}{\partial q_{mi}} + c^u \bar{\beta}_i^u + p_{mi}^* - \beta_i^u \gamma_{1i}^*\right] \times [q_{mi} - q_{mi}^*]$$
$$+\sum_{i=1}^{I}\sum_{j=1}^{J}\left[\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + f^{rec} + \gamma_{1i}^* - p_{ij}^*\right] \times [q_{ij} - q_{ij}^*]$$
$$+\sum_{i=1}^{I}\left[\beta_i^r \sum_{n=1}^{N} q_{ni}^* + \beta_i^u \sum_{m=1}^{M} q_{mi}^* - \sum_{j=1}^{J} q_{ij}^*\right]$$
$$\times[\gamma_{1i} - \gamma_{1i}^*] \geq 0 \ \forall (Q_1, Q_2, Q_3, \gamma_1) \in \mathbb{R}_+^{NI+MI+IJ+I}. \tag{5}$$

In inequality (5), γ_{1i} is the Lagrange multiplier associated with constraint (4) for manufacturer i, and γ_1 is the column vector of all the manufacturer's multipliers.

3.3 Retailers and their Equilibrium Conditions

Each retailer j must decide jointly on the quantity to order from the manufacturers and the selling price in order to cope with the random demand while seeking to maximize its expected profit. Each retailer j is faced with a handling cost, which may include the display and storage cost associated with the product. For simplicity, we assume a constant handling cost c_j; all our results extend without loss of generality to increasing and convex functions $c_j\left(\sum_{i=1}^{I} q_{ij}\right)$. To avoid triviality, we assume that $\lambda_j^+ \leq p_j$.

We assume that the demand for the product at each retailer j, $D_j(p_j, \epsilon_j)$, is random and depends on the price p_j and a random variable ϵ_j independent of p_j and defined on the range $[A_j, B_j]$. We assume the mean demand is specified by a function $y_j(p_j)$ that captures the dependency between demand and price:

$$E(D_j(p_j, \epsilon_j)) = y_j(p_j),$$

where $y_j(p_j)$ is continuous, nonnegative and three times differentiable. Let $y_j'(p_j)$, $y_j''(p_j)$ and $y_j'''(p_j)$ denote the first, second and third derivative of $y_j(p_j)$.

Assumption 1 *$y_j'(p_j) \leq 0$, $y_j''(p_j) \geq 0$, $p_j y_j(p_j)$ is concave in p_j and $p_j y_j'(p_j)$ is convex in p_j.*

The last concavity and convexity conditions amount to $2y_j'(p_j) + p_j y_j''(p_j) \leq 0$ and $2y_j''(p_j) + p_j y_j'''(p_j) \geq 0$. The concavity condition was also used by Kocabiyikoglu and Popescu (2011). It can be easily verified that the following demand functions satisfy the conditions of Assumption 1:

i) Linear: $y_j(p_j) = a_j - b_j p_j, a_j > 0, b_j > 0, p_j \leq \frac{a_j}{b_j}, j = 1, \cdots, J$.

ii) Logarithmic: $y_j(p_j) = a_j - b_j \ln(p_j + 1), a_j > 0, b_j > 0, p_j \leq e^{\frac{a_j}{b_j}} - 1$, $j = 1, \cdots, J$.

iii) Power: $y_j(p_j) = a_j - b_j p_j^{\gamma_j}, a_j > 0, b_j > 0, 0 < \gamma_j \leq 1, p_j \leq \left(\frac{a_j}{b_j}\right)^{\frac{1}{\gamma_j}}$, $j = 1, \cdots, J$.

Note that the conditions p_j smaller than some constants are only added to ensure positive demand.

In the literature, two types of demand functions are commonly used: the additive form where $D_j(p_j, \epsilon_j) = y_j(p_j) + \epsilon_j$, and the multiplicative form where $D_j(p_j, \epsilon_j) = y_j(p)\epsilon_j$ (Petruzzi and Dada 1999). For the additive demand model, we assume that $E(\epsilon_j) = 0$ and $y_j(\bar{p}_j) + A_j = 0$ where $\bar{p}_j$ is the maximum admissible level of p_j. As discussed in Xu et al. (2010), this last condition will ensure a positive demand in the range of the price interval $[0, \bar{p}_j]$. For the multiplicative demand model, we assume that $E(\epsilon_j) = 1$ and $y_j(\bar{p}_j) = 0$.

We assume that the random variable ϵ_j has a continuous distribution $F_j(x)$ with density $f_j(x)$. Define the failure rate function of the ϵ_j's distribution as:

$$r_j(x) = \frac{f_j(x)}{1 - F_j(x)},$$

and the generalized failure rate function as:

$$g_j(x) = \frac{x f_j(x)}{1 - F_j(x)}.$$

Assumption 2a *For each retailer j, the distribution of the random variable ϵ_j has* **increasing failure rate** *(IFR) and $\frac{1}{r_j(x)}$ is convex.*

Assumption 2b *For each retailer j, the distribution of the random variable ϵ_j has* **increasing generalized failure rate** *(IGFR) and $\frac{1}{g_j(x)}$ is convex.*

As discussed in Yao et al. (2006), the class of IFR distributions include: Uniform, Normal (as well as truncated Normal at zero), Exponential, Gamma (with shape parameter $s \geq 1$), Beta (with parameters (r, s) being both ≥ 1), and Weibull distribution (with shape parameter $s \geq 1$). The class of IGFR distributions generalizes that of IFR and include all previous distributions without parameters restrictions. It is also easy to check that all these distributions satisfy the conditions $2r_j''(x) - r_j(x) r_j'(x) \geq 0$ and $2g_j''(x) - g_j(x) g_j'(x) \geq 0$ which are required for the convexity of $\frac{1}{r_j(x)}$ and $\frac{1}{g_j(x)}$.

In the next subsections, we describe the equilibrium conditions of the retailers. We first focus on the additive demand case. We then turn to the multiplicative case.

3.3.1 Additive Demand Case

Let $s_j = \sum_{i=1}^{I} q_{ij}$ denote the total supply at retailer j obtained from all the manufacturers. If demand for the product does not exceed s_j, then the revenue of retailer j is $p_j D_j(p_j, \epsilon_j)$ and each of the $s_j - D_j(p_j, \epsilon_j)$ leftovers is disposed at the unit salvage value $\lambda_j^+ \leq p_j$. Alternatively, if demand exceeds s_j, then the revenue of retailer j is $p_j s_j$ and each of the $D_j(p_j, \epsilon_j) - s_j$ shortages incurs a per-unit shortage cost λ_j^-. Let $Q_j = (q_{ij})_{i=1}^{I}$. Then, the profit of retailer j, $W_j(Q_j, p_j)$, is the difference between sales revenue and the total costs:

$$W_j(Q_j, p_j) = \begin{cases} p_j D_j(p_j, \epsilon_j) - c_j s_j - \sum_{i=1}^{I} p_{ij}^* q_{ij} + \lambda_j^+ [s_j - D_j(p_j, \epsilon_j)] \text{ if } D_j(p_j, \epsilon_j) \leq s_j \\ p_j s_j - c_j s_j - \sum_{i=1}^{I} p_{ij}^* q_{ij} - \lambda_j^- [D_j(p_j, \epsilon_j) - s_j] \text{ if } D_j(p_j, \epsilon_j) > s_j \end{cases}$$

Defining $z_j = s_j - y_j(p_j)$, the profit $W_j(Q_j, p_j)$ can be computed as:

$$W_j(Q_j, p_j) = \begin{cases} p_j[y_j(p_j) + \epsilon_j] - c_j s_j - \sum_{i=1}^{I} p_{ij}^* q_{ij} + \lambda_j^+ [z_j - \epsilon_j)] \text{ if } \epsilon_j \leq z_j \\ p_j[y_j(p_j) + z_j] - c_j s_j - \sum_{i=1}^{I} p_{ij}^* q_{ij} - \lambda_j^- [\epsilon_j - z_j] \text{ if } \epsilon_j > z_j. \end{cases}$$

The expected values of leftover and shortage of retailer j, $e_j^+(z_j)$ and $e_j^-(z_j)$ are computed as:

$$e_j^+(z_j) = \int_{A_j}^{z_j} (z_j - x) f_j(x) dx,$$
$$e_j^-(z_j) = \int_{z_j}^{B_j} (x - z_j) f_j(x) dx.$$

Therefore, each retailer j seeks to maximize its expected profit, $\Pi_j(Q_j, p_j) = E\left(W_j(Q_j, p_j)\right)$:

$$\max \Pi_j(Q_j, p_j) = p_j y_j(p_j) - p_j e_j^-(z_j) + \lambda_j^+ e_j^+(z_j) - \lambda_j^- e_j^-(z_j) - c_j s_j - \sum_{i=1}^{I} p_{ij}^* q_{ij}. \tag{6}$$

Objective function (6) states that the expected profit of retailer j, which is the difference between the expected revenue and the sum of the expected leftover and shortage, the handling cost and the payout to the manufacturers, should be maximized.

Similar to Kocabiyikoglu and Popescu (2011), we define the lost-sales rate (LSR) elasticity for a given pair (s_j, p_j) as:

$$\eta_j^1(s_j, p_j) = -\frac{(p_j + \delta_j)\frac{\partial(1 - F_j(z_j))}{\partial p_j}}{1 - F_j(z_j)} = \frac{-(p_j + \delta_j) y_j'(p_j) f_j(z_j)}{1 - F_j(z_j)}$$
$$= -(p_j + \delta_j) y_j'(p_j) r_j(z_j),$$

where $\delta_j = \min\{\lambda_j^- - \lambda_j^+, 0\}$. The LSR elasticity measures the percentage change in the rate of lost sales with respect to the percentage change in price. For each pair (s_j, p_j), we define the set $\Gamma_j^1 = \{(Q_j, p_j) \in \mathbb{R}_+^{I+1} \mid \eta_j^1(s_j, p_j) \geq 1/2\}$. The following results demonstrates that the set Γ_j^1 is convex, the function $\Pi_j(Q_j, p_j)$ is jointly concave in Q_j and p_j and the optimal solution (Q_j^*, p_j^*) belongs to the set Γ_j^1 under the conditions of Assumptions 1 and 2a.

Lemma 1 *In the additive case, if conditions of Assumptions 1 and 2a are satisfied, then the set* Γ_j^1 *is convex.*

Proof See Appendix A. □

Theorem 1 *In the additive case, if conditions of Assumptions 1 and 2a are satisfied, then the function* $\Pi_j(Q_j, p_j)$ *is jointly concave in* Q_j *and* p_j *in the set* Γ_j^1.

Proof See Appendix B. □

Proposition 1 *In the additive case, if conditions of Assumptions 1 and 2a are satisfied, then the optimal solution* (Q_j^*, p_j^*) *belongs to the set* Γ_j^1.

Proof See Appendix C. □

Under the conditions of Theorem 1, the function $\Pi_j(Q_j, p_j)$ is jointly concave in Q_j and p_j and as pointed in Lemma 1, the set Γ_j^1 is convex therefore the optimality conditions for all retailers could be expressed simultaneously as the following variational inequality: Determine $(Q_3^*, P_1^*) \in \Gamma^1 \subset \mathbb{R}_+^{IJ+J}$ satisfying:

$$\begin{aligned}
&\sum_{i=1}^{I}\sum_{j=1}^{J}\left[-\left(\lambda_j^- + p_j^* - c_j - p_{ij}^*\right) + \left(p_j^* + \lambda_j^- - \lambda_j^+\right) F_j(z_j^*)\right] \times [q_{ij} - q_{ij}^*] \\
&+\sum_{j=1}^{J}\left[-y_j(p_j^*) + e_j^-(z_j^*) - y_j'(p_j^*)\left[(p_j^* + \lambda_j^- - \lambda_j^+)F_j(z_j^*) - \lambda_j^-\right]\right] \\
&\times[p_j - p_j^*] \geq 0,\ \forall (Q_3, P_1) \in \Gamma^1, \qquad (7)
\end{aligned}$$

where $z_j^* = s_j^* - y_j(p_j^*)$ and $\Gamma^1 = \bigotimes_{j=1}^{J} \Gamma_j^1$.

3.3.2 Multiplicative Demand Case

In the multiplicative demand case, the profit $W_j(Q_j, p_j)$ can be written as:

$$W_j(Q_j, p_j) = \begin{cases} p_j y_j(p_j)\epsilon_j - c_j s_j - \sum_{i=1}^{I} p_{ij}^* q_{ij} + \lambda_j^+ y_j(p_j)[z_j - \epsilon_j)] & \text{if } \epsilon_j \leq z_j \\ p_j y_j(p_j) z_j - c_j s_j - \sum_{i=1}^{I} p_{ij}^* q_{ij} - \lambda_j^- y_j(p_j)[\epsilon_j - z_j] & \text{if } \epsilon_j > z_j, \end{cases}$$

where $Q_j = (q_{ij})_{i=1}^{I}$ and $z_j = \frac{s_j}{y_j(p_j)}$. Therefore, the expected profit of retailer j is computed as:

$$\begin{aligned}
\max \Pi_j(Q_j, p_j) = &= p_j y_j(p_j)(1 - e_j^-(z_j)) + y_j(p_j)\left(\lambda_j^+ e_j^+(z_j) - \lambda_j^- e_j^-(z_j)\right) \\
&\quad - c_j s_j - \sum_{i=1}^{I} p_{ij}^* q_{ij}. \qquad (8)
\end{aligned}$$

Similar to the additive case, we define the lost-sales rate (LSR) elasticity for a given pair (s_j, p_j) as:

$$\begin{aligned}
\eta_j^2(s_j, p_j) &= -\frac{(p_j + \delta_j)\frac{\partial(1 - F_j(z_j))}{\partial p_j}}{1 - F_j(z_j)} = \frac{-(p_j + \delta_j) z_j y_j'(p_j) f_j(z_j)}{y_j(p_j)(1 - F_j(z_j))} \\
&= \frac{-(p_j + \delta_j) y_j'(p_j) g_j(z_j)}{y_j(p_j)},
\end{aligned}$$

and the set $\Gamma_j^2 = \{(Q_j, p_j) \in \mathbb{R}_+^{I+1} \mid \eta_j^2(s_j, p_j) \geq 1/2\}$.

The following results demonstrates that the set Γ_j^2 is convex, the function $\Pi_j(Q_j, p_j)$ is jointly concave in Q_j and p_j and the optimal solution (Q_j^*, p_j^*) belongs to the set Γ_j^2 under the conditions of Assumptions 1 and 2b.

Lemma 2 *In the multiplicative case, if conditions of Assumptions 1 and 2b are satisfied, then the set Γ_j^2 is convex.*

Proof See Appendix D. □

Theorem 2 *In the multiplicative case, if conditions of Assumptions 1 and 2b are satisfied, then the function $\Pi_j(Q_j, p_j)$ is jointly concave in Q_j and p_j in the set Γ_j^2.*

Proof See Appendix E. □

Proposition 2 *In the multiplicative case, if conditions of Assumptions 1 and 2b are satisfied, then the optimal solution (Q_j^*, p_j^*) belongs to the set Γ_j^2.*

Proof See Appendix F. □

Under the conditions of Theorem 2, the function $\Pi_j(Q_j, p_j)$ is jointly concave in Q_j and p_j and as pointed in Lemma 2, the set Γ_j^2 is convex therefore the optimality conditions for all retailers could be expressed simultaneously as the following variational inequality: Determine $(Q_3^*, P_1^*) \in \Gamma^2 \subset \mathbb{R}_+^{IJ+J}$ satisfying:

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\left[-\left(\lambda_j^- + p_j^* - c_j - p_{ij}^*\right) + \left(p_j^* + \lambda_j^- - \lambda_j^+\right) F_j(z_j^*)\right] \times [q_{ij} - q_{ij}^*]$$
$$+\sum_{j=1}^{J}\left[-y_j'(p_j^*)\left[(p_j^* + \lambda_j^- - \lambda_j^+)\left(1 - e_j^-(z_j^*) + z_j(F_j(z_j^*) - 1)\right) - \lambda_j^-\right]\right.$$
$$\left. - y_j(p_j^*)(1 - e_j^-(z_j^*))\right] \times [p_j - p_j^*], \geq 0, \ \forall (Q_3, P_1) \in \Gamma^2, \tag{9}$$

where $z_j^* = \frac{s_j^*}{y_j(p_j^*)}$ and $\Gamma^2 = \bigotimes_{j=1}^{J} \Gamma_j^2$.

3.4 Recovery Centers and their Equilibrium Conditions

Recovery centers are assumed to buy-back the amount of used product from consumers at various demand markets. Each recovery center m must decide jointly on the quantity to sell to the manufacturers and the buy-back price in order to cope with the random return while seeking to maximize its expected profit.

Corresponding to the recycling fees, recovery center m would obtain a subsidy equal to s^{rec} from the corresponding environment protection agency for each unit of recyclable product (see Sheu et al. 2005). Before selling the reusable materials to the manufacturers in the original supply chain, each recovery center m must pick up, clean, inspect and disassemble the amount of used product incurring a unit recycling cost of c_m^u.

