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Public Transit Corridor Assignment Assuming Congestion Due to Passenger Boarding and Alighting

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Abstract This paper proposes a formulation of deterministic equilibrium in a public transit corridor that takes into account the congestion effect as perceived directly in travel times. The identification of the relationship between flows and travel times includes time at transit stops for passenger boarding and alighting. A simple case is analyzed that demonstrates the existence of equilibria in which identical users adopt different travel strategies, and a method is supplied for determining such an equilibrium. To find the general case assignment for a corridor, an assignment algorithm based on incremental flow increases is also presented. Finally, the algorithm is implemented in a simple corridor. The results show that identical users faced with the same trip must be allowed to take different decisions for an equilibrium assignment to exist.

Keywords Public transit · Assignment · Congestion

1 Introduction

The last decade or so has witnessed the emergence in various of the world's cities of dedicated public transit corridors in which buses can pass each other in the approaches to a bus stop they do not intend to pull in at. This allows operators to run express services that omit certain stops and thereby demonstrably improve the functioning of the system by reducing travel times, operating costs and the fleet size

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needed to serve the corridor. Demand for such services is particularly high in systems with integrated fares, a common feature of modern public transit networks.

The proper design of an express bus system turns on the ability to predict the demand each bus service or line would capture and the costs it would incur. To identify demand, a model is required that would predict rider behavior as regards the choice of lines to make any given journey. Typically, riders minimize expected total travel time. Since we will confine ourselves to services whose only difference is the set of stops they serve, it is essential that vehicle travel time be modeled so as to include all phenomena related to serving or omitting a stop that can affect it. This implies that the impact of an additional passenger on travel time must be analyzed.

Existing studies on the design of express services are limited by their tendency to assume that travel times are independent of network flows. In Turnquist (1979), for example, express lines can benefit passengers by taking advantage of urban freeways paralleling dedicated corridors to provide faster routes between origin and destination. But such a mode restricts the demand structure by assuming that the destination of all trips is the corridor's final node, or that all journeys are generated at the initial node.

Based on empirical evidence, recent studies suggest some general guidelines for designing express bus services on the context of Bus Rapid Transit. Vuchic et al. (1994), and Goodman et al. (1997) characterize key elements of BRT, while Silverman et al (1998) and Schwarcz (2004) focus on specific design elements. Sun and Hickman (2005) studied the convenience of skipping bus stops in real time.

As regards the assignment of passengers in transit systems, early works such as those by Dial (1967), Fearnside and Draper (1971) and Le Clercq (1972) assumed riders simply chose the lines whose travel times were shortest. This approach not only underestimates the importance of congestion, it also supposes that passengers wait till their chosen line appears, an assumption valid only when there are no parallel routes serving the same origin–destination pair. Chriqui and Robillard (1975) presented a model that addressed this latter limitation by allowing riders to choose a subset of lines that minimized expected travel time, defined to include the trip itself plus waiting time. Although they did not take congestion into account, the assumption of a set of attractive lines has been widely accepted for modeling this type of rider assignment.

One of the earliest models that does incorporate the effect of congestion in transit assignment is found in Spiess (1984), who modeled the phenomenon as the discomfort suffered by the rider. However, this approach still excludes the impact of congestion on travel time. Gendreau (1984) accounted for congestion's travel time effects by including an analysis of bus boarding queues, but the complexity of the resulting model is such that equilibria cannot be found. The first study to successfully treat congestion was that of De Cea and Fernández (1993), who assumed that it translates into longer waiting times at bus stops and modeled it by defining an "effective frequency" of service that decreases with the rise in network flow levels. A disadvantage of this technique is that it supposes all passengers consider the same set of lines to be attractive, which in certain cases prevents the existence of a Wardrop equilibrium. Finally, Cominetti and Correa (2001) presented a model in which riders can choose between different sets of attractive lines, but always on the assumption that congestion results in waiting times at bus stops due to bus capacity constraints.

In this paper we propose a model in which deterministic non-homogeneous equilibria for identical transit users are permitted (that is, identical users can choose different sets of lines to make their journeys) and flow levels directly affect travel times, thus allowing us to study the advantages and disadvantages of having routes along a given corridor that do not serve certain stops.

The remainder of this article is organized as follows: Section 2 describes the passenger assignment problem, outlining the main characteristics of the proposed model and how it relates to earlier works. Section 3 builds a model to solve the problem as formulated in the previous section, presenting the necessary notation and then studying two specific cases. The first case is a simplified version of the problem that affords an understanding of bus network behavior and a characterization of its solution, while the second case addresses the problem in its fullest dimension and offers a detailed solution algorithm. Section 4 sets forth and analyzes the results obtained from a particular application of the algorithm, and finally, Section 5 summarizes the main conclusions of our study.

