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Incorporating Ridesharing in the Static Traffic Assignment Model

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Abstract This paper develops a combined mode choice and traffic assignment model that incorporates ridesharing as an option in a mode choice model, attempting to quantify the ridesharing market share in an equilibrium context. The mode choice model takes into account that the waiting time for a ride is dependent on the available drivers. The traffic assignment model is a static user equilibrium that interacts with the discrete choice model through level of service variables. An iterative algorithm was implemented and applied in a simple network and a more realistic network. The results indicate that the quantity of ride sharing drivers is a key parameter to the service success, and below a critical mass of drivers, it is unlikely that passengers will find the service valuable. It is also shown that ride sharing has the ability to reduce in-vehicle times for all the users, although passenger may suffer from longer door-to-door times, having to wait for their ride.

Keywords Ridesharing · Traffic assignment · Mode choice · Network equilibrium · Variable demand

1 Introduction

1.1 Carpooling and Ridesharing

Car commuting trips are a major source of congestion in transportation networks. The economic impact of congestion was estimated to be $ 78 billion in 2005 (Schrank and Lomax 2007), due to delays and wasted fuel. Recent data in the U.S. shows that drive alone continues to be the most popular mode of travel in many metropolitan areas,

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while carpooling share does not exceed 10 % of commuters (Bureau of Transportation Statistics 2010). As a consequence, car occupancies at peak hours tend to be very low.

The motivation for using empty car seats to increase occupancy rate is therefore understood. By exploiting the amount of empty seats that are already travelling daily, one could substantially improve the utilization of the urban transportation network, and potentially reduce congestion, emission and parking problems. Modes comprising sharing of rides or vehicles have been proposed and used with limited success over the years. Ridesharing is inherently a non-profit mode that brings together people with similar travel needs to share expenses.

Carpooling, for example, was a relatively popular mode in the 1980's in the US, but has significantly dropped in the more recent years (Pisarski 2006). Empirical studies on carpools show that it is still considered relatively inconvenient mode of transportation, in comparison with both public transit and private car. Ferguson (1997) analyzes the factors associated with carpooling decline in the US. The most important factors found were increased availability of vehicles, reduced gasoline prices, and higher educational level of commuters.

Traditional carpooling, however, is too limiting to accommodate the unconventional schedules of today's ridesharing demand, where flexible commuting times are an important need of many travelers (Levofsky and Greenberg 2001). The main drawback that limits the popularity is the relative difficulty of forming a carpool, and the lack of flexibility. Dynamic Ridesharing (DRS) tries to solve some of these drawbacks by leveraging technological advances like mobile devices, social networks, etc. (Agatz et al. 2010).

Dynamic Ridesharing is an automated service system that finds matches between users that share similar travel needs, in terms of time, origin and destination. A central management of the system replaces the user's need to form a carpool, and in principle enables drivers and riders to find a match on a short notice, or even en-route. Chan and Shaheen (2012) performed a classification of ridesharing modes. In their classification, dynamic ridesharing refers to carpools formed using internet based computerized ride matching (whether or not using smartphones and GPS). This mechanism can prevent information barriers for users who seek a ride share.

In the last decade, many start-up companies initiated DRS services (e.g., - Carticipate, SimRide or Avego). Ghoseiri et al. (2011) performed a thorough review of ridesharing experiments and online services. Clearly, the quality of service given by a dynamic ride-share provider depends on the characteristics of the environment in terms of participant geographic density, traffic patterns, and the available roadway and transit infrastructure. Some of these factors were analyzed by Hall and Qureshi (1997).

To create a ridesharing service, the main task of the service is to perform matching between drivers and passengers in an effective way. The methods and implementation of matching algorithms are investigated for example in Kleiner et al. (2011), and is outside the scope of this paper. For simplicity, we assume a greedy strategy of first-come-first-served, which may be easily implemented by a central service management system, although it may be sub-optimal.

An estimation of the ridesharing market share is given in Tsao and Lin (1999), using a synthetic entropy model. Using statistical data from Los Angeles area, they found that ridesharing has very little potential to reduce the number of commuter trips. On the other hand, a more recent study (Agatz et al. 2011) simulates travel data based on

Atlanta travel survey, and finds that when modern optimization methods are used, ridesharing performance is very good. It is also shown that urban sprawl is not limiting the creation of a sustainable population of ridesharing users.

The incorporation of ridesharing mode with other commute modes was investigated by Habib et al. (2011) for example. They showed how to model carpool in a mode choice model, not only as a distinct choice but also in combination with other modes. Another example is Huang et al. (2000), which explicitly investigate the mode choice between car drivers and passengers. Their suggestion for linear utility functions is used in this paper when considering ridesharing mode choice.

The utility for ridesharing passengers is uniquely affected by the need to wait for a ride. The modelling of waiting time was already treated in transit mode assignments. Wu et al. (1994) developed a traffic assignment solution to the problem of transit route-choice. Waiting time is an arc on the hyper-graph, and depends on the transit line frequency. Yang et al. (2000) used MM1 queue to model waiting time for taxi, which is more suitable for ridesharing, where there is no scheduled frequency, but rather random appearance of vehicles. The taxi model by Yang et al. (2000) presents the cyclic relationship between level of service parameters like waiting time and demand, which is also applicable in this paper.

1.2 Objectives and Scope

This paper will use traffic assignment methods to analyze the behavior of a ridesharing service in a given environment. Mathematical programming has become a standard tool to analyze traffic assignment problems. Extensions of these methods to incorporate elastic (variable) demand were developed by Florian (1977) and others. The variable demand traffic assignment program has a unique solution, as long as the demand function is monotonic. This can be demonstrated for simple cases of modal split, where the properties of the different modes are fixed. But showing these conditions for the general case, where mode utility functions can depend on the network flow quantities, is not easily accomplished.

In Dafermos (1980), variational inequality is used to analyze these cases, in which there may be interactions between links in the network, and between modes on the same link. These methods were successfully implemented in many traffic assignment examples, for example in Nguyen and Pallottino (1988). In Yang and Bell (1997), the variable demand traffic equilibrium program was extended to include links with limited capacity. This was used to show how toll levels could restrain demand in a congested network, but the same behavior is expected for ridesharing performance which is strictly limited by the availability of ridesharing drivers.

A more suitable description of this capacity limitation for passengers is given in Cepeda et al. (2006), where congested transit networks are considered, and a limited capacity is described by a zero frequency. They also investigate the existence of equilibrium in cases in which waiting time functions present infinite asymptotes.

When considering traffic assignment of ridesharing modes, supporting asymmetric utility function is a critical property. As an example, the number of ridesharing passengers in a given path will form a queue, therefore affecting their own utility via their waiting time, and also the utility function for drivers in this path. In order to ensure a solution to corresponding traffic assignment problem, the mode utility functions should form a contraction mapping. The exact conditions for existence and uniqueness of solution in the general asymmetric case were investigated by e.g., Magnanti and Perakis (2004) and may be used in the case of ridesharing modes as well.

Recent studies emphasize the potential benefit of ridesharing in various areas (Amey 2010; Deakin et al. 2010). In reality, though, the bootstrapping problem of creating a critical mass of users that will enable an efficient service still exists. In that sense, the factors that affect the mode choice behavior of users are of high importance. Therefore, this paper will investigate the behavior of ridesharing service by means of user equilibrium optimization, combining both the mode choice behavior of commuters, and the resulting traffic assignment patterns.

