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Optimizing the Location of a Production Firm

Zvi Drezner · Carlton H. Scott

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Abstract A facility needs to be located in the plane to sell goods to a set of demand points. The cost for producing an item and the actual transportation cost per unit distance are given. The planner needs to determine the best location for the facility, the price charged at the source (mill price) and the transportation rate per unit distance to be charged to customers. Demand by customers is elastic and assumed declining linearly with the total charge. For each customer two parameters are given: the demand at charge zero and the decline of demand per unit charge. The objective is to find a location for the facility in the plane, the mill price charged to customers and the unit transportation rate charged to customers such that the company's profit is maximized. The problem is formulated and an algorithm that finds the optimal solution is designed and tested on randomly generated problems.

Keywords Location · Economics · Elastic demand · Pricing

1 Introduction

The Weber problem (Weber 1909; Wesolowsky 1993; Drezner et al. 2002) is one of the most useful and researched problem in the location literature.

Z. Drezner (✉)
Steven G. Mihaylo College of Business and Economics,
California State University-Fullerton, Fullerton, CA 92834, USA
e-mail: zdrezner@fullerton.edu

C. H. Scott
Paul Merage School of Business, University of California-Irvine, Irvine, CA 92697, USA

It is based on a set of demand points generating demand for a facility and the best location of the facility that minimizes the total transportation cost to the facility is sought. Many papers investigate the problem in the plane with Euclidean distances. For a review see Drezner et al. (2002). Weiszfeld (1937) suggested an iterative procedure to solve the problem using Euclidean distances. When using rectilinear distances the problem is separable into two one dimensional problems and the solution in each dimension is the median point (Love et al. 1988; Drezner et al. 2002). This result is the source of the name 1-median problem that was generalized to the p-median problem when the location of *p* facilities is sought (Current et al. 2002). Hakimi (1964, 1965) proved that the solution in a network environment to the Weber (1-median) problem is on a node of the network which suggests a very simple algorithm for the solution to the Weber problem in a network environment. Therefore, researchers are interested only in developing solution approaches to the p-median problems which are not that simple (Daskin 1995; Current et al. 2002; Alp et al. 2003). p-median problems on the plane are sometimes called location-allocation problems because demand points are allocated among the facilities and the location of each facility is at the Weber solution based on the subset of demand points allocated to it (Cooper 1963; Beaumont 1980; Chen 1983; Love 1976; Love and Morris 1975; Sherali and Shetty 1977).

Moses (1958) introduced specific considerations related to the location of a production firm in his seminal paper about location and the theory of production. He claimed that "there is no need for much of the esoteric paraphernalia sometimes employed by location specialists", a statement that generated many follow-up papers generalizing his approach (Sakashita 1967; Bradfield 1971; Emerson 1973; Khalili et al. 1974; Osleeb and Cromley 1977; Miller and Jensen 1978). More recent papers on various aspects of the topic are by Martinich and Hurter (1982) and Tobin and Friesz (1986). The book by Hurter and Martinich (1989) provides a comprehensive analysis of the location of a production firm. They consider models that incorporate cost (such as electricity cost that may change by region) and transportation cost (linear in the distance). There are no customers in their models (both the facility and the demand points are part of the firm), and their objective is to minimize the cost to the firm.

Another stream of research considers the cost charged to customers as a variable in location models. The planner determines the cost to be charged to customers in addition to the location decision. Drezner and Wesolowsky (1996) investigated the location-allocation problem making the cost charged to users by a facility a function of the total number of users patronizing the facility. Users select the facility to patronize based on the facility charges and transportation cost. In a competitive environment such a situation may or may not lead to an equilibrium solution (Cournot-Nash or Stackelberg). Such models are discussed in Miller et al. (1992, 1993, 1996, 2007); Tobin et al. (1995); Tobin and Friesz (1986); Friesz et al. (1988a, b, 1989).

Some models assume that demand is declining when the distance to the facility increases. Drezner and Scott (2006) investigated the Weber problem including a queuing component assuming that demand is declining by the distance and therefore some of the demand is lost. Berman et al. (2003) investigated the problem that the reliability of the service, thus demand served, is declining with the distance to the facility. In the competitive facility literature there are many papers that assume that the demand is declining with the distance to the facility (for example, Huff 1964, 1966; Drezner 1994, 1995).