We assume that the return for the recyclable product associated to each recovery center m, $R_m(p_m, \epsilon_m)$, is random and depends on the buy-back price p_m and a random variable ϵ_m independent of p_m and defined on the range $[A_m, B_m]$. We assume the mean return is specified by a function $y_m(p_m)$ that captures the dependency between return and buy-back price:

$$E\left(R_m(p_m, \epsilon_m)\right) = y_m(p_m),$$

where $y_m(p_m)$ is continuous, nonnegative and two times differentiable. Lets denote $y'_m(p_m)$ and $y''_m(p_m)$ the first and second derivative of $y_m(p_m)$.

Assumption 3 $y'_m(p_m) \geq 0$, $y''_m(p_m) \leq 0$ *and* $p_m y_m(p_m)$ *is convex in* p_m.

The last convexity condition amount to $2y'_m(p_m) + p_m y''_m(p_m) \geq 0$. It can be easily verified that the following supply functions satisfy the conditions of Assumption 3:

i) Linear: $y_m(p_m) = b_m p_m - a_m = b_m(p_m - p_m^0)$, where $p_m^0 = a_m/b_m$, $b_m > 0$, $m = 1, \cdots, M$.
ii) Logarithmic: $y_m(p_m) = b_m \ln(p_m + 1) - a_m = b_m(\ln(p_m + 1) - \ln(p_m^0 + 1))$, where $\ln(p_m^0 + 1) = a_m/b_m$, $b_m > 0$, $m = 1, \cdots, M$.
iii) Power: $y_m(p_m) = b_m p_m^{\gamma_m} - a_m = b_m(p_m^{\gamma_m} - p_m^{0\,\gamma_m})$, where $(p_m^0)^{\gamma_m} = a_m/b_m$, $b_m > 0$, $0 < \gamma_m \leq 1$, $m = 1, \cdots, M$,

Note that p_m^0 is used to simplify notations and it denotes the price over which the return becomes positive.

As in the retailer problem, two types of return functions are used in our model: the additive form where $R_m(p_m, \epsilon_m) = y_m(p_m) + \epsilon_m$, and the multiplicative form where $R_m(p_m, \epsilon_m) = y_m(p_m)\epsilon_m$. For the additive return model, we assume that $E(\epsilon_m) = 0$ and $y_m(\bar{p}_m) = 0$ where $\bar{p}_m$ is the minimum admissible level of p_m. For the multiplicative return model, we assume that $E(\epsilon_m) = 1$ and $y_m(\bar{p}_m) = 0$.

In the next subsections, we describe the equilibrium conditions of the recovery centers. We first start with the additive return case. We then turn to the multiplicative case.

3.4.1 Additive Return Case

Let $q_m = \sum_{i=1}^{I} q_{mi}$ denote the total quantity shipped from recovery center m to all the manufacturers. If the actual transformed return quantity $\chi_m R_m$ is less than the total quantity q_m, there is a unit understocking cost equal to $\lambda_m^- + c_m^u + c^u \bar{\chi}_m$. This is because recovery center m has to make an emergency call to acquire and clean used products from other markets to compensate the shortage so as to satisfy the total order from manufacturers. Once disassembled, the unit disposal cost of recovery center m for sending the unusable materials to the landfill is $c^u \bar{\chi}_m$. If the actual transformed return quantity $\chi_m R_m$ is more than the quantity q_m, there is a salvage value equal to $\lambda_m^+ \leq \lambda_m^-$. Therefore, the expected total cost of recovery center m is given by:

$$\left(c_m^u + c^u \bar{\chi}_m\right) q_m + p_m y_m(p_m) - \lambda_m^+ e_m^+(z_m) + \lambda_m^- e_m^-(z_m), \tag{10}$$

where the expected values of excess supply and shortage of recovery center m, $e_m^+(z_m)$ and $e_m^-(z_m)$ are computed as:

$$\begin{aligned} e_m^+(z_m) &= \int_{z_m}^{B_m} (x - z_m) f_m(x) dx, \\ e_m^-(z_m) &= \int_{A_m}^{z_m} (z_m - x) f_m(x) dx, \end{aligned}$$

and $z_m = \frac{q_m}{\chi_m} - y_m(p_m)$. In equation (10), the total expected cost is the the sum of the recycling and disposal cost, the expected payout to the consumers, and the expected surplus and shortage. Given the above expected costs, each recovery center m seeks to maximize its expected profit:

$$\begin{aligned} \max \Pi_m(Q_m, p_m) = & \sum_{i=1}^{I} p_{mi}^* q_{mi} + s^{rec} y_m(p_m) - \left(c_m^u + c^u \bar{\chi}_m\right) q_m \\ & - p_m y_m(p_m) + \lambda_m^+ e_m^+(z_m) - \lambda_m^- e_m^-(z_m), \end{aligned} \tag{11}$$

where $Q_m = (q_{mi})_{i=1}^{I}$.

The following theorem demonstrates that the function $\Pi_m(q_{mi}, p_m)$ is jointly concave in Q_m and p_m under the conditions of Assumption 3.

Theorem 3 *In the additive case, if conditions of Assumption 3 are satisfied, then the function $\Pi_m(Q_m, p_m)$ is jointly concave in Q_m and p_m.*

Proof See Appendix G. □

Under the conditions of Theorem 3, the function $\Pi_m(q_{mi}, p_m)$ is jointly concave in Q_m and p_m and therefore the optimality conditions for all recovery centers could be expressed simultaneously as the following variational inequality : Determine $(Q_2^*, P_2^*) \in \mathbb{R}_+^{MI+M}$ satisfying:

$$\sum_{m=1}^{M}\sum_{i=1}^{I}\left[-p_{mi}^* + c_m^u + c^u\bar{\chi}_m + \frac{\lambda_m^+}{\chi_m} + \frac{\left(\lambda_m^- - \lambda_m^+\right)}{\chi_m}F_m(z_m^*)\right] \times [q_{mi} - q_{mi}^*]$$
$$+\sum_{m=1}^{M}\left[y_m(p_m^*) - y_m'(p_m^*)\left(s^{rec} - p_m^* + \lambda_m^+ + F_m(z_m^*)(\lambda_m^- - \lambda_m^+)\right)\right]$$
$$\times[p_m - p_m^*] \geq 0\,, \tag{12}$$

$\forall (Q_2, P_2) \in \mathbb{R}_+^{MI+M}$, where $z_m^* = \frac{q_m^*}{\chi_m} - y_m(p_m^*)$.

3.4.2 Multiplicative Return Case

In the multiplicative demand case, the expected profit of recovery center m is given as:

$$\max \Pi_m(Q_m, p_m) = \sum_{i=1}^{I} p_{mi}^* q_{mi} + s^{rec} y_m(p_m) - \left(c_m^u + c^u\bar{\chi}_m\right) q_m$$
$$-p_m y_m(p_m) + y_m(p_m)\left(\lambda_m^+ e_m^+(z_m) - \lambda_m^- e_m^-(z_m)\right), \tag{13}$$

where $Q_m = (q_{mi})_{i=1}^{I}$ and $z_m = \frac{q_m}{\chi_m y_m(p_m)}$.

The following theorem demonstrates that the function $\Pi_m(q_{mi}, p_m)$ is jointly concave in Q_m and p_m under the conditions of Assumption 3.

Theorem 4 *In the multiplicative case, if conditions of Assumption 3 are satisfied, then the function $\Pi_m(Q_m, p_m)$ is jointly concave in Q_m and p_m.*

Proof See Appendix H. □

Under the conditions of Theorem 4, the function $\Pi_m(q_{mi}, p_m)$ is jointly concave in Q_m and p_m and therefore the optimality conditions for all recovery centers could be expressed simultaneously as the following variational inequality : Determine $(Q_2^*, P_2^*) \in \mathbb{R}_+^{MI+M}$ satisfying:

$$\sum_{m=1}^{M}\sum_{i=1}^{I}\left[-p_{mi}^* + c_m^u + c^u\bar{\chi}_m + \frac{\lambda_m^+}{\chi_m} + \frac{\left(\lambda_m^- - \lambda_m^+\right)}{\chi_m}F_m(z_m^*)\right] \times [q_{mi} - q_{mi}^*] \tag{14}$$
$$+\sum_{m=1}^{M}\left[y_m(p_m^*) - y_m'(p_m^*)\left(s^{rec} - p_m^* + \lambda_m^-\right.\right.$$
$$\left.\left. -(\lambda_m^- - \lambda_m^+)\left(1 + e_m^-(z_m^*) - z_m^* F_m(z_m^*)\right)\right)\right] \times [p_m - p_m^*] \geq 0\,,$$

$\forall (Q_2, P_2) \in \mathbb{R}_+^{MI+M}$, where $z_m^* = \frac{q_m^*}{\chi_m y_m(p_m^*)}$.

4 Equilibrium Conditions of the CLSC

4.1 Equilibrium Conditions

In equilibrium, we must have that the total reusable materials recovery centers could sell to all manufacturers should not exceed a fraction r of the transformed shipments of the total retailers' supply:

$$\sum_{m=1}^{M}\sum_{i=1}^{I}\frac{1}{\chi_m}q_{mi}^{*} \leq r\sum_{i=1}^{I}\sum_{j=J}^{I}q_{ij}^{*}. \tag{15}$$

Note that r is the return ratio of used products at all demand markets and the transformation rate χ_m is allowed to depend on each recovery center m. Constraint (15) can be rewritten as:

$$\left[r\sum_{i=1}^{I}\sum_{j=J}^{I}q_{ij}^{*} - \sum_{m=1}^{M}\sum_{i=1}^{I}\frac{1}{\chi_m}q_{mi}^{*}\right] \times \left[\gamma_2 - \gamma_2^{*}\right] \geq 0, \forall \gamma_2 \geq 0, \tag{16}$$

where γ_2 is the Lagrange multiplier associated with constraint (15).

As in the supply chain equilibrium literature (e.g., Yang et al. 2009; Qiang et al. 2013), we must have that the sum of the optimality conditions for all raw material suppliers, as expressed by inequality (2), the optimality conditions for all manufacturers, as expressed by inequality (5), the optimality conditions for all retailers, as expressed by inequality (7) in the additive case and inequality (9) in the multiplicative case, and the optimality conditions for all recovery centers, as expressed by inequality (12) in the additive case and inequality (14) in the multiplicative case, must be satisfied.

Definition 1 (Closed-loop supply chain network equilibrium with random additive demand and return). The equilibrium state of the closed-loop supply chain with random additive demand and return is one where the flows between tiers of the network coincide, and the shipments and prices satisfy the sum of the optimality conditions (2), (5), (7), (12) and (16).

The summation of inequalities (2), (5), (7), (12) and (16) after algebraic simplification, yields the following result:

Theorem 5 *The equilibrium conditions governing the closed-loop supply chain model with random additive demand and return are equivalent to the solution of the* *variational inequality problem given by: determine* $(Q_1^*, Q_2^*, Q_3^*, P_1^*, P_2^*, \gamma_1^*, \gamma_2^*) \in \Omega$ *satisfying*

$$
\begin{aligned}
&\sum_{n=1}^{N}\sum_{i=1}^{I}\left[\frac{\partial f_i^r\left(\beta_i^r, \sum_{n=1}^{N} q_{ni}^*\right)}{\partial q_{ni}} + \frac{\partial f_n^r\left(\sum_{i=1}^{I} q_{ni}^*\right)}{\partial q_{ni}} + c^r\bar{\beta}_i^r - \beta_i^r\gamma_{1i}^* + \frac{\partial c_{ni}(q_{ni}^*)}{\partial q_{ni}}\right] \times [q_{ni} - q_{ni}^*] \\
&+ \sum_{m=1}^{M}\sum_{i=1}^{I}\left[\frac{\partial f_i^u\left(\beta_i^u, \sum_{m=1}^{M} q_{mi}^*\right)}{\partial q_{mi}} + c^u\bar{\beta}_i^u - \beta_i^u\gamma_{1i}^* + \frac{\partial c_{mi}(q_{mi}^*)}{\partial q_{mi}}\right. \\
&\qquad\qquad \left. + c_m^u + c^u\bar{\chi}_m + \frac{\lambda_m^+}{\chi_m} + \frac{(\lambda_m^- - \lambda_m^+)}{\chi_m}F_m(z_m^*) + \frac{\gamma_2^*}{\chi_m}\right] \times [q_{mi} - q_{mi}^*] \\
&+ \sum_{i=1}^{I}\sum_{j=1}^{J}\left[\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + f^{rec} + \gamma_{1i}^* - \left(\lambda_j^- + p_j^* - c_j\right)\right. \\
&\qquad\qquad \left. + \left(p_j^* + \lambda_j^- - \lambda_j^+\right)F_j(z_j^*) - r\gamma_2^*\right] \times [q_{ij} - q_{ij}^*] \\
&+ \sum_{j=1}^{J}\left[-y_j(p_j^*) + e_j^-(z_j^*) - y_j'(p_j^*)\left[(p_j^* + \lambda_j^- - \lambda_j^+)F_j(z_j^*) - \lambda_j^-\right]\right] \times [p_j - p_j^*] \\
&+ \sum_{m=1}^{M}\left[y_m(p_m^*) - y_m'(p_m^*)\left(s^{rec} - p_m^* + \lambda_m^+ + F_m(z_m^*)(\lambda_m^- - \lambda_m^+)\right)\right] \times [p_m - p_m^*] \\
&+ \sum_{i=1}^{I}\left[\beta_i^r\sum_{n=1}^{N} q_{ni}^* + \beta_i^u\sum_{m=1}^{M} q_{mi}^* - \sum_{j=1}^{J} q_{ij}^*\right] \times [\gamma_{1i} - \gamma_{1i}^*] \\
&+ \left[r\sum_{i=1}^{I}\sum_{j=J}^{I} q_{ij}^* - \sum_{m=1}^{M}\sum_{i=1}^{I}\frac{1}{\chi_m}q_{mi}^*\right] \times [\gamma_2 - \gamma_2^*] \geq 0, \;\; \forall (Q_1, Q_2, Q_3, P_1, P_2, \gamma_1, \gamma_2) \in \Omega,
\end{aligned}
\tag{17}
$$

where $\Omega = \{(Q_1, Q_2, Q_3, P_1, P_2, \gamma_1, \gamma_2) \in \mathbb{R}_+^{NI+MI+IJ+J+M+I+1} \,|\, (Q_3, P_1) \in \Gamma^1\}$.

Proof The formulation is developed using the standard variational inequality theory (e.g., Nagurney (2013)). □

Definition 2 (Closed-loop supply chain network equilibrium with random multiplicative demand and return). The equilibrium state of the closed-loop supply chain with random multiplicative demand and return is one where the flows between tiers of the network coincide, and the shipments and prices satisfy the sum of the optimality conditions (2), (5), (9), (14) and (16).

The summation of inequalities (2), (5), (9), (14) and (16) after algebraic simplification, yields the following result:

Theorem 6 *The equilibrium conditions governing the closed-loop supply chain model with random multiplicative demand and return are equivalent to the solution of the variational inequality problem given by: determine* $(Q_1^*, Q_2^*, Q_3^*, P_1^*, P_2^*, \gamma_1^*, \gamma_2^*) \in \Omega$ *satisfying*

$$\begin{aligned}
&\sum_{n=1}^{N}\sum_{i=1}^{I}\left[\frac{\partial f_i^r(\beta_i^r, \sum_{n=1}^{N} q_{ni}^*)}{\partial q_{ni}} + \frac{\partial f_n^r(\sum_{i=1}^{I} q_{ni}^*)}{\partial q_{ni}} + c^r\bar{\beta}_i^r - \beta_i^r\gamma_{1i}^* + \frac{\partial c_{ni}(q_{ni}^*)}{\partial q_{ni}}\right] \times [q_{ni} - q_{ni}^*] \\
&+ \sum_{m=1}^{M}\sum_{i=1}^{I}\left[\frac{\partial f_i^u\left(\beta_i^u, \sum_{m=1}^{M} q_{mi}^*\right)}{\partial q_{mi}} + c^u\bar{\beta}_i^u - \beta_i^u\gamma_{1i}^* + \frac{\partial c_{mi}(q_{mi}^*)}{\partial q_{mi}}\right. \\
&\qquad \left. + c_m^u + c^u\bar{\chi}_m + \frac{\lambda_m^+}{\chi_m} + \frac{(\lambda_m^- - \lambda_m^+)}{\chi_m}F_m(z_m^*) + \frac{\gamma_2^*}{\chi_m}\right] \times [q_{mi} - q_{mi}^*] \\
&+ \sum_{i=1}^{I}\sum_{j=1}^{J}\left[\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + f^{rec} + \gamma_{1i}^* - \left(\lambda_j^- + p_j^* - c_j\right)\right. \\
&\qquad \left. + \left(p_j^* + \lambda_j^- - \lambda_j^+\right)F_j(z_j^*) - r\gamma_2^*\right] \times [q_{ij} - q_{ij}^*] \\
&+ \sum_{j=1}^{J}\left[-y_j'(p_j^*)\left[(p_j^* + \lambda_j^- - \lambda_j^+)\left(1 - e_j(z_j^*) + z_j(F_j(z_j^*) - 1)\right) - \lambda_j^-\right]\right. \\
&\qquad \left. - y_j(p_j^*)(1 - e_j^-(z_j^*))\right] \times [p_j - p_j^*] \\
&+ \sum_{m=1}^{M}\left[y_m(p_m^*) - y_m'(p_m^*)\left(s^{rec} - p_m^* + \lambda_m^-\right.\right. \\
&\qquad \left.\left. - (\lambda_m^- - \lambda_m^+)\left(1 + e_m^-(z_m^*) - z_m^* F_m(z_m^*)\right)\right)\right] \times [p_m - p_m^*] \geq 0 \\
&+ \sum_{i=1}^{I}\left[\beta_i^r\sum_{n=1}^{N} q_{ni}^* + \beta_i^u\sum_{m=1}^{M} q_{mi}^* - \sum_{j=1}^{J} q_{ij}^*\right] \times [\gamma_{1i} - \gamma_{1i}^*] \\
&+ \left[r\sum_{i=1}^{I}\sum_{j=J}^{I} q_{ij}^* - \sum_{m=1}^{M}\sum_{i=1}^{I}\frac{1}{\chi_m}q_{mi}^*\right] \times [\gamma_2 - \gamma_2^*] \geq 0, \ \forall (Q_1, Q_2, Q_3, P_1, P_2, \gamma_1, \gamma_2) \in \Omega,
\end{aligned} \tag{18}$$

where $\Omega = \{(Q_1, Q_2, Q_3, P_1, P_2, \gamma_1, \gamma_2) \in \mathbb{R}_+^{NI+MI+IJ+J+M+I+1} \mid (Q_3, P_1) \in \Gamma^2\}$.

