Solving Over-production and Supply-guarantee Problems in Economic Equilibria

Bing-sheng He · Wei Xu · Hai Yang · Xiao-Ming Yuan

Abstract The classical Spatial Price Equilibrium of economic markets may result in over-production at supply markets and under-supply at demand markets. This paper considers the policy instruments of levying taxes at supply markets to avoid over-production and granting subsidy for traders to guarantee supply at demand markets. The decision process of determining appropriate rates of tax and subsidy is characterized by an implicit complementarity problem. Consequently, a direct type algorithm is applied to solve the complementarity model. Preliminary numerical results are also reported.

B.-s. He
Department of Mathematics, Nanjing University, Nanjing, People's Republic of China
e-mail: hebma@nju.edu.cn

W. Xu
School of Management and Engineering, Nanjing University,
Nanjing, People's Republic of China
e-mail: xuwei@nju.edu.cn

H. Yang
Department of Civil Engineering, The Hong Kong University of Science and Technology,
Hong Kong, People's Republic of China
e-mail: cehyang@ust.hk

X.-M. Yuan (✉)
Department of Mathematics, Hong Kong Baptist University, Hong Kong,
People's Republic of China
e-mail: xmyuan@hkbu.edu.hk

1 Introduction

We consider the economic markets of a certain commodity, which is produced
at the supply markets and consumed at the demand markets. For the realistic
case that the supply and demand markets are spatially separated, the commod-
ity needs to be shipped from supply markets to demand markets by traders.
Associated with the purchase prices at supply markets and the sale prices at
demand markets; and under the assumption of perfect competition, the well-
known Spatial Price Equilibrium (SPE) is ultimately attained, where the sale
price equals the purchase price plus the cost of shipment whenever there is
trade between a pair of supply and demand markets. We refer to Dafermos
and Nagurney (1989), Ferris and Pang (1997), Florian and Los (1982), Friesz
et al. (1984), Gabriel et al. (2003), Harker (1986), Lederer (2003), Miller et al.
(1996), Nagurney (1993), Reggiani et al. (2002), Samuelson (1952), Scarf and
Hansen (1973), Tobin and Friesz (1983) for the theoretical and computational
development on SPE.

Thus, once the purchase and sale prices are given, the optimal levels of
productions at supply markets, the shipment pattern operated by traders
and the virtual levels of supply at demand markets are determined by the
endogenous equilibrium conditions. Note that the equilibrium is attained
according to the profit-maximizing principles. Hence, the avaricious producers
have incentives to increase the amounts of productions as long as their profits
can be improved. This inevitable fact may result in over-production at supply
markets and thus cause negative consequences such as the over-exploitation
of raw resources and environmental pollution. At the same time, the rational
traders reduce the amounts of shipments whenever there is no profit to earn
between some pairs of supply and demand markets, caring nothing about
the necessary supply at some particular demand markets. This undesired
characteristic may lead to the lack of necessary supply at demand markets.
Therefore, when such situations occur, the governmental authority would
like to step into the markets to control the levels of production and supply
by implementing some policy instruments, with the aims at avoiding over-
production and guaranteeing supply.

In this paper, we consider the policy instruments of tax and subsidy to
accomplish these targets. More specifically, the governmental authority levies
certain amounts of taxes at supply markets so as to prevent producers from
over-production; whereas the authority grants subsidy for traders to encourage
more commodities to be shipped to demand markets so as to avoid under-
supply. For simplicity, we assume that both the purchase and sale prices are
fixed. Then, the question arising naturally is: how does the governmental
authority determine the rates of tax and subsidy so that the resulted SPE has
such levels of productions that do not exceed the alarm levels at supply markets

and such levels of supply that guarantee the necessary requirement at demand markets?

