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Consumers' preference modeling to price bundle offers in the telecommunications industry: a game with competition among operators

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Abstract Network operators are merging their services, such as fixed or wireless telephony, internet or television, into single offers, called bundles. It is essential to understand consumers' preferences to define the most profitable bundles, with their associated prices, especially in the fierce competitive current market. We start by defining a random linear utility model and then, analyze the competition between an integrated operator and new entrants proposing substitutable services. Each operator ignores the consumers' reservation prices for his offers and has to deal with uncertainties about the marketing strategies of competitors, due to potential different size and cost structure. A two-level game is introduced and solved by backward induction. In the second level, the operators determine their optimal offer prices for each possible combination of marketing strategies while the consumers select their most profitable purchasing processes; the natural framework is that of Bayesian game theory. Finally at the top level, knowing the outcome of the other level, the operators identify which marketing strategy to use between market share expansion, segment targeting or multi-level price discrimination, to maximize their expected utilities conditionally to their private informations.

Keywords Bundles · Discrete choice · Pricing · Bayesian game · Marketing strategies

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1 Introduction

Mobile and fixed telephony markets are becoming saturated in Western Europe. In the meantime, high speed internet, including or not television, has become more and more popular. Competition forces telecommunication operators to merge these different services into single offers, called *bundles*. As an illustration, *triple play offers*, combining telephony, television over internet and high speed internet access have flooded the market, but the interest of other combinations of services has to be studied, from a marketing point of view, as well as their associated price. Practically, it raises the problem of the service convergence. Indeed, the operators have to be present on every service market (i.e., fixed, mobile, internet), or it requires alliances to lease the service from competitors. Furthermore, the goals of the operators might be quite different, depending on their sizes and cost structures. For example, small operators might prefer to target specific market segments or expand their market shares as fast as possible, while big companies who already have wide market shares, would prefer to price discriminate between the segments.

Modeling users' preferences, to accurately understand their behaviours, is the focus of the first part of our article, in order to launch properly chosen offers on the market, in terms of content and price, to maximize the operator's revenue. Our choice, based on what was done by Chung and Rao [7], but with a specific application to telecommunication in mind, and with some refinements, is to use random utility functions, with parameters that have to be estimated. The goal is not only to determine optimal prices, but also to select the choice set of offers yielding the highest revenue. A key variable is the *reservation price*, which represents the price at which a user is indifferent between buying the considered offer or choosing any other alternative in the choice set. We consider a *linear* model based on *attributes*, i.e., select important characteristics for preferences. A random variable is added to the model to represent the error choices of the customers, this variable being drawn according to a discrete choice model. In order to find the coefficients (weights) of the different parameters in the linear model, we make use of a panel of customers, and, for a fixed number of segments, ask them to give *ratings* (or grades) for the presence of various attributes (such as trust in the Operator brand / loyalty, Quality of Service, etc.) in each *single* offer. From these ratings, a linear utility function is deduced, its coefficients being estimated. An algorithm based on Bayesian networks and Monte Carlo simulation is used to determine the parameters of the utility. The idea is to allow data augmentation to get a better approximation of the coefficients. The framework of Bayesian networks is a typically relevant tool, making use of a priori densities, which influence is negligible on the values the algorithm converges to, and helps in getting more accurate coefficient estimations. Market segmentation is also performed, by looking for the number of segments maximizing the log marginal likelihood of the parameters.

However, the price optimization analysis is performed partially neglecting the competition between operators; this competition might have drastic consequences on providers' revenues if not taken into account. This remark is valid for most previous works on preferences modeling (cf. Aydin and Ryan [3], Berry and Pakes [4], Khouja and Robbins [13], Tallury and Van Ryzin [20]) which did not incorporate competition, but also for the bundling literature (cf. Chung and Rao [7], Kephart et al. [12], Jeididi et al. [11]).

Our goal and contribution is therefore to add another level of game representing the competition between providers. Indeed, while an operator in a situation of monopole can choose the whole set of offers in order to optimize his revenue, this is not the case in a competitive framework, where competitors' offers are not (directly at least) controlled, but have an impact on the market share one can get. As a consequence, we introduce a model with two rival operators playing a horizontal game with no possible cooperation. Besides, one of the operators is *integrated*, i.e., he is already present on every market (fixed, mobile, internet) and has a big cost structure, while the other is a new entrant proposing *substitutable* services.[1] Specifically, we suppose that we have estimated the reservation price and attribute parameter densities on every market segment, for every offer. Besides, the number of segments is fixed to the optimal one determined as maximizing the log marginal likelihood. Each operator wants to optimally price his offers and to choose the marketing strategy maximizing his revenue. He considers three possible marketing strategies: (i) targeting a specific market segment (like virtual network operators which do not have their own infrastructure), (ii) expanding their market shares (like rather small operators), (iii) developing a multi-level pricing strategy, i.e., applying price discrimination among market segments (like big companies, cf. [7]). Using our random linear utility model for users, operators do not know the true reservation prices of the consumers but only the densities, nor the marketing strategy of competitors. How to use at best the uncertainties and the conflicting interests of the various actors, operators and customers, to help the operator to find the most profitable prices? Our contribution here is to study this kind of game in the case of two operators with a preference model derived from Chung and Rao [7]. The modeling framework used is that of a two-level game between the operators and customers. In the second level, for each possible combination of marketing strategies, we define a Bayesian game between the operators and the customers to determine the optimal offer prices (or randomized pricing strategies). However, it seems difficult to obtain analytically the randomized pricing strategies characterizing the equilibrium of the game, and simulation can therefore be used to get an approximation. This kind of procedure has already been applied in the literature, by few papers in other contexts. Holenstein [10], for instance, uses simulation to approach the equilibrium in an auction game, where the players' types and the action spaces of players are continuous. Similarly, Cai and Wurman [5] use Monte Carlo

[1] It means that the operators commercialize offers having the same technical properties; for example, Operator 1 might sell a bundle of wired-phone / internet while the other would propose a simple offer based on wired-phone. Both offers are in competition.

methods to approach Bayesian equilibria in sequential auction games. In our case, we play a game based on the computation of the best-response strategy profiles. The approach is myopic since it does not consider future consequences of the action choices, but the algorithm is checked to converge to a Bayesian equilibrium. Besides we impose some conditions in the algorithm convergence proof, to get the unicity of the Bayesian equilibrium. Then, at the top level, the operators identify the most profitable marketing strategy to use, i.e., the one maximizing their revenues. However, the integrated operator being the leader, acts first, then the new entrant selects her own strategy. It is as if a time-shift were introduced between both operators' marketing strategy selection.