For easy reference in the subsequent sections, variational inequalities (17) and (18) can be rewritten in standard variational inequality form as follows: determine $X^* \in \Omega$, such that

$$\langle \mathcal{F}(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \Omega, \tag{19}$$

where $X \equiv (Q_1, Q_2, Q_3, P_1, P_2, \gamma_1, \gamma_2)$ and $\mathcal{F}(x) \equiv (\mathcal{F}_{ni}, \mathcal{F}_{mi}, \mathcal{F}_{ij}, \mathcal{F}_j, \mathcal{F}_m, \mathcal{F}_i, \mathcal{F}_0)$ (with the specific components of $\mathcal{F}(x)$ being given by the respective functional terms preceding the multiplication signs in (17) or (18)).

4.2 Qualitative Properties

In this section, we provide some qualitative properties of the solution to variational inequality (19). In particular, we derive existence and uniqueness results and investigate properties of the function $\mathcal{F}$ that enters this variational inequality.

Since the feasible set is not compact, we cannot derive existence simply from the assumption of the continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee the existence of a solution.

Let $\Omega_b \equiv \{(Q_1, Q_2, Q_3, P_1, P_2, \gamma_1, \gamma_2) | 0 \leq (Q_1, Q_2, Q_3, P_1, P_2, \gamma_1, \gamma_2) \leq b\}$ where $b = (b_1, b_2, b_3, b_4, b_5, b_6, b_7) \geq 0$ and $Q_1 \leq b_1,\ Q_2 \leq b_2,\ Q_3 \leq b_3,\ P_1 \leq b_4,\ P_2 \leq b_5, \gamma_1 \leq b_6,$ and $\gamma_2 \leq b_7$. Indeed Ω_b is a bounded closed convex subset of $\mathbb{R}_+^{NI+MI+IJ+J+M+I+1}$.

Theorem 7 *(Existence- Additive case). Suppose that there exist positive constants R_1 and S_1 such that*

$$\begin{aligned}
&\frac{\partial f_i^r\left(\beta_i^r, \sum_{n=1}^{N} q_{ni}\right)}{\partial q_{ni}} + \frac{\partial f_n^r\left(\sum_{i=1}^{I} q_{ni}\right)}{\partial q_{ni}} + c^r \bar{\beta}_i^r - \beta_i^r \gamma_{1i} \\
&\qquad + \frac{\partial c_{ni}(q_{ni})}{\partial q_{in}} \geq R_1, \ \forall Q_1 \text{ with } q_{ni} \geq S_1, \ \forall n, \ i. \\
&\frac{\partial f_i^u\left(\beta_i^u, \sum_{m=1}^{M} q_{mi}\right)}{\partial q_{mi}} + c^u \bar{\beta}_i^u - \beta_i^u \gamma_{1i} + \frac{\partial c_{mi}(q_{mi})}{\partial q_{mi}} \\
&\qquad + c_m^u + c^u \bar{\chi}_m + \frac{\lambda_m^+}{\chi_m} + \frac{\left(\lambda_m^- - \lambda_m^+\right)}{\chi_m} F_m(z_m) \\
&\qquad + \frac{\gamma_2}{\chi_m} \geq R_1, \ \forall Q_2 \text{ with } q_{mi} \geq S_1, \ \forall m, \ i. \\
&\frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}} + f^{rec} + \gamma_{1i} - \left(\lambda_j^- + p_j - c_j\right) \\
&\qquad + \left(p_j + \lambda_j^- - \lambda_j^+\right) F_j(z_j) - r\gamma_2 \geq R_1, \forall Q_3 \text{ with } q_{ij} \geq S_1, \ \forall i, \ j. \\
&\qquad - y_j(p_j) + e_j^-(z_j) - y_j'(p_j) \left[(p_j + \lambda_j^- - \lambda_j^+) F_j(z_j) - \lambda_j^-\right] \\
&\qquad \geq R_1, \ \forall P_1 \text{ with } p_j \geq S_1, \ \forall j. \\
&y_m(p_m) - y_m'(p_m) \left(s^{rec} - p_m + \lambda_m^+ + F_m(z_m)(\lambda_m^- - \lambda_m^+)\right) \\
&\qquad \geq R_1, \ \forall P_2 \text{ with } p_m \geq S_1, \ \forall m.
\end{aligned}$$

Then variational inequality (17) *admits at least one solution.*

Proof Following the proof of Proposition 1 in Nagurney and Zhou (1993), it is possible to construct $b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6$ and b_7 large enough so that the variational inequality (17) will satisfy the following boundedness condition:

$$Q_1 \leq b_1,\ Q_2 \leq b_2,\ Q_3 \leq b_3,\ P_1 \leq b_4,\ P_2 \leq b_5,\ \gamma_1 \leq b_6,\ \gamma_2 \leq b_7. \quad (20)$$

Thus, variational inequality (17) admits at least one solution $X_b \in \Omega_b$, from the standard theory of variational inequalities, since Ω_b is compact and the functions are continuous. □

Theorem 8 *(Existence- Multiplicative case). Suppose that there exist positive constants R_2 and S_2 such that*

$$\begin{aligned}
&\frac{\partial f_i^r\left(\beta_i^r, \sum_{n=1}^{N} q_{ni}\right)}{\partial q_{ni}} + \frac{\partial f_n^r\left(\sum_{i=1}^{I} q_{ni}\right)}{\partial q_{ni}} + c^r\bar{\beta}_i^r - \beta_i^r\gamma_{1i} \\
&\quad + \frac{\partial c_{ni}(q_{ni})}{\partial q_{in}} \geq R_2,\ \forall Q_1 \text{ with } q_{ni} \geq S_2,\ \forall n,\ i. \\
&\frac{\partial f_i^u\left(\beta_i^u, \sum_{m=1}^{M} q_{mi}\right)}{\partial q_{mi}} + c^u\bar{\beta}_i^u - \beta_i^u\gamma_{1i} + \frac{\partial c_{mi}(q_{mi})}{\partial q_{mi}} \\
&\quad + c_m^u + c^u\bar{\chi}_m + \frac{\lambda_m^+}{\chi_m} + \frac{\left(p_m + \lambda_m^- - \lambda_m^+\right)}{\chi_m} F_m(z_m) \\
&\quad + \frac{\gamma_2}{\chi_m} \geq R_2,\ \forall Q_2 \text{ with } q_{mi} \geq S_2,\ \forall m,\ i. \\
&\frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}} + f^{rec} + \gamma_{1i} - \left(\lambda_j^- + p_j - c_j\right) \\
&\quad + \left(p_j + \lambda_j^- - \lambda_j^+\right) F_j(z_j) - r\gamma_2 \geq R_2,\ \forall Q_3 \text{ with } q_{ij} \geq S_2,\ \forall i,\ j. \\
&\quad - y_j'(p_j)\left[(p_j + \lambda_j^- - \lambda_j^+)\left(1 - e_j(z_j) + z_j(F_j(z_j) - 1)\right) - \lambda_j^-\right] \\
&\quad - y_j(p_j)(1 - e_j^-(z_j)) \geq R_2,\ \forall P_1 \text{ with } p_j \geq S_2,\ \forall j. \\
&y_m(p_m) - y_m'(p_m)\left(s^{rec} - p_m + \lambda_m^- - (\lambda_m^- - \lambda_m^+)\left(1 + e_m^-(z_m) - z_m F_m(z_m)\right)\right) \geq R_2, \\
&\quad \forall P_2 \text{ with } p_m \geq S_2,\ \forall m.
\end{aligned}$$

Then variational inequality (18) *admits at least one solution.*

Proof Similar to the one used in Theorem 7. □

We now explore additional qualitative properties of the vector function $\mathcal{F}$ that enters the variational inequality problem. In particular, we show that $\mathcal{F}$ is monotone (strictly monotone), which is fundamental in establishing the convergence of the algorithmic scheme used to solve variational inequality (19).

Theorem 9 *(Monotonicity) Assume the following cost functions, f_n^r, f_i^r, f_i^u, c_{ni}, c_{mi},and c_{ij} are convex. Further assume that the conditions in Theorem 1 (additive case) or Theorem 2 (multiplicative case) are satisfied for each $j, j = 1, .., J$ and the conditions in Theorem 3 (additive case) or Theorem 4 (multiplicative case) are satisfied for each $m, m = 1, ..., M$. Then the vector function $\mathcal{F}$ defined in (19) is monotone, that is,*

$$\langle \mathcal{F}(X') - \mathcal{F}(X''), X' - X'' \rangle \geq 0 \qquad \forall X', X'' \in \Omega. \tag{21}$$

Proof See Appendix I. □

Theorem 10 *(Uniqueness) Assume the following cost functions, f_n^r, f_i^r, f_i^u, c_{ni}, c_{mi}, c_{ij} and c_j are strictly convex. Further assume that the conditions in Theorem 1 (additive case) or Theorem 2 (multiplicative case) are satisfied for each $j, j = 1, .., J$ and the conditions in Theorem 3 (additive case) or Theorem 4 (multiplicative case) are satisfied for each $m, m = 1, ..., M$. Then variational inequality* (19) *admits a unique solution.*

Proof Under the strict convexity of the cost functions, f_n^r, f_i^r, f_i^u, c_{ni}, c_{mi},and c_{ij}, we can follow the proof of Theorem 9 to show that $\mathcal{F}(X)$ is strictly monotone. Under the strict monotonicity of $\mathcal{F}(X)$, uniqueness follows from the standard variational inequality theory. □

4.3 Solution Algorithm

The extragradient method of Khobotov (1987) is utilized to compute the solution of variational inequality (19). As discussed by Tinti (2005), the algorithm is guaranteed to converge if the function $\mathcal{F}$ that enters the variational inequality is pseudomonotone (and that a solution exists). Details on the numerical complexity of the extragradient algorithm are outlined in Monteiro and Svaiter (2010), Monteiro and Svaiter (2012) and Huang et al. (2012).

After solving variational inequality (19), we can recover the equilibrium prices p_{ni}^*, p_{mi}^* and p_{ij}^*.

Take the prices p_{ni}^*. Since the objective function (2) is continuously differentiable concave and the feasible set is convex, the Karush-Kuhn-Tucker optimality conditions, which represent optimality conditions for the variational inequality problem (see Facchinei et al. (1999)), take the form:

$$\left[\frac{\partial f_n^r\left(\sum_{i=1}^{I} q_{ni}^*\right)}{\partial q_{ni}} + \frac{\partial c_{ni}(q_{ni}^*)}{\partial q_{ni}} - p_{ni}^*\right] \geq 0,$$

$$\left[\frac{\partial f_n^r\left(\sum_{i=1}^{I} q_{ni}^*\right)}{\partial q_{ni}} + \frac{\partial c_{ni}(q_{ni}^*)}{\partial q_{ni}} - p_{ni}^*\right] q_{ni}^* = 0, \quad \forall i, n.$$

If there is a positive shipment quantity $q_{ni}^* > 0$ then

$$p_{ni}^* = \frac{\partial f_n^r\left(\sum_{i=1}^{I} q_{ni}^*\right)}{\partial q_{in}} + \frac{\partial c_{ni}(q_{ni}^*)}{\partial q_{ni}}. \tag{22}$$

Using the same argument, prices p_{mi}^* and p_{ij}^* are obtained as follows:

$$p_{mi}^* = \beta_i^u \gamma_{1i}^* - \frac{\gamma_2^*}{\chi_m} - \frac{\partial f_i^u\left(\beta_i^u, \sum_{m=1}^{M} q_{mi}^*\right)}{\partial q_{mi}} - \frac{\partial c_{mi}(q_{mi}^*)}{\partial q_{mi}} - c^u \bar{\beta}_i^u, \tag{23}$$

$$p_{ij}^* = \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + f^{rec} + \gamma_{1i}^* - r\gamma_2^*. \tag{24}$$

5 Numerical Examples

To illustrate the effects of randomness on the equilibrium solutions, we apply the extragradient algorithm to several numerical examples (e.g. Nagurney et al. (2002)). The algorithm is implemented in Matlab and has been successfully tested for validity on the numerical examples provided in Tinti (2005). The optimal solution of the deterministic model (Yang et al. 2009) will be used as a starting point in all numerical tests.

5.1 Impact of Model Parameters

Example 1.1 In the first basic example, we consider a closed-loop supply chain network consisting of two raw material suppliers, two manufacturers, two retailers and two recovery centers. The recycling fee, unit of subsidy and the unit cost of landfill are set to 8, 6 and 2, respectively ($f^{rec} = 8$, $s^{rec} = 6$, $c^r = c^u = 2$). Also, we set $\beta_u^i = \chi_m = 0.7$, $\beta_r^1 = 0.6$, $\beta_r^2 = 0.7$, and $r = 1$. The transaction cost functions, procurement cost functions, production cost functions and handling cost functions faced by all supply chain members are given by:

$$
\begin{aligned}
c_{ni}(q_{ni}) &= 0.5(q_{ni})^2 + 3.5(q_{ni}), \quad \forall n = 1,2,\ \forall i = 1,2.\\
c_{ij}(q_{ij}) &= 0.5(q_{ij})^2 + 3.5(q_{ij}), \quad \forall i = 1,2,\ \forall j = 1,2.\\
c_{mi}(q_{mi}) &= 0.5(q_{mi})^2 + 3.5(q_{mi}), \quad \forall m = 1,2,\ \forall i = 1,2.\\
f_n^r\left(\sum_{i=1}^{2} q_{ni}\right) &= 2.5\left(\sum_{i=1}^{2} q_{ni}\right)^2 + \sum_{i=1}^{2} q_{ni} + 2, \quad \forall n = 1,2.\\
f_i^r\left(\beta_i^r, \sum_{n=1}^{2} q_{ni}\right) &= 3\left(\beta_i^r \sum_{n=1}^{2} q_{ni}\right)^2 + \beta_i^r \sum_{n=1}^{2} q_{ni} + 5, \quad \forall i = 1,2.\\
f_i^u\left(\beta_i^r, \sum_{m=1}^{2} q_{mi}\right) &= 1.5\left(\beta_i^r \sum_{n=1}^{2} q_{ni}\right)^2 + 2\left(\beta_i^r \sum_{m=1}^{2} q_{mi}\right) + 2, \quad \forall i = 1,2.\\
c_j\left(\sum_{i=1}^{2} q_{ij}\right) &= 0.5\left(\sum_{i=1}^{2} q_{ij}\right)^2, \quad \forall j = 1,2.
\end{aligned}
$$

The unit penalties of having excess supply/demand of retailers are given by:

$$\lambda_j^+ = 2,\ \lambda_j^- = 2 \quad \forall j = 1,2.$$

The unit recycling cost, unit penalties of having excess supply/demand of recovery centers are given by:

$$c_m^u = 1,\ \lambda_m^+ = 4,\ \lambda_m^- = 100 \quad \forall m = 1,2.$$

The demand functions at retailer outlets are given by:

$$D_1(p_1, \epsilon_1) = \begin{cases} 290 - p_1 + \epsilon_1 & \text{additive model} \\ (290 - p_1)\epsilon_1 & \text{multiplicative model,} \end{cases}$$

$$D_2(p_2, \epsilon_2) = \begin{cases} 300 - p_2 + \epsilon_2 & \text{additive model} \\ (300 - p_2)\epsilon_2 & \text{multiplicative model,} \end{cases}$$

where, in this subsection, ϵ_j is uniformly distributed in $[-4, 4]$ (additive model) and uniformly distributed in $[0.8, 1.2]$ (multiplicative model), $j = 1, 2$.