We shall show that the desired rates of tax and subsidy with the aforementioned aims correspond to the solutions of a complementarity problem, and thus the procedure of approaching to the desired rates can be characterized by solving the complementarity model. Recall that the complementarity problem is a special case of the so-called variational inequalities (VI for short), which serve as the unified mathematical model for various equilibrium problems arising in diversing applications, see, e.g., Dafermos (1980, 1990), Ferris and Pang (1997), Friesz et al. (1983, 1993, 2006), Jofré et al. (2007), Konnov (2008), Nagurney (1993).

Recent decades have witnessed the extensive development on the computational aspects of economic equilibria since the pioneering works of Friesz et al. (1984), Scarf and Hansen (1973), Zangwill and Garcia (1981). For our derived complementarity model, the main computational challenge is its implicit nature in the sense that the underlying function, which is determined by the SPE (with fixed taxes and subsidy), has no explicit formulation available. This implicit feature of the complementarity model excludes the applications of such methods for solving VI that require the knowledge of derivatives or other analytic properties. Therefore, we will apply a direct method that uses only the function values during the iterative process of solving the complementarity model. As we shall explain in detail, it is reasonable to assume that the function values are available since the resulted levels of production, shipments and supply at SPE can be observed under the implementation of taxes and subsidy in practice. In addition, the attempt to reduce the numbers of iterations as many as possible is desired not only because of saving the computational loads, but also due that it reduces the times of adjusting the rates of tax and subsidy and thus makes the policy instruments more stable and acceptable. The algorithm we shall apply is demonstrated to have this advantage, verified by some preliminary numerical experiments.

The rest of the paper is organized as follows. In Section 2, we establish the mathematical model for SPE with tax and subsidy. Then, Section 3 provides the implicit complementarity model for the governmental authority to determine appropriate rates of tax and subsidy with the concerns of over-production and supply-guarantee. Section 4 focuses on the computational aspects of the implicit complementarity model. In particular, some numerical results are reported.

2 SPE with tax and subsidy

We assume that there are m supply markets and n demand markets involved in the production and consumption of the concerned commodity. The following notations are used:

S_i the i-th supply market, $i = 1, \ldots, m$;

D_j the j-th demand market, $j = 1, \ldots, n$;
x_{ij} the shipment from S_i to D_j;
s_i the level of production at S_i
d_j the level of supply at D_j
f_i the purchase price at S_i;
g_j the sale price at D_j;
t_{ij} the shipment cost between S_i and D_j.

Then it is conventional to assume that the following conservation conditions hold, see, e.g., Nagurney (1987):

$$
\begin{cases}
s_i = \sum_{j=1}^{n} x_{ij} & \text{for} \quad i = 1, \cdots, m; \\
d_j = \sum_{i=1}^{m} x_{ij} & \text{for} \quad j = 1, \cdots, n; \\
\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}.
\end{cases}
$$

It is well-known (see, e.g., Florian and Los 1982; Samuelson 1952) that the classical SPE is characterized by the following conditions:

$$
f_i + t_{ij} \begin{cases} \geq g_j, & \text{if } x_{ij} = 0, \\ = g_j, & \text{if } x_{ij} > 0 \,. \end{cases} \tag{2.1}
$$

Now, with the consideration of levying taxes at supply markets to avoid over-production and granting subsidy for traders to guarantee necessary supply, the new SPE can be obviously characterized by the following conditions:

$$
f_i + y_i + t_{ij} \begin{cases} \geq g_j + z_j, & \text{if } x_{ij} = 0, \\ = g_j + z_j, & \text{if } x_{ij} > 0 \,, \end{cases} \tag{2.2}
$$

where y_i is the tax levied at S_i and z_j is the subsidy for those traders who ship the commodity to D_j. It is easy to notice (see, e.g., Ferris and Pang 1997; Nagurney 1993) that (2.2) can be written as an equivalent complementarity problem: for any $i = 1, \ldots, m$ and $j = 1, \ldots, n$, we have

$$
x_{ij} \geq 0, \quad (f_i + y_i + t_{ij}) - (g_j + z_j) \geq 0, \quad x_{ij} \cdot \big((f_i + y_i + t_{ij}) - (g_j + z_j) \big) = 0. \tag{2.3}
$$