The purpose of this paper is to develop a mathematical model which will enable the investigation of the necessary conditions that lead to a successful adoption of ridesharing. A relatively simple discrete choice model for ridesharing vs. other modes will be proposed and incorporated into the user equilibrium framework using traffic assignment methods. As will be explained in detail in subsequent sections, the main contribution of the paper is the development of a combined mode choice and assignment model that accounts for flow-dependent waiting times.

The rest of this paper is organized as follows. The next section describes the model assumptions and develops a mathematical formulation of the problem. The subsequent section illustrates model application for simple networks, and also presents results for a real size network in various conditions. The last section discusses the results and outlines further research directions that will enhance model's capabilities.

2 Model Formulation

2.1 Model Assumptions

Consider a transportation road network. Users on this network are assumed to have a known and fixed demand for travel, and may choose between different modes of transportation. For simplicity, only three modes will be considered: (1) Ridesharing as a driver; and (2) Ridesharing as a passenger; and (3) Driving alone in a private car.

The ridesharing system is assumed to work as follows. Ridesharing drivers are individuals who chose to use their private car, while sharing their travel schedule origin and destination, to enable ride offering for passengers. Passengers are users who chose to travel with a suitable ride offered by the collective of ride sharing drivers. Regularly, part of the ridesharing drivers will have passengers on-board, while others may not have passengers. These are still considered ridesharing drivers, as they are registered in the ridesharing system, and play an important role in lowering the average waiting time for a potential passenger. The actual assignment is probabilistic, and this paper will focus on first order statistics that influence the users' mode choice behavior. Drive alone users will always drive alone, as they never expressed their willingness to take passengers. Note that when seeing a car with a solo driver, we cannot tell if this is a drive alone user or a ridesharing driver that did not have a passenger currently assigned.

This paper does not consider public transportation modes in order to keep the model simple enough, and to focus on the modal split between ridesharing and driving alone. In reality, however, the effect of this assumption on the penetration of ridesharing modes may be significant and require additional research. Implementing ridesharing as an alternative travel mode may also attract passengers from public transportation modes. This might reduce the economic efficiency of the affected public transportation lines, and may harm the quality of service given by the ridesharing drivers to a new passenger. In the scope of this paper, the interaction between ridesharing and public transportation modes will be omitted. On the other hand, if ridesharing drivers are paid for taking passengers, this effect may ensure that a stable population of ridesharing drivers is created and maintained, which is an important key for the success of ridesharing.

This paper assumes a full origin–destination (O-D) coupling between ridesharing drivers and passengers. This means that a passenger riding from a certain origin to his destination will only consider ridesharing drivers that are driving from the same origin and are heading to the same destination. This assumption will simplify the mathematical analysis in our traffic assignment model, while ruling out two important real-life travel options. The first one is the ability of a ride share to occur over only part of the driver's path. From the passenger's perspective, considering only drivers that have the same origin and destination is greatly limiting his options. To make full use of the travel options, we would like every driver that rides through the passenger's origin and destination to be a viable riding alternative. The second limitation is that a passenger has to reach his destination in a single ride. This might be especially limiting when we consider the natural hierarchy that often characterizes a metropolitan network. Another practical consideration is the number of passengers per vehicle. This paper assumes that every ridesharing driver may take only one additional passenger.

2.2 Mode Choice Model

This section presents the utility model assumed for the three available travel modes. In general, the utility for a ride can be formalized as:

$$U_{i,n} = V_i(X_i) + \varepsilon_{i,n} \tag{1}$$

Where $U_{i,n}$ is the utility function of alternative mode i for individual n traveling between origin–destination pair *od*. X_i are mode-specific variables which may include level-of-service variables (such as ride times and waiting times). $\varepsilon_{i,n}$ represents an error term.

Utility functions will be formed as a sum of components, as proposed in (Huang et al. 2000). For ridesharing passengers, the level-of-service will mainly depend on the waiting time until a suitable ride has arrived. For ridesharing drivers, an important factor will be the actual share of drivers that take a passenger. Clearly, not all drivers that expressed their willingness to take passengers will be assigned ones. Furthermore, our model estimates the average utility for the entire mode population regardless of the individual circumstances. Therefore, the utility function should linearly depend on the utilization of ridesharing drivers as follows. For ridesharing drivers (mode 1):

$$V_1(X_1) = \phi(X_1) + \psi_1(X_1) \tag{2}$$

For ridesharing passengers (mode 2):

$$V_2(X_2) = \psi_2(X_2) + \kappa(X_2) \tag{3}$$

For drive-alone (mode 3):

$$V_3(X_3) = \phi(X_3) \tag{4}$$

Where $\phi(X_i)$ is the utility associated with driving a car, which is relevant to mode 1 (ridesharing driver) and mode 3 (drive-alone). $\psi_2(X_2)$ is the cost for the passenger mode, $\kappa(X_2)$ is the cost associated with the passenger waiting time, and $\psi_1(X_1)$ is an additional cost associated with the ridesharing driver mode. This term is further expanded to take into account the chances of taking a passenger:

$$\psi_1(X_1) = \psi_1^0(X_1) + \mu_{od}\psi_1^\mu(X_1) \tag{5}$$

where $\psi_1^0(X_1)$ is the cost for joining the ridesharing driver mode (registering to the service, revealing travel plans, etc.), and $\psi_1^\mu(X_1)$ is the inconvenience caused by taking of passenger (sharing private space, stop for drop-off, etc.). μ_{od} is the average ridesharing utilization per O-D pair, expressed by:

$$\mu_{od} = \frac{f_2^{od}}{f_1^{od}} \tag{6}$$

Where f_1^{od} is the flow of ridesharing drivers, and f_2^{od} is the flow of ridesharing passengers.

Note that μ_{od} creates a cyclic dependency in the mode choice model. The travel utility function $\phi(X_1)$ is assumed to be a linear function as follows:

$$\phi(X_1) = \phi\left(t_r^{od}\right) = -\alpha - \beta_d t_r^{od} \tag{7}$$

The parameter α represents a fixed cost of driving a car, either driving alone or with a passenger. Some of these expenses may be related to long-term decisions, like car ownership. For simplicity, the paper focuses on the part of the cost that is associated with a specific ride, such as parking costs. Other fixed costs may be associated with the convenience of riding your own car, having the flexibility of deciding when and where to go, etc. Some of the factors may depend on the length of the ride, like the cost of gasoline, the user's value-of-time, etc. The parameter β_d indicates in-vehicle value of time.

Note that α can express any fixed difference in the utility between drivers and passengers. For ridesharing passengers, a fixed utility may be associated with being dependent of finding a suitable ride. This is especially true for commuters who choose this mode of travel while having a car, and are willing to trust the ridesharing service to address their travel needs not only for the specific morning ride to work, but also on their ride back. This factor highly depends on public attitude and on the reputation built by the ridesharing service being efficient and reliable.

