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A network economic game theory model of a service-oriented internet with choices and quality competition

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Abstract This paper develops a new dynamic network economic model of Cournot-Nash competition for a service-oriented Internet in the case of service differentiation and quality competition. Each service provider seeks to maximize its own profit by determining its service volumes and service quality. We utilize variational inequality theory for the formulation of the governing Nash equilibrium as well as for the computational approach. We then construct the projected dynamical systems model, which provides a continuous-time evolution of the service providers service volumes and service quality levels, and whose set of stationary points coincides with the set of solutions to the variational inequality problem. We recall stability analysis results using a monotonicity approach and construct a discrete-time version of the continuous-time adjustment process, which yields an algorithm, with closed form expressions at each iteration. The algorithm is then utilized to compute the solutions to several numerical examples. A sensitivity analysis is also conducted.

Keywords Game theory · Service differentiation · Quality competition · Service-oriented internet · Variational inequalities · Projected dynamical systems

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1 Introduction

The Internet has revolutionized the way in which we communicate, obtain news and information, and conduct a myriad of transactions from shopping to banking and financial payments. It has also transformed the entertainment landscape with movies that can be viewed online and music that can be listened to as well as downloaded and purchased. It has advanced and extended the reach of education, along with research, through online courses as well as new computing platforms and new collaborative networks.

Although the underlying technology associated with the existing Internet is rather well-understood, the economics of the associated services has been less studied. Much of the Internet success comes from its ability to support a wide range of service at the edge of the network. However, as argued in [34], the Internet offers little choice of data communication services inside the network. It is widely agreed that this limitation inhibits the deployment and use of new networking services, protocols, security designs, management frameworks, and other components that are essential to support the increasingly diverse systems, applications, and communication paradigms of the future Internet.

It is worth noting that the Internet architecture defines the fundamental principles of how data communication between different end-systems can be achieved [3] and [4]. While there are very clear specifications for different protocols used in the Internet (e.g., Internet Protocol (IP) [25], Transmission Control Protocol (TCP) [26]), there are no prescribed economic interactions between entities. The importance of the relationship between technology and economics became apparent in the early days of the commercial Internet (cf. [11]), but there has been a lack of common metrics for economic analysis in the Internet. Since the Internet is a network of networks, where different autonomous systems (AS) are managed by different administrative entities, economic relationships have evolved between these providers. For example, peering agreements between network providers specify how traffic is forwarded for mutual benefit ([2] and [36]).

Recently, the scope of the Internet has been expanded from merely providing connectivity between system to providing a variety of services. These services can be offered on the network edge [8] as well as in the core of the network [33]. The importance of quality of experience in the context of Internet services has been pointed out by van Moorsel [30]. Initial economic models for an Internet with services have been proposed by Zhang et al. [35]. Their work, however is limited to two providers with many simplifying assumptions. In our work, we present a much more detailed model for a service-oriented Internet that considers product differentiation with different quality levels and that does not require linear functions.

Recent research initiatives are calling for a fresh look at future Internet architectures and, hence, creating network economic models associated with the existing and prospective architectures are sorely needed. As discussed in [34], *choice* is a key aspect since it can drive innovation. Choice suggests that entities can select from a range of alternative services that may differ in functionality, performance, and cost. These choices appear at the level of application services provided by *service providers* (e.g., web content, streaming media, cloud computing services, etc.) or at the level of network services by *network providers* (e.g., high-bandwidth connections, in-network caching, etc.). For such an environment, it is necessary to put in place suitable economic processes to ensure that users can choose and reward good services and that the resulting competition drives innovation.

There is a considerable amount of literature studying the competition among network providers in the quality, quantity, and price of services. Faulhaber and Hogendorn [7], Shetty, Schwartz and Walrand [29], and Radonjic et al. [27] analyzed the capacity and pricing decisions made by network operators competing á la Cournot. Gibbons, Mason, and Steinberg [10], Niyato and Hossain [20], Parzy and Bogucka [24], and Zhang et al. [37] modeled a Cournot competition between wireless networks to find an efficient use of all available spectrum bandwidth.

However, only a small number of articles studied the quality of the services offered by service providers and their pricing mechanisms. Njoroge et al. [21] and Njoroge et al. [22] addressed the interconnection between network providers, the endogenous quality choice by network and service providers, and the market coverage in tandem. They studied the competition between service providers in the quality levels of their services and the competition between network providers in their quality levels and prices in a network with two network providers and multiple service providers.

Different from the literature, our proposed model considered the oligopolistic Cournot competition among numerous service providers who offer differentiated services with different quality levels and transport them to demand markets via multiple network providers. Another distinguishing feature of our work is modeling the competition among the service providers in a dynamic way by capturing the continuous time adjustment process of the equilibrium solution besides considering the static model. On top of that, in the numerical example, we evaluated the influence of the network transmission price on the equilibrium solution of the competition.

In this paper, we develop a dynamic network economic model which can be utilized as the foundation for exploring many relevant issues, including the pricing of services, the addition or the removal of service providers, as well as the network providers, and the users, who are represented by the consumers at the demand markets. One can also consider alternative cost functions, and demand price functions associated with the services and demand markets.

Our model is inspired, in part, by the recent contribution of [35], who emphasized the need for new network economic models of the Internet and derived a game theoretic formulation in the case of two service providers who were Cournot competitors, two network providers who were Bertrand competitors, and two users (see also [31]). In our framework, we do not restrict the number of service providers, or network providers, or users. However, we focus on the Cournot competition among the service providers but allow for alternative choices associated with network provision. In addition, we construct not only an equilibrium model, but also describe the underlying dynamics, along with some stability results, and a computational procedure.

This model allows for service differentiation, and distinct quality levels associated with the services offered by the service providers. Hence, the dynamic model allows for the tracking of the evolution not only of the volume of services provided but also the evolution of the quality levels. The methodologies that we utilize for the network economic model of the service-oriented Internet are variational inequality theory (cf. [12]) for the equilibrium version and projected dynamical systems theory (cf. [17]) for the dynamics.

The paper is organized as follows. In Section 2, we develop both the Cournot-Nash network economic equilibrium model with service differentiation and quality competition and its dynamic counterpart. In Section 3, we provide some qualitative results including some stability results. In Section 4, we present the algorithm, which yields closed form expressions for the service volumes and the quality levels, at each iteration. We then apply the algorithm to several numerical examples and conduct a sensitivity analysis in Section 5 to gain insights into the network economics and the evolutionary process. We summarize and present our conclusions in Section 6.

2 The network economic game theory model

In this Section, we develop a dynamic network economic model of a service-oriented Internet with quality competition. We assume that there are m service providers, with a typical service provider denoted by i, o network providers, which provide access to the services, with a typical one denoted by k, and n demand markets associated with the customers/clients for these services. A typical demand market is denoted by j. The service providers offer multiple different services such as movies for video streaming, music for downloading, news, etc. Users can choose among different service offerings (e.g., movie streaming from service provider i_1 vs. movie streaming from service provider i_2). Different network providers can be used for data communication over the Internet (i.e., "transport") between the service providers and users. In practice, multiple network providers are involved in a single connection over the Internet. However, for simplicity, our model only considers a single level of network provider. Also, while we consider different quality levels among service providers, we do not explicitly handle the quality level among network providers (but allow for differing costs).

The demand for a service is reflected in the demand price function at a demand market. We allow for consumers to differentiate among the services provided by the service providers in terms of the service quality. It is assumed that the service providers compete under the Cournot-Nash equilibrium concept of non-cooperative behavior and select both their service volumes (quantities) as well as the quality levels of their services. The consumers, in turn, signal their preferences for the services through the demand price functions associated with the demand markets. The demand price functions are, in general, functions of the demands for the services at all the demand markets as well as their quality levels and the fixed network transmission price.

We first develop the equilibrium model and derive the variational inequality formulation. We then describe the underlying dynamics associated with the service providers service outputs as well as quality levels and present the projected dynamical systems model whose set of stationary points corresponds to the set of solutions of the variational inequality problem.