We start to reformulate (2.3) into a compact mathematical form for the convenience of analysis. Let $e_n \in \mathbb{R}^n$ denote the column vector that each entry equals 1 and $I_n \in \mathbb{R}^{n \times n}$ denote the identity matrix. Utilizing the following notations:

$$
x = (x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots \ldots, x_{m1}, x_{m2}, \ldots, x_{mn})^T,
$$

$$
t = (t_{11}, t_{12}, \ldots, t_{1n}, t_{21}, t_{22}, \ldots, t_{2n}, \ldots \ldots, t_{m1}, t_{m2}, \ldots, t_{mn})^T,
$$

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix},$$

$$M = \begin{pmatrix} e_n^T \\ & e_n^T \\ & & \ldots\ldots \\ & & & e_n^T \end{pmatrix} \in \mathbb{R}^{m \times nm}, \qquad N = \begin{pmatrix} I_n & I_n & \cdots\cdots & I_n \end{pmatrix} \in \mathbb{R}^{n \times nm},$$

then the complementarity problem (2.3) is equivalent to:

$$x \geq 0, \qquad t(x) + M^T \left[f(Mx) + y \right] - N^T \left[g(Nx) + z \right] \geq 0, \qquad (2.4a)$$

$$x^T \left\{ t(x) + M^T \left[f(Mx) + y \right] - N^T \left[g(Nx) + z \right] \right\} = 0. \qquad (2.4b)$$

Alternatively, the VI reformulation of (2.4) is to find $(x, s, d) \in \Omega$ such that

$$\left(x' - x \right)^T t(x) + \left(s' - s \right)^T \left[f(s) + y \right] - \left(d' - d \right)^T \left[g(d) + z \right] \geq 0, \quad \forall \ (x', s', d') \in \Omega, \qquad (2.5)$$

where

$$\Omega = \left\{ (x, s, d) \in \mathbb{R}^{mn} \times \mathbb{R}^m \times \mathbb{R}^n \mid x \geq 0, \ Mx = s, \ Nx = d \right\}.$$

Although the structures of both (2.1) and (2.2) are pretty simple, the solvability of these SPE depends heavily on the involving price and cost functions, i.e., f, g and t. Throughout this paper, we consider the separable case for the purchase price, sale price and shipment cost:

1. the purchase price depends separably upon the level of production, i.e., $f_i = f_i(s_i)$;
2. the sale price depends separably upon the level of supply, i.e., $g_j = g_j(d_j)$;
3. the transaction cost depends separably upon the amount of shipments, i.e., $t_{ij} = t_{ij}(x_{ij})$.

We further make some reasonable and mild assumptions on functions f_i, g_j and t_{ij}.

Assumptions

A1 f_i ($\forall i \in \{1, 2, \cdots, m\}$) strongly increases with respect to s_i, i.e., there is a constant $\tau > 0$ such that

$$(s - \tilde{s})^T (f(s) - f(\tilde{s})) \geq \tau \|s - \tilde{s}\|^2, \quad \forall \ s, \tilde{s} \in \mathbb{R}_+^m. \qquad (2.6)$$

A2 g_j ($\forall j \in \{1, 2, \cdots, n\}$) strongly decreases with respect to d_j, i.e., there is a constant $\tau > 0$ such that

$$\left(d - \tilde{d} \right)^T \left(g(\tilde{d}) - g(d) \right) \geq \tau \|d - \tilde{d}\|^2, \quad \forall \ d, \tilde{d} \in \mathbb{R}_+^n. \qquad (2.7)$$

A3 $t_{ij}(x_{ij})$ ($\forall i \in \{1, 2, \cdots, m\}, \forall j \in \{1, 2, \cdots, n\}$) is a non-decreasing function with respect to the quantity of shipment x_{ij}:

$$(x - \tilde{x})^T (t(x) - t(\tilde{x})) \geq 0, \quad \forall \ x, \tilde{x} \in \mathbb{R}_+^{mn}. \tag{2.8}$$