The remainder of this paper is organized as follows. In Section 2, we review the basic notions on the random linear utility model which is an extension of [7] to the telecommunication industry. We introduce competition and uncertainty on the operator's profits via a two-level game in Section 3. For each combination of marketing strategies, a Bayesian game is defined in Section 4. The utilities of the different actors and the horizontal game are first settled. Then, we introduce the optimization problem to solve to get the optimal randomized pricing strategies. Since the equilibria cannot be computed analytically, we resort to use simulation and check the algorithm convergence in Subsection 4.4. Then, the first level of game is introduced in Section 5. Numerical illustrations are also provided and the algorithm complexity is discussed. Finally, we conclude and give some directions for future research in Section 6.

In all the article, Operator 1 will be designed as a male player, while Operator 2 will be female.[2]

1.1 Notations

Next we review the specific notation of this paper.

$\mathcal{K}$	set of operator's available families services (ex: available services about tv, Internet, or wireless or wired telephony)
b	a bundle
$b(k)$	the k^{th} component of the bundle, belonging to service family k
$\mathcal{B}$	operator's choice set containing the offers to commercialize
$\mathcal{N}_P$	panel of tested customers
N_P	cardinality of the panel set $\mathcal{N}_P$
Z_i	Random variable giving the segment customer i belongs to
a	an attribute

[2]This is a classical assumption in Game theory.

$\mathcal{A}^1$	class of type 1 attributes, fully comparable for the available services
$\mathcal{A}^2$	class of type 2 attributes, partially comparable for the available services
$\mathcal{A}^3$	class of type 3 attributes, non comparable for the available services
$\mathcal{A}$	total set of attributes
$w_i(k)$	nonnegative coefficient characterizing the importance of service k for customer i
$X_i\,(a, b\,(k)) \in \{0, 1, 2, ..., 10\}$	rating of customer i for the importance of attribute a in bundle b
$\mathcal{S}_i(a, b)$	weighted sum, in terms of the $w_i(k)$, of attribute a in bundle b for customer i
$\mathcal{D}_i(a, b)$	weighted dispersion of the ratings of services in bundle b on attribute a
$\mathcal{F}$	set of market segments
F	Maximum number of segments
$U_{ib\,\vert\,Z_i=f}$	customer i's utility for the bundle b, provided i belongs to market segment f
$BV(ib\,\vert\,Z_i = f)$	customer i's valuation for the bundle b, provided i belongs to market segment f, excluding price
$\beta_a(f, i)$	utility parameter representing the importance of attribute a for customer i on market segment f
$\gamma_a(f, i)$	utility parameter measuring the substitutability (< 0)/complementarity (≥ 0) of attribute a, for the client i in f
$\alpha_P(f, i)$	utility parameter of price sensitivity for client i on market segment f
$\psi(f, i)$	probability that customer i belongs to market segment f
Φ_i	vector of unknown coefficients for customer i
$\rho_{i1} \in [0; 1]$	degree of independence in unobserved utility among the alternatives in the choice set for customer i
ρ_{i0}	client i's probability to not buy anything
$\bar{P}(b)$	mean market price for the bundle b
$\mathcal{T}_{k,f}$	estimated density of the trust in Operator k brand on segment f
$R(i, b)$	reservation price of the customer $i \in \mathcal{N}$ for the bundle b
$\mathcal{N}$	consumer set on the market
N	cardinality of the consumer set
$\Delta(\mathcal{E})$	set of all the probability distributions on the generic set $\mathcal{E}$

$\vert\mathcal{E}\vert$	cardinality of the generic space $\mathcal{E}$
$\mathcal{B}_{\text{op}_i}$	set of offers commercialized by Operator $i = 1, 2$
T_i	consumer i's type space
t_i	consumer i's type
D_i	consumer i's action space
$P_{\text{op}}(b)$	Operator's retail price for the offer b
$\mathcal{C}$	cheating probability set
d_i	consumer i's action
T_{-i}	set of all the possible combinations of types for the consumers other than i
$u_i(d, t)$	consumer i's utility, the global action c and type t being chosen
$\sigma_i(d_i\vert t_i)$	probability that consumer i chooses the action d_i provided his type is t_i
$U_i(\sigma\vert t_i)$	player i's conditional expected utility under pricing strategy σ
T_{op}	operator's type space
$c_{\text{op}}(b)$	Operator's offer b cost
$\mathcal{P}$	discrete price set
$u_{\text{op}}(d, t)$,	operator's utility, the global action c and type t being chosen
n_1	upper bound of the low interest / income category
n_2	upper bound of the middle interest / income category
n_3	upper bound of the high interest / income category
$\bar{F}_{b,f}$	complementary cumulative distribution of segment f consumer maximum admissible price for offer b
s_{op_k}	Operator k's marketing strategy

2 Consumer preference modeling

This section summarizes the consumers' preference modeling, based on [7], which is used in the game between operators. Some additional details about this section are also provided in [14].

A choice set contains the set of offers of various sizes that an operator commercializes on the market. It will be denoted $\mathcal{B}$. A bundle b is a combination of elementary offers $b(k)$, for a number of communication supports k in $\mathcal{K}$. If as elementary offer in b belongs to the k-th family, we use the convention: $b(k) = 0$ and the consumers will not have to evaluate the attributes on the k-th component of the bundle b. For example, a bundle b can be made of an offer in terms of mobile communication, another one in terms of internet access, another one for tv and another one for wired-telephony. Any offer considering one, two or three of these technologies is also possible. Hence, the operator's choice set could be made of simple offers sold independently and bundles.