Finally, the return functions associated to recovery centers are given by:

$$R_1(p_1, \epsilon_1) = \begin{cases} -35 + p_1 + \epsilon_1 & \text{additive model} \\ (-35 + p_1)\epsilon_1 & \text{multiplicative model,} \end{cases}$$

$$R_2(p_2, \epsilon_2) = \begin{cases} -40 + p_2 + \epsilon_2 & \text{additive model} \\ (-40 + p_2)\epsilon_2 & \text{multiplicative model,} \end{cases}$$

where, in this subsection, ϵ_m is uniformly distributed in $[-4, 4]$ (additive model) and uniformly distributed in $[0.8, 1.2]$ (multiplicative model), $m = 1, 2$.

Table 1 displays the optimal equilibrium solutions, the total revenue of all supply chain members, the number of iterations required for the convergence of the extra-gradient algorithm, and the CPU time (on a Dell Laptop with Intel Core i5 @2.4 GHz) for both deterministic (model of Yang et al. (2009)) and random (our model)

Table 1 Deterministic vs. Random solutions

Variable	Deterministic	Random (additive) $\mathcal{U}[-4, 4]$	Random (additive) Weibull[2,2]	Random (additive) $\mathcal{N}(0, 4)$	Random (multiplicative) $\mathcal{U}[0.8, 1.2]$
$(q^*_{ni})_{n=1,2;i=1,2}$	$\begin{pmatrix} 7.65 & 8.92 \\ 7.65 & 8.92 \end{pmatrix}$	$\begin{pmatrix} 7.64 & 8.91 \\ 7.64 & 8.91 \end{pmatrix}$	$\begin{pmatrix} 7.64 & 8.91 \\ 7.64 & 8.91 \end{pmatrix}$	$\begin{pmatrix} 7.63 & 8.89 \\ 7.63 & 8.89 \end{pmatrix}$	$\begin{pmatrix} 7.36 & 8.59 \\ 7.36 & 8.59 \end{pmatrix}$
$(q^*_{mi})_{m=1,2;i=1,2}$	$\begin{pmatrix} 7.96 & 7.70 \\ 7.18 & 6.92 \end{pmatrix}$	$\begin{pmatrix} 7.95 & 7.69 \\ 7.18 & 6.92 \end{pmatrix}$	$\begin{pmatrix} 7.95 & 7.69 \\ 7.18 & 6.92 \end{pmatrix}$	$\begin{pmatrix} 7.95 & 7.68 \\ 7.18 & 6.92 \end{pmatrix}$	$\begin{pmatrix} 7.68 & 7.43 \\ 6.86 & 6.61 \end{pmatrix}$
$\left(q^*_{ij}\right)_{i=1,2;j=1,2}$	$\begin{pmatrix} 9.17 & 10.60 \\ 10.64 & 12.07 \end{pmatrix}$	$\begin{pmatrix} 9.17 & 10.61 \\ 10.64 & 12.07 \end{pmatrix}$	$\begin{pmatrix} 9.17 & 10.60 \\ 10.64 & 12.07 \end{pmatrix}$	$\begin{pmatrix} 9.15 & 10.60 \\ 10.62 & 12.06 \end{pmatrix}$	$\begin{pmatrix} 8.85 & 10.15 \\ 10.26 & 11.57 \end{pmatrix}$
$\left(z^*_j\right)_{j=1,2}$	$\begin{pmatrix} - \\ - \end{pmatrix}$	$\begin{pmatrix} -3.35 \\ -3.28 \end{pmatrix}$	$\begin{pmatrix} -1.19 \\ -1.16 \end{pmatrix}$	$\begin{pmatrix} -2.79 \\ -2.69 \end{pmatrix}$	$\begin{pmatrix} 0.84 \\ 0.84 \end{pmatrix}$
$(z^*_m)_{m=1,2}$	$\begin{pmatrix} - \\ - \end{pmatrix}$	$\begin{pmatrix} 1.55 \\ 1.59 \end{pmatrix}$	$\begin{pmatrix} 0.47 \\ 0.49 \end{pmatrix}$	$\begin{pmatrix} 1.07 \\ 1.10 \end{pmatrix}$	$\begin{pmatrix} 1.09 \\ 1.09 \end{pmatrix}$
$(p^*_{ni})_{n=1,2;i=1,2}$	$\begin{pmatrix} 95.00 & 96.27 \\ 95.00 & 96.27 \end{pmatrix}$	$\begin{pmatrix} 94.89 & 96.15 \\ 94.89 & 96.15 \end{pmatrix}$	$\begin{pmatrix} 94.87 & 96.14 \\ 94.87 & 96.14 \end{pmatrix}$	$\begin{pmatrix} 94.71 & 95.97 \\ 94.71 & 95.97 \end{pmatrix}$	$\begin{pmatrix} 91.60 & 92.82 \\ 91.60 & 92.82 \end{pmatrix}$
$(p^*_{mi})_{m=1,2;i=1,2}$	$\begin{pmatrix} 106.91 & 106.91 \\ 107.69 & 107.69 \end{pmatrix}$	$\begin{pmatrix} 102.43 & 102.43 \\ 103.20 & 103.20 \end{pmatrix}$	$\begin{pmatrix} 105.57 & 105.57 \\ 106.34 & 106.34 \end{pmatrix}$	$\begin{pmatrix} 103.76 & 103.76 \\ 104.53 & 104.53 \end{pmatrix}$	$\begin{pmatrix} 105.38 & 105.38 \\ 106.20 & 106.20 \end{pmatrix}$
$\left(p^*_{ij}\right)_{i=1,2;j=1,2}$	$\begin{pmatrix} 230.55 & 231.98 \\ 230.55 & 231.98 \end{pmatrix}$	$\begin{pmatrix} 227.27 & 228.71 \\ 227.27 & 228.71 \end{pmatrix}$	$\begin{pmatrix} 229.44 & 230.88 \\ 229.44 & 230.88 \end{pmatrix}$	$\begin{pmatrix} 227.95 & 229.40 \\ 227.95 & 229.40 \end{pmatrix}$	$\begin{pmatrix} 224.38 & 225.68 \\ 224.38 & 225.68 \end{pmatrix}$
$\left(p^*_j\right)_{j=1,2}$	$\begin{pmatrix} 270.18 \\ 277.33 \end{pmatrix}$	$\begin{pmatrix} 266.85 \\ 274.04 \end{pmatrix}$	$\begin{pmatrix} 269.00 \\ 276.17 \end{pmatrix}$	$\begin{pmatrix} 267.43 \\ 274.65 \end{pmatrix}$	$\begin{pmatrix} 267.21 \\ 274.21 \end{pmatrix}$
$(p^*_m)_{m=1,2}$	$\begin{pmatrix} 57.36 \\ 60.13 \end{pmatrix}$	$\begin{pmatrix} 55.79 \\ 58.56 \end{pmatrix}$	$\begin{pmatrix} 56.87 \\ 59.64 \end{pmatrix}$	$\begin{pmatrix} 56.26 \\ 59.03 \end{pmatrix}$	$\begin{pmatrix} 54.87 \\ 57.68 \end{pmatrix}$
$(\Pi^*_n)_{n=1,2}$	$\begin{pmatrix} 753.48 \\ 753.48 \end{pmatrix}$	$\begin{pmatrix} 751.56 \\ 751.56 \end{pmatrix}$	$\begin{pmatrix} 751.31 \\ 751.31 \end{pmatrix}$	$\begin{pmatrix} 748.58 \\ 748.58 \end{pmatrix}$	$\begin{pmatrix} 697.74 \\ 697.74 \end{pmatrix}$
$(\Pi^*_i)_{i=1,2}$	$\begin{pmatrix} 581.10 \\ 790.01 \end{pmatrix}$	$\begin{pmatrix} 588.29 \\ 785.57 \end{pmatrix}$	$\begin{pmatrix} 584.65 \\ 789.68 \end{pmatrix}$	$\begin{pmatrix} 585.47 \\ 786.27 \end{pmatrix}$	$\begin{pmatrix} 530.25 \\ 730.30 \end{pmatrix}$
$\left(\Pi^*_j\right)_{j=1,2}$	$\begin{pmatrix} 589.14 \\ 771.24 \end{pmatrix}$	$\begin{pmatrix} 573.96 \\ 755.53 \end{pmatrix}$	$\begin{pmatrix} 580.70 \\ 762.37 \end{pmatrix}$	$\begin{pmatrix} 559.80 \\ 740.56 \end{pmatrix}$	$\begin{pmatrix} 617.30 \\ 794.36 \end{pmatrix}$
$(\Pi^*_m)_{m=1,2}$	$\begin{pmatrix} 499.96 \\ 405.32 \end{pmatrix}$	$\begin{pmatrix} 350.70 \\ 263.70 \end{pmatrix}$	$\begin{pmatrix} 447.16 \\ 354.80 \end{pmatrix}$	$\begin{pmatrix} 385.35 \\ 296.08 \end{pmatrix}$	$\begin{pmatrix} 394.72 \\ 312.53 \end{pmatrix}$
Total revenue	5143.73	4820.86	5022.00	4850.69	4774.94
Nb. of iterations	1929	4449	1996	1799	10511
CPU (sec)	9.42	27.39	13.24	12.57	64.90

cases. It is easy to show that the optimality/equilibrium conditions are satisfied with good accuracy. In contrast with the deterministic case, incorporating randomness in the model induces retailers to decrease their quantity shipments (q^*_{ij}) implying shortage at each retail outlet ($z_j < 0$ in the additive case and $z_j < 1$ in the multiplicative case). Also, recovery centers are facing shortage ($z_m > 0$ in the additive case and $z_m > 1$ in the multiplicative case) based on the current shortage and salvage values. The optimal prices, the optimal quantities and the total revenue have all decreased due to uncertainty.

We have also tested the impact of demand and return distributions on the equilibrium solutions. We keep the same data as in Example 1.1 and for the additive model we tried different demands and return distributions. Table 1 displays the results for the uniform distribution $\mathcal{U}[-4, 4]$ with variance $\sigma^2 = 64/12 \approx 5.33$, the Weibull distribution (shifted to have a mean of 0) with parameters $\lambda = 2$, $k = 2$ and variance $\sigma^2 = 4 - \pi \approx 0.85$ and the normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 4$. Comparing the expected total profits of retailers and recovery centers (Π_j^*) and (Π_m^*), we observe higher total revenue in the Weibull distribution case and lower revenue in the uniform distribution case. This is due to the impact of the variance on the expected profits. The more variability in the distribution, the lower expected profits will be due to the increase in uncertainty.

In all future examples, all results reported correspond to the variational inequality (17) involving additive demand and return functions. We have tested the same examples using multiplicative demand and return functions and obtained comparable results.

Example 1.2 (*Variant 1 of Example 1.1*) In Example 1.2, we increase the a_js associated with both retailers by 50 but keep the remainder of the data as in Example 1.1. This implies that the demand associated with each retailer outlet increases.

The extragradient method requires 10729 iterations for convergence and yields the new equilibrium pattern shown in the third column of Table 2. Observe that with a higher a_j for each retailer, all quantity shipments increase since the demand increase at each retailer outlet and the optimal prices and expected profits also increase.

To study the effect of changing the parameter a_j on the optimal prices p_j^* and quantities q_{ij}^*, analytically, one notice that $\frac{\partial^2 \Pi_j}{\partial p_j \partial a_j} = F_j(z_j) + b_j(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j) \geq 0$ and $\frac{\partial^2 \Pi_j}{\partial s_j \partial a_j} = (p_j + \lambda_j^- - \lambda_j^+) f_j(z_j) \geq 0$. Therefore, the optimal price p_j^* and optimal quantity s_j^* both increase with a_j.

Example 1.3 (*Variant 2 of Example 1.1*) To construct Example 1.3, we keep the data as in Example 1.1, but now we decrease the a_ms associated with both recovery centers by 5. Hence, the return amount associated to each recovery center decreases.

The extragradient algorithm converges in 9119 iterations and yields the new equilibrium pattern shown in the fourth column of Table 2. When the return amount associated to each recovery center decreases, buy-back prices p_m increase and the reusable material shipments, q_{mi}^*, as well as the expected profits, π_m^*, decline due to the decrease of the return amount associated to each recovery center.

To study the effect of changing the parameter a_m on the optimal prices p_m^* and quantities q_{mi}^*, analytically, one notice that $\frac{\partial^2 \Pi_m}{\partial p_m \partial a_m} = 1 + b_m(\lambda_m^- - \lambda_m^+) f_m(z_m) \geq 0$ and $\frac{\partial^2 \Pi_m}{\partial q_m \partial a_m} = -(\lambda_m^- - \lambda_m^+) f_m(z_m)/\chi_m \leq 0$. Therefore, the optimal price p_m^* increases and optimal quantity q_m^* decreases with a_m.

Example 1.4 (*Variant 3 of Example 1.1*) In Example 1.4, we keep the same data as in Example 1.1 with the following change: we increase the weight associated with