In this paper we investigate a problem where demand depends on the mill price and transportation rate charged by the facility which depend on the distance to the facility. In addition to finding the best location for the facility, there are two parameters to be determined by the planner: the mill price and the transportation rate per unit distance charged by the facility. Demand by customers is elastic and thus depends on the total price charged for the good which depends on the distance between the demand point and the facility.

The paper is organized as follows. In Section 2 the problem is formulated and in Section 3 solution algorithms are presented. In Section 4 computational experiments are reported and we conclude in Section 5.

2 Formulation

A facility needs to be located in the plane to sell goods to a set of demand points. The facility charges the customers for the product a mill price P and a transportation rate T per unit distance. The cost for producing an item is C, and the transportation cost to the company is T_0 per unit distance. One would expect that for profit maximization $T \geq T_0$ but the optimal strategy for the company may be to subsidize the transportation rate. Demand by customers is elastic. For customer i demand is $w_i - \Delta w_i P_T$ where w_i is the y intercept of the elastic demand line and Δw_i is its negative slope. P_T is the total charged price. The problem is based on three variables: the location of the facility $X = (x, y)$, the mill price charged by the facility P and the transportation rate per unit distance charged by the company T. The problem is to find X, P and T such that the company's profit is maximized.

Let $d_i(X)$ be the distance between demand point i and the facility located at X. The cost charged to demand point i is $P_i(X) = P + T \times d_i(X)$. The demand generated at demand point i is $w_i - \Delta w_i P_i(X)$. The total profit of the company, $F(X, P, T)$, to be maximized, is therefore

$$F(X, P, T) = \sum_{i=1}^{n} \left[w_i - P\Delta w_i - T\Delta w_i d_i(X)\right]\left[P - C + (T - T_0)d_i(X)\right] \quad (1)$$

$$= A_1 P^2 + 2A_2 PT + A_3 T^2 + 2A_4 P + 2A_5 T + A_6 \quad (2)$$

where

$$A_1 = -\sum_{i=1}^{n} \Delta w_i; \quad A_2 = -\sum_{i=1}^{n} \Delta w_i d_i(X); \quad A_3 = -\sum_{i=1}^{n} \Delta w_i d_i^2(X)$$

$$A_4 = \frac{1}{2}\left\{\sum_{i=1}^{n} w_i + C\sum_{i=1}^{n} \Delta w_i + T_0 \sum_{i=1}^{n} \Delta w_i d_i(X)\right\}$$

$$A_5 = \frac{1}{2}\left\{\sum_{i=1}^{n} w_i d_i(X) + C\sum_{i=1}^{n} \Delta w_i d_i(X) + T_0 \sum_{i=1}^{n} \Delta w_i d_i^2(X)\right\};$$

$$A_6 = -C\sum_{i=1}^{n} w_i - T_0 \sum_{i=1}^{n} w_i d_i(X) \tag{3}$$

Theorem 1 $\left[\sum_{i=1}^{n} \Delta w_i d_i(X)\right]^2 \le \sum_{i=1}^{n} \Delta w_i \sum_{i=1}^{n} \Delta w_i d_i^2(X)$ *with equality holding only when all distances are equal to one another.*

Proof By the Schwartz inequality (Hardy et al. 1952): $\left[\sum_{i=1}^{n} \Delta w_i d_i(X)\right]^2 = \left[\sum_{i=1}^{n} \sqrt{\Delta w_i}\sqrt{\Delta w_i} d_i(X)\right]^2 \le \sum_{i=1}^{n} \Delta w_i \sum_{i=1}^{n} \Delta w_i d_i^2(X)$ with equality holding only when $K\sqrt{\Delta w_i} = \sqrt{\Delta w_i} d_i(X)$ for a constant $K > 0$. This is true only if $d_i(X) = K$ for all i. □