Customers are separated into market segments to better understand their behaviour. Let $\mathcal{F}$ be the segment set and F the maximum number of segments. Customer i's segment is unknown from the operators, we therefore define a random variable Z_i representing the segment customer i belongs to.

The random utility model represents user i's preferences at price $P(b)$, given that it belongs to segment f, as

$$U_{ib|Z_i=f}(P(b)) = V_{ib|Z_i=f}(P(b)) + \varepsilon(i, b), \; \forall i \in \mathcal{N}_P, \; \forall b \in \mathcal{B}, \; \forall f \in \mathcal{F}, \quad (1)$$

where vector $\varepsilon(i) = \big(\varepsilon(i, 1) \; \varepsilon(i, 2) \; ... \; \varepsilon(i, |\mathcal{B}|)\big)$ contains all the error coefficients for the client i, taking into account the uncertainty associated with customers' valuations.

On the other hand, $V_{ib|Z_i=f}$ is the deterministic part and is also decomposed (linearly) into

$$V_{ib|Z_i=f}(P(b)) = BV(ib|Z_i = f) + \alpha_P(f, i) P(b)$$

to separate between the valuation due to the price $P(b)$, with $\alpha_P(f, i) \leq 0$ coefficient expressing the customer i sensitivity towards price, and the valuation $BV(ib|Z_i = f)$ due to the bundle itself, excluding price.

Note that customer i's utility for the bundle b takes the form

$$V_{ib} = \sum_{f=1}^{F} \psi(f, i) \; V_{ib|Z_i=f}, \quad (2)$$

where $\psi(f, i)$, $i \in \mathcal{N}_P$, $f \in \mathcal{F}$, is the unknown probability that customer i belongs to the market segment f.

The valuation $BV(ib|Z_i = f)$ is itself supposed to be linear in terms of *attributes*. An attribute is an entity that defines customers' valuations properties and distinguishes between the offers. For example, classical attributes in telecommunication are the Quality of Service (QoS), the consumer trust in the operator brand / loyalty, etc. Attributes are in three different categories, depending on whether it can be compared on the service families: the class $\mathcal{A}^1$ contains fully comparable attributes, which appear in every service family; the class $\mathcal{A}^2$ contains partially comparable attributes, appearing in at least two service families; the class $\mathcal{A}^3$ contains non comparable attributes appearing in only one system family.

Given these definitions, the valuation is assumed to take the form:

$$BV(ib|Z_i = f) = \alpha_0(f,i) + \sum_{a_1 \in \mathcal{A}^1} \left[\beta_{a_1}(f,i)\mathcal{S}_i(a_1,b) + \gamma_{a_1}(f,i)\mathcal{D}_i(a_1,b)\right]$$
$$+ \sum_{a_2 \in \mathcal{A}^2} \left[\beta_{a_2}(f,i)\mathcal{S}_i(a_2,b) + \gamma_{a_2}(f,i)\mathcal{D}_i(a_2,b)\right]$$
$$+ \sum_{a_3 \in \mathcal{A}^3} \alpha_{a_3}(f,i)C_i(a_3,b). \tag{3}$$

Utility parameters $\mathcal{S}_i(a,b)$, $\mathcal{D}_i(a,b)$, $i \in \mathcal{N}_P$, $b \in \mathcal{B}$, $a \in \mathcal{A}^1 \cup \mathcal{A}^2$ and $C_i(a_3,b)$ are estimated from a panel of customers who have rated the importance of attributes: each customer i in the panel $\mathcal{N}_P$ is supposed to have provided a rating $X_i(a, b(k)) \in \{0, 1, 2, ..., 10\}$ to the presence of the attribute a in the simple offer $b(k)$. Similarly, each customer $i \in \mathcal{N}_P$ associates importance weights for every family. These coefficients are nonnegative and normalized, i.e.: $w_i(k) \geq 0$, $k = 1, 2, ..., |\mathcal{K}|$, and $\sum_{k=1}^{|\mathcal{K}|} w_i(k) = 1$, $\forall i \in \mathcal{N}_P$. Given these ratings, $\mathcal{S}_i(a,b)$ measures the ratings of customer i's relevant components (i.e., the individual offers $b(k)$) for offer b on attribute a,

$$\mathcal{S}_i(a,b) = \sum_{k=1}^{|\mathcal{K}|} w_i(k)\, \mathbf{1}_{(a,b(k))}\, X_i(a, b(k)), \quad a \in \mathcal{A}^1 \cup \mathcal{A}^2,\ b \in \mathcal{B}, \tag{4}$$

where $\mathbf{1}_{(a,b(k))} = 1$ if the attribute a importance can be evaluated in the component $b(k)$, and 0 otherwise. Another essential characteristic that we use as parameter is the weighted dispersion $\mathcal{D}_i(a,b)$ of relevant components for offer b and the average rating of offer b, on attribute a,

$$\mathcal{D}_i(a,b) = \sum_{k=1}^{|\mathcal{K}|} w_i(k)\, \mathbf{1}_{(a,b(k))} \left[X_i(a, b(k)) - \bar{X}_i(a,b)\right], \quad a \in \mathcal{A}^1 \cup \mathcal{A}^2,\ b \in \mathcal{B}, \tag{5}$$

with $\bar{X}_i(a,b) = \frac{\sum_{k=1}^{|\mathcal{K}|} \mathbf{1}_{(a,b(k))} X_i(a,b(k))}{\sum_{k=1}^{|\mathcal{K}|} \mathbf{1}_{(a,b(k))}}$ the average rating that customer i gives to the presence of attribute a in every single offer $b(k)$ composing the bundle b. Our last set of utility parameters, the $C_i(a_3,b)$, given by $C_i(a_3,b) = \sum_{k=1}^{|\mathcal{K}|} w_k(i) X_i(a, b(k))$, represents customer i's (weighted) perceived value for non-comparable attributes.