Please refer to Fig. 1 for the underlying structure of the network economic model with service differentiation.

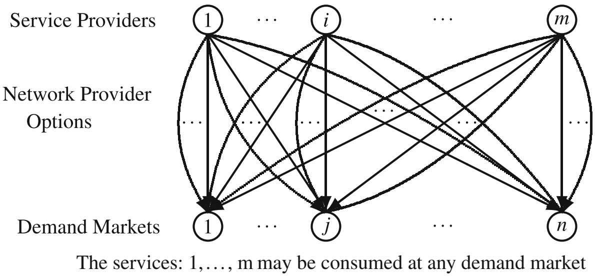

Fig. 1 The structure of the network economic problem

There is a distinct (but substitutable) service produced by each of the m service providers and consumed at the n demand markets. Let s_i denote the nonnegative service volume (output) produced by service provider i and let d_{ij} denote the demand for the service of provider i at demand market j. Let Q_{ijk} denote the nonnegative service volume of service provider i to demand market j transported by network provider k. We group the service quantities into the vector $s \in R^m_+$, the demands into the vector $d \in R^{mn}_+$, and the volumes of service transported from the service providers for the demand markets into the vector $Q \in R^{mno}_+$. Here q_i denotes the quality level, or, simply, the quality of service i, which is produced by service provider i. We group the quality levels of all service providers into the vector $q \in R^m_+$. All vectors here are assumed to be column vectors, except where noted.

All vectors here are assumed to be column vectors. The following conservation of flow equations must hold:

$$s_i = \sum_{j=1}^{n} \sum_{k=1}^{o} Q_{ijk}, \quad i = 1, \ldots, m; \tag{1}$$

$$d_{ij} = \sum_{k=1}^{o} Q_{ijk}, \quad i = 1, \ldots, m; \; j = 1, \ldots, n, \tag{2}$$

$$Q_{ijk} \geq 0, \quad i = 1, \ldots, m; \; j = 1, \ldots, n; \; k = 1, \ldots, o, \tag{3}$$

and, since the quality levels must also be nonnegative, we must also have that

$$q_i \geq 0, \quad i = 1, \ldots, m. \tag{4}$$

Hence, according to Eq. (1), the quantity of the service produced by each service provider is equal to the sum of the amounts of service transported to all the demand markets, and the quantity of a service provider's service consumed at a demand market, according to Eq. (2), is equal to the amount transported from the service provider to that demand market via all the network providers. Both the service volumes and the quality levels must be nonnegative.

We associate with each service provider i a *production* cost $\hat{f}_i$, and allow for the general situation where the production cost of i may depend upon the entire service pattern and on its own quality level, that is,

$$\hat{f}_i = \hat{f}_i(s, q_i), \quad i = 1, \ldots, m. \tag{5}$$

As noted earlier, consumers located at the demand markets respond not only to the volumes of services available but also to their quality levels. Hence, we allow the demand price at a demand market j associated with the service provided by service provider i, and denoted by ρ_{ij}, to depend, in general, upon the entire consumption pattern, as well as on all the levels of quality of all the services:

$$\rho_{ij} = \rho_{ij}(d, q, p), \quad i = 1, \dots, m; \, j = 1, \dots, n. \tag{6}$$

The generality of the expression in Eq. (6) allows for modeling and application flexibility. The demand price functions are, typically, assumed to be monotonically decreasing in service quantity but increasing in terms of service quality. In addition, the demand price functions in Eq. (6) also depend on the network transmission price p and we expect a negative relationship, that is, the higher the value of this transmission charge, the lower the price the consumers are willing to pay for a specific service at a specific demand market.

Let $\hat{c}_{ij}$ denote the total provision/transportation cost associated with providing transporting access to provider i's service for demand market j, which is given by the function:

$$\hat{c}_{ij} = \sum_{k=1}^{o} \hat{c}_{ijk}(Q_{ijk}), \quad i = 1, \dots, m; \, j = 1, \dots, n, \tag{7}$$

where $\hat{c}_{ijk}$ is the provision cost of providing access to provider i's service for demand market j through network provider k. In our model, it is the service providers that pay for the provision/transportation of the services. Functions (5), (6) and (7) are assumed to be continuous and continuously differentiable.

For service provider i, we group all its Q_{ijk} into vector Q_i. The strategic variables of service provider i are its service transport volumes $\{Q_i\}$ and its quality level q_i.

The profit or utility U_i of service provider i; $i = 1, \dots, m$, is, hence, given by the expression

$$U_i = \sum_{j=1}^{n} \rho_{ij} d_{ij} - \hat{f}_i - \sum_{j=1}^{n} \hat{c}_{ij}, \tag{8}$$

which is the difference between its total revenue and its total cost.

In view of Eqs. (1–8), one may write the profit as a function solely of the service provision/transportation pattern and quality levels, that is,

$$U = U(Q, q), \tag{9}$$

where U is the m-dimensional vector with components: $\{U_1, \dots, U_m\}$.

Let K^i denote the feasible set corresponding to service provider i, where $K^i \equiv \{(Q_i, q_i) | Q_i \geq 0, \text{ and } q_i \geq 0\}$ and define $K \equiv \prod_{i=1}^{m} K^i$.

We consider the oligopolistic market mechanism, in which the m service providers supply their services in a non-cooperative fashion, each one trying to maximize its own profit. We seek to determine a nonnegative service volume and quality level pattern (Q^*, q^*) for which the m service providers will be in a state of equilibrium as defined below. In particular, Nash [11, 12] generalized Cournot's concept of an equilibrium among several players, in what has been come to be called a non-cooperative game.

Definition 1 A Network Economic Cournot-Nash Equilibrium with Service Differentiation, Network provision Choices, and Quality Levels
A service transport volume and quality level pattern $(Q^*, q^*) \in K$ is said to constitute a Cournot-Nash equilibrium if for each service provider i,

$$U_i(Q_i^*, q_i^*, \hat{Q}_i^*, \hat{q}_i^*) \geq U_i(Q_i, q_i, \hat{Q}_i^*, \hat{q}_i^*), \quad \forall (Q_i, q_i) \in K^i, \tag{10}$$

where

$$\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_m^*); \text{ and } \hat{q}_i^* \equiv (q_1^*, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_m^*). \tag{11}$$

According to Eq. (10), an equilibrium is established if no service provider can unilaterally improve upon its profits by selecting an alternative vector of service volumes and quality level of its service. Alternative variational inequality formulations of the above equilibrium are:

Variational Inequality Formulations We now present alternative variational inequality formulations of the above Cournot-Nash equilibrium with service differentiation in the following theorem.

Theorem 1 *Assume that for each service provider i the profit function $U_i(Q, q)$ is concave with respect to the variables $\{Q_{i1}, \ldots, Q_{in}\}$, and q_i, and is continuous and continuously differentiable. Then $(Q^*, q^*) \in K$ is a Cournot-Nash equilibrium according to Definition 1 if and only if it satisfies the variational inequality*

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o} \frac{\partial U_i(Q^*, q^*)}{\partial Q_{ijk}} \times (Q_{ijk} - Q_{ijk}^*) - \sum_{i=1}^{m} \frac{\partial U_i(Q^*, q^*)}{\partial q_i} \times (q_i - q_i^*) \geq 0,$$
$$\forall (Q, q) \in K, \tag{12}$$

or, equivalently, $(s^*, d^*, Q^*, q^*) \in \mathcal{K}^1$ is an equilibrium production, transport, consumption, and quality level pattern if and only if it satisfies the variational inequality

$$\sum_{i=1}^{m} \frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial s_i} \times (s_i - s_i^*) - \sum_{i=1}^{m}\sum_{j=1}^{n} \rho_{ij}(d^*, q^*, p) \times (d_{ij} - d_{ij}^*)$$
$$+ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o} \left[\frac{\partial \hat{c}_{ijk}(Q_{ijk}^*)}{\partial Q_{ijk}} - \sum_{l=1}^{n} \frac{\partial \rho_{il}(d^*, q^*, p)}{\partial d_{ij}} \times d_{il}^* \right] \times (Q_{ijk} - Q_{ijk}^*)$$
$$+ \sum_{i=1}^{m} \left[\frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial q_i} - \sum_{l=1}^{n} \frac{\partial \rho_{il}(d^*, q^*, p)}{\partial q_i} \times d_{il}^* \right]$$
$$\times (q_i - q_i^*) \geq 0, \; \forall (s, d, Q, q) \in \mathcal{K}^1, \tag{13}$$

where $\mathcal{K}^1 \equiv \{(s, d, Q, q) | \; Q \geq 0, \; q \geq 0$, and Eqs. (1) *and* (2) *hold*$\}$.