The extra cost for ridesharing drivers is captured by $\psi_1^0(X_1)+\mu_{od}\psi_1^\mu(X_1)$, where $\psi_1^0(X_1)$ is a fixed cost not dependent on the actual ridesharing utilization, and $\mu_{od}\psi_1^\mu(X_1)$ depends on the ridesharing utilization. In this paper, the following utility functions will be used to describe these components:

$$\psi_1^0(X_1) = -\gamma_1 \tag{8}$$

$$\psi_1^\mu(X_1) = -\gamma_2 - \delta t_r^{od} + R_{od} \tag{9}$$

Where γ_1 is a fixed cost for joining mode 1 (ridesharing driver), γ_2 is a fixed cost for taking a passenger, δ is the additional value of time (VOT) for taking a passenger, and R_{od} is the payment for the ride. γ_1 will be used to express the extra fixed cost associated with being a ridesharing driver. The decision to join the ridesharing service and the willingness to take passengers incorporates an extra dis-utility identified also by (Horowitz and Sheth 1977) and (Dueker et al. 1977). Clearly, not every driver will agree to join the service and take passengers. To do so, a driver not only has to register to the service, but should also agree to reveal his travel plans to the system, and let the system assign passengers to ride with him. Some drivers may see this as an invasion to their privacy, especially having to share private space with a stranger. Others may look at the same process as an opportunity to meet interesting people and have a chat during the ride. So this cost may be highly individual and also depend on cultural attitude and trends.

Note that not all of these who agreed to be ridesharing drivers actually take passengers. The way the ridesharing system works is that a certain number of drivers declare their ride plans and express their willingness to take passengers. Normally, only some of them will be assigned with passengers, depending on the actual demand for rides. Although the willingness to declare travel plans and take passengers may incorporate a fixed cost for all ridesharing drivers, there is an extra cost for actually taking one. This cost creates a differentiation between users in the same travel mode. The passenger assignment decisions are part of the central ridesharing management, and therefore are not known to the users *a priori*, thus cannot be directly taken into account in the travel decisions. However, it can be assumed that after many days of using the system, it will reach an equilibrium in which a ridesharing driver of a specific O-D pair will get to know the chances of being assigned a passenger. This differentiation within the same mode resembles other situations in transportation, e.g., the chance of having an unoccupied seat on the bus.

For the ridesharing passengers, it is assumed the following utility functions:

$$\psi_2(X_2) = \psi_2\left(t_r^{od}, R_{od}\right) = -\beta_p t_r^{od} - R_{od} \tag{10}$$

$$\kappa(X_2) = \kappa(t_w) = -\beta_w t_w \tag{11}$$

The fixed utility for the passenger mode was already accounted for by the parameter α. The variable part of the passenger utility may depend both on in-vehicle time and on waiting time. The utility associated with in-vehicle time includes the subjective value-of-time for the user, and may also include the payment for the ride, if modelled as time-based rather than mileage-based. The parameter β_p represents the effective value of in-vehicle time for passengers.

The need to wait for a ride is an additional burden of being a passenger. Usually waiting time is associated with a larger cost per minute than in-vehicle time (e.g., Bajwa et al. 2008). The parameter β_w represents the value of waiting time. Waiting time itself depends on the specific network conditions. The actual waiting time for a passenger is obviously a random variable.

The appearance of ridesharing drivers and passengers is assumed as a random Poisson process, and passengers are assigned on a first-come-first-served basis. Under these assumptions the waiting time will be modelled as an MM1 queue, with an average waiting time of:

$$t_w = \frac{1}{\left(f_1^{od} - f_2^{od}\right)} \tag{12}$$

Where an average flow of f_2^{od} passengers are waiting for f_1^{od} drivers to take them.

In addition to the ride time, the waiting time is the second cost factor that creates a circular dependency in the mode choice model, as shown in Fig. 1. In addition to the subjective cost of waiting, a passenger may have to pay directly for the ride. Depending on the operational model of the specific ridesharing service, passengers are likely to be charged for the ride. R_{od} represents this charge.

The model specification can be summarized by the following utility functions:

$$V_1\left(t_r^{od}, \mu_{od}\right) = V_3\left(t_r^{od}\right) - \left(\gamma_1 + \mu_{od}\left(\gamma_2 + \delta t_r^{od} - R_{od}\right)\right) \tag{13}$$

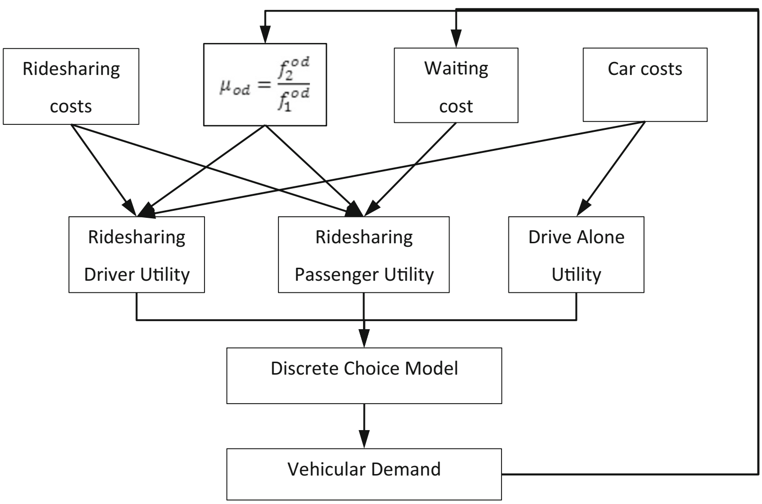

Fig. 1 Factors affecting the mode choice model

$$V_2\left(t_r^{od}, t_w\right) = -\left(\beta_p t_r^{od} + \beta_w t_w + R_{od}\right) \tag{14}$$

$$V_3\left(t_r^{od}\right) = -\left(\alpha + \beta_d t_r^{od}\right) \tag{15}$$

Given any utility function for the various modes under consideration, a discrete choice model would represent the way of selecting the travel mode based on these utilities. A straightforward model would be the multinomial logit model, which is simple to formalize, assuming that the error terms $\varepsilon_{i,n}$ are independent and identically distributed. A key disadvantage of this model is its inability to express correlation between alternatives. For example, using the multinomial logit model will result in a fixed ratio between ridesharing drivers and the ones who drive alone, regardless of the quantity of passengers. Different results are expected in reality, since changing the number of passengers will affect the ridesharing drivers' utility, and therefore change the ratio between these drivers, and the ones that drive alone.

Note that even the multinomial logit model can successfully handle the trade-off between drivers and passengers, by suitably defining a utility model that can express this trade-off. In other words, the change in the number of passengers can affect the ratio of drivers not only through the mode choice model, but also through the utility functions of the model. Consider, for example, the utility functions in Eqs. 13 to 15, and assume that the number of passengers changed due to a rise in β_p. Realizing that the utility function for ridesharing drivers directly depends on the passenger utilization, the flow ratio is going to change even under the simple logit model.