The function $F(X, P, T)$ Eq. (2) is concave because $A_1 \le 0$, $A_3 \le 0$, and $A_1 A_3 - A_2^2 \ge 0$ by Theorem 1. Substitute $P = U - \frac{A_2}{A_1} T$, then

$$\begin{aligned} F(X, P, T) &= A_1 U^2 - 2A_2 UT + \frac{A_2^2}{A_1} T^2 + 2A_2 UT - 2\frac{A_2^2}{A_1} T^2 \\ &\quad + A_3 T^2 + 2A_4 U - 2A_4 \frac{A_2}{A_1} T + 2A_5 T + A_6 \\ &= A_1 U^2 + \left(A_3 - \frac{A_2^2}{A_1}\right) T^2 + 2A_4 U + 2\left(A_5 - A_4 \frac{A_2}{A_1}\right) T + A_6 \end{aligned} \tag{4}$$

The maximization of $F(X, P, T)$ Eq. (4) is separable by U and T, and the maximum is achieved for

$$U = -\frac{A_4}{A_1}; \quad T = -\frac{A_5 - A_4 \frac{A_2}{A_1}}{A_3 - \frac{A_2^2}{A_1}} = -\frac{A_5 A_1 - A_4 A_2}{A_1 A_3 - A_2^2};$$

$$P = U - \frac{A_2}{A_1} T = \frac{A_5 A_2 - A_4 A_3}{A_1 A_3 - A_2^2} \tag{5}$$

The objective function to be maximized $F(X)$ after substituting the optimal values of P and T is by Eq. (4):

$$F(X) = -\frac{A_4^2}{A_1} - \frac{(A_5 - A_4\frac{A_2}{A_1})^2}{A_3 - \frac{A_2^2}{A_1}} + A_6 = -\frac{A_1A_5^2 + A_3A_4^2 - 2A_2A_4A_5}{A_1A_3 - A_2^2} + A_6 \tag{6}$$

Following substitution of Eq. (3) and extensive manipulations:

$$P = \frac{C}{2} + \frac{\sum_{i=1}^n w_i \sum_{i=1}^n \Delta w_i d_i^2(X) - \sum_{i=1}^n w_i d_i(X) \sum_{i=1}^n \Delta w_i d_i(X)}{2\left[\sum_{i=1}^n \Delta w_i \sum_{i=1}^n \Delta w_i d_i^2(X) - \left\{\sum_{i=1}^n \Delta w_i d_i(X)\right\}^2\right]} \tag{7}$$

$$T = \frac{T_0}{2} + \frac{\sum_{i=1}^n w_i d_i(X) \sum_{i=1}^n \Delta w_i - \sum_{i=1}^n w_i \sum_{i=1}^n \Delta w_i d_i(X)}{2\left[\sum_{i=1}^n \Delta w_i \sum_{i=1}^n \Delta w_i d_i^2(X) - \left\{\sum_{i=1}^n \Delta w_i d_i(X)\right\}^2\right]} \tag{8}$$

It is interesting to note that for a given location X the optimal value of P Eq. (7) is independent of the value of T_0 and the optimal value of T Eq. (8) is independent on the value of C. They are implicitly dependent on these values because when C or T_0 are changed, the value of the optimal location X may change.

Evaluating the objective function $F(X)$ by substitution of Eq. (3) and simplifying the expressions is quite long and tedious. The result is:

$$\begin{aligned} F(X) =& \\ & \frac{C^2}{4}\sum_{i=1}^n \Delta w_i + \frac{T_0^2}{4}\sum_{i=1}^n \Delta w_i d_i^2(X) - \frac{C}{2}\sum_{i=1}^n w_i - \frac{T_0}{2}\sum_{i=1}^n w_i d_i(X) + \frac{CT_0}{2}\sum_{i=1}^n \Delta w_i d_i(X) \\ & + \frac{\sum_{i=1}^n \Delta w_i \left\{\sum_{i=1}^n w_i d_i(X)\right\}^2 + \left\{\sum_{i=1}^n w_i\right\}^2 \sum_{i=1}^n \Delta w_i d_i^2(X) - 2\sum_{i=1}^n w_i \sum_{i=1}^n w_i d_i(X) \sum_{i=1}^n \Delta w_i d_i(X)}{4\left[\sum_{i=1}^n \Delta w_i \sum_{i=1}^n \Delta w_i d_i^2(X) - \left\{\sum_{i=1}^n \Delta w_i d_i(X)\right\}^2\right]} \end{aligned} \tag{9}$$

Note that the denominator of the last term is positive by Theorem 1 unless all distances are equal to one another.