2 Description of the problem

2.1 Main characteristics

To simplify the passenger assignment problem we consider a single-corridor network which is used by various bus lines with known frequencies and stops and has a known and fixed trip matrix (origin–destination, hereafter also "OD"). We also assume that bus stop time due to the mere effect of decelerating and accelerating can be considered part of (known) travel time between two successive stops, and that the time spent at the stop itself depends on passenger boarding and alighting flows. This latter time factor is taken to be directly proportional to the number of passengers boarding and alighting. The model we propose allows us to predict how passengers are assigned to the available services and therefore also the number of passengers carried by each service on each of its arcs.

In Chriqui and Robillard (1975), the passenger assignment model posited the existence of a set of attractive lines for each OD pair that minimizes expected total waiting time on the assumption that a user at a stop takes the first bus belonging to a line in the set. Cominetti and Correa (2001) showed that if congestion due to vehicle capacity limits is taken into account, the attractive set is no longer unique for each OD pair, and different passengers traveling on the same pair must adopt different attractive sets simultaneously in order for an equilibrium to exist.

The assignment model proposed in this paper does not include capacity restraints. Although in some cases such an omission may be unrealistic, it is justifiable in the present context because the model is intended as part of a larger design problem for which, at least in the initial stages, it is useful to consider an ideal system that can operate with buses of whatever size may be necessary.

Another assumption in our model is that for the sake of simplicity, the various services charge the same fare and all other relevant attributes are also similar (comfort, safety, etc.) so that users choose exclusively on the basis of travel time. Passengers are also assumed to have perfect information about the services and value all their attributes in identical manner. In that sense, passengers are assumed identical. Explicitly incorporating these additional factors is a simple matter, though they would have to be converted into their equivalents in time units.

Finally, the model does not allow passengers to change buses at any point on the way to their destination. This constraint imposes a condition on the design of the network in that for each OD pair where there is a flow (in principle, any pair) there must be at least one line connecting them.

3 Formulation and solution of the model

3.1 Notation

We begin building our model of the assignment problem by assuming a unidirectional public transit corridor with n bus stops. Together they comprise a set P, ordered in the same sequence the stops p are encountered by the buses traveling the corridor ($p=1,\ldots,n$). The set of stops defines the set of origin–destination pairs W in the corridor, whose elements w correspond to all pairs of stops i,j in P such that $i<j$.

A set of lines (or services) denoted L operates along the corridor, each line identified as l_j where j is a number between 1 and m. The set of all non-empty subsets of lines belonging to L is called S. Thus, each element $s\in S$ is a possible subset of lines. We will denote $\{i_1,\ldots,i_n\}$ as the subset s of lines $l_{i1},\ldots,l_{in}$. Every passenger traveling a given OD pair has a "strategy" consisting of a subset of lines all of which stop at the passenger's origin and destination, and which are considered to be attractive in the sense that the first bus to arrive from that subset will be the one taken, and expected travel time will be minimized.

The lines in a subset do not necessarily all serve the same set of stops, but it will be assumed that each line's set of stops is fixed and known. Also, the demand for trips in the corridor is known for each w, is denoted T_w and is measured in passenger units per time unit.

Each line has two known attributes that determine how trips are assigned to it: frequency, and fixed travel time between each pair of stops served. Frequency is represented by f_i for each line l_i, and is measured as the number of buses per time unit, whereas fixed travel time for an OD pair w on line l_i is denoted c_i^w. This latter factor is the bus's travel time for the pair, taking into account the intermediate stops it serves but excluding actual dwell time at the stops for taking on and letting off passengers.

De Cea and Fernández (1993) formulate the equilibrium assignment using a vector of variables $V = \left\{v_i^w\right\}$ representing the total flow traveling a pair w on line l_i. However, when passengers boarding and alighting times are considered, these variables are not enough for the equilibrium to be characterized. As will be shown, in this case, the output variables for the assignment problem model comprise the vector $Y = \left\{y_s^w\right\}$, each of whose elements is defined as the flow of passengers traveling an OD pair who consider the subset of lines s to be attractive.

The relationship between the variables y_s^w and v_i^w is given by

$$v_i^w = \sum_{s \in \wp_i} y_s^w \frac{f_i}{\sum_{j \in s} f_i}, \text{ where } s \in \wp_i \Leftrightarrow i \in s \tag{1}$$

This equation is derived by assuming that the flow y_s^w is distributed between the various lines in s in inverse proportion to their frequencies.

A passenger's travel time is the sum of actual time on the vehicle and waiting time at the bus stop. Following the notation defined by Chriqui and Robillard (1975), also employed by De Cea and Fernández (1993), and widely used in many transit assignment models, the expected travel time (ETT) for passengers traveling the OD pair w who consider s to be attractive is expressed by:

$$ETT_s^w = \frac{\lambda + \sum_{i \in s} f_i t_i^w(V)}{\sum_{i \in s} f_i} \tag{2}$$

where $t_i^w(V)$ is the travel time for OD pair w on line i. This travel time indicator takes into account actual dwell times at each stop, and thus includes boarding and alighting times. It therefore depends on the trip assignments to all other pairs that could have used line i and thereby affected its value. The constant term λ represents two effects: passengers' valuation of waiting time relative to travel time, and the proportion of the average interval between buses actually experienced as average waiting time.