In particular, some existing methods for solving SPE (e.g., Dafermos and Nagurney 1989; Marcotte et al. 1992; Nagurney 1986) assume that these functions are explicit with linear structures:

$$f_i(s_i) = \xi_i + a_i s_i, \qquad a_i \geq 0; \tag{2.9a}$$

$$g_j(d_j) = \eta_j - b_j d_j, \qquad b_j \geq 0; \tag{2.9b}$$

$$t_{ij}(x_{ij}) = \zeta_{ij} + c_{ij} x_{ij}, \qquad c_{ij} \geq 0, \tag{2.9c}$$

where $a_i, b_j, c_{ij}, \xi_i, \eta_j$ and ζ_{ij} are known constants. Note that (2.3) with (2.9) is a symmetric monotone linear complementarity problem, which amounts to solving a convex quadratic programming with non-negative constraints (see, e.g., Fletcher 1987).

3 The implicit complementarity model

This section is devoted to modelling the decision process of seeking appropriate rates of tax and subsidy to an implicit complementarity problem.

We first emphasize the fact that when the SPE (2.2) (with the given y and z) is attained, the corresponding levels of production and supply and the shipment pattern can be observed. In other words, we do not need to solve the VI (2.5) since its solution (x, s, d) can be observed once y and z are given. Note that with the assumptions $A1$–$A3$, the VI (2.5) has the unique solution (x, s, d) for each given pair of (y, z). However, although (x, s, d) depends on (y, z) in a functional manner, the explicit expression of this function and its analytic properties are "hidden" to us.

On the other hand, let the vector $s^{\max}$ denote the alarm levels of over-production at supply markets ($s_i^{\max}$ is the alarm level at S_i); and the vector $d^{\min}$ denote the alarm levels of under-supply at demand markets ($d_j^{\min}$ is the alarm level at D_j). We assume that both $s^{\max}$ and $d^{\min}$ are determined by the governmental authority in advance. Obviously, the consistent relationship

$$\sum_{i=1}^m s_i^{\max} \geq \sum_{j=1}^n d_j^{\min}$$

should be satisfied.

Then the target of the governmental authority is to determine appropriate y^* and z^* such that the corresponding levels of production and supply at SPE (2.2), denoted by s^* and d^*, should satisfy

$$s^* \leq s^{\max} \quad \text{and} \quad d^* \geq d^{\min}. \tag{3.1}$$

Mathematically, the governmental authority is to find $y^* \in \mathbb{R}^m$ and $z^* \in \mathbb{R}^n$ such that

$$y^* \geq 0, \quad s^{\max} - s^* \geq 0, \quad y^{*T}\left(s^{\max} - s^*\right) = 0, \tag{3.2}$$

and

$$z^* \geq 0, \quad d^* - d^{\min} \geq 0, \quad z^{*T}\left(d^* - d^{\min}\right) = 0, \tag{3.3}$$

where (s^*, d^*) are determined by the VI problem (2.5).

The complementarity emerging in both (3.2) and (3.3) deserves further explanations. If $s_i^{\max} - s_i^* > 0$, i.e., the level of production at S_i does not exceed the alarm level, then no tax should be levied, i.e., $y_i^* = 0$. Otherwise, the level of production keeps decreasing while the rate of tax increases, until the SPE is eventually attained, which implies that $y_i^* > 0$ and $s_i^{\max} - s_i^* = 0$. This is exactly the complementarity in (3.2), and similar explanations apply to the complementarity in (3.3).