2.1 A special case

If we assume that $\Delta w_i = \alpha w_i$ for some positive α, the optimization problem is simplified. This assumption is equivalent to assuming that demand vanishes at the same charge for all customers. This is a reasonable assumption if the service area is homogenic and customers behave in a similar fashion. It is also plausible that the elasticity of demand is known for the area as a whole and not for each demand point and thus such an assumption is necessary for the application of the model.

The optimal value of P is by Eq. (7):

$$P = \frac{C}{2} + \frac{1}{2\alpha} \tag{10}$$

The optimal value of T by Eq. (8) is

$$T = \frac{T_0}{2} \tag{11}$$

Note that for this special case the optimal P and T do not depend on the location of the facility. The following result can be directly obtained by substituting Eqs. (10) and (11) into Eq. (1). We show it by evaluating Eq. (9) for this special case:

$$\begin{aligned}F(X) &= \frac{\alpha C^2 - 2C}{4}\sum_{i=1}^{n} w_i + \frac{\alpha T_0^2}{4}\sum_{i=1}^{n} w_i d_i^2(X) - \frac{(1-\alpha C)T_0}{2}\sum_{i=1}^{n} w_i d_i(X) + \frac{\sum_{i=1}^{n} w_i}{4\alpha} \\ &= \frac{\alpha T_0^2}{4}\sum_{i=1}^{n} w_i d_i^2(X) - \frac{T_0(1-\alpha C)}{2}\sum_{i=1}^{n} w_i d_i(X) + \frac{(1-\alpha C)^2}{4\alpha}\sum_{i=1}^{n} w_i \\ &= \alpha\sum_{i=1}^{n} w_i\left\{\frac{T_0}{2}d_i(X) - \frac{1-\alpha C}{2\alpha}\right\}^2 = \frac{\alpha T_0^2}{4}\sum_{i=1}^{n} w_i\left\{d_i(X) - \frac{1-\alpha C}{\alpha T_0}\right\}^2\end{aligned} \tag{12}$$

By Eqs. (10) and (11) the condition $d_i(X) \le \frac{1-\alpha C}{\alpha T_0}$ is equivalent to $2\alpha T d_i(X) \le 2(1-\alpha P)$, or $w_i - \Delta w_i(P + T d_i(X)) \ge 0$. This means that the demand is non-negative. It is therefore necessary that $d_i(X) \le \frac{1-\alpha C}{\alpha T_0}$ otherwise the demand is negative which is impossible. Therefore, the value of α must be small enough not to entail negative demand otherwise the data does not accurately reflect actual customer behavior. In conclusion, we must require that $d_i(X) \le \frac{1-\alpha C}{\alpha T_0}$. Without restrictions on the facility location, the best location of the facility by this formulation is at infinity.

Optimizing Eq. (9) is complicated because of the ratio in the last term of the expression. We therefore propose a different approach to maximizing $F(X, P, T)$.

2.2 An alternative approach

We first find the best price P for a given location X *and* a transportation rate T.

$$\begin{aligned}F(X, P, T) = & -\left\{\sum_{i=1}^{n} \Delta w_i\right\} P^2 \\ & + \left\{\sum_{i=1}^{n} \left[w_i - T\Delta w_i d_i(X) + C\Delta w_i - \Delta w_i(T - T_0)d_i(X)\right]\right\} P \\ & + \sum_{i=1}^{n} \left[w_i - \Delta w_i T d_i(X)\left[\left[-C + (T - T_0)d_i(X)\right]\right. \\ = & -aP^2 + bP + c \end{aligned} \tag{13}$$

Since $a > 0$, the maximum of $F(X, P, T)$ is achieved for

$$\begin{aligned}P = \frac{b}{2a} & = \frac{\sum_{i=1}^{n}\left[w_i + C\Delta w_i - (2T - T_0)\Delta w_i d_i(X)\right]}{2\sum_{i=1}^{n} \Delta w_i} \\ & = \frac{C}{2} + \frac{\sum_{i=1}^{n} w_i}{2\sum_{i=1}^{n} \Delta w_i} - (2T - T_0)\frac{\sum_{i=1}^{n} \Delta w_i d_i(X)}{2\sum_{i=1}^{n} \Delta w_i}\end{aligned} \tag{14}$$

with a maximum value of the objective function of $F(X, T) = \frac{b^2}{4a} + c$.