The coefficients composing vector

$$\Phi_i := \left(\{\beta_a(f,i)\}_{a,f},\ \{\gamma_a(f,i)\}_{a,f},\ \{\alpha_a(f,i)\}_f,\ \{\alpha_0(f,i)\}_f,\ \{\alpha_P(f,i)\}_f\right)$$

are defined to balance the relative importance of the above utility parameters and have to be determined.

The way to estimate those coefficients is based on the use of the hierarchical Bayesian framework studied by Poirier [18] for data augmentation and better approximation, due to the lack of information of the operator about customers. We assume a priori laws on distributions and look by simulation to which values the simulation converges. It is known that the result is robust to the choice of *a priori* laws. The approach therefore enables to learn iteratively these preference coefficients. It consists in sampling vector Φ_i given the segment $Z_i = f$ customer i belongs to. Market segmentation is also performed.

Error terms follow distributions so that the the model is the nested logit model, by using a cumulated distribution function of form

$$F\left(\varepsilon(i)\right) = \exp\left(-\exp(-\varepsilon(i,0))\right)\left(\exp\left(-\sum_{b\in\mathcal{B}}\exp\left(-\frac{\varepsilon(i,b)}{\rho_{i1}}\right)\right)\right)^{\rho_{i1}},$$

where $\rho_{i1} \in [0;1]$ measures the degree of independence in unobserved utility among the alternative in the choice set $\mathcal{B}$. Since the cumulated distribution function associated with the empty set error is $F(\varepsilon(i,0)) = \exp(-\exp(-\varepsilon(i,0)))$, the empty set error is drawn according to a Logit density

$$\varepsilon(i,0) \sim \exp(-\varepsilon(i,0) - \exp(-\varepsilon(i,0))).$$

From these errors, classical discrete choice theory (cf. Train [21]) derives the expressions of the choice probabilities. The consumer i probability to not buy anything at all is:

$$P_{i0}(\rho_{i0},\rho_{i1}) = \frac{1}{1+\exp\left(-V_{i0}\left(\sum_{j\in\mathcal{B}}\exp\left(\frac{V_{ij}(P(j))}{\rho_{i1}}\right)\right)^{\rho_{i1}}\right)}, \tag{6}$$

where, by assumption $\rho_{i0} := U_{i0}$ is the client i's utility of not buying anything and, $V_{i0} = U_{i0} - \varepsilon(i,0) := \rho_{i0} - \varepsilon(i,0)$. Also, the consumer i probability to buy a bundle $b \in \mathcal{B}$ is

$$P_{ib}(\rho_{i0},\rho_{i1}) = \frac{\exp\left(\frac{V_{ib}(P(b))}{\rho(i,1)}\right)\left(\sum_{b'\in\mathcal{B}}\exp\left(\frac{V_{ib'}(P(b'))}{\rho(i,1)}\right)\right)^{\rho_{i1}-1}}{\left(\sum_{b'\in\mathcal{B}}\exp\left(\frac{V_{ib'}(P(b'))}{\rho_{i1}}\right)\right)^{\rho_{i1}} + \exp(V_{i0})}. \tag{7}$$

The *reservation price* $R(i,b)$ is the offer b price at which customer i is indifferent between buying offer b and choosing another alternative in the choice set. Formally, the reservation price for the offer b can be defined as follows[3]

$$U_{ib|Z_i=f}(R(i,b)) = \max\left\{\max_{b'\in\mathcal{B},b'\neq b,b'\neq 0} U_{ib'|Z_i=f}(P(b'));\ U_{i0|i\in f}\right\}.$$

It can be infered using the estimation of utilities described just above. Finally, using non-parametric statistical approaches, we estimate the attribute

[3]Including in $\mathcal{B}$ the empty set, leading to a null utility.

of consumer trust in the operator brand / loyalty and the reservation price distributions, on each market segment and for each offer of the considered choice set [14].

Numerical illustration We consider two rival operators. Operator 1 (for instance, Orange) sells simple offers of wired-telephony and internet, and a bundle of wired-telephony / internet / television. Operator 2 commercializes a simple offer of mobile, a double-play offer of wired-telephony / internet and a triple-play offer of mobile / internet / television. The choice set $\mathcal{B}$ is now made of the offers commercialized by both operators. We test our algorithm on data issued from a survey realized by Orange in 2006. The consumer panel was made of 1014 families, i.e., 2058 individual consumers. It consists of a set of questions aiming at understanding individual subjective preferences and behaviours towards telecommunication, multimedia and leisure activities. The optimal number of segments is four: segment 1 coincides with trendy and sociable teenagers, who have high communication needs, very important leisure and are enthusiastic to discover new technologies; the segment 2 is made of technological addicts, i.e., young couples using internet in all ways, who have home centered entertainments (cf. DVDs, MP3, iPod, etc.); on the segment 3, we meet essentially work/life jugglers. They are mainly middle age, very busy, high social grade persons, using all means of communications. Their technological adoption is driven by work; finally, the segment 4 contains the low-income technological adverse seniors, who use exclusively wired-phone, and are not interested in technology.

At the top of Fig. 1, we have plotted the distribution of the consumer trust in Operator 1 brand / loyalty, for segment 1 (left) and 3 (right). We observe that the teenagers are less confident than the middle-age active couples who already know this integrated operator. Indeed, teens give priority to the brand image, churn a lot, feel attracted by new virtual network operators offering fashionable images and cheap services. At the bottom of Fig. 1, the complementary cumulative distribution function of the consumer maximum admissible price on segment 3 and 4 are represented for the wired-telephony / internet / TV bundle. We note that the segment 3 seems very interested and is ready to pay a high price to get such an offer; whereas the segment 4 seems reluctant to this triple-play offer adoption.