2.3 Combined Mode Choice and Assignment Problem

This section develops the traffic assignment model that incorporates ridesharing as a mode choice alternative. Consider a road network, with a fixed demand Q_{od} defined for every origin–destination pair in the network. Throughout the analysis of the model, it is assumed that the system has reached a steady state. We will therefore refer to static traffic assignment variables that represent the equilibrium values of flows. The total demand per O-D pair will be divided into the three modes of transportation:

$$Q_{od} = \sum_i f_i^{od} \tag{16}$$

Where f_i^{od} represents the person trips for mode i traveling between od. To analyze the traffic assignment in this model, this paper uses notations and methods, which parallels the analysis of transit modal split using variable demand traffic assignment models, as in Florian (1977) and Sheffi (1985). The variable car demand between od is related to the vehicular flow as follows:

$$q_{od} = f_1^{od} + f_3^{od} \tag{17}$$

This includes the ridesharing drivers (mode 1) and drive alone drivers (mode 3). The flow of drivers defined by Eq. (17) may be further divided according to path choice. For every path k between O-D, a different flow of drivers exists, such that:

$$q_{od} = \sum_k f_{1k}^{od} + \sum_k f_{3k}^{od} \tag{18}$$

This paper assumes a constant ratio between mode 1 and mode 3 for every path k. This is equivalent to assume that mode choice and route choice are independent decisions, and therefore uncorrelated. This assumption simplifies the traffic assignment model, since only the variables (f_{1k}^{od}+f_{3k}^{od}) needs to be handled. The flow for each mode can be easily computed by:

$$f_{1k}^{od} = \frac{f_1^{od}}{f_1^{od} + f_3^{od}} \left(f_{1k}^{od} + f_{3k}^{od} \right) \tag{19}$$

It is further assumed that a passenger does not choose a specific driver, and also do not choose a specific route. Passengers are assigned to drivers on a first-come-first-served basis, without any correlation to the route choice of that driver. Therefore, the distribution of passengers between routes will also reflect the distribution of drivers of the same O-D pair between their available routes.

To combine the mode choice model into the framework of the traffic assignment, we will define the following vector variables. The flow vector F^{od} is defined as:

$$F^{od} = \begin{pmatrix} f_1^{od} \\ f_2^{od} \\ f_3^{od} \end{pmatrix} \tag{20}$$

The utility function V^{od} is represented as:

$$V^{od} = \begin{pmatrix} V_1^{od}\left(t_r^{od}, F^{od}\right) \\ V_2^{od}\left(t_r^{od}, F^{od}\right) \\ V_3^{od}\left(t_r^{od}, F^{od}\right) \end{pmatrix} = V\left(t_r^{od}, F^{od}\right) \tag{21}$$

The mode choice model can also be written in a vector form. For example, the following vector represents the flow vector according to the multinomial logit model:

$$F^{od} = M\left(V^{od}\right) = \begin{pmatrix} e^{V_1^{od}\left(t_r^{od}, F^{od}\right)} \\ e^{V_2^{od}\left(t_r^{od}, F^{od}\right)} \\ e^{V_3^{od}\left(t_r^{od}, F^{od}\right)} \end{pmatrix} \frac{Q_{od}}{\sum_i e^{V_i\left(t_r^{od}, F^{od}\right)}} \tag{22}$$

Combining the utility functions with the mode choice model, the following relationship is created:

$$F^{od} = MV\left(t_r^{od}, F^{od}\right) \tag{23}$$

Since the ride time is flow dependent, this relationship does not have an analytic form. However, there are enough constraints to solve for F^{od} given t_r^{od}. The solution of this relationship can be named as "the ridesharing demand function":

$$F^{od} = D\left(t_r^{od}\right) \tag{24}$$

The function D determines the modal split for every value of ride time. Specifically, we can look at the second component of D which relates to the average flow of passenger, and take the inverse function:

$$t_r^{od} = D_2^{-1}\left(f_2^{od}\right) \tag{25}$$

This can be treated as a synthetic impedance function for the passenger flow. The combined modal split between car riders and driver traffic assignment on the network can then be formulated similar to Sheffi (1985) and Patriksson (2004) by the following mathematical program:

$$Min\ Z\left(x, f_2^{od}\right) = \sum_a \int_0^{x_a} t_a(w)dw + \sum_{od} \int_0^{f_2^{od}} \left(D_2^{od}\right)^{-1}(w)dw \tag{26}$$

Subject to:

$$\sum_k \left(f_{1k}^{od} + f_{3k}^{od}\right) + f_2^{od} = Q_{od} \tag{27}$$

$$\sum_{odk} \left(f_{1k}^{od} + f_{3k}^{od}\right)\delta_{ak}^{od} = x_a \tag{28}$$

$$f_{1k}^{od} \geq 0 \qquad f_{2k}^{od} \geq 0 \qquad f_{3k}^{od} \geq 0 \qquad Q_{od} \geq 0$$

The program is formulated in terms of the variable passenger demand per O-D pair f_2^{od}, and the variable car flow per network link x_a, where the constraints verify that the variable car flow is indeed a feasible solution. δ_{ak}^{od} is 1 if route k between od passes through link a and 0 otherwise.

Note that in general, additional constraints may be imposed by the mode choice model. For example, the cost model described above use MM1 queue to model the passenger waiting time. In this model it is necessary that:

$$f_1^{od} > f_2^{od} \tag{29}$$

These constraints will further limit the feasible area for the program solution, as discussed in the subsequent section.

The first-order conditions for the solution are detailed in Sheffi (1985) and are briefly outlined here. Assume that a solution for this mathematical program does exist, for each of the O-D pairs. For the values that solve this program, we can therefore write the following identity:

$$Y = F^{od} - MV\left(t_r^{od}, F^{od}\right) = 0 \tag{30}$$

And consequently,

$$dY = J_y\left(F^{od}\right)dF^{od} + \frac{\partial Y}{\partial t_r^{od}}dt_r^{od} = 0 \tag{31}$$

Where $J_y(F^{od})$ is the Jacobian matrix of the functional Y, and can be found by:

$$J_y\left(F^{od}\right) = I - J_M(V)J_V(F) \tag{32}$$

$J_M(V)$ and $J_V(F)$ are respectively the Jacobian matrices of the specific utility functions (V) and discrete choice model (M) under consideration. The flow vs. ride time derivative functions may therefore be obtained by:

$$\frac{dF^{od}}{dt_r^{od}} = -J_Y\left(F^{od}\right)^{-1}\frac{\partial Y}{\partial t_r^{od}} = J_Y\left(F^{od}\right)^{-1}J_M(V)\frac{dV}{dt_r^{od}} \tag{33}$$

The resulting function is the demand function of each of the modes in our system with respect to the ride time. It should be tested for monotonicity in order to prove uniqueness of model under consideration.

2.4 Algorithm Implementation

The algorithm is based on the Method of Successive Averages (MSA), but with an additional loop. The algorithm iterates between computing the utility functions needed for the mode choice model, and solving the traffic assignment program. To initialize, free flow travel times are used to calculate the initial modal split. In the modal split phase of the algorithm, the travel times are assumed to be fixed (based on the previous iteration values), and for these values the algorithm needs to solve the cost model equations and find the demand values (passenger and vehicle travel demand) that will satisfy the cost model equations.

The cost model should be solved for every O-D pair in the network separately. These solutions are indeed separable due to our basic assumption that couples only drivers and passengers from the same origin and destination. In a more complicated model in which drivers from one origin can take passengers en-route (e.g., Li et al. 2014), this phase will require a network-level solution. The iterative scheme used for solving the problem is illustrated in Fig. 2.