To simplify we write

$$\frac{b}{2\sqrt{a}} = \sum_{i=1}^{n}\left[-\gamma_i d_i(X) + \delta_i\right]$$

and

$$c = \sum_{i=1}^{n}\left[-\epsilon_i + \phi_i d_i(X) - \theta_i d_i^2(X)\right].$$

where:

$$\gamma_i = \frac{\Delta w_i\,[2T - T_0]}{2\sqrt{\sum_{i=1}^{n} \Delta w_i}}; \quad \delta_i = \frac{w_i + C\Delta w_i}{2\sqrt{\sum_{i=1}^{n} \Delta w_i}} \tag{15}$$

$$\epsilon_i = Cw_i; \quad \phi_i = CT\Delta w_i + w_i(T - T_0); \quad \theta_i = \Delta w_i T(T - T_0) \tag{16}$$

Therefore, the objective function $F(X, T)$, to be maximized, is:

$$F(X,T) = \left\{\sum_{i=1}^{n}\left[-\gamma_i d_i(X) + \delta_i\right]\right\}^2 + \sum_{i=1}^{n}\left[-\epsilon_i + \phi_i d_i(X) - \theta_i d_i^2(X)\right]$$
$$= \left\{\sum_{i=1}^{n}\gamma_i d_i(X)\right\}^2 - \sum_{i=1}^{n}\theta_i d_i^2(X) + \sum_{i=1}^{n}\left[\phi_i - 2\gamma_i\sum_{i=1}^{n}\delta_i\right] d_i(X)$$
$$+ \left\{\sum_{i=1}^{n}\delta_i\right\}^2 - \sum_{i=1}^{n}\epsilon_i \quad (17)$$

Theorem 2 *For a given location X, the objective function $F(X, T)$ is concave in T.*

Proof $F(X, T)$ for a given X is a second order polynomial in T. The coefficient of T^2 is:

$$\frac{\left[\sum_{i=1}^{n}\Delta w_i d_i(X)\right]^2}{\sum_{i=1}^{n}\Delta w_i} - \sum_{i=1}^{n}\Delta w_i d_i^2(X)$$

and the theorem follows by Theorem 1. □

Define

$$L = \sum_{i=1}^{n}\delta_i; \quad K = \sum_{i=1}^{n}\epsilon_i; \quad \xi_i = \phi_i - 2L\gamma_i$$
$$F_1(X,T) = \left\{\sum_{i=1}^{n}\gamma_i d_i(X)\right\}^2; \quad F_2(X,T) = \sum_{i=1}^{n}\theta_i d_i^2(X)$$
$$F_3(X,T) = \sum_{\xi_i \geq 0}\xi_i d_i(X); \quad F_4(X,T) = -\sum_{\xi_i < 0}\xi_i d_i(X)$$

and the objective function is:

$$F(X,T) = F_1(X,T) - F_2(X,T) + F_3(X,T) - F_4(X,T) + L^2 - K \quad (18)$$

3 Solution approach

We propose to find the optimal X for a given T and determine the best T by a golden section search (Zangwill 1969) on the value of T. This approach yields the optimal solution by Theorem 2. Note that for the special case analyzed in Section 2.1 the optimal value of T is known by Eq. (11) and thus there is no need for the golden section search.

We propose to maximize $F(X, T)$ for a given T by the Big Triangle Small Triangle (BTST) approach proposed in Drezner and Suzuki (2004).

3.1 The BTST approach

A feasible region which consists of a finite number of convex polygons is given.

Phase 1: Each convex polygon is triangulated using the Delaunay triangulation. The vertices of the triangles are the demand points and the vertices of the convex polygon. The union of the triangulations is the initial set of triangles.