Outputs of Section 2 are used in the description of the Bayesian game which occurs between the different players. However, the consumers do not know the selling prices of the offers used in the random utility (cf. Eq. 1), since it is the aim of the forthcoming sections to determine optimal pricing strategies for the operators. Consequently, to run the preference algorithm, the consumers associate with each offer b a mean market price, $\bar{P}(b)$ (this is a classical assumption in economy [1]). The mean market price of the offer b is the mean of the prices at which every offer belonging to the same family combination has been sold in the past. For example, in the telecommunication market, since 2000, six versions of the triple-play offer made of the combination of a

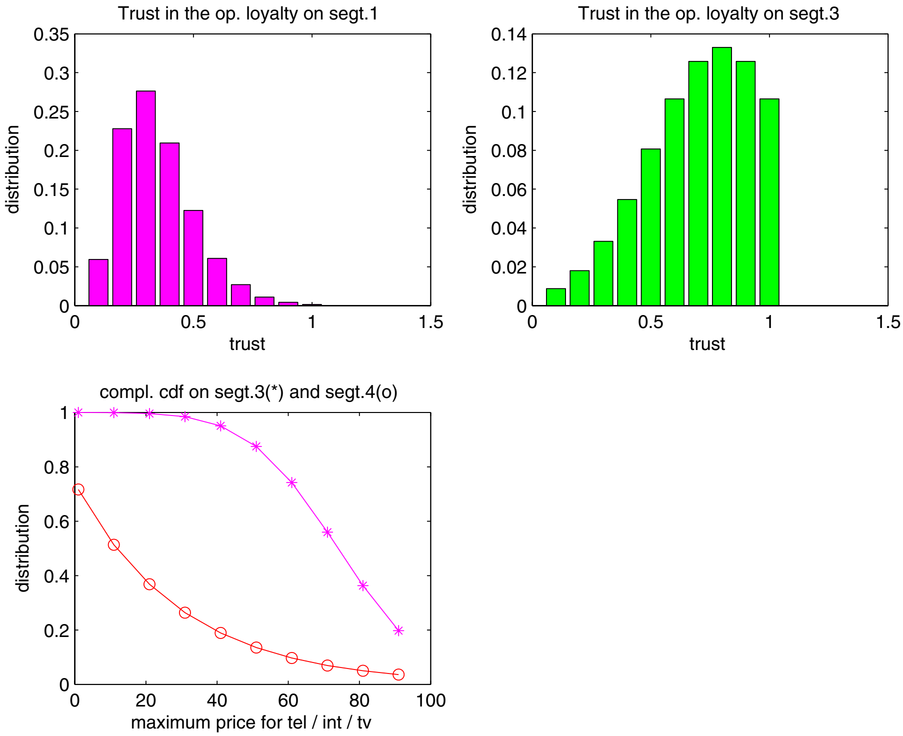

Fig. 1 Distributions of consumer trust in Operator 1 brand / loyalty and consumer maximum admissible price for tel./internet/tv depending on some market segments

wired-phone, internet and tv access offers, have been commercialized. To get the wired-phone / internet / tv bundle mean price, we compute the mean of the six prices of the previously commercialized bundles. We can proceed the same way for every combination of elementary offers. If the offer b is totally new on the market, $\bar{P}(b)$ is fixed arbitrarily at 0.

In the remainder of the article, we assume that two rival operators have determined their choice sets. Outputs of Section 2 (listed below) will be used to define our two-level game settings

- the consumer trust in both operator's brand / loyalty distribution on each market segment is known; it is $\mathcal{T}_{1,f}$ for Operator 1 on segment f and $\mathcal{T}_{2,f}$ for Operator 2 on segment f. These informations are public and known by both providers.
- the market segmentation is known, i.e., each operator knows to which segment consumer i belongs.
- The complementary cumulative distribution functions of the consumers' maximum admissible prices, on each market segment f and for every offer b: $\bar{F}_{b,f}$, are known by both operators.

3 Pricing under horizontal competition between an integrated operator and a new entrant

The aim of this section is to present the two-level game occuring between operators. In order to proceed, we need to model their goals, and we assume that they have three possible marketing strategies:

(i) either they just target a market segment. This situation is typical of Mobile Virtual Network Operators (MVNOs) in telecommunications, which are carriers providing users with mobile services without their own license for bandwidth, but renting that bandwidth to other providers [15];
(ii) either they can expand their market shares to become larger operators and therefore get a higher revenue on a longer term. Market share expansion strategy is usually used by new market entrants. To fetch numerous market shares, the operator has first to lower his offer price to seduce as many consumers as possible;
(iii) or they choose to discriminate prices among segments for increasing their benefits (typical revenue management situation of large operators). Such a strategy is profitable only for operators having a sufficiently large market share.

But those operators have to deal with many uncertainties on the market. First they do not know the true reservation prices of customers, but only their distributions. They also ignore the *profit* (i.e., the difference between the selling price and the real cost of an offer) of their competitors, as well as their marketing strategies (highlighted above).

In the remainder of the paper, in order to simplify the analysis, we just consider two operators in competition, but it can be easily extended to a larger number and the impact of such an extension on the game solution approaching algorithm will be studied in Subsection 4.4.

How to model interactions among all those actors? Again, given that we assume that they all act selfishly, Game theory is the natural framework. Additionally, due to the uncertainties involved, Bayesian game theory (cf. [17]) becomes more specifically the relevant tool.

Practically, we assume that Operator 1 is an integrated provider, like Orange for instance. It means that he is already well-known on the wired-phone, mobile phone and internet markets. Such an operator has already a wide market share and is powerful due to his high budget; besides consumers trust gladly in his brand / loyalty. On the contrary, Operator 2 is a new entrant on the telecommunication market. It is either a new telecommunication operator who has just bought a new license or a firm already well-known on some other markets (for instance, a beverage, an airline-ticket seller, or a tv channel) who wants to improve his brand value by extending his proposed services and becoming a virtual network operator [15].

Formally, the game between providers is made of two levels. The first level can be identified with a Stackelberg game on both operators' marketing strategies under uncertainties; Operator 1 being more powerful or having some sort of advantage, acts first (i.e., he is the leader). In the second level, for each combination of marketing strategies, we solve a Bayesian game whose output gives us the equilibria, i.e., the operators' randomized pricing strategies and the consumers' randomized purchasing processes maximizing the players' expected utilities conditionally to their types.

3.1 Description of the game between an integrated operator and a new entrant under uncertainties

Operator 1 and Operator 2 have defined $\mathcal{B}_{\mathrm{op}_1}$ and $\mathcal{B}_{\mathrm{op}_2}$ as their choice sets.