Proof Equation 12 follows directly from [2] and [4].

In order to obtain Eq. (13) from Eq. (12), we note that, in light of Eqs. (1) and (2): for each i, j, and k,

$$-\frac{\partial U_i}{\partial Q_{ijk}} = \left[\frac{\partial \hat{f}_i}{\partial Q_{ijk}} + \frac{\partial \hat{c}_{ijk}}{\partial Q_{ijk}} - \rho_{ij}\frac{\partial d_{ij}}{\partial Q_{ijk}} - \sum_{l=1}^{n}\frac{\partial \rho_{il}}{\partial Q_{ijk}} \times d_{il}\right]$$
$$= \left[\frac{\partial \hat{f}_i}{\partial s_i}\frac{\partial s_i}{\partial Q_{ijk}} + \frac{\partial \hat{c}_{ijk}}{\partial Q_{ijk}} - \rho_{ij} - \sum_{l=1}^{n}\frac{\partial \rho_{il}}{\partial d_{ij}}\frac{\partial d_{ij}}{\partial Q_{ijk}} \times d_{il}\right]$$
$$= \left[\frac{\partial \hat{f}_i}{\partial s_i} + \frac{\partial \hat{c}_{ijk}}{\partial Q_{ijk}} - \rho_{ij} - \sum_{l=1}^{n}\frac{\partial \rho_{il}}{\partial d_{ij}} \times d_{il}\right], \quad (14)$$

and for each i,

$$-\frac{\partial U_i}{\partial q_i} = \left[\frac{\partial \hat{f}_i}{\partial q_i} - \sum_{l=1}^{n}\frac{\partial \rho_{il}}{\partial q_i} \times d_{il}\right]. \quad (15)$$

Multiplying the right-most expression in Eq. (14) by $(Q_{ijk} - Q^*_{ijk})$ and summing the resultant over all i, j, and k; similarly, multiplying the right-most expression in Eq. (15) by $(q_i - q_i^*)$ and summing the resultant over all i yields, respectively:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial \hat{f}_i}{\partial s_i} + \frac{\partial \hat{c}_{ijk}}{\partial Q_{ijk}} - \rho_{ij} - \sum_{l=1}^{n}\frac{\partial \rho_{il}}{\partial d_{ij}} \times d_{il}\right] \times (Q_{ijk} - Q^*_{ijk}) \quad (16)$$

and

$$\sum_{i=1}^{m}\left[\frac{\partial \hat{f}_i}{\partial q_i} - \sum_{l=1}^{n}\frac{\partial \rho_{il}}{\partial q_i} \times d_{il}\right] \times (q_i - q_i^*). \quad (17)$$

Finally, summing (16) and (17) and then using constraints (1) and (2), yields variational inequality (13). □

We now put the above oligopolistic market equilibrium problem with service differentiation and quality levels into standard variational inequality form, that is,

Determine $X^* \in \mathcal{K} \subset R^N$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \quad (18)$$

where F is a given continuous function from $\mathcal{K}$ to R^N, and $\mathcal{K}$ is a closed and convex set.

We define the $(mno + m)$-dimensional vector $X \equiv (Q, q)$ and the $(mn + m)$-dimensional row vector $F(X) = (F^1(X), F^2(X))$ with the (i, j, k)-th component, F^1_{ijk}, of $F^1(X)$ given by

$$F^1_{ijk}(X) \equiv -\frac{\partial U_i(Q, q)}{\partial Q_{ijk}}, \quad (19)$$

the i-th component, F^2_i, of $F^2(X)$ given by

$$F^2_i(X) \equiv -\frac{\partial U_i(Q, q)}{\partial q_i}, \quad (20)$$

and with the feasible set $\mathcal{K} \equiv K$. Then, clearly, variational inequality (13) can be put into standard form (18).

In a similar manner, one can establish that variational inequality (14) can also be put into standard variational inequality form (18).

For additional background on the variational inequality problem, we refer the reader to the book [5].

The Projected Dynamical System Model We now propose a dynamic adjustment process for the evolution of the service providers' service volumes and service quality levels. Observe that, for a current service volume and quality level pattern at time t, $X(t) = (Q(t), q(t))$, $-F^1_{ijk}(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial Q_{ijk}}$ given by Eq. (19), is the marginal utility (profit) of service provider i with respect to its transport of services to demand market j via k. Similarly, $-F^2_i(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial q_i}$, given by Eq. (20), is the provider's marginal utility (profit) with respect to its quality level. In this framework, the rate of change of the service flow between a provider and demand market pair using k, (i, j, k), is in proportion to $-F^1_{ijk}(X)$, as long as the service volume Q_{ijk} is positive. Namely, when $Q_{ijk} > 0$,

$$\dot{Q}_{ijk} = \frac{\partial U_i(Q, q)}{\partial Q_{ijk}}, \tag{21}$$

where $\dot{Q}_{ijk}$ denotes the rate of change of Q_{ijk}. However, when $Q_{ijk} = 0$, the nonnegativity condition (3) forces the service volume Q_{ijk} to remain zero when $\frac{\partial U_i(Q,q)}{\partial Q_{ijk}} \leq 0$. Hence, in this case, we are only guaranteed of having possible increases of the transport volume. Namely, when $Q_{ijk} = 0$,

$$\dot{Q}_{ijk} = \max\{0, \frac{\partial U_i(Q, q)}{\partial Q_{ijk}}\}. \tag{22}$$

We may write Eqs. (21) and (22) concisely as:

$$\dot{Q}_{ijk} = \begin{cases} \frac{\partial U_i(Q,q)}{\partial Q_{ijk}}, & \text{if} \quad Q_{ijk} > 0 \\ \max\left\{0, \frac{\partial U_i(Q,q)}{\partial Q_{ijk}}\right\}, & \text{if} \quad Q_{ijk} = 0. \end{cases} \tag{23}$$

As for the quality levels, when $q_i > 0$, then

$$\dot{q}_i = \frac{\partial U_i(Q, q)}{\partial q_i}, \tag{24}$$

where $\dot{q}_i$ denotes the rate of change of q_i; otherwise:

$$\dot{q}_i = \max\left\{0, \frac{\partial U_i(Q, q)}{\partial q_i}\right\}, \tag{25}$$

since q_i must be nonnegative.

Combining Eqs. (24) and (25), we may write:

$$\dot{q}_i = \begin{cases} \frac{\partial U_i(Q,q)}{\partial q_i}, & \text{if} \quad q_i > 0 \\ \max\left\{0, \frac{\partial U_i(Q,q)}{\partial q_i}\right\}, & \text{if} \quad q_i = 0. \end{cases} \tag{26}$$

Applying Eq. (23) to all service provider and demand market pairs (i, j); $i = 1, \ldots, m$; $j = 1, \ldots, n$, and all network providers $k = 1, \ldots, o$, and applying

Eq. (26) to all service providers i; $i = 1, \dots, m$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes of the service transport volumes and quality levels, in vector form, as:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \tag{27}$$

where, since $\mathcal{K}$ is a convex polyhedron, according to [3], $\Pi_{\mathcal{K}}(X, -F(X))$ is the projection, with respect to $\mathcal{K}$, of the vector $-F(X)$ at X defined as

$$\Pi_{\mathcal{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathcal{K}}(X - \delta F(X)) - X}{\delta} \tag{28}$$

with $P_{\mathcal{K}}$ denoting the projection map:

$$P(X) = \text{argmin}_{z \in \mathcal{K}} \|X - z\|, \tag{29}$$

and where $\| \cdot \| = \langle x, x \rangle$. Hence, $F(X) = -\nabla U(Q, q)$, where $\nabla U(Q, q)$ is the vector of marginal utilities with components given by Eqs. (19) and (20).