According to Wardrop's first principle, a flow vector Y^* represents an equilibrium if and only if for each OD pair there is a minimum expected travel time corresponding to each subset of lines that is attractive for some user and therefore experiences some passenger flow. In mathematical terms,

$$y_s^w > 0 \Rightarrow ETT_s^w(V(Y*)) = E\widehat{T}T^w(V(Y*)) \tag{3}$$

where $E\widehat{T}T^w(V)$ is defined as $\min_{s \in S} \mathrm{ETT}_s^w(V)$. The simultaneous complement of Eqs. (1), (2) and (3) constitute the equilibrium model that determines user assignment in the corridor.

3.2 Wardrop assignment for an isolated OD pair

The nature of the proposed model and its behavior can be better understood by considering a simple case of equilibrium on an isolated OD pair w served by a set of lines with their respective times and frequencies. This example will allow us to derive an analytic expression for the equilibrium flows on each line. Since w here is fixed and unique, we will not use a superindex in the various terms indexed to it.

To formulate this version of the model we must determine the function t_i (V) that appears in the expression for $\mathrm{ETT}_s^w(V)$. In this case t_i (V) is modeled as:

$$t_i(V) = c_i + \frac{2\,\mathrm{tba}v_i}{f_i} \tag{4}$$

where the first term on the right-hand side (c_i) is the constant part of travel time and the second term is the total boarding/alighting time. The constant tba represents a single rider's boarding/alighting time while v_i is the flow (in pax/h) carried on line i. The term v_i/f_i is the expected number of passengers that will travel on each line i bus.

The assignment problem with congestion is isomorphic to a conventional assignment problem. Assume that the network nodes are the origin and destination stops of the corridor, and that for each strategy (that is, for each subset s) there exists an arc between the two nodes. The cost of traveling each arc in the network is analogous to the ETT of each strategy, and the flow along any arc corresponds to each variable y_s^w in the problem. The only difference is that now the cost function for each arc is not separable as a function of its own flow but rather depends on the arcs with which it shares one or more services.

To solve such a Wardrop equilibrium problem there are various resources available whose choice will depend on the characteristics of the network. A first option might be to find the optimization problem equivalent to the assignment. A factor that will ensure the existence of this problem is a cost function whose Jacobian matrix (for the ETT values in this case) is symmetric, implying that the network interactions are also symmetric. In this simple case the Jacobian is seen to have this property, so the assignment can indeed be obtained from the equivalent optimization model. Another option for solving the problem would be to use a diagonalization algorithm (Florian 1977; Abdulaal and LeBlanc 1979), a version of the iterative Jacobi method for solving non-linear equations (Pang and Chan 1982). Both of these methods, however, fail to exploit certain special characteristics of the problem that facilitate the derivation of a more direct solution.

Suppose we know the lines l_1, l_2,..., l_n that serve a given OD pair. For ease of explanation we further assume they are ordered such that $c_1<c_2<\ldots<c_n$. It will also be useful to define the subsets $s_i:=\{l_1, l_2,\ldots, l_i\}$.

We want to assign a total flow of T trips to the OD pair. To understand the network flow assignment, suppose that initially the network has no load (i.e., no flow) and begins to load slowly, all care being taken so that with each flow increment the assignment continues to satisfy the equilibrium conditions. The network flow at any stage in the process is given by the variable x, where $x<T$.

Now consider the first increase in the level of flow. Clearly, there will always exist a flow level sufficiently small that its attractive lines are the same as those that would obtain if the problem involved no congestion. According to Chriqui and Robillard (1975), this attractive line subset will always be of the form $\{l_1, l_2,\ldots, l_i\}$, implying that if a given line belongs to the subset, all lines with lower fixed travel time will necessarily belong to it as well. We will call l_q the line with the highest fixed travel cost in the subset, and the latter will thus be denoted s_q.

As the arc corresponding to the optimal strategy without congestion begins to load, its ETT increases and the ETTs of other strategy arcs in the network begin to rise by varying amounts. At a given moment the ETT of an arc with no flow (recall that the arcs correspond to line subsets) may come into equality with that of the arc that is loading (y_{sq}). From that point on we cannot be sure that as y_{sq} continues to mount the network will remain in equilibrium; indeed, except for special cases it normally will not. Also, the first arc to become attractive during this stage (that is, its

ETT equalizes with that of the loading arc) is the one corresponding to subset s_{q+1}, and this occurs when the network flow reaches a given threshold.