Phase 2: Calculate an upper bound, UB, and a lower bound, LB, for each triangle. Let the largest LB be $\overline{LB}$. Discard all triangles for which $UB \leq \overline{LB}(1+\epsilon)$.

Phase 3: Choose the triangle with the largest LB and divide it into four small triangles by connecting the centers of its sides (see Fig. 1 where the large triangle is split into four small triangles and the smaller lower left triangle is further split into four smaller triangles). Calculate UB and LB for each triangle, and update the $\overline{LB}$ if necessary. The large triangle and all triangles for which $UB \geq \overline{LB}(1+\epsilon)$ are discarded.

Stopping Criterion: The branch and bound is terminated when there are no triangles left. The solution $\overline{LB}$ is within a relative accuracy of ϵ from the optimum.

Note that: (i) A lower bound in a triangle is the value of the objective function at any point in the triangle (such as the center of gravity). (ii) Since the triangulation is based on the demand points as vertices, no demand point is in the interior of a triangle. This is also true for all triangles generated in the process.

In order to implement the BTST approach we need an upper bound for the value of the objective function in a triangle. To construct such an upper bound we express the function $F(X, T)$ by Eq. (17) as a difference between two convex functions. An upper bound based on a difference between two convex functions proved successful in many papers (Drezner 2007; Drezner and Drezner 2004; Tuy et al. 1995; Drezner and Nickel 2009). For the analysis

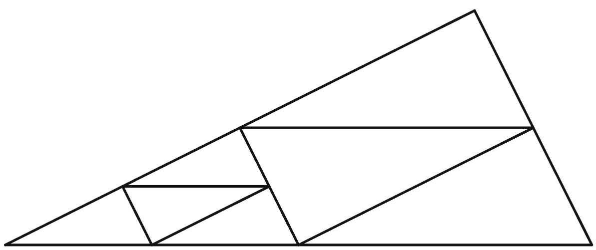

Fig. 1 Split of a triangle into four small triangles

below we assume a given T and a given triangle. We find an upper bound for $F(X, T)$ at all the points of the triangle.

- The function $F_1(X, T)$ is convex because γ_i are either all positive (when $D \geq T_0/2$) or all negative (when $D \leq T_0/2$).
- The function $F_2(X, T)$ is convex for $T \geq T_0$ and concave for $T \leq T_0$.
- The functions $F_3(X, T)$ and $F_4(X, T)$ are both convex.

In summary, for a given T, $F(X, T) = G_1(X, T) - G_2(X, T)$ is expressed as a difference between convex functions in X $G_1(X, T)$ and $G_2(X, T)$:

For $T \geq T_0$:

$$G_1(X, T) = F_1(X, T) + F_3(X, T) \tag{19}$$

$$G_2(X, T) = F_2(X, T) + F_4(X, T) \tag{20}$$

For $T < T_0$:

$$G_1(X, T) = F_1(X, T) - F_2(X, T) + F_3(X, T) \tag{21}$$

$$G_2(X, T) = F_4(X, T) \tag{22}$$

To complete the construction of the upper bound, the function $G_2(X, T)$ is bounded by a tangent plane at a point in the triangle. Specifically, let X_0 be a point in the triangle such as its center of gravity. By the convexity of $G_2(X, T)$, $G_2(X, T) \geq G_2(X_0, T) + \nabla_X G_2(X_0, T)(X - X_0) = H(X, T)$. Therefore, $F(X, T) \leq G_1(X, T) - H(X, T)$. Since $H(X, T)$ is a linear function of X, $G_1(X, T) - H(X, T)$ is a convex function of X which obtains its maximum on one of the three vertices of the triangle. The values of $G_1(X, T) - H(X, T)$ at the three vertices of the triangle are calculated, and the upper bound in the triangle UB is the maximum among these three values.

4 Computational experiments

Programs in Fortran[1] using double precision arithmetic were coded, compiled by Intel 9.0 FORTRAN Compiler, and ran on a 2.8GHz Pentium IV desk top computer with 256MB RAM.