We now interpret the ODE (27) in the context of the network economic model with service differentiation and quality competition. First, note that ODE (27) ensures that the service volumes and quality levels are always nonnegative. Indeed, if one were to consider, instead, the ordinary differential equation: $\dot{X} = -F(X)$, or, equivalently, $\dot{X} = \nabla U(X)$, such an ODE would not ensure that $X(t) \geq 0$, for all $t \geq 0$, unless additional restrictive assumptions were to be imposed. ODE (27), however, retains the interpretation that if X at time t lies in the interior of $\mathcal{K}$, then the rate at which X changes is greatest when the vector field $-F(X)$ is greatest. Moreover, when the vector field $-F(X)$ pushes X to the boundary of the feasible set $\mathcal{K}$, then the projection $\Pi_{\mathcal{K}}$ ensures that X stays within $\mathcal{K}$. Hence, the service volumes and quality levels are always nonnegative.

Recall now the definition of $F(X)$ for the network economic model, in which case the dynamical system (27) states that the rate of change of the service transport volumes and quality levels is greatest when the firms' marginal utilities (profits) are greatest. If the marginal utilities with respect to the transport volumes are positive, then the service providers will increase their volumes; if they are negative, then they will decrease them. The same adjustment behavior holds for the service quality levels. This type of behavior is rational from an economic standpoint. Therefore, ODE (27) is a reasonable continuous adjustment process for the network economic problem with service differentiation.

Although the use of the projection on the right-hand side of ODE (27) guarantees that the service flows and the quality levels are always nonnegative, it also raises the question of existence of a solution to ODE (27), since this ODE is nonstandard due to its discontinuous right-hand side. Dupuis and Nagurney [3] developed the fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by Eq. (28). We cite the following theorem from that paper.

Theorem 2 *X^* solves the variational inequality problem* (18) *if and only if it is a stationary point of the ODE* (27)*, that is,*

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)). \tag{30}$$

This theorem demonstrates that the necessary and sufficient condition for a service flow and quality level pattern $X^* = (Q^*, q^*)$ to be a Cournot-Nash equilibrium, according to Definition 1, is that $X^* = (Q^*, q^*)$ is a stationary point of the adjustment process defined by ODE (27), that is, X^* is the point at which $\dot{X} = 0$.

Consider now the competitive economic system consisting of the service providers, who, in order to maximize their utilities, adjust their service flow and quality level patterns by instantly responding to the marginal utilities, according to Eq. (27). The following questions naturally arise and are of interest. Does the utility gradient process defined by Eq. (27), approach a Cournot-Nash equilibrium, and how does it approach an equilibrium? Also, for a given Cournot-Nash equilibrium, do all the disequilibrium service flow and quality level patterns that are close to this equilibrium always stay near by? Motivated by these questions, we now present the stability analysis of Cournot-Nash equilibrium, under the above utility gradient process.

As noted earlier, the stability of Cournot-Nash equilibrium has been well-studied in the history of oligopoly theory. Among others, [1] investigated the asymptotical stability of Cournot-Nash equilibrium (see also [15]). In that paper, in place of the projection operator, $\Pi_{\mathcal{K}}$, a discontinuous matrix function, γ, was used to multiply the utility gradient on the right-hand side of the ODE (but in a much simpler model than developed here), to ensure that the adjustment process would evolve within the nonnegative orthant. Okuguchi and Szidarovszky [13] also studied the asymptotical stability of the utility gradient process at the Cournot-Nash equilibrium, under the assumptions of linear price functions and quadratic cost functions, and with no quality levels as strategic variables or with the spatial dimension.

3 Stability under monotonicity

We now turn to the questions raised in the previous section, that is, whether and under what conditions does the adjustment process defined by ODE (27) approaches a Cournot-Nash equilibrium? We first note that Lipschitz continuity of $F(X)$ (cf. [3] and [10]) guarantees the existence of a unique solution to Eq. (31) below, where we have that $X^0(t)$ satisfies ODE (27) with initial service transport volume and quality level pattern (Q^0, q^0). In other words, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X^0, \tag{31}$$

with $X^0(0) = X^0$. For convenience, we will sometimes write $X^0 \cdot t$ for $X^0(t)$.

For the definitions of stability and monotonicity, the stability properties of the gradient process under various monotonicity conditions, and the associated proofs, please refer to [10].

We now turn to establishing existence and uniqueness results of the equilibrium pattern by utilizing the theory of variational inequalities.

In the context of the network economic problem, where $F(X)$ is the vector of negative marginal utilities as in Eqs. (18–20), we point out that if the utility functions are twice differentiable and the Jacobian of the negative marginal utility functions (or, equivalently, the negative of the Hessian matrix of the utility functions) is positive definite, then the corresponding $F(X)$ is strictly monotone.

In a practical oligopoly model, it is reasonable to expect that the utility of any service provider i, $U_i(Q, q)$, would decrease whenever its output has become sufficiently large, that is, when U_i is differentiable, $\frac{\partial U_i(Q,q)}{\partial Q_{ijk}}$ is negative for sufficiently large Q_{ijk}, because $q_i \geq Q_{ijk}$, for all j; the same holds for sufficiently large q_i. Hence, the following assumption is not unreasonable:

Assumption 1 Suppose that in our network economic model there exists a sufficiently large M, such that for any (i, j, k),

$$\frac{\partial U_i(Q, q)}{\partial Q_{ijk}} < 0, \tag{32}$$

for all service transport volume patterns Q with $Q_{ijk} \geq M$ and that there exists a sufficiently large $\bar{M}$, such that for any i,

$$\frac{\partial U_i(Q, q)}{\partial q_i} < 0, \tag{33}$$

for all quality level patterns q with $q_i \geq \bar{M}$.

We now give an existence result.

Proposition 1 *Any network economic problem, as described above, that satisfies Assumption 1 possesses at least one equilibrium service transport volume and quality level pattern.*

Proof The proof follows from Proposition 1 in [18]. □

We now present the uniqueness result, the proof of which follows from the basic theory of variational inequalities (cf. [5]).

Proposition 2 *Suppose that F is strictly monotone at any equilibrium point of the variational inequality problem defined in Eq.* (18)*. Then it has at most one equilibrium point.*

The following theorem is a natural extension/adaptation of Theorem 6.10 in [12] *(see also* [17]*) to the more general network economic oligopoly problem formulated here with service differentiation and quality competition.*

Theorem 3 *(i). If $-\nabla U(Q, q)$ is monotone, then every network economic Cournot-Nash equilibrium, provided its existence, is a global monotone attractor for the utility gradient process.*

(ii). If $-\nabla U(Q, q)$ is strictly monotone, then there exists at most one network economic Cournot-Nash equilibrium. Furthermore, provided existence, the unique spatial Cournot-Nash equilibrium is a strictly global monotone attractor for the utility gradient process.

(iii). If $-\nabla U(Q, q)$ is strongly monotone, then there exists a unique network economic Cournot-Nash equilibrium, which is globally exponentially stable for the utility gradient process.

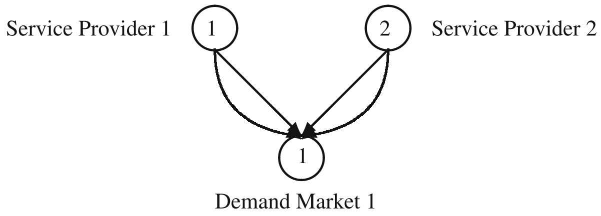

Fig. 2 Example 1

We now present two examples in order to illustrate some of the above concepts and results.

Example 1 Consider a network oligopoly problem consisting of two service providers, two network providers, and one demand market, as depicted in Fig. 2.