Uncertainties appear in:

- the true consumers' reservation prices (only the distributions are known)
- both operators' profits (i.e., the difference between the selling prices and the real costs of the offers)
- the rival's marketing strategy

1. *Selection of the operators' marketing strategies* conditionally to their true types (cheating probabilities), Operator 1 chooses his marketing strategy and then, the new entrant selects hers between segment targeting, market share expansion and multi-level price discrimination.
2. *Determination of the retail prices conditionally to each type* the operators determine simultaneously the randomized pricing strategies maximizing their expected utilities conditionally to their types (cheating probabilities). At the same time, depending on their intrinsic preferences, consumers buy offers or nothing.

In the second level, for each possible combination of marketing strategies, a Bayesian game between the operators and the consumers is introduced. Indeed, each operator has incomplete information about his rival's type (the ratio between the real costs and the retail prices of the operator's offers) and about the consumers' preferences. The operators want to find the randomized pricing strategy for their offers, maximizing their utilities. Besides, the operators' payoffs depend on consumers' choices. Indeed, consumers buy the most valued offer (i.e., the one having the highest reservation price in case they are high income / interest) or the offer generating the greatest benefit (i.e., the greatest difference between the selling price and the reservation price; in case they are low income / interest). As outputs of the Bayesian games, we get equilibria, i.e., randomized purchasing process for the consumers and randomized pricing strategies for both operators; it then enables us to infer the actors' expected utilities, conditionally to each type.

In the first level, Operator 1 has still uncertainty about Operator 2's type and about the marketing strategy that she will use (and reciprocally). However, Operator 1 being integrated, is more powerful; it enables him to move first. The Stackeberg game under uncertainties is then solved by backward induction. Operator 1 considers what the best response of the follower (Operator 2) is, i.e., how she will respond once she has observed the marketing strategy of the leader. The leader (Operator 1) then picks a marketing strategy that maximizes his expected utility conditionally to his true type, anticipating the predicted response of the follower. The follower actually observes this and in equilibrium picks the expected marketing strategy as a response.

A Bayesian game needs to be solved for each combination of marketing strategies. In this article, we only deal with a small number of combinations (exactly, nine); but in general the two-level game may become very tedious to solve when applied to a large marketing strategy space. Practically, it is possible to restrict the level 1 exploration to the most realistic marketing strategy combinations only. For instance, if Operator 1 has already a diversified large market share, he would not use a segment targeting strategy and would prefer a price-discrimination strategy; if Operator 2 is an already well-known beverage manufacturer, airline ticket reseller or powerful tv-channel, who wants to propose telecommunication services to increase her brand value, she would rather use a segment targeting strategy; finally if Operator 2 is a new totally unknown entrant provider with a small budget, she would rather use a market share expansion approach. Hence, all these economical considerations enable us to narrow the marketing strategy state space.

In Section 4, we study the Bayesian game (second level) for a fixed combination of marketing strategies.

4 Second level: description of the Bayesian game

We let $\mathcal{N}$ be the set of consumers in the game. Using the inference model of Section 2, each operator has determined the set of offers that he (she) wants to commercialize: $\mathcal{B}_{\text{op}_1}$ for Operator 1 and $\mathcal{B}_{\text{op}_2}$ for Operator 2. We assume that Operator 1 sells a simple offer of mobile, a double-play offer of wired-telephony / internet and a triple-play offer of mobile / internet / tv. Operator 2's choice set is made of: simple offers of wired-telephony, internet, and a bundle of wired-telephony /internet / tv. The total set of offers proposed on the market is then denoted as: $\mathcal{B} = \mathcal{B}_{\text{op}_1} \cup \mathcal{B}_{\text{op}_2}$. The critical point in this model is that both operators' services are *substitutable*. For instance, simple offers of wired-telephony and internet issued from Operator 1's choice set might be cannibalized (cf. [12] who studies cannibalization in a product line) by Operator 2's bundle of wired-phone / internet. Identically, Operator 1's bundle of mobile / internet / tv might be replaced by Operator 2's bundle of wired-phone / internet / tv, provided the consumer agrees to change his wired-phone against a mobile one. Hence, to survive in such a competitive framework, is it better for the operators to target a specific market segment, which would provide a guaranteed revenue, or try to capture as much consumers as possible?

Each operator ignores the other's marketing strategy (i.e., market share expansion, segment targeting or multi-level price discrimination) and some private information remains hidden (i.e., both operators' profit levels and each consumer's true reservation price).

To solve our two-level game, we proceed by backward induction. We therefore start by assuming that the operators' marketing strategies are fixed. Then, for each marketing strategy combination, a Bayesian game is defined and solved. As outputs, we get the operators' optimal randomized pricing strategies and the associated expected utilities conditional to their types.

Consumer i's and Operator's action sets will be denoted D_i and D_{op} respectively. The global action set containing all the possible action combinations for each player is $D = \times_{i\in\mathcal{N}} D_i \times D_{\text{op}_1} \times D_{\text{op}_2}$. Consumer i's type space is T_i, whereas operator's one is called T_{op}. The type spaces contain all the actors' private informations. The global type space, made of all the possible type combination for each player, is $T = \times_{i\in\mathcal{N}} T_i \times T_{\text{op}_1} \times T_{\text{op}_2}$.

Consumer i's subjective probability about the other players' types is a function defined as

$$p_i : T_i \to T_{-i} \times T_{\text{op}_1} \times T_{\text{op}_2}$$
$$t_i \mapsto p_i\left(\times_{j\in\mathcal{N}_{-i}} t_j, t_{\text{op}_1}, t_{\text{op}_2} | t_i\right)$$

where $\mathcal{N}_{-i}$ is the set of all the consumers other than i and T_{-i} is the set of all the possible combinations of types for the consumers other than consumer i. Identically, Operator j's subjective probability about the other players' types is

$$p_{\text{op}_j} : T_{\text{op}_j} \to \times_{i\in\mathcal{N}} T_i \times T_{\text{op}_k}$$
$$t_{\text{op}_j} \mapsto p_{\text{op}_j}\left(\times_{i\in\mathcal{N}} t_i, t_{\text{op}_k} | t_{\text{op}_j}\right)$$

where $j, k \in \{1; 2\}$ and $j \neq k$. Finally, u_i and u_{op} denote consumer i and operator's utility.