If only the newly active arc continues to load once this threshold is reached, the equilibrium will hold until the ETT of a new arc reaches $\widehat{ETT}$ at some point. This new arc will be s_{q+2}. The process just described will continue until x equalizes with T. If the flow that activates arc s_{q+i}, and which we will call the flow threshold, is denoted x_i, it can then be calculated as follows:

$$x_i = \frac{1}{2tba}\left(\sum_{j=1}^{q+i-1} f_j\left(c_{q+i} - c_j\right) - \lambda\right) \tag{5}$$

This expression enables us to identify the equilibrium assignment for any desired level of demand; knowing the thresholds, we can easily determine the flow values for each arc. A simple method of calculating the flow in each line for any network flow level T would be to draw a diagram such as the one shown here in Fig. 1. In the diagram different values of T generate varying flows for each strategy and thus for each line. The arc flows calculated for these thresholds satisfy the Wardrop equilibrium conditions, as was demonstrated in Larrain (2006).

The assignment obtained through this method is not unique as regards the flows y_s; in fact, examples in which equilibrium can be attained with various values of Y

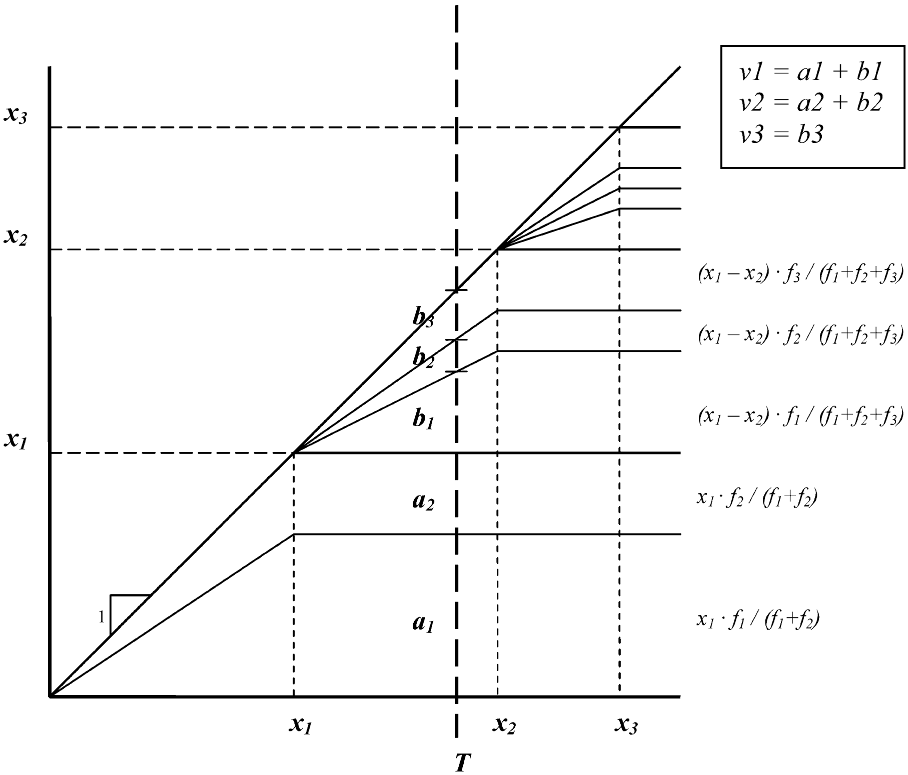


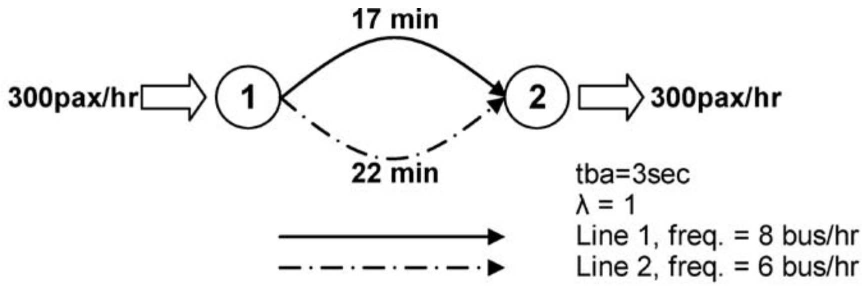
Fig. 1 Passenger assignment when s_q contains two lines

for a given flow level are easily found. However, the solution does seem to be unique in terms of the flows *V*, formal proof of which is currently being developed.

This equilibrium assignment for the isolated OD pair case, implies that identical users make different decisions (i.e. consider a different set of attractive lines) for the same trip. A simple example where this type of equilibrium is the only possible one is provided in Fig. 2. Thus, when on-board travel times consider a congestion term, the analysis must allow passengers to adopt different attractive line sets in order for an equilibrium to exist. This shows the importance of expressing the equilibrium in terms of the set of variables *Y*.