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 + s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 + s_2 + q_2^2 + 37,$$

the total transportation cost functions are:

$$\hat{c}_{111} = 0.5Q_{111}^2 + 0.4Q_{111}, \quad \hat{c}_{112} = 0.7Q_{112}^2 + 0.5Q_{112},$$

$$\hat{c}_{211} = 0.6Q_{211}^2 + 0.4Q_{211}, \quad \hat{c}_{212} = 0.4Q_{212}^2 + 0.2Q_{212},$$

and the demand price functions are:

$$\rho_{11}(d, q) = 100 - d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2 - p,$$
$$\rho_{21}(d, q) = 100 - 0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2 - p,$$

where $p = 30$. The utility function of service provider 1 is, hence:

$$U_1(Q, q) = \rho_{11}d_{11} - \hat{f}_1 - \hat{c}_{111} - \hat{c}_{112},$$

whereas the utility function of service provider 2 is:

$$U_2(Q, q) = \rho_{21}d_{21} - \hat{f}_2 - \hat{c}_{211} - \hat{c}_{212}.$$

The Jacobian matrix of -$\nabla U(Q, q)$, denoted by $J(Q_{11}, Q_{21}, q_1, q_2)$, is

$$J(Q_{111}, Q_{112}, Q_{211}, Q_{212}, q_1, q_2) = \begin{pmatrix} 5 & 4 & 0.4 & 0.4 & -0.3 & -0.05 \\ 4 & 5.4 & 0.4 & 0.4 & -0.3 & -0.05 \\ 0.6 & 0.6 & 8.2 & 7 & -0.1 & -0.5 \\ 0.6 & 0.6 & 7 & 7.8 & -0.1 & -0.5 \\ -0.3 & -0.3 & 0 & 0 & 4 & 0 \\ 0 & 0 & -0.5 & -0.5 & 0 & 2 \end{pmatrix}.$$

This Jacobian matrix is positive-definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, -$\nabla U(Q, q)$ is strongly monotone (see also [5]). Thus, both the existence and uniqueness of the solution to variational inequality (13) with respect to this example are guaranteed. The equilibrium solution is: $Q_{111}^* = 8.40$, $Q_{112}^* = 5.93$, $Q_{211}^* = 3.18$, $Q_{212}^* = 5.01$, $q_1^* = 1.08$, and $q_2^* = 2.05$, and it is globally exponentially stable.

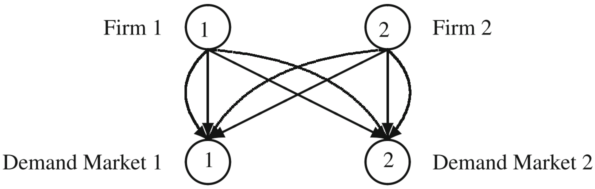

Fig. 3 Example 2

Example 2 We now present another example with the network depicted in Fig. 3.

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 + s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 + s_2 + q_2^2 + 37,$$

the total transportation cost functions are:

$$\begin{aligned}
\hat{c}_{111} &= 0.5Q_{111}^2 + 0.4Q_{111}, \quad \hat{c}_{112} = 0.7Q_{112}^2 + 0.5Q_{112} \\
\hat{c}_{211} &= 0.6Q_{211}^2 + 0.4Q_{211}, \quad \hat{c}_{212} = 0.4Q_{212}^2 + 0.2Q_{212}, \\
\hat{c}_{121} &= 0.3Q_{121}^2 + 0.1Q_{121}, \quad \hat{c}_{122} = 0.5Q_{122}^2 + 0.3Q_{122}, \\
\hat{c}_{221} &= 0.4Q_{221}^2 + 0.3Q_{221}, \quad \hat{c}_{222} = 0.4Q_{222}^2 + 0.2Q_{222},
\end{aligned}$$

and the demand price functions are:

$$\begin{aligned}
\rho_{11}(d, q) &= 100 - d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2 - p, \\
\rho_{12}(d, q) &= 100 - 2d_{12} - d_{22} + 0.4q_1 + 0.2q_2 - p, \\
\rho_{21}(d, q) &= 100 - 0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2 - p, \\
\rho_{22}(d, q) &= 100 - 0.7d_{12} - 1.7d_{22} + 0.01q_1 + 0.6q_2 - p,
\end{aligned}$$

where $p = 30$. The utility function of firm 1 is:

$$U_1(Q, q) = \rho_{11}d_{11} + \rho_{12}d_{12} - \hat{f}_1 - (\hat{c}_{111} + \hat{c}_{121} + \hat{c}_{112} + \hat{c}_{122})$$

with the utility function of firm 2 being:

$$U_2(Q, q) = \rho_{21}d_{21} + \rho_{22}d_{22} - \hat{f}_2 - (\hat{c}_{211} + \hat{c}_{221} + \hat{c}_{212} + \hat{c}_{222}).$$

The Jacobian of $-\nabla U(Q, q)$, denoted by $J\big(Q_{111}, Q_{112}, Q_{121}, Q_{122}, Q_{211}, Q_{212}, Q_{221}, Q_{222}, q_1, q_2\big)$, is

$$J\big(Q_{111}, Q_{112}, Q_{121}, Q_{122}, Q_{211}, Q_{212}, Q_{221}, Q_{222}, q_1, q_2\big)$$

$$= \begin{pmatrix} 5 & 4 & 2 & 2 & 0.4 & 0.4 & 0 & 0 & -0.3 & -0.05 \\ 4 & 5.4 & 2 & 2 & 0.4 & 0.4 & 0 & 0 & -0.3 & -0.05 \\ 2 & 2 & 6.6 & 6 & 0 & 0 & 1 & 1 & -0.4 & -0.2 \\ 2 & 2 & 6 & 7 & 0 & 0 & 1 & 1 & -0.4 & -0.2 \\ 0.6 & 0.6 & 0 & 0 & 8.2 & 7 & 4 & 4 & -0.1 & -0.5 \\ 0.6 & 0.6 & 0 & 0 & 7 & 7.8 & 4 & 4 & -0.1 & -0.5 \\ 0 & 0 & 0.7 & 0.7 & 4 & 4 & 8.2 & 7.4 & -0.01 & -0.6 \\ 0 & 0 & 0.7 & 0.7 & 4 & 4 & 7.4 & 8.2 & -0.01 & -0.6 \\ -0.3 & -0.3 & -0.4 & -0.4 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & -0.5 & -0.6 & -0.6 & 0 & 2 \end{pmatrix}.$$

Clearly, this Jacobian matrix is also positive-definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, -$\nabla U(Q, q)$ is strongly monotone (cf. [6]). Thus, both the existence and uniqueness of the solution to variational inequality (13) with respect to this example are also guaranteed. Moreover, the equilibrium solution (stationary point) is: $Q^*_{111} = 6.97$, $Q^*_{112} = 4.91$, $Q^*_{121} = 2.40$, $Q^*_{122} = 3.85$, $Q^*_{211} = 3.58$, $Q^*_{212} = 1.95$, $Q^*_{221} = 2.77$, $Q^*_{222} = 2.89$,$q^*_1 = 1.52$, $q^*_2 = 3.08$, and it is globally exponentially stable.

The stationary points of both Examples 1 and 2 were computed using the Euler method, which is induced by the general iterative scheme of [3]. In the next Section, we present the induced closed form expressions at each iteration, along with convergence results.

4 The algorithm

As mentioned in the introduction, the projected dynamical system yields continuous-time adjustment processes. However, for computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories is needed.

We now recall the Euler method, which is induced by the general iterative scheme of [3]. Specifically, iteration τ of the Euler method (see also [10]) is given by:

$$X^{\tau+1} = P_{\mathcal{K}}(X^{\tau} - a_{\tau} F(X^{\tau})), \tag{34}$$

where $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ and F is the function that enters the variational inequality problem (19).

As shown in [3] and [10], for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$. Specific conditions for convergence of this scheme as well as various applications to the solutions of other oligopoly models can be found in [6–10].