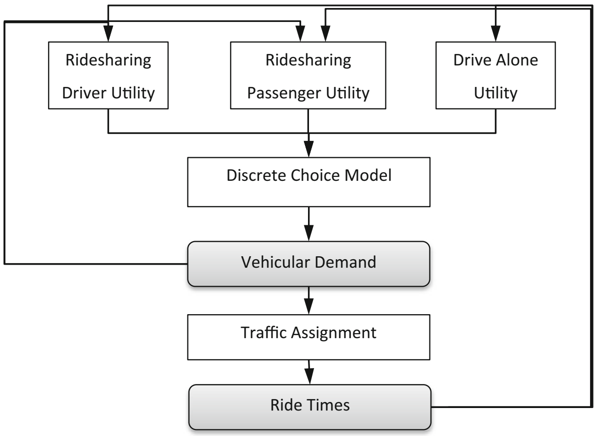

Fig. 2 Iterative scheme for solving the mode choice and traffic assignment model

Note that modal split calculation is itself a numerical process which requires an iterative process, since the cost equations depend on the actual demand variables through either waiting time or utilization variable. After solving the cost equations, the second phase of the algorithm solves a traffic assignment problem with these demand values as a fixed demand problem, using convex combinations or any other method. The entire process will converge when the cost model equations will return the same values as in the previous iterations.

3 Results

3.1 Simple Case: Single O-D Pair

This section presents results for a simple network to illustrate the equilibrium properties of the combined modal split and assignment problem. This model allows to find an analytical solution of the problem. Consider a network with a single origin and a single destination. The user equilibrium in this network can be obtained either directly using a variable demand function, or by adding an excess-demand link that represent the passenger flow.

Consider the utility functions represented in Eqs. (13) to (15), with the following parameters:

$$\gamma_1 = \gamma \;\; ; \;\; \gamma_2 = 0\,; \delta = 0 \;\; ; \;\; R_{od} = 0 \tag{34}$$

The simplified utility functions are then expressed as follows:

$$V_1\left(t_r^{od}, \mu_{od}\right) = -\left(\alpha + \beta_d t_r^{od}\right) - \gamma \tag{35}$$

$$V_2\left(t_r^{od}, t_w\right) = -\left(\beta_p t_r^{od} + \beta_w t_w\right) \tag{36}$$

$$V_3\left(t_r^{od}\right) = -\left(\alpha + \beta_d t_r^{od}\right) \tag{37}$$

The ratio between car flow (mode 3 and mode 1) is expressed as follows:

$$\frac{f_3^{od}}{f_1^{od}} = \frac{e^{V_3^{od}}}{e^{V_1^{od}}} = e^{V_3^{od} - V_1^{od}} = e^{\gamma} \tag{38}$$

Assuming the simple multinomial logit model (MNL), the demand for each mode is expressed as follows:

$$f_1^{od} = \frac{q_{od}}{1 + e^{\gamma}} \tag{39}$$

$$f_2^{od} = Q_{od} - q_{od} \tag{40}$$

$$f_3^{od} = \frac{e^{\gamma}}{1 + e^{\gamma}} q_{od} \tag{41}$$

Assuming a Nested Logit (NL) model with two nests: Driver (mode 1 and mode 3) and Passenger (mode 2), the probability P_D of being a driver for a given OD pair is given by (we omit the index OD for simplicity):

$$P_D = \frac{e^{V_D}}{e^{V_D} + e^{V_2}} = \frac{1}{1 + e^{V_2 - V_D}} \tag{42}$$

Where the composed utility V_D is expressed by:

$$V_D = \mu \ln\left(e^{V_1} + e^{V_3}\right) \tag{43}$$

And the probability of being a ridesharing driver is then given by:

$$P_1 = \left(P_{1|D}\right) P_D = \frac{e^{V_1}}{e^{V_1} + e^{V_3}} \frac{e^{V_D}}{e^{V_D} + e^{V_2}} = \left(\frac{1}{1 + e^{V_3 - V_1}}\right)\left(\frac{1}{1 + e^{V_2 - V_D}}\right) \tag{44}$$

In general, the inverse function for the NL model cannot be obtained analytically and has to be evaluated numerically. However, for the simplified utility functions (Eqs. 35 to 37), it is possible to develop the excess demand function for the passenger mode ($D_2^{od}(f_2^{od})$) as follows.

Equation (44) produces the following expression:

$$P_1^{od} = \frac{f_1^{od}}{Q_{od}} = \frac{Q_{od} - f_2^{od}}{Q_{od}(1+e^{\gamma})} = \left(\frac{1}{1+e^{\gamma}}\right)\left(\frac{1}{1+e^{V_2-V_D}}\right) \tag{45}$$

Using the simplified utility functions (Eqs. 35 to 37), we can write:

$$V_2 - V_D = \ln\left(\frac{f_2^{od}}{Q_{od} - f_2^{od}}\right) = -\beta_p t_r^{od} - \beta_w t_w + \mu\alpha + \mu\beta_d t_r^{od} + \mu\gamma - \mu\ln(1+e^{\gamma}) \tag{46}$$

After some manipulations, the excess demand function can be expressed by:

$$t_r^{od} = D_{od}\left(f_2^{od}\right) = t_p + \frac{1}{\theta} ln\left(\frac{f_2^{od}}{Q_{od} - f_2^{od}}\right) + \frac{K_w}{Q_{od} - \epsilon f_2^{od}} \tag{47}$$

Where:

$$\theta = \mu\beta_d - \beta_p \tag{48}$$

$$t_p = \frac{\mu}{\theta}(ln(1+e^{\gamma}) - \alpha - \gamma) \tag{49}$$

$$K_w = \frac{\beta_w(1+e^{\gamma})}{\theta} \tag{50}$$

$$\epsilon = 2 + e^{\gamma} \tag{51}$$

The above relationships are illustrated in Fig. 3, which shows the components of the function ($D_2^{od}(f_2^{od})$). The function is depicted for α=1; ϵ=5;θ=0.2; β_w= 5; μ=0.9. The first term of the function (t_p) can be interpreted as a constant time difference between the passenger mode and the driving mode. The second term of the function is the equivalent impedance of the logit mode choice function. These first two terms are depicted as 'Logit Component' in the graph. The last term, which is unique to ridesharing passengers, represent the time that passengers have to wait for a suitable driver (in this case, a driver that has the same O-D pair as the passenger). It is a rising hyperbolic term that adds a vertical asymptote to the utility function, such that the maximum flow of passengers is $^{Q}/_{\epsilon}$. It is also worth noting that this function is strictly monotonic (both terms are increasing with increasing passenger flows). This implied that the objective function in Eq. (26) is strictly convex. Additional constraints are derived from:

$$\epsilon f_2^{od} < Q_{od} \tag{52}$$

Which are equivalent to the constraints $f_1^{od} > f_2^{od}$. Since the new constraints are linear, the solution space remains convex, and therefore the program has a unique solution.

The influence of the total demand on the equilibrium is illustrated as follows. The network is solved for different levels of demand, varying from 20 to 60 vehicles per minute, when the capacity of the single physical link is 30 vehicles per minute. The model parameters used for these runs were:

$$\alpha = -3\ ; \gamma = \ln(8)\ ; \beta_d = 0.06\ ;\ \beta_p = 0.03;\ \beta_w = 0.12\ ; \overline{q} = 20\left[\frac{veh}{min}\right]$$

For each of the demand values, the network is solved twice: once as a single mode network for reference, and second with ridesharing. Figure 4 present how performance metrics change with this varying demand. All metrics but the last one refers to the left vertical axis. The average passenger share is the share of passengers out of all the travel demand in the network. The figure shows that congestion gives rise to more passengers, but there is an asymptote that limits this rise. For the values of parameters used (e^{γ}=8), the maximum passenger share is 10 % (corresponds to 80 % of drive alone users, and an additional 10 % of ride sharing drivers). Additional runs with heavier loads show this asymptote empirically.