Table 2 Impact of changing the model parameters

Variable	Example 1.1	Example 1.2	Example 1.3	Example 1.4	Example 1.5	Example 1.6
$(q^*_{ni})_{n=1,2;i=1,2}$	$\begin{pmatrix} 7.64 & 8.91 \\ 7.64 & 8.91 \end{pmatrix}$	$\begin{pmatrix} 9.28 & 10.78 \\ 9.28 & 10.78 \end{pmatrix}$	$\begin{pmatrix} 7.51 & 8.76 \\ 7.51 & 8.76 \end{pmatrix}$	$\begin{pmatrix} 7.78 & 9.07 \\ 7.78 & 9.07 \end{pmatrix}$	$\begin{pmatrix} 7.46 & 8.70 \\ 7.46 & 8.70 \end{pmatrix}$	$\begin{pmatrix} 6.37 & 6.80 \\ 6.37 & 6.80 \\ 6.37 & 6.80 \end{pmatrix}$
$(q^*_{mi})_{m=1,2;i=1,2}$	$\begin{pmatrix} 7.95 & 7.69 \\ 7.18 & 6.92 \end{pmatrix}$	$\begin{pmatrix} 9.55 & 9.24 \\ 8.78 & 8.47 \end{pmatrix}$	$\begin{pmatrix} 7.80 & 7.54 \\ 7.03 & 6.78 \end{pmatrix}$	$\begin{pmatrix} 8.10 & 7.83 \\ 7.33 & 7.07 \end{pmatrix}$	$\begin{pmatrix} 7.74 & 7.49 \\ 6.97 & 6.71 \end{pmatrix}$	$\begin{pmatrix} 8.04 & 7.82 \\ 7.27 & 7.05 \end{pmatrix}$
$\left(q^*_{ij}\right)_{i=1,2;j=1,2}$	$\begin{pmatrix} 9.17 & 10.61 \\ 10.64 & 12.07 \end{pmatrix}$	$\begin{pmatrix} 11.27 & 12.71 \\ 13.03 & 14.47 \end{pmatrix}$	$\begin{pmatrix} 8.97 & 10.41 \\ 10.42 & 11.86 \end{pmatrix}$	$\begin{pmatrix} 9.37 & 10.80 \\ 10.86 & 12.29 \end{pmatrix}$	$\begin{pmatrix} 8.89 & 10.33 \\ 10.33 & 11.77 \end{pmatrix}$	$\begin{pmatrix} 10.37 & 11.81 \\ 11.62 & 13.06 \end{pmatrix}$
$\left(z^*_j\right)_{j=1,2}$	$\begin{pmatrix} -3.35 \\ -3.28 \end{pmatrix}$	$\begin{pmatrix} -3.33 \\ -3.27 \end{pmatrix}$	$\begin{pmatrix} -3.36 \\ -3.29 \end{pmatrix}$	$\begin{pmatrix} 0.14 \\ 0.13 \end{pmatrix}$	$\begin{pmatrix} -3.37 \\ -3.30 \end{pmatrix}$	$\begin{pmatrix} -3.28 \\ -3.21 \end{pmatrix}$
$(z^*_m)_{m=1,2}$	$\begin{pmatrix} 1.55 \\ 1.59 \end{pmatrix}$	$\begin{pmatrix} 2.19 \\ 2.24 \end{pmatrix}$	$\begin{pmatrix} 1.85 \\ 1.89 \end{pmatrix}$	$\begin{pmatrix} 1.61 \\ 1.65 \end{pmatrix}$	$\begin{pmatrix} -1.58 \\ -1.56 \end{pmatrix}$	$\begin{pmatrix} 1.59 \\ 1.64 \end{pmatrix}$
$(p^*_{ni})_{n=1,2;i=1,2}$	$\begin{pmatrix} 94.89 & 96.15 \\ 94.89 & 96.15 \end{pmatrix}$	$\begin{pmatrix} 114.08 & 115.58 \\ 114.08 & 115.58 \end{pmatrix}$	$\begin{pmatrix} 93.40 & 94.64 \\ 93.40 & 94.64 \end{pmatrix}$	$\begin{pmatrix} 96.53 & 97.82 \\ 96.53 & 97.82 \end{pmatrix}$	$\begin{pmatrix} 92.74 & 93.98 \\ 92.74 & 93.98 \end{pmatrix}$	$\begin{pmatrix} 76.70 & 77.13 \\ 76.70 & 77.13 \\ 76.70 & 77.13 \end{pmatrix}$
$(p^*_{mi})_{m=1,2;i=1,2}$	$\begin{pmatrix} 102.43 & 102.43 \\ 103.20 & 103.20 \end{pmatrix}$	$\begin{pmatrix} 113.43 & 113.43 \\ 114.20 & 114.20 \end{pmatrix}$	$\begin{pmatrix} 107.53 & 107.53 \\ 108.29 & 108.29 \end{pmatrix}$	$\begin{pmatrix} 103.46 & 103.46 \\ 104.23 & 104.23 \end{pmatrix}$	$\begin{pmatrix} 109.70 & 109.70 \\ 110.47 & 110.47 \end{pmatrix}$	$\begin{pmatrix} 103.20 & 103.20 \\ 103.97 & 103.97 \end{pmatrix}$
$\left(p^*_{ij}\right)_{i=1,2;j=1,2}$	$\begin{pmatrix} 227.27 & 228.71 \\ 227.27 & 228.71 \end{pmatrix}$	$\begin{pmatrix} 263.81 & 265.24 \\ 263.81 & 265.24 \end{pmatrix}$	$\begin{pmatrix} 228.49 & 229.93 \\ 228.49 & 229.93 \end{pmatrix}$	$\begin{pmatrix} 230.52 & 231.94 \\ 230.52 & 231.94 \end{pmatrix}$	$\begin{pmatrix} 229.00 & 230.44 \\ 229.00 & 230.44 \end{pmatrix}$	$\begin{pmatrix} 220.80 & 222.24 \\ 220.80 & 222.24 \end{pmatrix}$
$\left(p^*_j\right)_{j=1,2}$	$\begin{pmatrix} 266.85 \\ 274.04 \end{pmatrix}$	$\begin{pmatrix} 312.37 \\ 319.56 \end{pmatrix}$	$\begin{pmatrix} 267.25 \\ 274.44 \end{pmatrix}$	$\begin{pmatrix} 269.91 \\ 277.04 \end{pmatrix}$	$\begin{pmatrix} 267.41 \\ 274.60 \end{pmatrix}$	$\begin{pmatrix} 264.74 \\ 271.93 \end{pmatrix}$
$(p^*_m)_{m=1,2}$	$\begin{pmatrix} 55.79 \\ 58.56 \end{pmatrix}$	$\begin{pmatrix} 59.64 \\ 62.41 \end{pmatrix}$	$\begin{pmatrix} 60.07 \\ 62.84 \end{pmatrix}$	$\begin{pmatrix} 56.15 \\ 58.92 \end{pmatrix}$	$\begin{pmatrix} 58.33 \\ 61.10 \end{pmatrix}$	$\begin{pmatrix} 56.06 \\ 58.83 \end{pmatrix}$
$(\Pi^*_n)_{n=1,2}$	$\begin{pmatrix} 751.56 \\ 751.56 \end{pmatrix}$	$\begin{pmatrix} 1105.22 \\ 1105.22 \end{pmatrix}$	$\begin{pmatrix} 726.92 \\ 726.92 \end{pmatrix}$	$\begin{pmatrix} 779.23 \\ 779.23 \end{pmatrix}$	$\begin{pmatrix} 716.26 \\ 716.26 \end{pmatrix}$	$\begin{pmatrix} 474.71 \\ 474.71 \\ 474.71 \end{pmatrix}$
$(\Pi^*_i)_{i=1,2}$	$\begin{pmatrix} 588.29 \\ 785.57 \end{pmatrix}$	$\begin{pmatrix} 883.13 \\ 1136.95 \end{pmatrix}$	$\begin{pmatrix} 552.44 \\ 760.43 \end{pmatrix}$	$\begin{pmatrix} 615.08 \\ 817.32 \end{pmatrix}$	$\begin{pmatrix} 538.09 \\ 750.52 \end{pmatrix}$	$\begin{pmatrix} 739.45 \\ 972.68 \end{pmatrix}$
$\left(\Pi^*_j\right)_{j=1,2}$	$\begin{pmatrix} 573.96 \\ 755.53 \end{pmatrix}$	$\begin{pmatrix} 869.42 \\ 1089.48 \end{pmatrix}$	$\begin{pmatrix} 550.05 \\ 728.11 \end{pmatrix}$	$\begin{pmatrix} 70.35 \\ 245.36 \end{pmatrix}$	$\begin{pmatrix} 540.18 \\ 716.77 \end{pmatrix}$	$\begin{pmatrix} 709.10 \\ 909.28 \end{pmatrix}$
$(\Pi^*_m)_{m=1,2}$	$\begin{pmatrix} 350.70 \\ 263.70 \end{pmatrix}$	$\begin{pmatrix} 540.02 \\ 436.19 \end{pmatrix}$	$\begin{pmatrix} 327.41 \\ 243.78 \end{pmatrix}$	$\begin{pmatrix} 366.93 \\ 278.36 \end{pmatrix}$	$\begin{pmatrix} 333.46 \\ 233.55 \end{pmatrix}$	$\begin{pmatrix} 362.76 \\ 274.59 \end{pmatrix}$
Total revenue	4820.86	7165.64	4616.05	3951.86	4545.08	5391.98
Nb. of iterations	4449	10729	9119	13301	14516	2926
CPU (sec)	27.39	64.76	54.99	79.89	86.92	19.08

undersupply at all retail outlets from 2 to 250. Also, we set the weights associated with oversupply at all retail outlets to 0. Hence, we now have that $\lambda_j^+ = 0$, $\lambda_j^- = 250$ for $j = 1, 2$.

The extragradient method for this example requires 13301 iterations for convergence and yields the equilibrium pattern shown in the fifth column of Table 2. When the penalty associated with shortage increases and there is no salvage value on oversupply by each retailer, we obtain positive values of z_j implying oversupply at each retail outlet. The total expected profit of each retailer has significantly decreased due to the high values of the shortage penalties.

To study the effect of changing the parameter λ_j^- on the optimal prices p_j^* and quantities q_{ij}^*, analytically, one notice that $\frac{\partial^2 \Pi_j}{\partial p_j \partial \lambda_j^-} = b_j(1 - F_j(z_j)) \geq 0$ and

$\frac{\partial^2 \Pi_j}{\partial s_j \partial \lambda_j^-} = 1 - F_j(z_j) \geq 0$. Therefore, the optimal price p_j^* and optimal quantity s_j^* both increase with λ_j^-. The same analysis applies to λ_j^+.

Example 1.5 (*Variant 4 of Example 1.1*) The fifth numerical example is constructed from the first example with the data retained but with following change: we increase the weight associated with undersupply at all recovery centers from 100 to 250. Also, we set the weights associated with oversupply at all recovery centers to 0. Hence, we now have that $\lambda_m^+ = 0$, $\lambda_m^- = 250$ for $m = 1, 2$.

The extragradient method for this example requires 14516 iterations for convergence and yields the equilibrium pattern shown in the sixth column of Table 2. When the penalty associated with shortage increases and there is no salvage value on oversupply by each recovery center, we obtain negative values of z_m implying oversupply at each recovery center. Note that the total expected profit of each recovery center has decreased due to the high values of the shortage penalties.

To study the effect of changing the parameter λ_m^- on the optimal prices p_m^* and quantities q_{mi}^*, analytically, one notice that $\frac{\partial^2 \Pi_m}{\partial p_m \partial \lambda_m^-} = b_m F_m(z_m) \geq 0$ and $\frac{\partial^2 \Pi_m}{\partial q_m \partial \lambda_m^-} = -F_m(z_m)/\chi_m \leq 0$. Therefore, the optimal price p_m^* increases and optimal quantity q_m^* decreases with λ_m^-. The same analysis applies to λ_m^+.

Example 1.6 The sixth numerical example consists of three raw material suppliers, two manufacturers, two retailers and two recovery centers. we retain the same functions and parameters as in Example 1.1 but now we add data for the third raw material supplier. In particular, we assume that the procurement costs and the transaction costs associated with the new supplier are of the same form as given above for the other raw material suppliers.

The extragradient algorithm converges in 2926 iterations and yields the new equilibrium pattern displayed in the seventh column of Table 2. Note that, in comparison to the results in Example 1, with the addition of a new raw material supplier, shipment quantities, q_{ni}^*, prices, p_{ni}^*, and expected profits, π_n^*, are now lower due to the competition.

5.2 Impact of Demand Functions

In this subsection, we test the impact of demand functions on the equilibrium solutions. We keep the same data as in Example 1.1. For the demand functions, we consider three examples. Example 3.1 corresponds to the linear demand case: $y_1(p_1) = 290 - p_1$; $y_2(p_2) = 300 - p_2$. Example 3.2 illustrates the logarithmic demand case: $y_1(p_1) = 290 - 50\ln(p_1 + 1)$; $y_2(p_2) = 300 - 50\ln(p_2 + 1)$. Example 3.3 corresponds to the power demand function: $y_1(p_1) = 290 - 17p_1^{0.5}$; $y_2(p_2) = 300 - 17p_2^{0.5}$. Table 3 displays the optimal equilibrium solutions, the total revenue of all supply chain members, the number of iterations required for the convergence of the extragradient algorithm, and the CPU time for these three examples. Comparing

Table 3 Impact of demand and return functions

Variable	Example 3.1 Linear Demand	Example 3.2 Logarithmic Demand	Example 3.3 Power Demand	Example 4.1 Linear Return	Example 4.2 Logarithmic Return	Example 4.3 Power Return
$(q^*_{ni})_{n=1,2;i=1,2}$	$\begin{pmatrix} 7.64 & 8.91 \\ 7.64 & 8.91 \end{pmatrix}$	$\begin{pmatrix} 5.42 & 6.36 \\ 5.42 & 6.36 \end{pmatrix}$	$\begin{pmatrix} 6.66 & 7.78 \\ 6.66 & 7.78 \end{pmatrix}$	$\begin{pmatrix} 7.64 & 8.91 \\ 7.64 & 8.91 \end{pmatrix}$	$\begin{pmatrix} 7.62 & 8.88 \\ 7.62 & 8.88 \end{pmatrix}$	$\begin{pmatrix} 8.29 & 9.65 \\ 8.29 & 9.65 \end{pmatrix}$
$(q^*_{mi})_{m=1,2;i=1,2}$	$\begin{pmatrix} 7.95 & 7.69 \\ 7.18 & 6.92 \end{pmatrix}$	$\begin{pmatrix} 5.28 & 5.09 \\ 4.52 & 4.33 \end{pmatrix}$	$\begin{pmatrix} 6.97 & 6.74 \\ 6.20 & 5.97 \end{pmatrix}$	$\begin{pmatrix} 7.95 & 7.69 \\ 7.18 & 6.92 \end{pmatrix}$	$\begin{pmatrix} 8.11 & 7.85 \\ 6.92 & 6.66 \end{pmatrix}$	$\begin{pmatrix} 8.44 & 8.16 \\ 7.96 & 7.68 \end{pmatrix}$
$\left(q^*_{ij}\right)_{i=1,2;j=1,2}$	$\begin{pmatrix} 9.17 & 10.61 \\ 10.64 & 12.07 \end{pmatrix}$	$\begin{pmatrix} 5.71 & 7.65 \\ 6.78 & 8.72 \end{pmatrix}$	$\begin{pmatrix} 7.72 & 9.47 \\ 9.01 & 10.76 \end{pmatrix}$	$\begin{pmatrix} 9.17 & 10.61 \\ 10.64 & 12.07 \end{pmatrix}$	$\begin{pmatrix} 9.11 & 10.55 \\ 10.57 & 12.01 \end{pmatrix}$	$\begin{pmatrix} 10.00 & 11.44 \\ 11.58 & 13.02 \end{pmatrix}$
$\left(z^*_j\right)_{j=1,2}$	$\begin{pmatrix} -3.35 \\ -3.28 \end{pmatrix}$	$\begin{pmatrix} -1.97 \\ -1.38 \end{pmatrix}$	$\begin{pmatrix} -2.95 \\ -2.78 \end{pmatrix}$	$\begin{pmatrix} -3.35 \\ -3.28 \end{pmatrix}$	$\begin{pmatrix} -3.35 \\ -3.28 \end{pmatrix}$	$\begin{pmatrix} -3.29 \\ -3.22 \end{pmatrix}$
$(z^*_m)_{m=1,2}$	$\begin{pmatrix} 1.55 \\ 1.59 \end{pmatrix}$	$\begin{pmatrix} 0.48 \\ 0.52 \end{pmatrix}$	$\begin{pmatrix} 1.15 \\ 1.20 \end{pmatrix}$	$\begin{pmatrix} 1.55 \\ 1.59 \end{pmatrix}$	$\begin{pmatrix} 1.61 \\ 1.68 \end{pmatrix}$	$\begin{pmatrix} 0.16 \\ 0.19 \end{pmatrix}$
$(p^*_{ni})_{n=1,2;i=1,2}$	$\begin{pmatrix} 94.89 & 96.15 \\ 94.89 & 96.15 \end{pmatrix}$	$\begin{pmatrix} 68.82 & 69.77 \\ 68.82 & 69.77 \end{pmatrix}$	$\begin{pmatrix} 83.32 & 84.45 \\ 83.32 & 84.45 \end{pmatrix}$	$\begin{pmatrix} 94.89 & 96.15 \\ 94.89 & 96.15 \end{pmatrix}$	$\begin{pmatrix} 94.60 & 95.87 \\ 94.60 & 95.87 \end{pmatrix}$	$\begin{pmatrix} 102.51 & 103.86 \\ 102.51 & 103.86 \end{pmatrix}$
$(p^*_{mi})_{m=1,2;i=1,2}$	$\begin{pmatrix} 102.43 & 102.43 \\ 103.20 & 103.20 \end{pmatrix}$	$\begin{pmatrix} 84.03 & 84.03 \\ 84.80 & 84.80 \end{pmatrix}$	$\begin{pmatrix} 95.66 & 95.66 \\ 96.42 & 96.42 \end{pmatrix}$	$\begin{pmatrix} 102.43 & 102.43 \\ 103.20 & 103.20 \end{pmatrix}$	$\begin{pmatrix} 103.51 & 103.51 \\ 104.70 & 104.70 \end{pmatrix}$	$\begin{pmatrix} 78.69 & 78.69 \\ 79.18 & 79.18 \end{pmatrix}$
$\left(p^*_{ij}\right)_{i=1,2;j=1,2}$	$\begin{pmatrix} 227.27 & 228.71 \\ 227.27 & 228.71 \end{pmatrix}$	$\begin{pmatrix} 173.26 & 175.20 \\ 173.26 & 175.20 \end{pmatrix}$	$\begin{pmatrix} 204.91 & 206.66 \\ 204.91 & 206.66 \end{pmatrix}$	$\begin{pmatrix} 227.27 & 228.71 \\ 227.27 & 228.71 \end{pmatrix}$	$\begin{pmatrix} 227.64 & 229.08 \\ 227.64 & 229.08 \end{pmatrix}$	$\begin{pmatrix} 221.99 & 223.43 \\ 221.99 & 223.43 \end{pmatrix}$
$\left(p^*_j\right)_{j=1,2}$	$\begin{pmatrix} 266.85 \\ 274.04 \end{pmatrix}$	$\begin{pmatrix} 246.33 \\ 281.85 \end{pmatrix}$	$\begin{pmatrix} 252.83 \\ 265.49 \end{pmatrix}$	$\begin{pmatrix} 266.85 \\ 274.04 \end{pmatrix}$	$\begin{pmatrix} 266.97 \\ 274.16 \end{pmatrix}$	$\begin{pmatrix} 265.13 \\ 272.31 \end{pmatrix}$
$(p^*_m)_{m=1,2}$	$\begin{pmatrix} 55.79 \\ 58.56 \end{pmatrix}$	$\begin{pmatrix} 49.35 \\ 52.12 \end{pmatrix}$	$\begin{pmatrix} 53.42 \\ 56.19 \end{pmatrix}$	$\begin{pmatrix} 55.79 \\ 58.56 \end{pmatrix}$	$\begin{pmatrix} 54.01 \\ 57.45 \end{pmatrix}$	$\begin{pmatrix} 47.04 \\ 49.88 \end{pmatrix}$
$(\Pi^*_n)_{n=1,2}$	$\begin{pmatrix} 751.56 \\ 751.56 \end{pmatrix}$	$\begin{pmatrix} 379.88 \\ 379.88 \end{pmatrix}$	$\begin{pmatrix} 571.22 \\ 571.22 \end{pmatrix}$	$\begin{pmatrix} 751.56 \\ 751.56 \end{pmatrix}$	$\begin{pmatrix} 746.83 \\ 746.83 \end{pmatrix}$	$\begin{pmatrix} 883.81 \\ 883.81 \end{pmatrix}$
$(\Pi^*_i)_{i=1,2}$	$\begin{pmatrix} 588.29 \\ 785.57 \end{pmatrix}$	$\begin{pmatrix} 260.33 \\ 379.84 \end{pmatrix}$	$\begin{pmatrix} 433.58 \\ 599.06 \end{pmatrix}$	$\begin{pmatrix} 588.29 \\ 785.57 \end{pmatrix}$	$\begin{pmatrix} 575.00 \\ 775.02 \end{pmatrix}$	$\begin{pmatrix} 737.12 \\ 877.65 \end{pmatrix}$
$\left(\Pi^*_j\right)_{j=1,2}$	$\begin{pmatrix} 573.96 \\ 755.53 \end{pmatrix}$	$\begin{pmatrix} 767.54 \\ 1488.64 \end{pmatrix}$	$\begin{pmatrix} 638.71 \\ 955.09 \end{pmatrix}$	$\begin{pmatrix} 573.96 \\ 755.53 \end{pmatrix}$	$\begin{pmatrix} 566.55 \\ 747.03 \end{pmatrix}$	$\begin{pmatrix} 683.19 \\ 879.95 \end{pmatrix}$
$(\Pi^*_m)_{m=1,2}$	$\begin{pmatrix} 350.70 \\ 263.70 \end{pmatrix}$	$\begin{pmatrix} 111.34 \\ 52.52 \end{pmatrix}$	$\begin{pmatrix} 251.31 \\ 174.69 \end{pmatrix}$	$\begin{pmatrix} 350.70 \\ 263.70 \end{pmatrix}$	$\begin{pmatrix} 414.12 \\ 288.34 \end{pmatrix}$	$\begin{pmatrix} 208.65 \\ 135.00 \end{pmatrix}$
Total revenue	4820.86	3819.97	4194.88	4820.86	4859.72	5289.17
Nb. of iterations	4449	7353	11052	4449	2059	9656
CPU (sec)	27.39	46.15	66.08	27.39	12.78	58.56

the quantities q^*_{ij}, we observe lower values in the logarithmic demand case and higher values in the linear demand case. This is due to the impact of the rate of decrease of the expected demand $y_j(p_j)$ with respect to the retailer price p^*_j. The faster demand decreases with respect to the retailer price p^*_j, the lower quantities q^*_{ij} retailers will order from manufacturers. A decrease in q^*_{ij} will result in a decrease in the quantities q^*_{mi} and q^*_{ni}, a decrease in the prices p^*_{mi}, p^*_{ni}, p^*_{ij} and p^*_m as well as a decrease in the expected profits Π^*_n, Π^*_i and Π^*_m and the total revenue. Note that the expected profits Π^*_j have increased based on the current model parameters.