Explicit Formulae for the Euler Method Applied to the Network Economic Model of the Internet The elegance of this procedure for the computation of solutions to our network economic model of the Internet with service differentiation and quality levels can be seen in the following explicit formulae. In particular, we have the following closed form expression for all the service volume $i = 1, \ldots, m$; $j = 1, \ldots, n$; $k = 1, \ldots, o$:

$$Q_{ijk}^{\tau+1} = \max\left\{0,\ Q_{ijk}^{\tau} + a_\tau \left(\rho_{ij}\left(d^\tau, q^\tau, p\right) + \sum_{l=1}^{n} \frac{\partial \rho_{il}\left(d^\tau, q^\tau, p\right)}{\partial d_{ij}} d_{il}^{\tau} - \frac{\partial \hat{f}_i\left(s^\tau, q_i^\tau\right)}{\partial s_i} - \frac{\partial \hat{c}_{ijk}(Q_{ijk}^{\tau})}{\partial Q_{ijk}}\right)\right\}, \tag{35}$$

and the following closed form expression for all the quality levels $i = 1, \ldots, m$:

$$q_i^{\tau+1} = \max\left\{0,\ q_i^{\tau} + a_\tau \left(\sum_{l=1}^{n} \frac{\partial \rho_{il}\left(d^\tau, q^\tau, p\right)}{\partial q_i} d_{il}^{\tau} - \frac{\partial \hat{f}_i\left(s^\tau, q_i^\tau\right)}{\partial q_i}\right)\right\} \tag{36}$$

with the demands being updated according to:

$$d_{ij}^{\tau+1} = \sum_{k=1}^{o} Q_{ijk}^{\tau+1}; \quad i = 1, \ldots, m;\ j = 1, \ldots, n, \tag{37}$$

and the supplies being updated according to:

$$s_i^{\tau+1} = \sum_{j=1}^{n} \sum_{k=1}^{o} Q_{ijk}^{\tau+1}, \quad i = 1, \ldots, m. \tag{38}$$

We now provide the convergence result. The proof is direct from Theorem 5.8 in [10].

Theorem 4 *In the network economics problem of the Internet with service differentiation and quality levels let $F(X) = -\nabla U(Q, q)$ be strictly monotone at any equilibrium pattern and assume that Assumption 1 is satisfied. Also, assume that F is uniformly Lipschitz continuous. Then there exists a unique equilibrium service volume and quality level pattern $(Q^*, q^*) \in K$ and any sequence generated by the Euler method as given by Eq. (34) above, where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$ converges to (Q^*, q^*).*

In the next Section, we apply the Euler method to compute solutions to numerical network oligopoly problems.

5 Numerical examples

We implemented the Euler method, as described in Section 3, using Matlab. The convergence criterion was $\epsilon = 10^{-6}$; that is, the Euler method was considered to have converged if, at a given iteration, the absolute value of the difference of each service volume and each quality level differed from its respective value at the preceding iteration by no more than ϵ.

The sequence $\{a_\tau\}$ was: $.1\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \dots\right)$. We initialized the algorithm by setting each service volume $Q_{ijk} = 2.5$, $\forall i, j, k$, and by setting the quality level of each firm $q_i = 0.00$, $\forall i$.

Example 1 Revisited In Section 3, we discussed stability analysis and presented results for two numerical examples. We now provide additional results for these examples.

The Euler method required 72 iterations for convergence to the equilibrium pattern for Example 1 described in Section 3. A graphical depiction of the iterates, consisting of the service volumes and the quality levels is given, respectively, in Fig. 4. The utility/profit of firm 1 was 567.35 and that of firm 2 was 216.94. □

Example 2 Revisited For Example 2 described in Section 3, in which there are two service providers, two network providers, and two demand markets, the Euler method required 84 iterations for convergence. A graphical depiction of the service volume and quality level iterates is given, respectively, in Fig. 5. The profit of firm 1 was 547.60, whereas that of firm 2 was 292.79.

The trajectories in Figs. 4 and 5 provide a discrete-time evolution of the service volumes and quality levels of the service providers as they respond to the feedback from the consumers as to the demands for the services and the quality levels from the preceding iteration (time period).

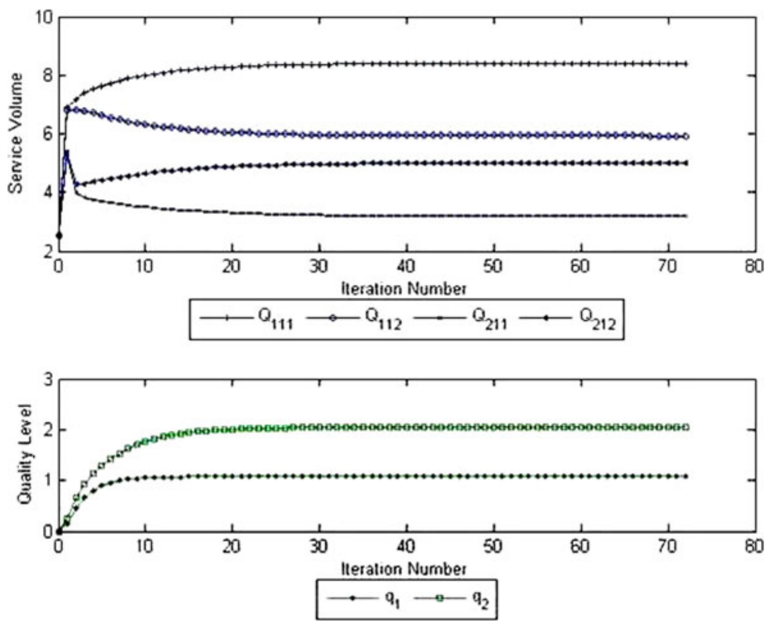

Fig. 4 Service volumes and quality levels for Example 1

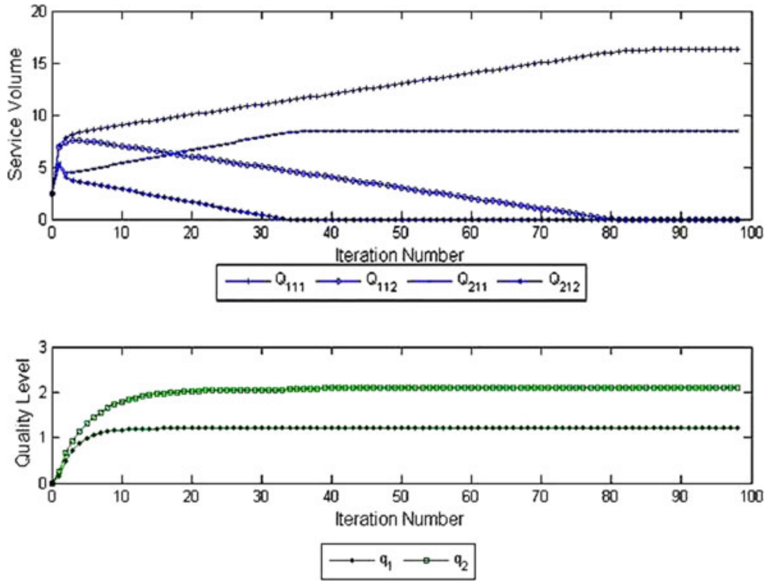

Fig. 5 Service volumes and quality levels for Example 2

We note that we verified the properties of the Jacobian matrix above in order to also evaluate the stability of the utility gradient process as well as to check whether conditions for convergence of the algorithm are satisfied. One should realize, however, that the algorithm does not require strong monotonicity of minus the gradient of the utility functions for convergence. Moreover, if the algorithm converges, it converges to a stationary point of the projected dynamical systems; equivalently, to a solution of the variational inequality problem governing the Nash-Cournot equilibrium conditions for our network oligopoly model.

□

Example 3 The third numerical network oligopoly example consisted of two firms and three demand markets, as depicted in Fig. 6.