3.3 Wardrop assignment for a public transit corridor

Our study of a simple case has elucidated some important characteristics of the assignment problem. First, it confirmed the need to use different flows for each strategy on each OD pair. And second, it demonstrated that as the level of flow through the network increases, the number of attractive alternatives for different users rises. The method used in the simple example for determining the thresholds also provides the inspiration for an algorithm we will now use in a more complex



$$ETT_{\{1\}} = \frac{f_1 t_1 + \lambda}{f_1} = 17 + \frac{60}{15} + \frac{0.1 \cdot v_1}{15} \qquad ETT_{\{2\}} = \frac{f_2 t_2 + \lambda}{f_2} = 22 + \frac{60}{12} + \frac{0.1 \cdot v_2}{12}$$

$$ETT_{\{1,2\}} = \frac{f_1 t_1 + f_2 t_2 + \lambda}{f_1 + f_2} = \frac{15 \cdot 17 + 12 \cdot 22}{15+12} + \frac{60}{15+12} + \frac{0.1 \cdot (v_1 + v_2)}{15+12}$$

Case	Only l_1	l_1 and l_2	Equilibrium
Flows	v_1 = 300 v_2 = 0	v_1 = 166,7 v_2 = 133,3	v_1 = 233.3 v_2 = 66.7 ($y_{\{1\}}$=$y_{\{1,2\}}$=150)
$ETT_{\{1\}}$	23.00	**22.11**	**22.56**
$ETT_{\{2\}}$	27.00	28.11	27.55
$ETT_{\{1,2\}}$	**22.56**	22.56	**22.56**

Fig. 2 Example of equilibrium on a single OD pair

case to find the Wardrop equilibrium for a congested transit corridor in which we seek to make assignments among various OD pairs.

As with the simple case, we must derive an expression for travel time t_i^w which appears in the ETT function. Whereas in the previous situation the only flow affecting the travel time of a line was its own flow, in this more complex case there are flows of the same line in other network OD pairs that also affect travel time. This can be expressed in the following manner:

$$t_i^w = c_i^w + \frac{tba \sum_{k \in W} \alpha_{wk} v_i^k}{f_i} \tag{6}$$

In this formula the term α_{wk} measures the effect on travel time for the OD pair w of an additional flow unit on a line for pair k. For example, if $k=w$ then $\alpha_{wk}=2$, which is the situation in the simple case of the previous section. On the other hand, if w and k do not overlap then $\alpha_{wk}=0$.

To identify the equilibrium assignment, an attractive approach is to extend the threshold concept of the simple example, although the analysis will be much more complex. Whereas with a single OD pair the equilibrium flow level of an arc is given by a single variable x, in the complex case it is expressed as a flow vector $(x_1, x_2,\ldots, x_w,\ldots, x_m)$ where m is the number of OD pairs in the corridor. The threshold values for a given OD pair will depend on the flow levels for the other pairs. The threshold flows of a corridor should thus be represented by hyperplanes within an m-dimensional space. Since values for the thresholds cannot be found a priori, we propose an algorithm for finding the network user equilibrium based on the strategy employed for determining the assignment in the simple case with a single OD pair. In broad terms the algorithm, to be described in detail below, proceeds in two phases: the search for the "direction of movement" of flow and the determination of the size of that movement.

3.3.1 Algorithm (phase 1)

Phase 1 of the algorithm begins by considering an incomplete assignment of the demand matrix that satisfies the equilibrium conditions, that is, the equilibrium that would be sought if the demand matrix were the assigned one. We then identify how network flows should change if the flow for a given OD pair is increased so that under the new assignment the equilibrium conditions continue to be satisfied.

The first step is therefore to arbitrarily choose an OD pair for which to increase the flow to meet the demand increase. The pair must be one for which x_w, calculated as $\sum_{i \in L} v_i^w$, is initially less than T_w (if there were no pairs for which x_w was less than T_w, the assignment would be complete.) The chosen pair is denoted $\widetilde{w}$.

Once the flow in w has been increased from $x_{\widetilde{w}}$ to $x_{\widetilde{w}} + \varepsilon$ (ε sufficiently small), we must determine which arcs for this and other pairs will see their flows affected. To refer to a given arc we will use the notation (s, w), which identifies it in terms of its attractive line strategy and its OD pair. Before the flow in $\widetilde{w}$ was increased, the network was in Wardrop equilibrium. In this situation there exist some arcs whose ETT is equal to the minimum for its pair, and only these arcs can carry flow both before and after the increase in demand (by the small amount ε). These arcs will be referred to as active, even if they carry no flow. Each iteration of the algorithm will only modify the flows in these arcs. The change of flow Δy_r^k will be of particular interest.

To determine how to redistribute the network flows y_s^w without altering the equilibrium, we must ascertain how changes in these flows impact each ETT_s^w. In Larrain (2006) it was demonstrated that the relationship between flow changes in network arcs (Δy_r^k) and the consequent change in an arc's travel time is a linear one given by the following expression:

$$\Delta \mathrm{ETT}_s^w = \sum_{r \in S} \sum_{k \in W} \alpha_{wk} \beta_{sr} \, \Delta y_r^k \tag{7}$$

where $\beta_{sr} := \sum_{i \in s \cap r} f_i / \left(\sum_{i \in s} f_i \sum_{i \in r} f_i \right)$. The calculation can be made for all possible combinations of subsets before launching the algorithm, as also with α_{wk}.