4.1 Notion on Bayesian equilibria under fixed marketing strategies

The Bayesian game can be defined as follows, according to the formalism used by Myerson [17]

$$\Gamma^b = \left(\mathcal{N}, \text{Op}_1, \text{Op}_2; D; T; (p_i)_{i\in\mathcal{N}}, p_{\text{op}_1}, p_{\text{op}_2}; (u_i)_{i\in\mathcal{N}}, u_{\text{op}_1}, u_{\text{op}_2}\right).$$

A randomized purchasing and pricing strategy profile for the Bayesian game Γ^b, is any σ of the form $\sigma = \left\{\left(\sigma_i(d_i|t_i)\right)_{d_i\in D_i, t_i\in T_i} \forall i \in \mathcal{N}, \left(\sigma_{\text{op}_j}(d_{\text{op}_j}| t_{\text{op}_j})\right)_{d_{\text{op}_j}\in D_{\text{op}_j}, t_{\text{op}_j}\in T_{\text{op}_j}} \forall j = 1, 2\right\}$ which satisfies the following constraints (cf. [17]):

- for each consumer $i \in \mathcal{N}$, $\sigma_i(d_i|t_i) \geq 0$ and $\sum_{d_i\in D_i} \sigma_i(d_i|t_i) = 1$;
- for each Operator $j = 1, 2$, $\sigma_{\text{op}_j}\left(d_{\text{op}_j}|t_{\text{op}_j}\right) \geq 0$ and $\sum_{d_{\text{op}_j}\in D_{\text{op}_j}} \sigma_{\text{op}_j}\left(d_{\text{op}_j}| t_{\text{op}_j}\right) = 1$.

In such a strategy profile, $\sigma_i(d_i|t_i)$ (resp. $\sigma_{\text{op}_j}(d_{\text{op}_j}|t_{\text{op}_j})$) represents the conditional probability that consumer i (resp. Operator j) would choose the action

d_i (resp. d_{op_j}), provided his type is t_i (resp. t_{op_j}). In the strategy profile σ, the randomized pricing strategy associated with the type t_{op_j} of Operator j is:

$$\sigma_{\mathrm{op}_j}\big(.|t_{\mathrm{op}_j}\big) \;=\; \Big(\sigma_{\mathrm{op}_j}\big(d_{\mathrm{op}_j}|t_{\mathrm{op}_j}\big)\Big)_{d_{\mathrm{op}\,j}\in D_{\mathrm{op}\,j}}$$

and consumer i's randomized purchasing strategy associated with the type t_i is:

$$\sigma_i\,(.|t_i) = (\sigma_i(d_i|t_i))_{d_i\in D_i}\,.$$

We introduce $\Delta(D_i)$ (resp. $\Delta(D_{\mathrm{op}})$) as the sets of all the probability distributions on consumer i (resp. Operator)'s action space D_i (resp. D_{op}).

A Bayesian equilibrium of the game Γ^b, is any combination of randomized purchasing and pricing strategies such that, for any consumer i and Operator j, and for any type $t_i \in T_i,\ t_{\mathrm{op}_j} \in T_{\mathrm{op}_j}$ (cf. [17]) the conditional expected utilities $U_i(\sigma|t_i)$ and $U_{\mathrm{op}_j}(\sigma|t_{\mathrm{op}_j})$ are maximized. For consumer i, the optimization problem takes the form

$$\sigma_i(.|t_i) \in \arg\max_{\tau_i\in\Delta(D_i)} \sum_{t_{-i},t_{\mathrm{op}_1},t_{\mathrm{op}_2}\in T_{-i}\times T_{\mathrm{op}_1}\times T_{\mathrm{op}_2}} p_i\left(t_{-i}, t_{\mathrm{op}_1}, t_{\mathrm{op}_2}|t_i\right)$$
$$\times\sum_{d\in D}\left(\prod_{j\in\mathcal{N}-i}\sigma_j\left(d_j|t_j\right)\sigma_{\mathrm{op}_1}\left(d_{\mathrm{op}_1}|t_{\mathrm{op}_1}\right)\times\sigma_{\mathrm{op}_2}\left(d_{\mathrm{op}_2}|t_{\mathrm{op}_2}\right)\right)\tau_i\left(d_i\right)\ u_i(d,t),\, i\in\mathcal{N}; \tag{8}$$

Once Eq. 8 has been solved, consumer i expected utility becomes

$$U_i(\sigma|t_i) = \sum_{t_{-i},t_{\mathrm{op}_1},t_{\mathrm{op}_2}\in T_{-i}\times T_{\mathrm{op}_1}\times T_{\mathrm{op}_2}} p_i\big(t_{-i}, t_{\mathrm{op}_1}, t_{\mathrm{op}_2}|t_i\big)$$
$$\times\sum_{d\in D}\left(\prod_{j\in\mathcal{N}-i}\sigma_j\big(d_j|t_j\big)\sigma_{\mathrm{op}_1}\big(d_{\mathrm{op}_1}|t_{\mathrm{op}_1}\big)\sigma_{\mathrm{op}_2}\big(d_{\mathrm{op}_2}|t_{\mathrm{op}_2}\big)\right)\sigma_i(d_i|t_i)u_i(d,t).$$