This example was built from Example 2 with the production cost functions of the original two demand markets expanded and the original demand price functions as

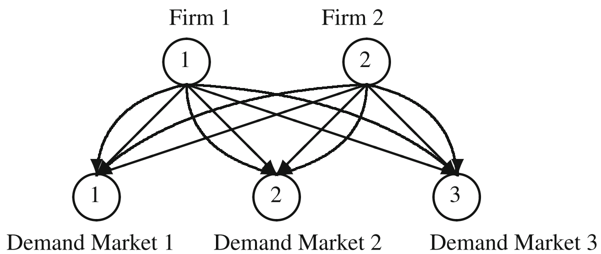

Fig. 6 Example 3

well. The new demand market, demand market 3, is farther than demand markets 1 and 2. We also added new data for the new firm. The complete data for this example are given below.

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 + s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 + s_2 + q_2^2 + 37.$$

The total transportation cost functions are:

$$\begin{aligned}
\hat{c}_{111} &= 0.5Q_{111}^2 + 0.4Q_{111}, & \hat{c}_{112} &= 0.7Q_{112}^2 + 0.5Q_{112} \\
\hat{c}_{211} &= 0.6Q_{211}^2 + 0.4Q_{211}, & \hat{c}_{212} &= 0.4Q_{212}^2 + 0.2Q_{212}, \\
\hat{c}_{121} &= 0.3Q_{121}^2 + 0.1Q_{121}, & \hat{c}_{122} &= 0.5Q_{122}^2 + 0.3Q_{122}, \\
\hat{c}_{221} &= 0.4Q_{221}^2 + 0.3Q_{221}, & \hat{c}_{222} &= 0.4Q_{222}^2 + 0.2Q_{222}, \\
\hat{c}_{131} &= Q_{131}^2 + 0.5Q_{131}, & \hat{c}_{132} &= Q_{132}^2 + 0.6Q_{132}, \\
\hat{c}_{231} &= 0.8Q_{231}^2 + 0.5Q_{231}, & \hat{c}_{232} &= Q_{232}^2 + 0.7Q_{232},
\end{aligned}$$

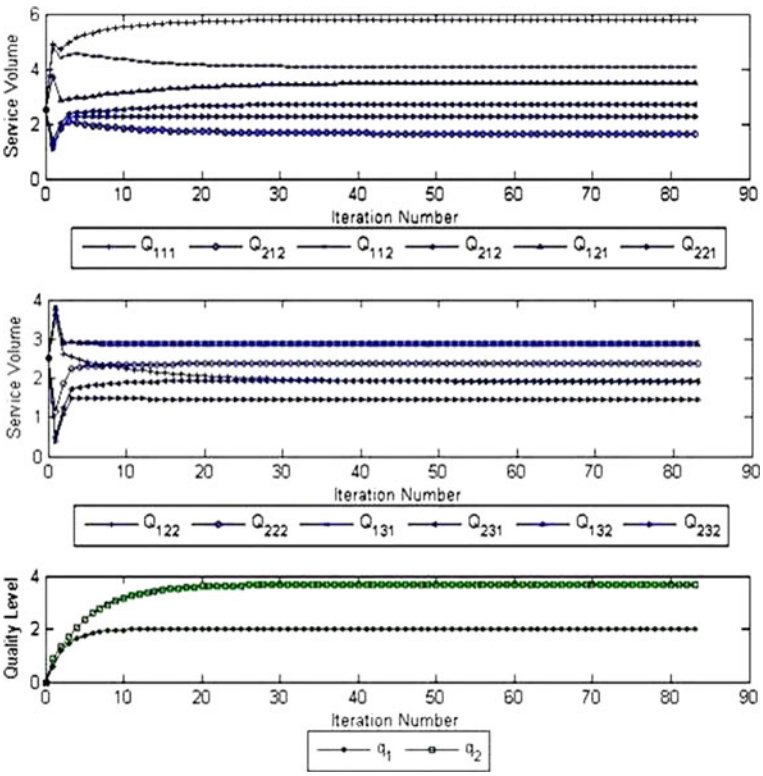

Fig. 7 Service volumes and quality levels for Example 3

and the demand price functions are:

$$
\begin{aligned}
\rho_{11}(d,q) &= 100 - d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2 - p,\\
\rho_{12}(d,q) &= 100 - 2d_{12} - d_{22} + 0.4q_1 + 0.2q_2 - p,\\
\rho_{13}(d,q) &= 100 - 1.7d_{13} - 0.7d_{23} + 0.5q_1 + 0.1q_2 - p,\\
\rho_{21}(d,q) &= 100 - 0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2 - p,\\
\rho_{22}(d,q) &= 100 - 0.7d_{12} - 1.7d_{22} + 0.01q_1 + 0.6q_2 - p,\\
\rho_{23}(d,q) &= 100 - 0.9d_{13} - 2d_{23} + 0.2q_1 + 0.7q_2 - p,
\end{aligned}
$$

The utility function expressions of firm 1, firm 2, and firm 3 are, respectively:

$$U_1(Q,q) = \rho_{11}d_{11} + \rho_{12}d_{12} + \rho_{13}d_{13} - \hat{f}_1 - (\hat{c}_{111} + \hat{c}_{121} + \hat{c}_{112} + \hat{c}_{122} + \hat{c}_{131} + \hat{c}_{132})$$

with the utility function of firm 2 being:

$$U_2(Q,q) = \rho_{21}d_{21} + \rho_{22}d_{22} + \rho_{23}d_{23} - \hat{f}_2 - (\hat{c}_{211} + \hat{c}_{221} + \hat{c}_{212} + \hat{c}_{222} + \hat{c}_{231} + \hat{c}_{232}).$$

The Jacobian of $-\nabla U(Q,q)$, denoted by $J\,(Q_{111},\ Q_{112},\ Q_{121},\ Q_{122},\ Q_{131},\ Q_{132},$ $Q_{211}, Q_{212}, Q_{221}, Q_{222}, Q_{231}, Q_{232}, q_1, q_2)$, is

$$J\,(Q_{111}, Q_{112}, Q_{121}, Q_{122}, Q_{131}, Q_{132}, Q_{211}, Q_{212}, Q_{221}, Q_{222}, Q_{231}, Q_{232}, q_1, q_2)$$

$= [J(Q_{111}, Q_{112}, Q_{121}, Q_{122}, Q_{131}, Q_{132}, Q_{211})|J_2(Q_{212}, Q_{221}, Q_{222}, Q_{231}, Q_{232},$ $q_1, q_2)]$, where

$$
J_1 = \begin{pmatrix}
5 & 4 & 2 & 2 & 2 & 2 & 0.4\\
4 & 5.4 & 2 & 2 & 2 & 2 & 0.4\\
2 & 2 & 6.6 & 6 & 2 & 2 & 0\\
2 & 2 & 6 & 7 & 2 & 2 & 0\\
2 & 2 & 2 & 2 & 7.4 & 5.4 & 0\\
2 & 2 & 2 & 2 & 5.4 & 7.4 & 0\\
0.6 & 0.6 & 0 & 0 & 0 & 0 & 8.2\\
0.6 & 0.6 & 0 & 0 & 0 & 0 & 7\\
0 & 0 & 0.7 & 0.7 & 0 & 0 & 4\\
0 & 0 & 0.7 & 0.7 & 0 & 0 & 4\\
0 & 0 & 0 & 0 & 0.9 & 0.9 & 4\\
0 & 0 & 0 & 0 & 0.9 & 0.9 & \\
-0.3 & -0.3 & -0.4 & -0.4 & -0.5 & -0.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -0.5
\end{pmatrix}.
$$

$$J_2 = \begin{pmatrix} 0.4 & 0 & 0 & 0 & 0 & -0.3 & -0.05 \\ 0.4 & 0 & 0 & 0 & 0 & -0.3 & -0.05 \\ 0 & 1 & 1 & 0 & 0 & -0.4 & -0.2 \\ 0 & 1 & 1 & 0 & 0 & -0.4 & -0.2 \\ 0 & 0 & 0 & 0.7 & 0.7 & -0.5 & -0.1 \\ 0 & 0 & 0 & 0.7 & 0.7 & -0.5 & -0.1 \\ 7 & 4 & 4 & 4 & 4 & -0.1 & -0.5 \\ 7.8 & 4 & 4 & 4 & 4 & -0.1 & -0.5 \\ 4 & 8.2 & 7.4 & 4 & 4 & -0.01 & -0.6 \\ 4 & 7.4 & 8.2 & 4 & 4 & -0.01 & -0.6 \\ 4 & 4 & 4 & 9.6 & 8 & -0.2 & -0.7 \\ 4 & 4 & 4 & 8 & 10 & -0.2 & -0.7 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ -0.5 & -0.6 & -0.6 & -0.7 & -0.7 & 0 & 2 \end{pmatrix}.$$

The above Jacobian matrix J is positive-definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, $-\nabla U(Q, q)$ is strongly monotone. Thus, both the existence and uniqueness of the solution to variational inequality (13) with respect to this example are guaranteed.