5.3 Impact of Return Functions

In this subsection, we test the impact of return functions on the equilibrium solutions. We keep the same data as in Example 1.1. For the return functions, we consider three examples. Example 4.1 corresponds to the linear return case: $y_1(p_1) = p_1 - 35$;

$y_2(p_2) = p_2 - 40$. Example 4.2 illustrates the logarithmic return case: $y_1(p_1) = 50\ln(p_1+1) - 179.176 = 50(\ln(p_1+1) - \ln(36))$; $y_2(p_2) = 50\ln(p_1+1) - 185.679 = 50(\ln(p_1+1) - \ln(41))$. Example 4.3 corresponds to the power return function: $y_1(p_1) = 25(p_1^{0.5} - 35^{0.5})$; $y_2(p_2) = 30(p_1^{0.5} - 40^{0.5})$. Table 3 displays the optimal equilibrium solutions, the total revenue of all supply chain members, the number of iterations required for the convergence of the extragradient algorithm, and the CPU time for these three examples. Comparing the quantities q_{mi}^*, we observe higher values in the power return case and lower values in the linear return case. This is due to the impact of the rate of increase of the expected return $y_m(p_m)$ with respect to the buy-back price p_m^*. The faster the return increases with respect to the price p_m^*, the higher quantities q_{mi}^* manufacturers will order from recovery centers. An increase in q_{mi}^* will result in an increase in the quantities q_{ij}^* and q_{ni}^*, an increase in the price p_{ni}^*, as well as an increase in the expected profits Π_n^*, Π_j^* and Π_i^* and the total revenue. Note that the prices p_{ij}^*, p_{mi}^* and p_m^* and the expected profits Π_m^* have decreased based on the current model parameters.

6 Conclusion

In this paper, a new closed-loop supply chain network equilibrium model was developed allowing us to consider random demands and random returns in a closed-loop system consisting of raw material suppliers, manufacturers, retailers and recovery centers that collect the recycled product directly from consumers at demand markets.

Using additive and multiplicative functions to model randomness in demand and return, we derived the equilibrium conditions of all supply chain members and then showed they satisfy a variational inequality problem. Qualitative properties were discussed and a solution algorithm based on the extragradient method was suggested to solve the model.

Numerical examples illustrated the flexibility of the model and showed the effects of randomness on the equilibrium shipments and expected profits. With our model, one can fine-tune the model parameters, the demand and return distributions, as well as the demand and return functions to quantify the effects on the equilibrium shipments, prices and expected profits in the closed-loop supply chain, which can also generate some implications for policy makers. Moreover, although the cost functions in our examples are hypothetical, we believe that some interesting managerial implications are reported.

This work establishes the foundation for closed-loop supply chain (CLSC) network problems in the case of random demands and returns. The proposed model can serve as an experimental tool to assist managers and policy makers in the long-term operation of closed-loop supply chains under the concerns of uncertainty, environment and net revenues. Future research may include the development of a multiperiod CLSC network model that considers inventory management, and the integration of multiple products, quality depreciation of recycled materials and the quality of return collected from the demand markets into the closed-loop supply chain network.

Appendix A: Proof of Lemma 1

Knowing that $y_j'(p_j) < 0$, one sees that $-(p_j + \delta_j)y_j'(p_j)r_j(z_j) \geq \frac{1}{2}$ is equivalent to $M_j(s_j, p_j) \leq 0$ where $M_j(s_j, p_j) = \frac{1}{r_j(z_j)} + 2(p_j + \delta_j)y_j'(p_j)$. It is easy to verify that:

$$
\begin{aligned}
\frac{\partial M_j}{\partial s_j} &= \frac{-r_j'(z_j)}{r_j(z_j)^2}, \\
\frac{\partial M_j}{\partial p_j} &= \frac{r_j'(z_j)y_j'(p_j)}{r_j(z_j)^2} + 2(p_j + \delta_j)y_j''(p_j) + 2y_j'(p_j), \\
\frac{\partial^2 M_j}{\partial s_j^2} &= \frac{2r_j'(z_j)^2 - r_j(z_j)r_j''(z_j)}{r_j(z_j)^3}, \\
\frac{\partial^2 M_j}{\partial s_j p_j} &= -y_j'(p_j)\frac{2r_j'(z_j)^2 - r_j(z_j)r_j''(z_j)}{r_j(z_j)^3}, \\
\frac{\partial^2 M_j}{\partial p_j^2} &= y_j'(p_j)^2 \frac{2r_j'(z_j)^2 - r_j(z_j)r_j''(z_j)}{r_j(z_j)^3} \\
&\quad + y_j''(p_j)\frac{r_j'(z_j)}{r_j(z_j)^2} + 4y_j''(p_j) + 2(p_j + \delta_j)y_j'''(p_j).
\end{aligned}
$$

Note that $\frac{\partial^2 M_j}{\partial s_j^2} \geq 0$ if $2(r_j'(z_j))^2 - r_j(z_j)r_j''(z_j) \geq 0$. The determinant of the hessian of M_j, given by $\Delta_j = \frac{\partial^2 M_j}{\partial s_j^2}\frac{\partial^2 M_j}{\partial p_j^2} - \left(\frac{\partial^2 M_j}{\partial s_j p_j}\right)^2$ simplifies to

$$
\Delta_j = \frac{2r_j'(z_j)^2 - r_j(z_j)r_j''(z_j)}{r_j(z_j)^3}\left(y_j''(p_j)\frac{r_j'(z_j)}{r_j(z_j)^2} + 4y_j''(p_j) + 2(p_j + \delta_j)y'''(p_j)\right).
$$

From Assumptions 1 and 2a, $2(r_j'(z_j))^2 - r_j(z_j)r_j''(z_j) \geq 0$, $r_j'(z_j) \geq 0$, $2y_j''(p_j) + p_j y'''(p_j) \geq 0$ and $y_j''(p_j) \geq 0$ therefore Δ_j has the same sign as $2y_j''(p_j) + (p_j + \delta_j)y'''(p_j)$. For $y'''(p_j) \geq 0$, one notice that $2y_j''(p_j) + (p_j + \delta_j)y'''(p_j) \geq 0$ Since $y''(p_j) \geq 0$ and $p_j + \delta_j \geq p_j - \lambda_j^+ \geq 0$. Next for $y'''(p_j) \leq 0$, $2y_j''(p_j) + (p_j + \delta_j)y'''(p_j) \geq 0$ since $2y_j''(p_j) + p_j y'''(p_j) \geq 0$ by Assumption 1 and $\delta_j y'''(p_j) \geq 0$ since $\delta_j \leq 0$ by definition. Hence $\Delta_j \geq 0$ implying that the function M_j is convex which in turn implies that the set Γ_j^1 is convex.

Appendix B: Proof of Theorem 1

First, without loss of generality, assume that $p^*_{Ij} = \min_j\{p^*_{ij}\}$. Optimization problem (6) can then be formulated as follows:

$$
\begin{aligned}
\max \Pi_j(\tilde{Q}_j, s_j, p_j) &= p_j y_j(p_j) - (p_j + \lambda_j^-)e_j^-(z_j) + \lambda_j^+ e_j^+(z_j) \\
&\quad - c_j s_j - p^*_{Ij} s_j - \sum_{i=1}^{I-1}(p^*_{ij} - p^*_{Ij})q_{ij} \\
&= \Pi_j^1(s_j, p_j) + \Pi_j^2(\tilde{Q}_j),
\end{aligned}
$$

where $\tilde{Q}_j = (q_{ij})_{i=1}^{I-1}$, $\Pi_j^1(s_j, p_j) = p_j y_j(p_j) - (p_j + \lambda_j^-)e_j^-(z_j) + \lambda_j^+ e_j^+(z_j) - c_j s_j - p^*_{Ij} s_j$ and $\Pi_j^2(\tilde{Q}_j) = -\sum_{i=1}^{I-1}(p^*_{ij} - p^*_{Ij})q_{ij}$.

Clearly, the function $\Pi_j^2(\tilde{Q}_j)$ is concave. We need to prove that $\Pi_j^1(s_j, p_j)$ is concave. One easily verifies that the first derivative of Π_j^1 with respect to s_j and p_j are given by

$$
\begin{aligned}
\frac{\partial \Pi_j^1}{\partial s_j} &= \lambda_j^- + p_j - c_j - p^*_{Ij} - \left(p_j + \lambda_j^- - \lambda_j^+\right) F_j(z_j). \\
\frac{\partial \Pi_j^1}{\partial p_j} &= y_j(p_j) - e_j^-(z_j) + y_j'(p_j)\left[(p_j + \lambda_j^- - \lambda_j^+)F_j(z_j) - \lambda_j^-\right].
\end{aligned}
$$

Straightforward computations show that:

$$
\begin{aligned}
\frac{\partial^2 \Pi_j^1}{\partial s_j^2} &= -\left(p_j + \lambda_j^- - \lambda_j^+\right) f_j(z_j), \\
\frac{\partial^2 \Pi_j^1}{\partial s_j \partial p_j} &= y_j'(p_j)\left(p_j + \lambda_j^- - \lambda_j^+\right) f_j(z_j) + 1 - F_j(z_j), \\
\frac{\partial^2 \Pi_j^1}{\partial p_j^2} &= -y_j'(p_j)^2\left(p_j + \lambda_j^- - \lambda_j^+\right) f_j(z_j) + 2y_j'(p_j)F_j(z_j) \\
&\quad + y_j''(p_j)\left[\left(p_j + \lambda_j^- - \lambda_j^+\right) F_j(z_j) - \lambda_j^-\right].
\end{aligned}
$$

Let H_j denotes the hessian matrix associated with $\Pi_j^1(s_j, p_j)$. The matrix H_j is computed as:

$$
\begin{pmatrix} H_{s_j s_j} & H_{p_j s_j} \\ H_{p_j s_j} & H_{p_j p_j} \end{pmatrix},
$$

with $H_{s_js_j} = -\left(p_j + \lambda_j^- - \lambda_j^+\right) f_j(z_j)$, $H_{p_js_j} = H_{s_jp_j} = y_j'(p_j)$ $\left(p_j + \lambda_j^- - \lambda_j^+\right) f_j(z_j) + 1 - F_j(z_j)$ and $H_{p_jp_j} = -y_j'(p_j)^2 \left(p_j + \lambda_j^- - \lambda_j^+\right)$ $f_j(z_j) + 2y_j'(p_j)F_j(z_j) + y_j''(p_j)\left[\left(p_j + \lambda_j^- - \lambda_j^+\right) F_j(z_j) - \lambda_j^-\right]$.

We will show that the matrix $-H_j$ is positive definite. Clearly, $-H_{s_js_j} = \left(p_j + \lambda_j^- - \lambda_j^+\right) f_j(z_j) > 0$. If we show that $det(-H_j) \geq 0$ then $-H_{p_jp_j} \geq \frac{\left(H_{p_js_j}\right)^2}{-H_{s_js_j}} \geq 0$. $det(-H_j)$ is calculated as:

$$\begin{aligned}
det(-H_j) &= -y_j''(p_j)(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j) \left[\left(p_j + \lambda_j^- - \lambda_j^+\right) F_j(z_j) - \lambda_j^-\right] \\
&\quad -2(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j) y_j'(p_j) F_j(z_j) \\
&\quad -2(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j) y_j'(p_j)(1 - F_j(z_j)) \\
&\quad -(1 - F_j(z_j))^2 \\
&= (p_j + \lambda_j^- - \lambda_j^+) f_j(z_j) \Big[-2y_j'(p_j) F_j(z_j) \\
&\qquad\qquad -y_j''(p_j)\left[\left(p_j + \lambda_j^- - \lambda_j^+\right) F_j(z_j) - \lambda_j^-\right]\Big] \\
&\quad +2(1 - F_j(z_j))^2 \left[\frac{-y_j'(p_j)(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j)}{1 - F_j(z_j)} - \frac{1}{2}\right] \\
&= (p_j + \lambda_j^- - \lambda_j^+) f_j(z_j) F_j(z_j) \left[-2y_j'(p_j) - p_j y_j''(p_j)\right] \\
&\quad +(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j) y_j''(p_j) \left[\lambda_j^-(1 - F_j(z_j)) + \lambda_j^+ F_j(z_j)\right] \\
&\quad +2(1 - F_j(z_j))^2 \left[-y_j'(p_j)(p_j + \lambda_j^- - \lambda_j^+) r_j(z_j) - \frac{1}{2}\right].
\end{aligned}$$

The first term in the above is nonnegative because $2y_j'(p_j) + p_j y_j''(p_j) \leq 0$. The second term is also nonnegative because $y_j''(p_j) \geq 0$. The last term is nonnegative since it can be written as $2(1 - F_j(z_j))^2 \left[\eta_j^1(s_j, p_j) - 1/2\right] + 2(1 - F_j(z_j))^2 \max(\lambda_j^- - \lambda_j^+, 0)(-y_j'(p_j) r_j(z_j))$ and $\eta_j^1(s_j, p_j) \geq 1/2$. Hence, we have $det(-H_j) \geq 0$ implying that the matrix $-H_j$ is positive definite. Therefore, Π_j^1 is concave in the set Γ_j^1.

Appendix C: Proof of Proposition 1

We need to show that the optimal vector (s_j^*, p_j^*) belongs to the set Γ_j^1. In other words, we will show that $\eta_j^1(s_j, p_j) \geq 1/2$ when $\frac{\partial \Pi_j^1}{\partial p_j} = 0$.

$$\frac{\partial \Pi_j^1}{\partial p_j} = 0 \Longleftrightarrow y_j'(p_j) = \frac{e_j^-(z_j) - y_j(p_j)}{(p_j + \lambda_j^- - \lambda_j^+) F_j(z_j) - \lambda_j^-}.$$

Since $y_j'(p_j) \leq 0$ and $y_j(p_j) - e_j^-(z_j) \geq 0$ one see that when $\frac{\partial \Pi_j^1}{\partial p_j} = 0$ one has $(p_j + \lambda_j^- - \lambda_j^+)F_j(z_j) - \lambda_j^- > 0$.

$$
\begin{aligned}
\eta_j^1(s_j, p_j) \geq 1/2 &\iff -(p_j+\delta_j)y_j'(p_j)r_j(z_j) - \frac{1}{2} \geq 0 \\
&\iff -(p_j+\delta_j)y_j'(p_j) - \frac{1}{2r_j(z_j)} \geq 0 \\
&\iff \frac{p_j+\delta_j}{(p_j+\lambda_j^- - \lambda_j^+)F_j(z_j) - \lambda_j^-}\Big(y_j(p_j) - e_j^-(z_j)\Big) - \frac{1}{2r_j(z_j)} \geq 0 \\
&\iff \frac{p_j+\delta_j}{(p_j+\lambda_j^- - \lambda_j^+)F_j(z_j) - \lambda_j^-}\left[y_j(p_j) - e_j^-(z_j) - \frac{F_j(z_j)}{2r_j(z_j)}\right] \\
&\qquad + \frac{1}{(p_j+\lambda_j^- - \lambda_j^+)F_j(z_j) - \lambda_j^-} \\
&\qquad \times \left[\frac{\lambda_j^-(1-F_j(z_j)) + \min(\lambda_j^-, \lambda_j^+)F_j(z_j)}{2r_j(z_j)}\right] \geq 0.
\end{aligned}
$$

The last inequality holds if $y_j(p_j)+\kappa_j(z_j) \geq 0$ where $\kappa_j(z_j) = -e_j^-(z_j) - \frac{F_j(z_j)}{2r_j(z_j)}$. Note that $\kappa_j(A_j) = A_j$ and $\kappa_j'(z_j) = \frac{1-F_j(z_j)}{2} + \frac{r'(z_j)F(z_j)}{2r_j(z_j)^2} \geq 0$ because $r_j'(z_j) \geq 0$. This yields $\kappa_j(z_j) \geq A_j$ for all $z_j \in [A_j, B_j]$. Then $y_j(p_j) + \kappa_j(z_j) \geq y_j(\bar{p}_j) + A_j = 0$. Therefore, $\eta_j^1(s_j, p_j) \geq 1/2$ when $\frac{\partial \Pi_j^1}{\partial p_j} = 0$.