Problems with ten values of n demand points between 10 and 10,000 were tested. Ten problems were randomly generated for each value of n for a total of 100 problems. The coordinates of demand points were randomly generated in a unit square. The parameters used are: $C = 1$, $T_0 = 2$ and $w_i \in [10, 20]$. Δw_i was calculated as αw_i with α generated in a range. Two ranges for α were tested $\alpha \in [0.05, 0.10]$ and $\alpha \in [0.05, 0.15]$. These values of α are small enough so that demand is positive for a facility located anywhere in the square. The solution by BTST for a given T was calculated to a relative accuracy of $\epsilon = 10^{-10}$. The

[1] We thank Atsuo Suzuki for his Fortran program that finds the triangulation based on Sugihara and Iri (1994) subroutines first developed in Ohya et al. (1984).

golden section search for the optimal value of T was conducted on the range [-10, 10] to an accuracy of 10^{-5}. This means that exactly 30 applications of the BTST are required for a solution of one problem.

Since a constant value of α for all demand points leads to a solution of $T = \frac{T_0}{2}$ Eq. (11), we also found the solution assuming that $T = \frac{T_0}{2}$ and compared it with the optimal solution. This experiment also provides the run times required for the solution of the special case analyzed in Section 2.1. Since such a solution requires only one application of the BTST, it is about 30 times faster.

The results are depicted in Tables 1 and 2. In Table 1 the optimal solutions for these problems are given. For each value of n the minimum, maximum and average values for P, T and the run times in seconds are given.

For any fixed α the optimal $T = \frac{T_0}{2} = 1$. For $\alpha \in [0.05, 0.10]$ the midrange is $\alpha = 0.075$ which leads to $P = 7.167$ and for $\alpha \in [0.05, 0.15]$ the midrange is $\alpha = 0.1$ which leads to $P = 5.5$. By examining Table 1 we conclude that as the number of demand points increases, the optimal values of P and T indeed approach the values of the special case by Eqs. (11) and (10) by using for α the midrange of the interval.

Run times increase by about the square of the value of n and do not vary much for problems with the same n. They are very short considering that 30 applications of BTST are required for the solution of each problem. Problems with $n = 10,000$ demand points were optimally solved in less than nine minutes.

Table 1 Computational results for the mill price and the transportation rate

n	P			T			CPU time (sec.)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
$\alpha \in [0.05, 0.10]$									
10	6.631	9.525	7.780	−4.921	5.494	−0.356	0.00	0.05	0.03
20	6.119	8.446	7.462	−3.472	4.817	0.431	0.03	0.11	0.05
50	6.471	8.300	7.222	−2.013	3.947	0.960	0.06	0.12	0.09
100	6.749	7.969	7.212	−1.109	2.575	0.898	0.17	0.20	0.19
200	6.823	7.472	7.238	−0.195	2.331	0.830	0.44	0.64	0.52
500	7.087	7.365	7.224	0.271	1.348	0.781	1.73	2.12	1.91
1,000	7.034	7.294	7.201	0.272	1.481	0.852	5.94	6.42	6.20
2,000	7.044	7.265	7.175	0.598	1.403	0.993	21.75	22.75	22.05
5,000	7.115	7.218	7.155	0.799	1.188	1.028	129.53	135.31	130.75
10,000	7.096	7.199	7.147	0.856	1.222	1.055	510.69	515.66	512.95
$\alpha \in [0.05, 0.15]$									
10	5.382	8.738	6.840	−7.827	4.449	−2.491	0.02	0.05	0.03
20	4.261	7.063	5.889	−4.354	5.496	0.207	0.02	0.06	0.04
50	4.745	6.917	5.596	−2.790	4.256	0.861	0.08	0.14	0.10
100	5.021	6.437	5.559	−1.447	2.833	0.866	0.17	0.23	0.20
200	5.115	5.852	5.585	−0.400	2.494	0.798	0.45	0.64	0.52
500	5.411	5.728	5.566	0.190	1.397	0.749	1.77	2.20	1.93
1,000	5.352	5.644	5.539	0.184	1.539	0.831	6.00	6.50	6.24
2,000	5.363	5.612	5.510	0.549	1.457	0.991	21.83	22.91	22.22
5,000	5.442	5.557	5.487	0.774	1.212	1.031	129.94	135.89	131.32
10,000	5.421	5.537	5.478	0.839	1.250	1.061	512.34	517.58	514.91