The conditions that must be satisfied in order to maintain equilibrium when flow in the chosen in the OD pair is increased are set out below:

1) For every OD pair w, all arcs that carry flow after the iteration must have the same ETT equal to $E\widehat{T}T^w$. For a sufficiently small flow this is equivalent to requiring that $\Delta \mathrm{ETT}_s^w$ be equal for all active arcs in the pair.
2) For every OD pair $w \neq \widetilde{w}$, flow variations must add up to 0; for $\widetilde{w}$, they must add up to ε (a sufficiently small number).

A third condition must be added to the preceding two to rule out negative flows in the network:

3) If $y_s^w = 0$ (despite being active) then $\Delta y_s^w > 0$.

Given that ε is sufficiently small, it is required in this phase that if (w, s) is a non-active arc, the condition $\Delta y_s^w = 0$ must hold. Conditions 1, 2 and 3 associated with the variables Δy_s^w for active arcs constitute a system of equations with the same number of variables as equations. The system is built as follows:

- For each OD pair, the equation below is added to the system:

$$\sum_{r \in S} \Delta y_r^w = \begin{cases} \varepsilon & w = \widetilde{w} \\ 0 & w \neq \widetilde{w} \end{cases} \tag{8}$$

  This equation ensures that Condition 2 above is satisfied. Since each OD pair has only one reference arc, this condition is applied once for each pair. Note that some variables in the summation were removed from the system and are given a value of 0; in other words, they should not figure in the summation.

- For each active variable of an OD pair whose current flow is positive ($y_s^w > 0$), a ETT equality is added to the system:

$$\Delta \mathrm{ETT}_s^w - \Delta \mathrm{ETT}_{\bar{s}(w)}^w = 0 \tag{9}$$

  Here, $\bar{s}(w)$ is an arc associated with an active positive flow variable for the pair (if the pair has no arc with flow, the one with the fewest lines can be chosen).

- If Δy_s^w is associated with an active arc with no flow ($y_s^w = 0$), the following equations must be added to the system:

$$\Delta \mathrm{ETT}_s^w - \Delta \mathrm{ETT}_{\bar{s}(w)}^w \geq 0 \tag{10}$$

$$\Delta y_s^w \geq 0 \tag{11}$$

$$\Delta y_s^w \Big(\Delta \mathrm{ETT}_s^w - \Delta \mathrm{ETT}_{\bar{s}(w)}^w \Big) = 0 \tag{12}$$

If an active arc has no flow, there are two alternatives: the flow change is positive and its ETT is equal to the minimum for the pair, or its flow change is zero and its ETT does not fall below the group's minimum level. These alternatives are represented by the three expressions above. In the case where the flow change is zero, the arc is no longer active at the next iteration. The solution to the resulting system of equations can be found using conventional mathematical techniques.

The outcome of this phase is the values of the set of variables Δy_s^w that enable an additional flow unit (ε) to be assigned while still maintaining the network in equilibrium. Given the characteristics of the resulting system of equations, all the variables are expressed linearly in ε. Thus, since we are interested in determining the assignment for non-small values of ε, we must attempt to identify the maximum value that ε can take without any constraints on the problem being activated by the changes in the variables. The second phase of the algorithm aims at precisely that.

3.3.2 Algorithm (phase 2)

The system of equations in Phase 1 of the algorithm defined the direction of movement of flow for each arc in the system as a function of the flow increase in the chosen OD pair. The objective of Phase 2 is to determine how much movement there can be in that direction without losing network equilibrium.

The maximum size of ε is determined by the first of three situations described below:

1) In some OD pair, an inactive arc becomes active; that is, its ETT becomes equal to that of the active arcs.
2) In some OD pair, the flow in an arc (initially strictly positive) whose Δy_s^w is negative falls to 0.
3) The total flow of arc $\widetilde{w}$, that is, $x_{\widetilde{w}}$, reaches the flow level $T_{\widetilde{w}}$ it is desired to assign.

Each of these three conditions determines an upper bound for ε, to be called z_1, z_2 and z_3, respectively. The value we seek is denoted z and defined as $z=\min(z_1, z_2, z_3)$.

To calculate z_1, note first that $\Delta\mathrm{ETT}_s^w$ is a (linear) function of the flow variations. We must therefore determine, for each arc, the value of ε that yields a ETT_s which attains the minimum for the OD pair. The arc with the smallest ε becomes the critical arc that determines z_1.

Calculating z_2 is simple; its value is just the smallest flow whose associated change is negative divided by the absolute value of the change. Finally, calculating z_3 trivial, being equal to the difference between $T_{\tilde{w}}$ and $x_{\tilde{w}}$.

Once z has been determined, the network flows are updated by adding to each of them the value given by $z\Delta y_s^w / \varepsilon$. Since the new assignment is an equilibrium for all values of ε smaller than or equal to z, it is a (partial) Wardrop equilibrium for the network, and provides the next point of departure for a new stage of the algorithm until the desired equilibrium where the OD matrix has been completely assigned is finally reached.