Appendix D: Proof of Lemma 2

Knowing that $z_j = s_j/y_j(p_j) \geq 0$, one sees that $\frac{-(p_j+\delta_j)y_j'(p_j)g_j(z_j)}{y_j(p_j)} \geq \frac{1}{2}$ is equivalent to $N_j(s_j, p_j) \leq 0$ where $N_j(s_j, p_j) = \frac{y_j(p_j)}{g_j(z_j)} + 2(p_j+\delta_j)y_j'(p_j)$. It is easy to check that:

$$
\begin{aligned}
\frac{\partial N_j}{\partial s_j} &= \frac{-g_j'(z_j)}{g_j(z_j)^2}, \\
\frac{\partial N_j}{\partial p_j} &= \frac{z_j g_j'(z_j)y_j'(p_j)}{g_j(z_j)^2} + \frac{y_j(p_j)}{g_j(z_j)} + 2(p_j+\delta_j)y_j''(p_j) + 2y_j'(p_j), \\
\frac{\partial^2 N_j}{\partial s_j^2} &= \frac{2g_j'(z_j)^2 - g_j(z_j)g_j''(z_j)}{g_j(z_j)^3 y_j(p_j)}, \\
\frac{\partial^2 N_j}{\partial s_j p_j} &= -z_j y_j'(p_j)\frac{2g_j'(z_j)^2 - g_j(z_j)g_j''(z_j)}{g_j(z_j)^3 y_j(p_j)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 N_j}{\partial p_j^2} &= z_j^2 y_j'(p_j)^2 \frac{2g_j'(z_j)^2 - g_j(z_j)g_j''(z_j)}{g_j(z_j)^3 y_j(p_j)} + \frac{y_j''(p_j)}{g_j(z_j)} \\
&\quad + y_j''(p_j)\frac{z_j g_j'(z_j)}{g_j(z_j)^2} + 4y_j''(p_j) + 2(p_j+\delta_j)y_j'''(p_j).
\end{aligned}
$$

Note that $\frac{\partial^2 N_j}{\partial s_j^2} \geq 0$ if $2(g_j'(z_j))^2 - g_j(z_j)g_j''(z_j) \geq 0$. The determinant of the hessian of N_j is given by

$$
\Theta_j = \frac{2g_j'(z_j)^2 - g_j(z_j)g_j''(z_j)}{g_j(z_j)^3 y_j(p_j)} \left(y_j''(p_j)\frac{z_j g_j'(z_j)}{g_j(z_j)^2} + \frac{y_j''(p_j)}{g_j(z_j)} + 4y_j''(p_j) + 2(p_j+\delta_j)y'''(p_j) \right).
$$

From Assumptions 1 and 2b, $2(g_j'(z_j))^2 - g_j(z_j)g_j''(z_j) \geq 0$, $g_j'(z_j) \geq 0$, $2y_j''(p_j) + p_j y'''(p_j) \geq 0$ and $y_j''(p_j) \geq 0$ therefore Θ_j has the same sign as $2y_j''(p_j) + (p_j+\delta_j)y'''(p_j)$. For $y'''(p_j) \geq 0$, one notice that $2y_j''(p_j) + (p_j + \delta_j)y'''(p_j) \geq 0$ since $y''(p_j) \geq 0$ and $p_j + \delta_j \geq p_j - \lambda_j^+ \geq 0$. Next for $y'''(p_j) \leq 0$, $2y_j''(p_j) + (p_j+\delta_j)y'''(p_j) \geq 0$ since $2y_j''(p_j) + p_j y'''(p_j) \geq 0$ by Assumption 1 and $\delta_j y'''(p_j) \geq 0$ since $\delta_j \leq 0$ by definition. Hence $\Theta_j \geq 0$ implying that the function N_j is convex. We can then conclude that the set Γ_j^2 is convex.

Appendix E: Proof of Theorem 2

As in the additive case, optimization problem (8) can be reformulated as:

$$
\begin{aligned}
\max \Pi_j(\tilde{Q}_j, s_j, p_j) &= p_j y_j(p_j)(1 - e_j^-(z_j)) + y_j(p_j) \\
&\quad \times \left(\lambda_j^+ e_j^+(z_j) - \lambda_j^- e_j^-(z_j)\right) - c_j s_j - p_{Ij}^* s_j \\
&\quad - \sum_{i=1}^{I-1} (p_{ij}^* - p_{Ij}^*) q_{ij} \\
&= \Pi_j^1(s_j, p_j) + \Pi_j^2(\tilde{Q}_j),
\end{aligned}
$$

where $\tilde{Q}_j = (q_{ij})_{i=1}^{I-1}$, $\Pi_j^1(s_j, p_j) = p_j y_j(p_j)(1 - e_j^-(z_j)) + y_j(p_j)\left(\lambda_j^+ e_j^+(z_j) - \lambda_j^- e_j^-(z_j)\right) - c_j s_j - p_{Ij}^* s_j$ and $\Pi_j^2(\tilde{Q}_j) = -\sum_{i=1}^{I-1}(p_{ij}^* - p_{Ij}^*)q_{ij}$.

Clearly, the function $\Pi_j^2(\tilde{Q}_j)$ is concave. We need to prove that $\Pi_j^1(s_j, p_j)$ is concave. Let us consider the first derivative of Π_j^1 taken with respect to s_j and p_j:

$$\begin{aligned}
\frac{\partial \Pi_j^1}{\partial s_j} &= -p_j\left(F_j(z_j)-1\right)+\lambda_j^+ F_j(z_j)-\lambda_j^-\left(F_j(z_j)-1\right)-c_j-p_{Ij}^*\\
&= \lambda_j^- + p_j - c_j - p_{Ij}^* - \left(p_j+\lambda_j^- - \lambda_j^+\right)F_j(z_j).\\
\frac{\partial \Pi_j^1}{\partial p_j} &= y_j(p_j)(1-e_j^-(z_j)) + p_j y_j'(p_j)(1-e_j^-(z_j)) + p_j z_j y_j'(p_j)(F_j(z_j)-1)\\
&\quad + y_j'(p_j)\Big(\lambda_j^+ e_j^+(z_j) - \lambda_j^- e_j^-(z_j)\Big) - z_j y_j'(p_j)\left(\lambda_j^+ F_j(z_j) - \lambda_j^-(F_j(z_j)-1)\right)\\
&= y_j(p_j)(1-e_j^-(z_j)) + y_j'(p_j)\\
&\quad \times\left[(p_j+\lambda_j^- - \lambda_j^+)[(1-e_j^-(z_j)) + z_j(F_j(z_j)-1)] - \lambda_j^-\right].
\end{aligned}$$

Straightforward computations show that:

$$\begin{aligned}
\frac{\partial^2 \Pi_j^1}{\partial s_j^2} &= -\frac{1}{y_j(p_j)}\left(p_j+\lambda_j^- - \lambda_j^+\right)f_j(z_j),\\
\frac{\partial^2 \Pi_j^1}{\partial s_j \partial p_j} &= \frac{z_j}{y_j(p_j)} y_j'(p_j)\left(p_j+\lambda_j^- - \lambda_j^+\right)f_j(z_j) + 1 - F_j(z_j),\\
\frac{\partial^2 \Pi_j^1}{\partial p_j^2} &= -\frac{z_j^2 y_j'(p_j)^2}{y_j(p_j)}\left(p_j+\lambda_j^- - \lambda_j^+\right)f_j(z_j) + 2y_j'(p_j)\Big[(1-e_j^-(z_j)) + z_j(F_j(z_j)-1)\Big]\\
&\quad + y_j''(p_j)\left[\left(p_j+\lambda_j^- - \lambda_j^+\right)\left((1-e_j^-(z_j)) + z_j(F_j(z_j)-1)\right) - \lambda_j^-\right].
\end{aligned}$$

Let H_j denotes the hessian matrix associated with $\Pi_j^1(s_j, p_j)$. The matrix H_j is computed as:

$$\begin{pmatrix} H_{s_j s_j} & H_{p_j s_j} \\ H_{p_j s_j} & H_{p_j p_j} \end{pmatrix},$$

with $H_{s_j s_j} = -\frac{1}{y_j(p_j)}\left(p_j+\lambda_j^- - \lambda_j^+\right)f_j(z_j)$, $H_{p_j s_j} = H_{s_j p_j} = \frac{z_j}{y_j(p_j)} y_j'(p_j)\left(p_j+\lambda_j^- - \lambda_j^+\right)f_j(z_j) + 1 - F_j(z_j)$ and

$$\begin{aligned}
H_{p_j p_j} &= -\frac{z_j^2 y_j'(p_j)^2}{y_j(p_j)}\left(p_j+\lambda_j^- - \lambda_j^+\right)f_j(z_j) + 2y_j'(p_j)\Big[(1-e_j^-(z_j)) + z_j(F_j(z_j)-1)\Big]\\
&\quad + y_j''(p_j)\left[\left(p_j+\lambda_j^- - \lambda_j^+\right)\left((1-e_j^-(z_j)) + z_j(F_j(z_j)-1)\right) - \lambda_j^-\right]\\
&= -\frac{z_j^2 y_j'(p_j)^2}{y_j(p_j)}\left(p_j+\lambda_j^- - \lambda_j^+\right)f_j(z_j) + 2y_j'(p_j)\int_{A_j}^{z_j} x f_j(x)dx\\
&\quad + y_j''(p_j)\left[\left(p_j+\lambda_j^- - \lambda_j^+\right)\int_{A_j}^{z_j} x f_j(x)dx - \lambda_j^-\right].
\end{aligned}$$

We will show that the matrix $-H_j$ is positive definite. Clearly, $-H_{s_j s_j} = \frac{1}{y_j(p_j)}\left(p_j + \lambda_j^- - \lambda_j^+\right) f_j(z_j)$ is positive. If $det(-H_j) > 0$ then $-H_{p_j p_j} > \frac{\left(H_{p_j s_j}\right)^2}{-H_{s_j s_j}} > 0$.

$$
\begin{aligned}
det(-H_j) = &-\frac{y_j''(p_j)}{y_j(p_j)}(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j)\left[\left(p_j + \lambda_j^- - \lambda_j^+\right)\int_{A_j}^{z_j} x f_j(x)dx - \lambda_j^-\right] \\
&-2\frac{y_j'(p_j)}{y_j(p_j)}(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j)\int_{A_j}^{z_j} x f_j(x)dx \\
&-2\frac{y_j'(p_j)}{y_j(p_j)}(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j) z_j\left(1 - F_j(z_j)\right) - (1 - F_j(z_j))^2 \\
= &\frac{(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j)}{y_j(p_j)}\int_{A_j}^{z_j} x f_j(x)dx\left[-2y_j'(p_j) - p_j y_j''(p_j)\right] \\
&+\frac{(p_j + \lambda_j^- - \lambda_j^+) f_j(z_j)}{y_j(p_j)} y_j''(p_j)\left[\lambda_j^- \int_{z_j}^{B_j} x f_j(x)dx + \lambda_j^+ \int_{A_j}^{z_j} x f_j(x)dx\right] \\
&+2(1 - F_j(z_j))^2\left[\frac{-y_j'(p_j)(p_j + \lambda_j^- - \lambda_j^+) g_j(z_j)}{y_j(p_j)} - \frac{1}{2}\right].
\end{aligned}
$$

The first term in the above is nonnegative because $2y_j'(p_j) + p_j y_j''(p_j) \leq 0$. The second term is also nonnegative because $y_j''(p_j) \geq 0$. The last term is nonnegative since it can be written as $2(1 - F_j(z_j))^2\left[\eta_j^2(s_j, p_j) - 1/2\right] + 2(1 - F_j(z_j))^2 \max(\lambda_j^- - \lambda_j^+, 0)\frac{-y_j'(p_j) g_j(z_j)}{y_j(p_j)}$ and $\eta_j^2(s_j, p_j) \geq 1/2$. Hence, we have $det(-H_j) \geq 0$ implying that the matrix $-H_j$ is positive definite. Therefore, Π_j^1 is concave in the set Γ_j^2.

Appendix F: Proof of Proposition 2

We need to show that the optimal vector (s_j^*, p_j^*) belongs to the set Γ_j^2. In other words, we need to show that $\eta_j^2(s_j, p_j) \geq 1/2$ when $\frac{\partial \Pi_j^1}{\partial p_j} = 0$.

$$
\begin{aligned}
y_j'(p_j) &= \frac{-y_j(p_j)(1 - e_j^-(z_j))}{\left(p_j + \lambda_j^- - \lambda_j^+\right)\left[(1 - e_j^-(z_j)) + z_j(F_j(z_j) - 1)\right] - \lambda_j^-} \\
&= \frac{-y_j(p_j)\left(\int_{A_j}^{z_j} x f_j(x)\, dx + z_j(1 - F_j(z_j))\right)}{\left(p_j + \lambda_j^- - \lambda_j^+\right)\int_{A_j}^{z_j} x f_j(x)\, dx - \lambda_j^-}.
\end{aligned}
$$

Since $y_j'(p_j) \leq 0$ and $\int_{A_j}^{z_j} x f_j(x)\,dx + z_j(1 - F_j(z_j)) \geq 0$ one see that when $\frac{\partial \Pi_j^1}{\partial p_j} = 0$ one has $\left(p_j + \lambda_j^- - \lambda_j^+\right) \int_{A_j}^{z_j} x f_j(x)\,dx - \lambda_j^- > 0$.

$$
\begin{aligned}
\eta_j^2(s_j, p_j) \geq 1/2 \iff & \frac{-(p_j + \delta_j) y_j'(p_j) g_j(z_j)}{y_j(p_j)} - \frac{1}{2} \geq 0 \\
\iff & \frac{-(p_j + \delta_j) y_j'(p_j)}{y_j(p_j)} - \frac{1}{2 g_j(z_j)} \geq 0 \\
\iff & \frac{(p_j + \delta_j) \left[\int_{A_j}^{z_j} x f_j(x)\,dx + z_j(1 - F_j(z_j)) - \frac{\int_{A_j}^{z_j} x f_j(x)\,dx}{2 g_j(z_j)} \right]}{\left(p_j + \lambda_j^- - \lambda_j^+\right) \int_{A_j}^{z_j} x f_j(x)\,dx - \lambda_j^-} \\
& + \frac{\lambda_j^- \int_{z_j}^{B_j} x f_j(x)\,dx + \min(\lambda_j^-, \lambda_j^+) \int_{A_j}^{z_j} x f_j(x)\,dx}{2 g_j(z_j) \left[\left(p_j + \lambda_j^- - \lambda_j^+\right) \int_{A_j}^{z_j} x f_j(x)\,dx - \lambda_j^- \right]} \geq 0.
\end{aligned}
$$

The last inequality holds if $\psi_j(z_j) \geq 0$ where $\psi_j(z_j) = \int_{A_j}^{z_j} x f_j(x)\,dx + z_j(1 - F_j(z_j)) - \frac{\int_{A_j}^{z_j} x f_j(x)\,dx}{2 g_j(z_j)}$. Note that $\psi_j(A_j) = A_j$ and $\psi_j'(z_j) = \frac{1 - F_j(z_j)}{2} + \frac{g_j'(z_j) \int_{A_j}^{z_j} x f_j(x)\,dx}{2 g_j(z_j)^2} \geq 0$ because $g_j'(z_j) \geq 0$. This yields $\psi_j(z_j) \geq A_j$ for all $z_j \in [A_j, B_j]$. Then $\psi_j(z_j) \geq 0$. Therefore, $\eta_j^2(s_j, p_j) \geq 1/2$ when $\frac{\partial \Pi_j^1}{\partial p_j} = 0$.