Table 2 Computational results for $T = T_0/2$

n	P			% below maximum			CPU time (sec.)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
$\alpha \in [0.05, 0.10]$									
10	6.684	8.021	7.363	0.021	2.443	0.706	0.00	0.02	0.00
20	6.991	7.686	7.297	0.002	1.011	0.215	0.00	0.02	0.00
50	7.027	7.462	7.181	0.002	0.524	0.184	0.00	0.02	0.00
100	7.073	7.357	7.174	0.005	0.235	0.070	0.00	0.02	0.01
200	7.086	7.349	7.190	0.001	0.114	0.027	0.02	0.03	0.02
500	7.032	7.243	7.168	0.000	0.035	0.011	0.06	0.08	0.06
1,000	7.110	7.206	7.163	0.000	0.034	0.008	0.19	0.22	0.21
2,000	7.132	7.234	7.173	0.000	0.010	0.004	0.72	0.77	0.73
5,000	7.137	7.204	7.162	0.000	0.003	0.001	4.30	4.50	4.36
10,000	7.148	7.183	7.161	0.000	0.003	0.001	17.02	17.23	17.12
$\alpha \in [0.05, 0.15]$									
10	4.976	6.527	5.739	0.080	7.238	2.268	0.00	0.02	0.00
20	5.305	6.108	5.654	0.007	2.633	0.704	0.00	0.02	0.00
50	5.345	5.840	5.517	0.005	1.301	0.489	0.00	0.02	0.00
100	5.395	5.718	5.509	0.012	0.600	0.180	0.00	0.02	0.01
200	5.409	5.708	5.527	0.002	0.284	0.070	0.02	0.03	0.02
500	5.350	5.587	5.501	0.000	0.086	0.027	0.05	0.08	0.07
1,000	5.436	5.544	5.496	0.000	0.085	0.021	0.20	0.22	0.21
2,000	5.461	5.576	5.508	0.001	0.025	0.011	0.72	0.77	0.74
5,000	5.467	5.542	5.495	0.000	0.006	0.002	4.33	4.52	4.38
10,000	5.480	5.518	5.493	0.000	0.008	0.002	17.05	17.25	17.13

In Table 2 the results of using $T = \frac{T_0}{2} = 1$ are given. Run times, as expected, are about 30 times shorter. It took less than 8 min to solve all 200 problems reported in Table 2. The optimal values of P are closer to the results by Eq. (10) then those found for the optimal value of T. Note that the optimal value of P for $T = \frac{T_0}{2}$ is independent of the location of the facility and can be calculated by Eq. (14) because the last term vanishes. The value of the objective function was quite close to optimality. There was one case where the objective function was more than 7% below the optimum but for most problems it was very close.

5 Conclusions

In this paper we optimally solved the problem of locating a facility and determining the mill price and transportation rate it charges the customers to maximize profit. Demand by customers is elastic and declines according to the total cost charged by the facility. Computational experiments demonstrated the effectiveness of the solution approach.

The analysis provided in this paper yields several interesting results.

Once the location of the facility is known (or given): the optimal mill price charged by the facility is independent on the company's transportation rate, and the optimal transportation rate charged by the facility is independent on the company's mill price.

When the elasticity of demand is the same for all customers: Both the optimal transportation rate and the mill price charged by the facility are independent of the location of the facility. The transportation rate charged by the facility is exactly half the transportation cost to the facility. This means that the transportation rate is subsidized by 50%. This explains the "free shipping" approach adopted by many companies. The mill price that should be charged by the facility can be directly calculated by a simple formula Eq. (10).

As future research we propose to analyze different elasticity assumptions such as a convex decline function of the demand. It may be possible to solve such problems by assuming a linear decline (the derivative of the demand decline function) when the facility is located in the neighborhood of the present facility location and solve the problem iteratively based on the techniques presented in this paper. This means using the values of w_i and Δw_i as functions of the distance from the facility but fix them at the distances of the present iterate. Note that even when the original problem satisfies the special case requirement (the same elasticity curve for all customers), the resulting optimization procedure for each iteration does not have this property because the elasticity curve depends on the distance from the facility and therefore is different for different demand points. Therefore, the general solution algorithm is required.

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