By denoting

$$u = \begin{pmatrix} y \\ z \end{pmatrix} \quad \text{and} \quad F(u) = \begin{pmatrix} s^{\max} - s(u) \\ d(u) - d^{\min} \end{pmatrix}, \tag{3.4}$$

Equations (3.2)–(3.3) can be compactly rewritten into the following complementarity problem:

$$u^* \geq 0, \quad F(u^*) \geq 0, \quad u^{*T}F(u^*) = 0. \tag{3.5}$$

We reiterate that $F(u)$ is a "black-box" function since the explicit expression and the related analytical properties of $s(u)$ and $d(u)$ (determined by (2.5)) are "hidden". In this sense, the complementarity problem (3.5) is "implicit". Note that the implicit complementarity model (3.5) is the dual problem of the constrained SPE problem with the constraint (3.1), which has many practical applications such as the road pricing scheme for mitigating traffic congestion.

Overall, the decision process of the governmental authority to seek desired rates of tax and subsidy for avoiding over-production and under-supply is characterized by the implicit complementarity model (3.5).

4 Computation

The implicity of F excludes the possibility of applying those existing methods for VI that require the derivative information or other analytic properties of F to solve the model (3.5). Therefore, it is practical to apply the so-called direct type methods which only need the function values of F during the whole iteration to solve the model (3.5). Recall that the function values of F can be observed in practice. In this section, we first prove some mathematical properties of F, then present the algorithm that is efficient for solving the model (3.5). Some simulation experiments are finally conducted and the numerical results are reported.

4.1 Mathematical properties

Theorem 1 *Under Assumptions A1–A3, $F(u)$ is monotone and Lipschitz continuous. In detail, we have*

$$(\bar{u} - \tilde{u})^T (F(\bar{u}) - F(\tilde{u})) \geq \tau \|F(\bar{u}) - F(\tilde{u})\|^2, \quad \forall \ \bar{u}, \tilde{u} \in \mathbb{R}^{m+n}_+ \qquad (4.1)$$

and

$$\|F(\bar{u}) - F(\tilde{u})\| \leq (1/\tau)\|\bar{u} - \tilde{u}\|, \quad \forall \ \bar{u}, \tilde{u} \in \mathbb{R}^{m+n}_+. \qquad (4.2)$$

Proof Let $(\bar{x}, \bar{s}, \bar{d})$ and $(\tilde{x}, \tilde{s}, \tilde{d})$ be the solutions of problem (2.5) by setting $(y, z) = (\bar{y}, \bar{z})$ and $(y, z) = (\tilde{y}, \tilde{z})$, respectively. Setting $(y, z) = (\bar{y}, \bar{z})$ and $(x', s', d') = (\tilde{x}, \tilde{s}, \tilde{d})$ in (2.5), one obtains

$$(\tilde{x} - \bar{x})^T t(\bar{x}) + (\tilde{s} - \bar{s})^T \left[f(\bar{s}) + \bar{y} \right] - \left(\tilde{d} - \bar{d} \right)^T \left[g(\bar{d}) + \bar{z} \right] \geq 0. \qquad (4.3)$$

Similarly, setting $(y, z) = (\tilde{y}, \tilde{z})$ and $(x', s', d') = (\bar{x}, \bar{s}, \bar{d})$ in (2.5), we have

$$(\bar{x} - \tilde{x})^T t(\tilde{x}) + (\bar{s} - \tilde{s})^T \left[f(\tilde{s}) + \tilde{y} \right] - \left(\bar{d} - \tilde{d} \right)^T \left[g(\tilde{d}) + \tilde{z} \right] \geq 0. \qquad (4.4)$$

Adding (4.3) and (4.4) together, we get

$$(\bar{y} - \tilde{y})^T (\tilde{s} - \bar{s}) + (\bar{z} - \tilde{z})^T \left(\bar{d} - \tilde{d} \right)$$

$$\geq (\bar{x} - \tilde{x})^T (t(\bar{x}) - t(\tilde{x})) + (\bar{s} - \tilde{s})^T (f(\bar{s}) - f(\tilde{s})) - \left(\bar{d} - \tilde{d} \right)^T \left(g(\bar{d}) - g(\tilde{d}) \right).$$