Table 1 Computed Optimal Service Volumes, quality levels, and profits as p increases

	$p = 10$	$p = 20$	$p = 30$	$p = 40$	$p = 50$	$p = 60$	$p = 70$
Q_{111}	7.48	6.64	5.80	4.96	4.12	3.28	2.44
Q_{112}	5.27	4.67	4.07	3.47	2.87	2.27	1.67
Q_{121}	4.48	3.99	3.50	3.01	2.53	2.04	1.55
Q_{122}	2.49	2.20	1.90	1.61	1.32	1.02	0.73
Q_{131}	3.75	3.33	2.91	2.48	2.06	1.64	1.22
Q_{132}	3.70	3.28	2.86	2.43	2.01	1.59	1.17
Q_{211}	2.16	1.90	1.65	1.40	1.14	0.89	0.64
Q_{212}	3.49	3.11	2.73	2.35	1.97	1.58	1.20
Q_{221}	2.92	2.59	2.25	1.92	1.58	1.24	0.91
Q_{222}	3.05	2.71	2.38	2.04	1.71	1.37	1.03
Q_{231}	2.50	2.22	1.94	1.66	1.38	1.11	0.83
Q_{232}	1.90	1.68	1.45	1.23	1.01	0.79	0.56
q_1	2.59	2.29	2.00	1.71	1.42	1.13	0.83
q_2	4.74	4.21	3.67	3.14	2.60	2.07	1.53
Profit of firm 1	1117.81	871.72	655.28	468.49	311.36	183.89	86.08
Profit of firm 2	576.43	441.90	324.18	223.28	139.20	71.94	21.49

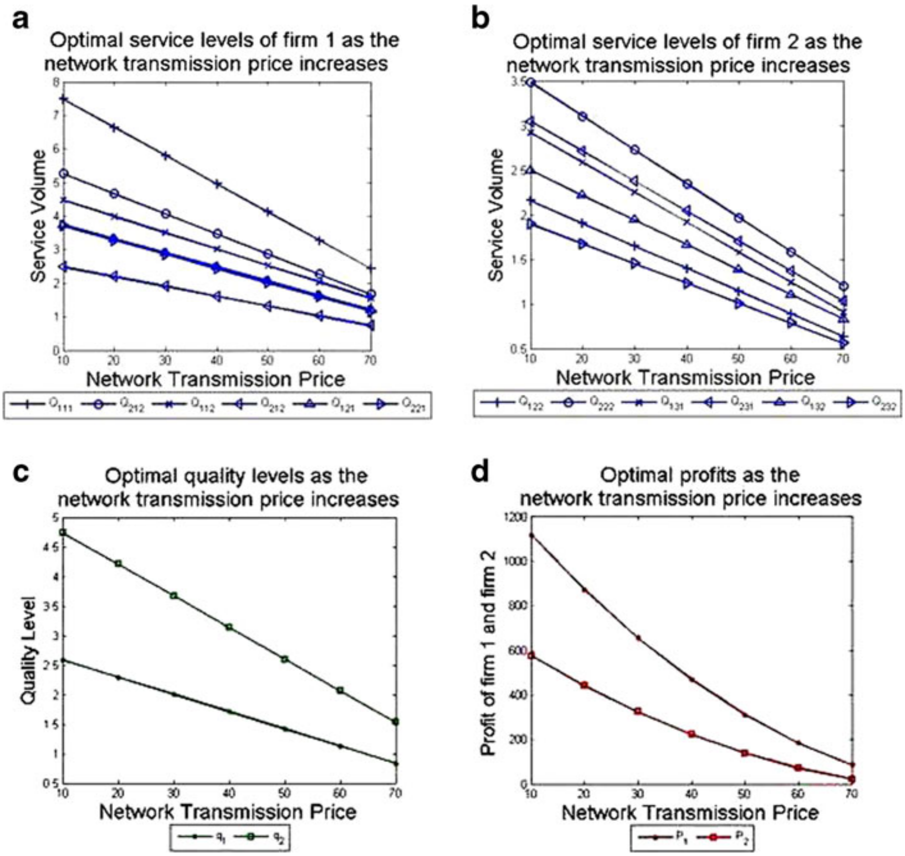

Fig. 8 Sensitivity analysis for Example 3

The Euler method converged to the equilibrium solution: $Q^*_{111} = 5.80$, $Q^*_{112} = 4.07$, $Q^*_{121} = 3.50$, $Q^*_{122} = 1.90$, $Q^*_{131} = 2.91$, $Q^*_{132} = 2.86$, $Q^*_{211} = 1.65$, $Q^*_{212} = 2.73$, $Q^*_{221} = 2.25$, $Q^*_{222} = 2.38$, $Q^*_{231} = 1.94$, $Q^*_{232} = 1.45$, $q^*_1 = 2.00$, $q^*_2 = 3.67$, in 84 iterations, and the equilibrium solution is globally exponentially stable. The profits of the firms were: $U_1 = 655.28$ and $U_2 = 324.18$. Graphical depictions of the product shipment and the quality level iterates are given, respectively, in Fig. 7.

In addition, with the above examples, we wish to illustrate the types of problems with not unrealistic features and underlying functions that can be theoretically effectively analyzed as to their qualitative properties and also their solutions computed.

Sensitivity Analysis for Example 3 After obtaining the equilibrium solution to Example 3, we are interested in the following question: how the changes in the network transmission price p influence the equilibrium solutions and the profit? We then conduct a sensitivity analysis based on the data given in Example 3, and attain the following results in Table 1 and Fig. 8.

As indicated in Fig. 8, service volumes, quality levels and the profits are negatively related to the network transmission price. The reason should be as following. As the network transmission price becomes higher, consumers would purchase less from the network providers as well as the service providers, which leads to the decreasing in service volumes. Then, as the service volumes decrease, there would be less incentive for the firms to improve their quality levels, so the quality levels would also decrease.

6 Conclusions

We developed a new dynamic network economic game theory model of a service-oriented Internet. The model handles service differentiation and includes service providers, production cost functions and demand price functions that capture both demand for the substitutable services as well as their quality levels. The model is a Cournot-Nash model in which the strategic variables of each service provider are its services volumes as well as the quality level of its service. We derived the governing equilibrium conditions and provided alternative variational inequality formulations.

We then proposed a continuous-time adjustment process and showed how our projected dynamical systems model guarantees that the service volumes and quality levels remain nonnegative. We provided qualitative properties of existence and uniqueness of the dynamic trajectories and also gave conditions, using a monotonicity approach, for stability analysis and associated results. We described an algorithm, which yields closed form expressions for the service volumes and quality levels at each iteration, and applied it to solve numerical examples and a sensitivity analysis.

Our network economic model of a service-oriented Internet contributes to the literature in a way that does not limit the number of service providers and network providers, or require specific functional forms. Moreover, this model captures quality levels both on the supply side as well as on the demand side, with linkages through the provision costs, yielding an integrated economic network framework. We hope that the ideas and results in this paper can serve as the basis for future explorations on the network economics of a service-oriented Internet.

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