The convergence of the algorithm has not been formally proven; it does, however, appear intuitively to do so. Regarding the uniqueness of the solution, it can be easily shown that it is not unique in terms of Y (an example can be found in Larrain 2006). However, the solution appears to be unique in terms of V since uniqueness has prevailed in many examples studied no matter in which order the arcs are loaded in the first phase of the algorithm. As in the case of classical user equilibrium that can be expressed uniquely in terms of arc flows but not in terms of path flows, although path flows must be used to model and solve analytically the problem, the same appears to be true here about V and Y.

4 Application of the algorithm

4.1 The network

In this section the algorithm is implemented in a four-stops corridor where three lines operate (lines 1, 2 and 3). Lines 1 and 3 stop in all four bus stops, while line 3 skips bus stop 2. The network configuration, including trip times and frequencies, is detailed in Fig. 3, while its OD matrix is given in Table 1.

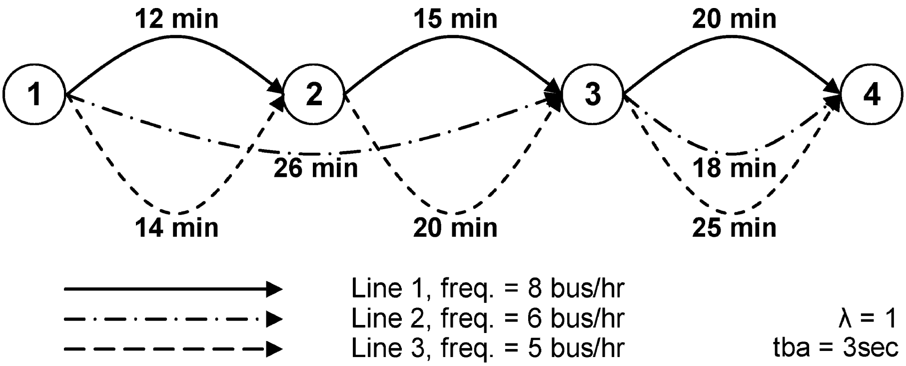


Fig. 3 Network configuration for application in section 4

Table 1 OD matrix for application in section 4

w	T_w (pax/h)
12	300
23	200
34	300
13	600
24	400
14	200

4.2 Executing the algorithm

Before executing the algorithm, parameters α_{wk} and β_{sr} must be computed. Their values for this case are shown in Fig. 4. The first few iterations of the algorithm are straightforward so they have been omitted from this text. When few arcs carry flow, is very likely that the system of equations resulting from Phase 1 considers a single equation, implying that flow could be added to the critical link as long as the threshold determined in Phase 2 of the algorithm is not reached. Thus, following the order in which OD pairs appear in the matrix, after six iterations the assignment shown in Table 2 is reached. Table 3 displays its associated ETT matrix. This assignment is not the seek equilibrium since it only considers four out of the six OD pair flows.

The next iteration is now presented in more detail. In Phase 1 the OD pair 2–4 (currently with no flow assigned) is chosen to increment its assigned flow. The following system of equations is reached:

$$\Delta y^{3-4}_{\{1,2\}} + \Delta y^{3-4}_{\{1,2,3\}} = 0 \tag{13}$$

$$0.038\,\Delta y^{3-4}_{\{1,2\}} - 0.009\,\Delta y^{2-4}_{\{1,3\}} = 0 \tag{14}$$

$$\Delta y^{2-4}_{\{1,3\}} = \varepsilon \tag{15}$$

The solution of this system is $\Delta y^{3-4}_{\{1,2\}} = 0.231\varepsilon$, $\Delta y^{3-4}_{\{1,2,3\}} = -0.231\varepsilon$ and $\Delta y^{2-4}_{\{1,3\}} = \varepsilon$. In Phase 2 of the algorithm the maximum valid value of ε is obtained, in this case 346.63 pax/h, (a greater value of ε would imply a negative flow in

α_{wk}		k: 12	23	34	13	24	14
w	12	2	0	0	1	0	1
	23	0	2	0	1	1	0
	34	0	0	2	0	1	1
	13	2	2	0	2	1	1
	24	0	2	2	1	2	1
	14	2	2	2	2	2	2

βsr		r: {1}	{2}	{3}	{12}	{13}	{23}	{123}
s	{1}	0.13	0	0	0.07	0.08	0	0.05
	{2}	0	0.17	0	0.07	0	0.09	0.05
	{3}	0	0	0.2	0	0.08	0.09	0.05
	{12}	0.07	0.07	0	0.07	0.04	0.04	0.05
	{13}	0.08	0	0.08	0.04	0.08	0.03	0.05
	{23}	0	0.09	0.09	0.04	0.03	0.09	0.05
	{123}	0.05	0.05	0.05	0.05	0.05	0.05	0.05