It follows from (2.7)–(2.8) that

$$(\bar{y} - \tilde{y})^T (\tilde{s} - \bar{s}) + (\bar{z} - \tilde{z})^T (\bar{d} - \tilde{d}) \geq \tau \left(\|\bar{s} - \tilde{s}\|^2 + \|\bar{d} - \tilde{d}\|^2 \right),$$

or in a compact vector form

$$\begin{pmatrix} \bar{y} - \tilde{y} \\ \bar{z} - \tilde{z} \end{pmatrix}^T \begin{pmatrix} \tilde{s} - \bar{s} \\ \bar{d} - \tilde{d} \end{pmatrix} \geq \tau \left\| \begin{matrix} \tilde{s} - \bar{s} \\ \bar{d} - \tilde{d} \end{matrix} \right\|^2. \qquad (4.5)$$

Note that $u = \begin{pmatrix} y \\ z \end{pmatrix}$ and $F(\bar{u}) - F(\tilde{u}) = \begin{pmatrix} \tilde{s} - \bar{s} \\ \bar{d} - \tilde{d} \end{pmatrix}$ (see (3.4)), the first assertion follows from (4.5) immediately. The second assertion follows from (4.1) and Cauchy-Schwarz inequality directly.   □

4.2 Algorithm

Among the effective direct type methods for solving VI are the classical extra-gradient method and its variants, see, e.g., He and Liao (2002), He et al. (2004), Khobotov (1987), Korpelevich (1976). Note that the extra-gradient type methods require the monotonicity and Lipschitz continuity of the underlying mapping to guarantee the convergence. Therefore, thanks to Theorem 1, the extra-gradient type methods are applicable for solving the

implicit complementarity model (3.5). We here apply the improved extra-gradient method integrating optimal step-sizes in He et al. (2004) to solve the model (3.5).

Let $\mathbb{R}_+^{m+n}$ denote the positive orthant of $\mathbb{R}^{m+n}$ and v^+ denote the projection of v on $\mathbb{R}_+^{m+n}$, that is,

$$v^+ = \mathrm{argmin}\{\|v - u\| \mid u \in \mathbb{R}_+^{m+n}\}.$$

Then, for the current rates of tax and subsidy u^k, the governmental authority updates the rates according to the following steps:

The algorithm for adjusting rates of tax and subsidy

Step 1. Prediction step

Select a $\beta_k > 0$, such that

$$\beta_k \| F(u^k) - F(\tilde{u}^k)\| \le \nu \|u^k - \tilde{u}^k\|,$$
$$0 < \nu < 1, \qquad (\text{usually } \nu \in [0.9, 0.95]), \tag{4.6}$$

where

$$\tilde{u}^k = \left[u^k - \beta_k F(u^k)\right]^+. \tag{4.7}$$

Step 2. Correction step

Adjust the rates of tax and subsidy via the scheme

$$u^{k+1} = \left[u^k - \alpha_k \beta_k F\left(\tilde{u}^k\right)\right]^+, \tag{4.8}$$

where

$$\alpha_k = \gamma \frac{\left(u^k - \tilde{u}^k\right)^T q\left(u^k, \beta_k\right)}{\|q\left(u^k, \beta_k\right)\|^2}, \quad \gamma \in [1, 2), \quad (\text{usually } \gamma \in [1.8, 1.95])$$

and

$$q\left(u^k, \beta_k\right) = u^k - \tilde{u}^k - \beta_k \left[F\left(u^k\right) - F\left(\tilde{u}^k\right)\right]. \tag{4.9}$$