Appendix G: Proof of Theorem 3

As in the retailer case, without loss of generality, we assume that $p_{mI}^* = \max_i \{p_{mi}^*\}$. Optimization problem (11) can be then reformulated as:

$$
\begin{aligned}
\max \Pi_m(\tilde{Q}_m, q_m, p_m) &= \sum_{i=1}^{I-1} (p_{mi}^* - p_{mI}^*) q_{mi} + p_{mI}^* q_m + s^{rec} y_m(p_m) - \left(c_m^u + c^u \bar{\chi}_m\right) q_m \\
&\quad - p_m y_m(p_m) + \lambda_m^+ e_m^+(z_m) - \lambda_m^- e_m^-(z_m), \\
&= \Pi_m^1(q_m, p_m) + \Pi_m^2(\tilde{Q}_m),
\end{aligned}
$$

where $\tilde{Q}_m = (q_{mi})_{i=1}^{I-1}$, $\Pi_m^1(q_m, p_m) = p_{mI}^* q_m + s^{rec} y_m(p_m) - \left(c_m^u + c^u \bar{\chi}_m\right) q_m - p_m y_m(p_m) + \lambda_m^+ e_m^+(z_m) - \lambda_m^- e_m^-(z_m)$ and $\Pi_m^2(\tilde{Q}_m) = \sum_{i=1}^{I-1} (p_{mi}^* - p_{mI}^*) q_{mi}$.

Clearly, the function $\Pi_m^2(\tilde{Q}_m)$ is concave. We need to show that $\Pi_m^1(q_m, p_m)$ is concave. The first derivative of Π_m^1 taken with respect to q_m and p_m and are given as:

$$\begin{aligned}
\frac{\partial \Pi_m^1}{\partial q_m} &= p_{mI}^* - c_m^u - c^u \bar{\chi}_m + \frac{\lambda_m^+}{\chi_m}(F_m(z_m) - 1) - \frac{\lambda_m^-}{\chi_m} F_m(z_m) \\
&= p_{mI}^* - c_m^u - c^u \bar{\chi}_m - \frac{\lambda_m^+}{\chi_m} - \frac{\left(\lambda_m^- - \lambda_m^+\right)}{\chi_m} F_m(z_m). \\
\frac{\partial \Pi_m^1}{\partial p_m} &= s^{rec} y_m'(p_m) - p_m y_m'(p_m) - y_m(p_m) \\
&\quad + y_m'(p_m)\lambda_m^+(1 - F_m(z_m)) + y_m'(p_m)\lambda_m^- F_m(z_m) \\
&= -y_m(p_m) + y_m'(p_m)\left(s^{rec} - p_m + \lambda_m^+ + F_m(z_m)(\lambda_m^- - \lambda_m^+)\right).
\end{aligned}$$

Straightforward computations show that:

$$\begin{aligned}
\frac{\partial^2 \Pi_m^1}{\partial q_m^2} &= -\frac{1}{\chi_m^2}\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m), \\
\frac{\partial^2 \Pi_m^1}{\partial q_m \partial p_m} &= \frac{y_m'(p_m)}{\chi_m}\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m), \\
\frac{\partial^2 \Pi_m^1}{\partial p_m^2} &= -y_m'(p_m)^2\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m) - 2y_m'(p_m) \\
&\quad + y_m''(p_m)\left[\left(\lambda_m^- - \lambda_m^+\right) F_m(z_m) + s^{rec} - p_m + \lambda_m^+\right].
\end{aligned}$$

Let H_m denotes the hessian matrix associated with $\Pi_m^1(q_m, p_m)$. The matrix H_m is calculated as:

$$\begin{pmatrix} H_{q_m q_m} & H_{p_m q_m} \\ H_{p_m q_m} & H_{p_m p_m} \end{pmatrix},$$

with $H_{q_m q_m} = -\frac{1}{\chi_m^2}\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m)$, $H_{p_m q_m} = H_{q_m p_m} = \frac{y_m'(p_m)}{\chi_m}\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m)$ and $H_{p_m p_m} = -y_m'(p_m)^2\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m) - 2y_m'(p_m) + y_m''(p_m) \left[\left(\lambda_m^- - \lambda_m^+\right) F_m(z_m) + s^{rec} - p_m + \lambda_m^+\right]$.

We will show that the matrix $-H_m$ is positive definite. Clearly, $-H_{q_m q_m} = \frac{1}{\chi_m^2}\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m) > 0$. If $det(-H_m) \geq 0$ then $-H_{p_m p_m} \geq \frac{(H_{p_m q_m})^2}{-H_{q_m q_m}} \geq 0$.

$$\begin{aligned}
\chi_m^2 det(-H_m) &= -(\lambda_m^- - \lambda_m^+) f_m(z_m)\left(\left(\lambda_m^- - \lambda_m^+\right) F_m(z_m) + s^{rec} - p_m + \lambda_m^+\right) y_m''(p_m) \\
&\quad + 2(\lambda_m^- - \lambda_m^-) f_m(z_m) y_m'(p_m) \\
&= (\lambda_m^- - \lambda_m^+) f_m(z_m)\left[2y_m'(p_m) + p_m y_m''(p_m)\right] \\
&\quad - (\lambda_m^- - \lambda_m^+) f_m(z_m) y_m''(p_m)\left[\lambda_m^- F_m(z_m) + \lambda_m^+(1 - F_m(z_m)) + s^{rec}\right].
\end{aligned}$$

The first part of the last term is nonnegative because $2y_m''(p_m) + p_m y_m''(p_m) \geq 0$. The second part is also nonnegative because $y_m''(p_m) \leq 0$. Hence, we have $det(-H_m) \geq 0$ implying that the matrix $-H_m$ is positive definite. Therefore, Π_m^1 is concave.

Appendix H: Proof of Theorem 4

Similar to the additive case, optimization problem (13) can be reformulated as:

$$\begin{aligned}\max \Pi_m(\tilde{Q}_m, q_m, p_m) &= \sum_{i=1}^{I-1}(p^*_{mi} - p^*_{mI})q_{mi} + p^*_{mI}q_m + s^{rec}y_m(p_m) - \left(c^u_m + c^u\bar{\chi}_m\right)q_m \\ &\quad - p_m y_m(p_m) + y_m(p_m)\left(\lambda^+_m e^+_m(z_m) - \lambda^-_m e^-_m(z_m)\right) \\ &= \Pi^1_m(q_m, p_m) + \Pi^2_m(\tilde{Q}_m),\end{aligned}$$

where $\tilde{Q}_m = (q_{mi})_{i=1}^{I-1}$, $\Pi^1_m(q_m, p_m) = p^*_{mI}q_m + s^{rec}y_m(p_m) - \left(c^u_m + c^u\bar{\chi}_m\right)q_m - p_m y_m(p_m) + y_m(p_m)\left(\lambda^+_m e^+_m(z_m) - \lambda^-_m e^-_m(z_m)\right)$ and $\Pi^2_m(\tilde{Q}_m) = \sum_{i=1}^{I-1}(p^*_{mi} - p^*_{mI})q_{mi}$.

Clearly, the function $\Pi^2_m(\tilde{Q}_m)$ is concave. We need to show that $\Pi^1_m(q_m, p_m)$ is concave. The first derivative of Π^1_m taken with respect to q_m and p_m and are computed as:

$$\begin{aligned}\frac{\partial \Pi^1_m}{\partial q_m} &= p^*_{mI} - c^u_m - c^u\bar{\chi}_m + \frac{\lambda^+_m}{\chi_m}(F_m(z_m) - 1) - \frac{\lambda^-_m}{\chi_m}F_m(z_m) \\ &= p^*_{mI} - c^u_m - c^u\bar{\chi}_m - \frac{\lambda^+_m}{\chi_m} - \frac{\left(\lambda^-_m - \lambda^+_m\right)}{\chi_m}F_m(z_m). \\ \frac{\partial \Pi^1_m}{\partial p_m} &= -y_m(p_m) - y'_m(p_m)p_m + y'_m(p_m)s^{rec} \\ &\quad + y'_m(p_m)\left[\left(\lambda^+_m e^+_m(z_m) - \lambda^-_m e^-_m(z_m)\right) - \lambda^+_m z_m(F_m(z_m) - 1) + \lambda^-_m z_m F_m(z_m)\right] \\ &= -y_m(p_m) + y'_m(p_m)\left[s^{rec} - p_m + \lambda^-_m - (\lambda^-_m - \lambda^+_m)\left((1 + e^-_m(z_m)) - z_m F_m(z_m)\right)\right].\end{aligned}$$

Straightforward computations show that:

$$\begin{aligned}\frac{\partial^2 \Pi^1_m}{\partial q^2_m} &= -\frac{1}{\chi^2_m y_m(p_m)}\left(\lambda^-_m - \lambda^+_m\right)f_m(z_m), \\ \frac{\partial^2 \Pi^1_m}{\partial q_m \partial p_m} &= \frac{y'_m(p_m)z_m}{\chi_m y_m(p_m)}\left(\lambda^-_m - \lambda^+_m\right)f_m(z_m), \\ \frac{\partial^2 \Pi^1_m}{\partial p^2_m} &= -\frac{z^2_m y'_m(p_m)^2}{y_m(p_m)}\left(\lambda^-_m - \lambda^+_m\right)f_m(z_m) - 2y'_m(p_m) \\ &\quad + y''_m(p_m)\left[s^{rec} - p_m + \lambda^-_m - (\lambda^-_m - \lambda^+_m)\left((1 + e^-_m(z_m)) - z_m F_m(z_m)\right)\right].\end{aligned}$$

Let H_m denotes the hessian matrix associated with $\Pi^1_m(p_m, p_m)$. The matrix H_m is calculated as:

$$\begin{pmatrix} H_{q_m q_m} & H_{p_m q_m} \\ H_{p_m q_m} & H_{p_m p_m} \end{pmatrix},$$

with $H_{q_m q_m} = -\frac{1}{\chi_m^2 y_m(p_m)}\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m)$, $H_{q_m p_m} = H_{p_m q_m}\frac{y_m'(p_m) z_m}{\chi_m^2 y_m(p_m)}\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m)$ and $H_{p_m p_m} = -\frac{z_m^2 y_m'(p_m)^2}{y_m(p_m)}\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m) - 2y_m'(p_m) + y_m''(p_m)\left[s^{rec} - p_m + \lambda_m^- - (\lambda_m^- - \lambda_m^+)\left((1+e_m^-(z_m)) - z_m F_m(z_m)\right)\right]$.

We will show that the matrix $-H_m$ is positive definite. Clearly, $-H_{q_m q_m} = \frac{1}{\chi_m^2 y_m(p_m)}\left(\lambda_m^- - \lambda_m^+\right) f_m(z_m) > 0$. If $det(-H_m) > 0$ then $-H_{p_m p_m} \geq \frac{(H_{p_m q_m})^2}{-H_{q_m q_m}} \geq 0$.

$$
\begin{aligned}
\chi_m^2 det(-H_m) &= -(\lambda_m^- - \lambda_m^+) f_m(z_m)\left(\lambda_m^- + s^{rec} - p_m - (\lambda_m^- - \lambda_m^+)\right.\\
&\quad \left.\left((1+e_m^-(z_m)) - z_m F_m(z_m)\right)\right)\frac{y_m''(p_m)}{y_m(p_m)} + 2(\lambda_m^- - \lambda_m^+) f_m(z_m)\frac{y_m'(p_m)}{y_m(p_m)}\\
&= -(\lambda_m^- - \lambda_m^+) f_m(z_m)\\
&\quad \times\left(\lambda_m^- + s^{rec} - p_m - (\lambda_m^- - \lambda_m^+)\int_{z_m}^{B_m} x f_m(x) dx\right)\frac{y_m''(p_m)}{y_m(p_m)}\\
&\quad +2(\lambda_m^- - \lambda_m^+) f_m(z_m)\frac{y_m'(p_m)}{y_m(p_m)}\\
&= \frac{(\lambda_m^- - \lambda_m^+) f_m(z_m)}{y_m(p_m)}\left[2y_m'(p_m) + p_m y_m''(p_m)\right]\\
&\quad -\frac{(\lambda_m^- - \lambda_m^+) f_m(z_m)}{y_m(p_m)} y_m''(p_m)\\
&\quad \times\left[\lambda_m^- \int_{A_m}^{z_m} x f_m(x)\, dx + \lambda_m^+ \int_{z_m}^{B_m} x f_m(x)\, dx + s^{rec}\right].
\end{aligned}
$$

The first part of the last term is nonnegative because $2y_m''(p_m) + p_m y_m''(p_m) \geq 0$. The second part is also nonnegative because $y_m''(p_m) \leq 0$. Hence, we have $det(-H_m) \geq 0$ implying that the matrix $-H_m$ is positive definite. Therefore, Π_m^1 is concave.

Appendix I: Proof of Theorem 9

We will only consider the case with additive demand and returns. The proof is similar in the multiplicative case. The expression $\langle \mathcal{F}(X') - \mathcal{F}(X''), X' - X''\rangle \geq 0$ is equivalent to (after some algebraic simplification):

$$
\sum_{n=1}^{N}\sum_{i=1}^{I}\left[\frac{\partial f_i^r\left(\beta_i^r, \sum_{n=1}^{N} q'_{ni}\right)}{\partial q_{ni}} - \frac{\partial f_i^r\left(\beta_i^r, \sum_{n=1}^{N} q''_{ni}\right)}{\partial q_{ni}} + \frac{\partial f_n^r\left(\sum_{i=1}^{I} q'_{ni}\right)}{\partial q_{ni}} - \frac{\partial f_n^r\left(\sum_{i=1}^{I} q''_{ni}\right)}{\partial q_{ni}}\right.
$$
$$
\left. + \frac{\partial c_{ni}(q'_{ni})}{\partial q_{ni}} - \frac{\partial c_{ni}(q''_{ni})}{\partial q_{ni}}\right] \times [q'_{ni} - q''_{ni}]
$$

$$
\begin{aligned}
&+\sum_{m=1}^{M}\sum_{i=1}^{I}\left[\frac{\partial f_i^u\left(\beta_i^u,\sum_{m=1}^{M}q'_{mi}\right)}{\partial q_{mi}}-\frac{\partial f_i^u\left(\beta_i^u,\sum_{m=1}^{M}q''_{mi}\right)}{\partial q_{mi}}+\frac{\partial c_{mi}(q'_{mi})}{\partial q_{mi}}-\frac{\partial c_{mi}(q''_{mi})}{\partial q_{mi}}\right]\\
&\qquad\times\left[q'_{mi}-q''_{mi}\right]\\
&+\sum_{i=1}^{I}\sum_{j=1}^{J}\left[\frac{\partial c_{ij}(q'_{ij})}{\partial q_{ij}}-\frac{\partial c_{ij}(q''_{ij})}{\partial q_{ij}}\right]\times[q'_{ij}-q''_{ij}]\\
&+\sum_{i=1}^{I}\sum_{j=1}^{J}\left[\left(p'_j+\lambda_j^- -\lambda_j^+\right)F_j(z'_j)-\left(p''_j+\lambda_j^- -\lambda_j^+\right)F_j(z''_j)\right]\times[q'_{ij}-q''_{ij}]\\
&+\sum_{j=1}^{J}\Big[-y_j(p'_j)+e_j^-(z'_j)-y'_j(p'_j)\left[(p'_j+\lambda_j^- -\lambda_j^+)F_j(z'_j)\right]\\
&\qquad-\left(y_j(p''_j)+e_j^-(z''_j)-y'_j(p''_j)\left[(p''_j+\lambda_j^- -\lambda_j^+)F_j(z''_j)\right]\right)\Big]\times[p'_j-p''_j]\\
&+\sum_{m=1}^{M}\sum_{i=1}^{I}\left[\frac{(\lambda_m^- -\lambda_m^+)}{\chi_m}F_m(z'_m)-\frac{(\lambda_m^- -\lambda_m^+)}{\chi_m}F_m(z''_m)\right]\times[q'_{mi}-q''_{mi}]\\
&+\sum_{m=1}^{M}\Big[y_m(p'_m)-y'_m(p'_m)\left(s^{rec}-p'_m+\lambda_m^+\right)-y'_m(p'_m)F_m(z'_m)\left(\lambda_m^- -\lambda_m^+\right)\\
&\qquad-\left(y_m(p''_m)-y'_m(p''_m)\left(s^{rec}-p''_m+\lambda_m^+\right)-y'_m(p''_m)F_m(z''_m)\left(\lambda_m^- -\lambda_m^+\right)\right)\Big]\\
&\qquad\times\left[p'_m-p''_m\right]\ge 0
\end{aligned}
$$

which is equivalent to $(I)+(II)+(III)+(IV)+(V)+(VI)+(VII)\ge 0$. Based on the convexity of the cost functions, f_n^r, f_i^r, f_i^u, c_{ni}, c_{mi},and c_{ij}, one can have $(I)\ge 0$, $(II)\ge 0$, and $(III)\ge 0$. The proof of $(IV)+(V)\ge 0$ can be derived from Theorem 1 since the function Π_j is concave for each $j,\ j=1,...,J$. Similarly, we can also show that $(VI)+(VII)\ge 0$ under the assumptions of Theorem 3. Therefore $\mathcal{F}(X)$ is monotone.

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