The congestion reduction describes the relative reduction in the network's average v/c (showing here the only link in this network). In this simple network, it simply follows the share of passengers.

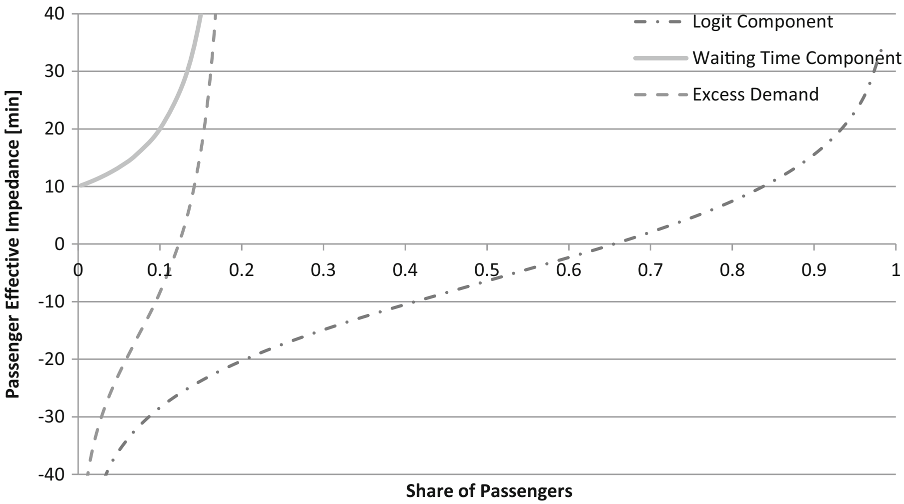

Fig. 3 Components of the Ridesharing demand impedance function

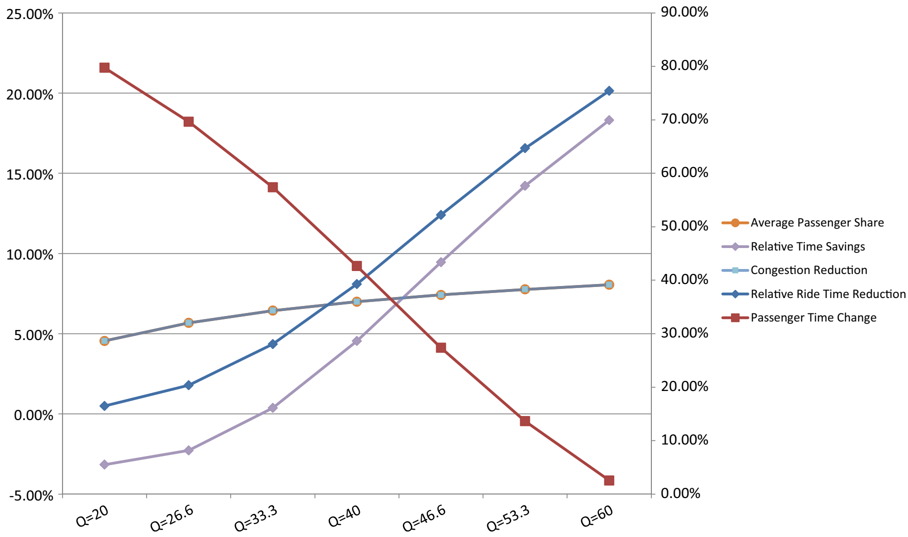

Fig. 4 Ridesharing performance metrics

The 'Relative Ride Time Reduction' metric reflects the shorter ride times in the network, compared to the single mode solution. When demand level is low, there is virtually no reduction in ride times, but when the demand level is increased, even the small percentage of passengers makes a large difference in ride times. This is due to the BPR-type link impedance function, which is very sensitive to flow at high demand:

$$t = t_0\left(1 + 0.6\left(\frac{V}{C}\right)^4\right) \tag{53}$$

Where V/C is the volume/capacity ratio for a given link and t_0 is the free-flow travel time.

The 'Relative Time Savings' metric measures the relative changes in the cumulative system time T which refers to the entire amount of time spent by the users on the road, not just the ride time (again, the reference point is the single mode traffic assignment). Fig. 4 shows that the system sometimes creates losses: when demands are low, the few passengers that did chose this mode are suffering from very large waiting time, since the available number of drivers is not high enough.

As demand builds up, more and more people become passengers, so there are more persons waiting. But ridesharing drivers are now more common, so waiting time per passenger considerably drop. Compared to the total system time, the effect of waiting time gets smaller as demand grows.

Another important key metric, is what happened to the passengers' door-to-door time. Note that for this metric, the vertical axis is the right one in Fig. 4. Knowing that the system on average is beneficial does not mean that all the individuals gained. Recall that the passengers are the ones that 'pay' the waiting time. The 'Passenger Time Change' metric compares the new passenger time (composed of the reduced ride time with the waiting time) to the original ride time in this system without having ridesharing as an option (single mode solution). Fig. 4 shows that although ride times were considerably reduced with ridesharing, passengers still lose time, under all the experimented demand levels.

Note that having these passengers choose this mode despite that fact is not a paradox. We modelled their value of time as lower than the drivers' ($\beta_d > \beta_p$), so in terms of utility, they have not worsened their conditions. One can claim that the model is not calibrated to reflect the actual attitude towards time gains or losses, especially when passengers can be assumed as yesterday's drivers.

3.2 'Critical Mass' of Ridesharing Drivers

One of the issues that hinder the successful implementation of ridesharing services is the need to create a critical mass of ridesharing drivers. The amount of drivers (per O-D pair) is the basis for the passenger waiting times, and without reasonable waiting times for passengers, a pool of passengers will not form. To focus on the effect of ridesharing on the traffic, let us assume that users that register to the service own a car, and they are willing to leave their car at home and commute as passengers, if their utility does not decrease. Suppose that p is the percentage of the users that are registered to the ridesharing service (passengers and drivers). Then assuming the simple utility functions, the passengers' flow will be:

$$pQ_{od} = f_1^{od} + f_2^{od} = \frac{Q_{od} - f_2^{od}}{1 + e^{\gamma}} + f_2^{od} \tag{54}$$

$$f_2^{od} = \left(\frac{p - \frac{1}{(1 + e^{\gamma})}}{1 - \frac{1}{(1 + e^{\gamma})}} \right) Q_{od} \tag{55}$$

Since flows cannot be negative, we can clearly observe a critical mass of users that is necessary to obtain a positive flow of passengers. This is graphically presented in Fig. 5. According to the current assumptions of this model, a resulting critical users share of p_{min} is needed before a positive flow of passengers will start to form:

$$p_{min} = \frac{1}{(1 + e^{\gamma})} \tag{56}$$

It is interesting to note that in this model, not only a minimum share of users is formed, but also a maximum share of users. As increasingly more users add to the ridesharing service, there are also more passengers. From Eqs. (54) and (55) it can be shown that when p rises, the number of passengers rises faster than the number of drivers. When p reaches a critical maximum value, the number of passengers will equal