Fig. 4 Values of α_{wk} and β_{sr} for application in section 4

Table 2 Flows after six iterations

y_s^w		w					
		1–2	2–3	3–4	1–3	2–4	1–4
s	{1}	0.00	0.00	0.00	0.00	0.00	0.00
	{2}	0.00	0.00	0.00	0.00	0.00	0.00
	{3}	0.00	0.00	0.00	0.00	0.00	0.00
	{12}	0.00	0.00	220.00	600.00	0.00	0.00
	{13}	300.00	200.00	0.00	0.00	0.00	0.00
	{23}	0.00	0.00	0.00	0.00	0.00	0.00
	{123}	0.00	0.00	80.00	0.00	0.00	0.00

$y^{2-4}_{\{1,3\}}$). Thus, arc flows can be increased in $\Delta y^{3-4}_{\{1,2\}} = 80\,\text{pax/h}$, $\Delta y^{3-4}_{\{1,2,3\}} = -80\,\text{pax/h}$ and $\Delta y^{2-4}_{\{1,3\}} = 346.63\,\text{pax/h}$ without losing the equilibrium conditions and yielding an assignment in which the first four OD pair flows and part of the fifth one are assigned in the network. Once two similar iterations are completed, the global equilibrium assignment presented in Tables 4 and 5 is reached.

It can be observed that the assignment satisfies user equilibrium conditions since each OD pair demand is divided only among fastest available strategies which ensure that no user has an incentive to change his decision. Also notice the case of the demand of OD pair 3–4 where 212.31 pax/h wait for buses of lines 1 and 2 only while 87.69 pax/h take the first bus of any line.

5 Conclusions

This study has revealed a number of important points regarding public transit corridor assignment when it is assumed that travel time on each line depends on its number of users. The principal finding is that there exist equilibrium situations in which identical users can choose different travel strategies to make the same journey. Also, the number of different simultaneous strategies in an equilibrium for a given origin–destination pair can be fairly large. Such situations can be easily constructed by applying the formulas in Section 3.2.

Traditional public transit assignment models often do not take into account congestion, or if they do, they assume it to be associated with the limited capacity

Table 3 ETTs after six iterations

ETT_s^w		w					
		1–2	2–3	3–4	1–3	2–4	1–4
s	{1}	23.95	26.18	29.49	42.63	48.17	64.62
	{2}	1,012.14	1,012.14	29.99	40.29	1,014.14	60.28
	{3}	28.31	33.54	37.42	49.85	58.96	75.27
	{12}	443.18	444.45	25.42	37.34	457.87	58.48
	{13}	21.01	24.40	27.93	40.79	47.71	64.10
	{23}	559.49	561.87	27.91	39.18	574.51	61.64
	{123}	330.84	333.16	25.42	37.47	349.74	59.74

Table 4 Equilibrium assignment flows for the network in section 4

y_s^w		w					
		1–2	2–3	3–4	1–3	2–4	1–4
s	{1}	0.00	0.00	0.00	0.00	0.00	0.00
	{2}	0.00	0.00	0.00	0.00	0.00	0.00
	{3}	0.00	0.00	0.00	0.00	0.00	0.00
	{12}	0.00	0.00	212.31	600.00	0.00	200.00
	{13}	300.00	200.00	0.00	0.00	400.00	0.00
	{23}	0.00	0.00	0.00	0.00	0.00	0.00
	{123}	0.00	0.00	87.69	0.00	0.00	0.00

of the transit vehicles. In many transit systems around the world this type of congestion in fact rarely arises; the type analyzed here, by contrast, is inherent in almost any system. The approach developed here is intended to provide a starting point for complex transit network assignment models as well as for network design models.

In more specific terms, the most direct application for the model we have presented is in a transit route design model for a public transit corridor. The model addresses the lower level of the optimization problem in such a design model. At an intermediate level a method would have to be found to optimize frequencies for a given operation scheme, while at the highest level the model would have to optimize the line routes. To employ this type of model the ability of users to change lines in order to reach their destination would have to be incorporated. The authors are currently working on this issue.

A possible extension of this study that could be of interest in this area of research would be to find a way of broadening the algorithm not only to allow movement from one equilibrium to another for a set of fixed data, but also to predict how flows would be reassigned when frequencies are varied. Such a tool would considerably simplify the transit corridor design problem. Another worthwhile line of inquiry would be to determine how to incorporate line capacity constraints to absorb the flows assigned by this method, or to allow passengers transferring between services to complete their trip.

Table 5 Equilibriums assignment ETTs for the network in section 4

ETT_s^w		w					
		1–2	2–3	3–4	1–3	2–4	1–4
s	{1}	24.66	27.72	31.73	44.88	51.95	69.12
	{2}	1,012.86	1,012.14	30.69	41.00	1,014.84	61.69
	{3}	28.31	35.08	39.00	51.38	62.08	78.38
	{12}	443.89	445.33	27.00	38.93	460.33	61.65
	{13}	21.45	25.93	29.91	42.77	51.23	68.07
	{23}	559.88	562.57	29.01	40.27	576.31	63.83
	{123}	331.37	334.21	27.00	39.05	357.37	62.89

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