Remark 1 If $0 < \beta_k \le \nu\tau$, then from (4.2), we have

$$\beta_k \| F\left(u^k\right) - F\left(\tilde{u}^k\right)\| \le \beta_k \frac{1}{\tau}\|u^k - \tilde{u}^k\| \le \nu \|u^k - \tilde{u}^k\|,$$

and thus the condition (4.6) is satisfied. Since we do not know the value of $\tau > 0$ in prior, some strategies of adjusting $\{\beta_k\}$ self-adaptively are necessary to accelerate the convergence of the above algorithm, e.g., He and Liao (2002), He et al. (2004). Denoting

$$r_k := \frac{\beta_k \| F(u^k) - F(\tilde{u}^k)\|}{\|u^k - \tilde{u}^k\|},$$

then we adjust $\{\beta_k\}$ via the following scheme:

$$\beta_{k+1} = \begin{cases} \dfrac{0.5v}{r_k}\beta_k & \text{if } r_k > v; \\[2mm] \dfrac{0.9v}{r_k}\beta_k & \text{if } r_k \leq \mu < v, \qquad (\text{usually } \mu \in [0.4, 0.5]) \\[2mm] \beta_k & \text{otherwise.} \end{cases} \qquad (4.10)$$

Theorem 2 *The sequence $\{u^k\}$ generated by the scheme* (4.6)–(4.9) *converges to a solution of the model* (3.5)*, i.e., the desired rates of tax and subsidy is computable.*

Proof The proof is omitted, see e.g., He et al. (2004) for the detail.   □

4.3 Numerical result

Note that the involved projection of the above algorithm is pretty easy to compute. Therefore, the main computational load in the decision process of seeking desired rates of tax and subsidy is to observe the function values of $F(u)$. Obviously, the observations are money- or time-consuming. Hence, we prefer to reduce the numbers of observing function values of $F(u)$ as many as possible. In addition, one iterate of the sequence $\{u^k\}$ means equivalently an adjustment of the rates of tax and subsidy. Apparently, the policy instruments with high frequency of adjustment are not preferable. Hence, we also need to reduce the number of iterations of the above algorithm as many as possible. We are about to demonstrate that the above algorithm has these advantages, thus justify that it is applicable for the governmental authority to find the appropriate rates of tax and subsidy with the concerns of over-production and supply-guarantee.

For the purpose of simulation, we generate f, g and t as the way described in (2.9), where the involving coefficients are generated randomly in the following intervals:

$$a_i \in (1, 2), \quad b_j \in (1, 2), \quad c_{ij} \in (0.1, 0.2), \quad \xi_i \in (300, 400),$$

$$\eta_j \in (600, 700), \quad \zeta_{ij} \in (10, 20).$$

We assume that $m = 20$ and $n = 50$. Therefore, the number of variables of the model (3.5) is 70. Furthermore, let the elements of $s^{\max}$ and $d^{\min}$ be uniformly 150 and 40, respectively. Let the iteration start with $u^0 = 0$ and $e(u^k) := \|\min\{u^k, F(u^k)\}\|_\infty$ be the error measure when solving the problem (3.5). Let $v = 0.9$ and $\mu = 0.5$. All the codes were written by Matlab 7.0 and were run in an IBM T42 laptop.

The following table lists the total numbers of computing function values (No. F) and its corresponding errors ($e(u^k)$) during the iterative procedure.

k	0	2	4	6	8	10	12	14
No. F	0	5	9	13	17	21	25	29
$e(u^k)$	7.6e+1	6.6e+1	5.8e+0	5.0e−1	2.5e−2	3.4e−4	3.6e−5	2.5e−6

According to this table, averagely, we need to calculate the function values of $F(u)$ twice at each iteration. This satisfactory performance is mainly due to the self-adaptive strategy of adjusting β_k. In addition, recall that the iterations start from the initial iterate $u^0 = 0$. However, in the practical decision, the governmental authority can determine a better initial iterate based on experiences and relevant information. Therefore, in real-life decisions, the performance of the implemented algorithm is expected to be better than what we reported.

Note that if we take $\alpha_k \equiv 1$ in the correction step (4.8) (which is feasible, as proved in He and Liao 2002; He et al. 2004), the algorithm reduces to Khobotov's extra-gradient method (Khobotov 1987), whose average numbers of computing function values and CPU time are about 8 times more than those in the above table.

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