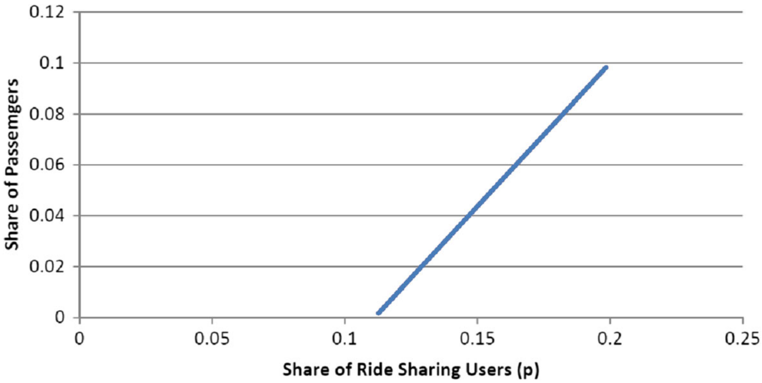

Fig. 5 Critical mass of drivers in ridesharing mode

that of the ridesharing drivers. This is impossible in our model since it leads to infinite waiting time for a ride. This share is calculated by developing an expression to the waiting time as a function of p as follows:

$$f_1^{od} = \frac{f_3^{od}}{e^{\gamma}} = \frac{1-p}{e^{\gamma}} Q_{od} \tag{57}$$

$$t_w = \frac{1}{\left(f_1^{od} - f_2^{od}\right)} = \frac{e^{\gamma}}{Q_{od}(2-(2+e^{\gamma})p)} \tag{58}$$

This function is a rising hyperbolic with an asymptote at:

$$p_{max} = \frac{2}{2+e^{\gamma}} \tag{59}$$

Finally, at the minimum critical share, the waiting time is expressed as follows:

$$t_w(p_{min}) = \frac{1+e^{\gamma}}{Q_{od}} \tag{60}$$

This is an expected result which can be used as a sanity-check for our analysis. For the minimum ridesharing share there are no passengers yet, and from all the cars on the road, one per every $1+e^{\gamma}$ is a ridesharing driver, as a result of our cost model.

3.3 Full Scale Network Runs

After demonstrating the basic features of the models on a simple case, this section discusses selected results on a large scale real-life network. The Winnipeg network available in the EMME software and used in Bekhor et al. (2008) is used for demonstrating the results.

Results were received by implementing convex combinations method on program (26) with the constraints (27),(28) and (29), using the same cost model described earlier (Eqs. (12) ..(15), (34)).

Similar to the simple network, the full-scale features of the entire network performance are compared for two model runs (with and without ridesharing). Figure 6 shows the metric previously introduced (average v/c ratio) and its reduction when introducing ridesharing mode. The same parameter values as in the simple network case were applied.

The horizontal axis represents the scaling of travel demand with respect to the reference data. The figure shows that until the average v/c ratio in the network reaches a value of about 2.0, ridesharing hardly has any effect on reducing the congestion. As noted in Bekhor et al. (2008), the Winnipeg network is moderately congested. When the travel demand is scaled up, the effect is more pronounced as ridesharing successfully removes a larger share of the congestion by reducing the vehicular demand.

Figure 7 shows some of the metrics that describe the overall network performance under ridesharing (system time, average v/c, average passenger share and relative ride time reduction). First, note the relatively low share of ridesharing passengers. Unlike the simple network case, here the passenger share does not reach the saturation (at a share of 20 % using the current parameter values passengers will run out of drivers to take them. Namely $f_1^{od}=f_2^{od}$), but rather stays on level of less than 10 % at the maximum.

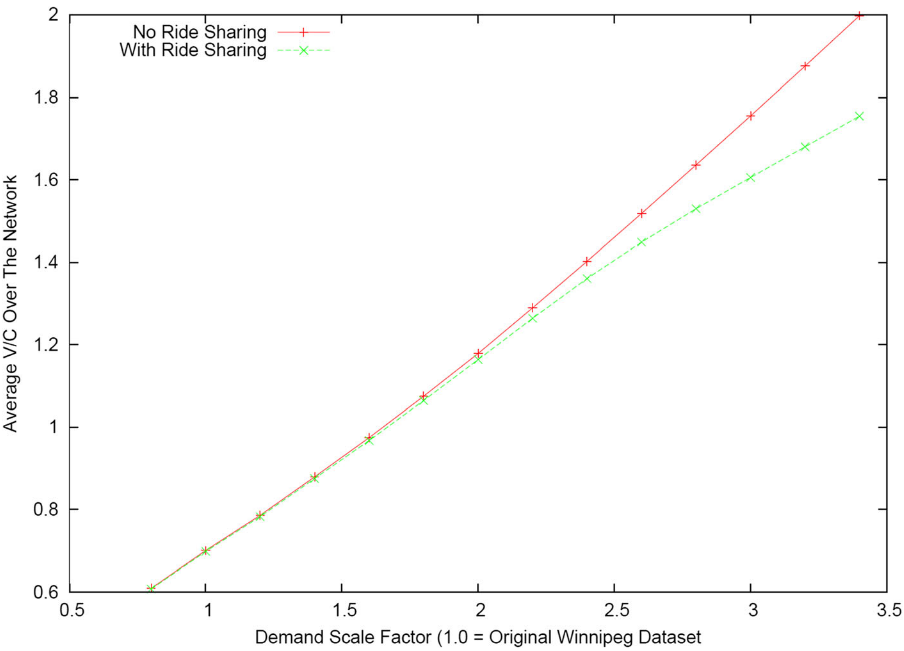

Fig. 6 Congestion Improvement in Winnipeg network

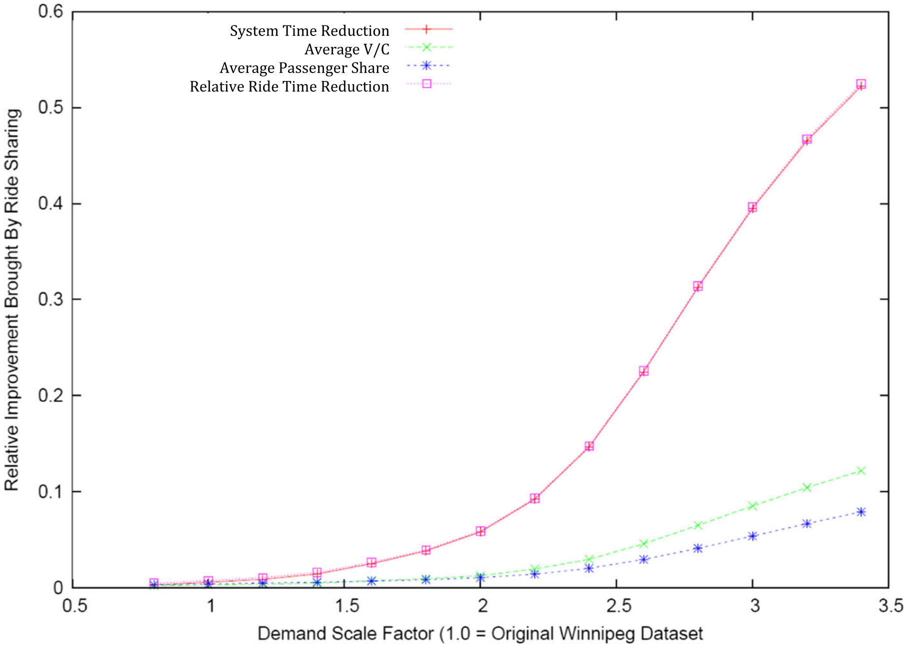

Fig. 7 Performance metrics of ridesharing in the Winnipeg network

To measure the average congestion in the network, we introduced a metric called "Average Congestion Ratio" which reflects the weighted average congestion over the network links. This metric was calculated by:

$$\frac{\sum_a v_a \left(\frac{v_a}{c_a}\right)}{\sum_a v_a} \tag{61}$$

It is a weighted average of the v/c in all the network links, where the weight corresponds to the link flow. In Fig. 6 we can see that the average congestion ratio (weighted v/c) was reduced by an amount that is larger than the share of passengers. This is surprising, since we expect the average congestion reduction to follow the passenger share.

This is due to a bias in the weighted average calculation: network links with higher flows are weighted more in the average congestion calculation. Moreover, it is likely that most of the longer and most congested paths in this network run through main transportation trunks. This creates a positive bias that tends to enlarge this specific metric.

The figure also shows the reduction in ride time (on average), and the reduction in total system time, due to ridesharing introduction. Both follow the same values, which can reach considerable levels of savings, mostly due to the combination of heavy network loading.

Note that although the total system time decreases, the door-to-door time for passengers increases, because of increasing waiting time, as shown in Fig. 8. While ride time can only be lowered with ridesharing, passenger also have a waiting time

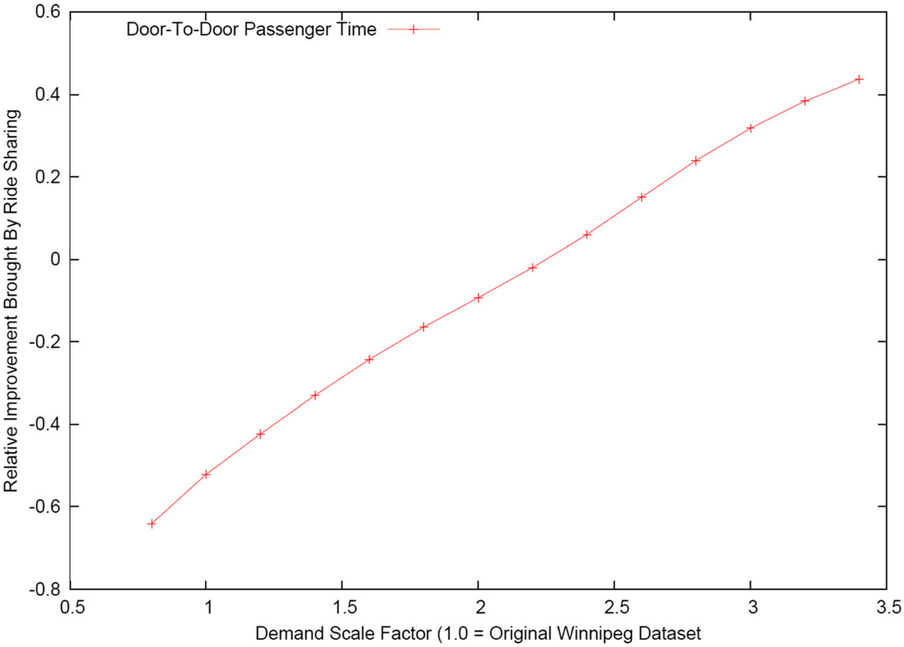

Fig. 8 Passenger door-to-door time in Winnipeg network

added to their balance. The figure shows that up to scaling the demand values by a factor of at least 2.2, passengers' time improvement is negative. The reason they chose to be passengers despite this fact is due to the mode choice model specification (i.e., they have lower VOT), but the door-to-door time can serve as a "sanity" metric to verify that this mode choice is not implausible, especially when migration between modes is considered.

4 Discussion

The theoretical potential of ridesharing in reducing congestion, emissions and travel costs cannot be ignored. This paper developed a quantitative framework that will enable the analysis of the potential benefit from ridesharing in road networks under various conditions.

The static equilibrium network approach was assumed to model the effect of ridesharing in the network. Discrete choice models were applied to model ridesharing modes of travels, and combined traffic assignment methods to model ridesharing as a viable travelling mode. This enabled the investigation of ridesharing performance, and its dependency on both the mode choice parameters, and the network conditions.

One of the most important factors for creating a successful ridesharing service is to create a large population of drivers. This population is critical to create reasonable waiting times for passengers. An individual who is initially a car driver may consider changing modes and become a ridesharing passenger. But for this kind of decision to happen and be stable, the waiting time (both the expected and the experienced) should not be too high. Our model predicts the share of both drivers and passengers based on discrete choice models (assuming fixed parameter values). In cases where the pool of potential ridesharing drivers is not large enough, the equilibrium solution will show very long waiting times. This result may suggest that long waiting times are an obstacle to the adoption of ridesharing, but further research is needed to investigate a causal relationship.

The paper incorporated ridesharing modes into the classical traffic assignment programs. This framework can be used to analyze ridesharing usage and its effect on congestion in various networks, scenarios and parameters. The paper illustrated the role of ridesharing mode as an additional way of restraining congestion, similar to the effect of adding bypass routes to a network.

In order to present a simple framework for solving a network with ridesharing modes, we used several simplifying assumptions. Some of these are too limiting to fully describe the real life behavior of ridesharing modes in actual scenarios. The most limiting assumption is probably the coupling of ridesharing driver and passengers that have the same origin and destination. In practical situation, the need for a passenger to find a ridesharing driver with the same O-D will be very limiting, hindering the migration of users from drivers to passengers.

Another line of research is to investigate the ‘first-come-first-served’ rule for the drivers. In this assumption, passengers that are located upstream with respect to a specific path have the advantage of more drivers from the origin. Alternative ways could try and pre-assign some of the drivers to passengers that are waiting downstream. In this case, the static assignment model used in this paper has to be replaced by a dynamic assignment model, as indicated in Peeta and Ziliaskopoulos (2001).

Another issue that should be handled is the resource management of empty seats. In this paper, a ridesharing driver could be either occupied or not. In case of en-route riding, the model will have to track which route segment the driver has taken a passenger, and for which segments there is a free seat for another passenger. We can also expand the occupancy to more than one passenger per car, and this can even create more options for picking up and dropping of passengers.

We expect that incorporating en-route ridesharing into the model will also have an effect on drivers’ route choice. Recall that ridesharing can be viewed as amending the network with additional bypass routes. In that sense, we can expect some drivers to find it beneficial to change their route in order to take a passenger, if they are rewarded for taking one.

Passengers may also consider the option of combining rides, similar to transfers in public transportation. A passenger from a small suburb will have a hard time finding a driver if he insists on a destination fit. A better strategy for him would be to find a ride out of the suburb to either the metropolitan CBD, or some more central junction, and then choose between (hopefully many) available rides to its destination. Adding this feature to the model will require some strategy planning for the passengers. Similar to public transportation, a passenger may choose between waiting for an infrequent but direct ride (from origin to destination), and more frequent but with multiple rides that can reach to his destination.

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