Each Operator j needs to solve the optimization problem

$$\sigma_{\mathrm{op}_j}\big(.|t_{\mathrm{op}_j}\big) \in \arg\max_{\tau_{\mathrm{op}\,j}\in\Delta\left(D_{\mathrm{op}\,j}\right)} \sum_{t_1,t_2,\ldots,t_N,t_{\mathrm{op}_k}\in T_1\times T_2\times\ldots\times T_N\times T_{\mathrm{op}_k}} p_{\mathrm{op}_j}\big(t_1, t_2, \ldots, t_N, t_{\mathrm{op}_k}|t_{\mathrm{op}_j}\big)$$
$$\times\sum_{d\in D}\left(\prod_{i\in\mathcal{N}}\sigma_i\big(d_i|t_i\big)\times\sigma_{\mathrm{op}_k}\big(d_{\mathrm{op}_k}|t_{\mathrm{op}_k}\big)\right)\tau_{\mathrm{op}_j}\big(d_{\mathrm{op}_j}\big)\ u_{\mathrm{op}_j}(d,t); \tag{9}$$

where $j, k \in \{1; 2\}$, $j \neq k$. Once Eq. 9 has been solved, Operator j expected utility can be infered as follows

$$U_{\mathrm{op}_j}(\sigma | t_{\mathrm{op}_j}) = \sum_{t_1, t_2, \ldots, t_N, t_{\mathrm{op}_k} \in T_1 \times T_2 \times \ldots \times T_{\mathrm{op}_k}} p_{\mathrm{op}_j}(t_1, t_2, \ldots, t_N, t_{\mathrm{op}_k} | t_{\mathrm{op}_j}) \times \sum_{d \in D} \left(\prod_{j \in \mathcal{N}} \sigma_j(d_j | t_j) \sigma_{\mathrm{op}_k}(d_{\mathrm{op}_k} | t_{\mathrm{op}_k}) \right) \sigma_{\mathrm{op}_j}(d_{\mathrm{op}_j} | t_{\mathrm{op}_j})\, u_{\mathrm{op}_j}(d, t).$$

4.2 Definition of the consumers' game setting

We define the consumers' type spaces, action sets, utilities and subjective beliefs to introduce our Bayesian game.

4.2.1 Consumers' type space

Consumer i's subjective valuation of the offer b is contained in his reservation price $R(i, b)$. However, the operators ignore individual consumer reservation prices. Using the survey results obtained for a representative panel of consumers (cf. Section 2), the operators can infer the reservation price distribution for each offer b of the global choice set, and on each market segment.

The associated complementary cumulative distribution function for offer b on segment f indicates the probability that consumers are ready to pay $P(b)$ as maximal price to buy the offer; this measure will be denoted: $\bar{F}_{b,f}(P(b))$ (cf. Fig. 1 for an illustration).

To simplify the analysis, we define for each offer b and on each market segment f, three categories containing maximum admissible prices for the offer b (cf. Fig. 2). It means that if the offer b retail price is above consumer i's maximum admissible price, he will not buy it.

- The first class ($\mathcal{L}_1 = [0; n_1[$) is defined for admissible prices between 0 and n_1 (the maximum admissible price being n_1 for this class); it is called

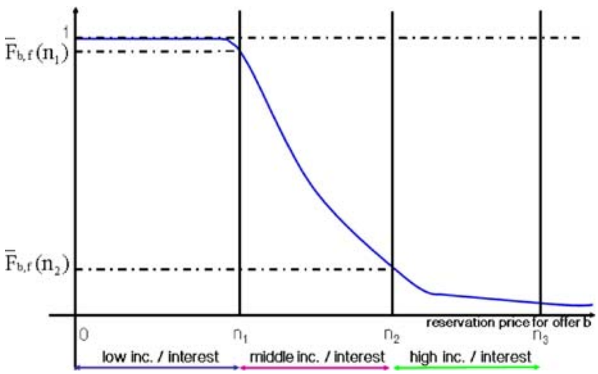

Fig. 2 Consumer's type for the offer b when he belongs to segment f

the *low income / interest category*, since consumers whose reservation price belongs to it are not ready to pay very much to buy the offer b, either because they are not interested in it, or because they lack money.

- The second class ($\mathcal{L}_2 = [n_1; n_2[$) contains admissible prices between n_1 and n_2 (the maximum admissible price being n_2 for this class); it is called the *middle income / interest category*.
- The third class ($\mathcal{L}_3 = [n_2; n_3]$) bounds are n_2 and n_3 (n_3 being the maximum admissible price for this class can be chosen arbitrarily large); it is the *high income / interest category*, since the consumers whose reservation price belongs to the interval $[n_2; n_3]$ valuate the offer b very much or are not money-sparing.

The parameters $n_1, \; n_2, \; n_3 \in \mathbb{R}_+$ are exogeneous to the game. If we consider a consumer i who belongs to segment f, then the complementary cumulative distribution function associated to this consumer's maximum admissible price for the offer b, is depicted in Fig. 2. The three income / interest classes have been defined exogeneously.

Actually, the consumer i type space T_i, coincides with these three classes, i.e., $T_i = \left\{ t_i = (t_i(1), t_i(2), ..., t_i(|\mathcal{B}|)) | t_i(b) \in \{\mathcal{L}_1; \mathcal{L}_2; \mathcal{L}_3\} \right\}$. It means that for each offer in the global choice set $\mathcal{B}$, consumer i's private information is his income / interest category. We note that a low income consumer might be very interested in an offer and hence, belongs to the category 3; while a high income consumer might be very reluctant to some technologies and belongs to category 1 for some services.

4.2.2 Consumers' action set

Each customer is characterized by the segment he belongs to, $f \in \mathcal{F}$. He can choose between all the offers in $\mathcal{B}$, and the possibility of not buying anything. Customer i's action space is therefore defined as

$$D_i = \left\{ d_i = \left(d_i(1), \; d_i(2), \; ..., \; d_i(|\mathcal{B}|) \right)^t \in \{0; 1\}^{|\mathcal{B}|} \right\},$$

where the $|\mathcal{B}|$-dimensional vector d_i has b-th coordinate $d_i(b) = 1$ if customer i buys offer b, and zero otherwise. Besides, x^t is the transpose algebraic operator of vector $x \in \mathbb{R}^n, \; n \geq 0$.

4.2.3 Consumers' utilities

We let $P_{\text{op}_1}(b_1)$ and $P_{\text{op}_2}(b_2)$ be Operator 1's and Operator 2's retail prices for offers belonging to their choice sets: $b_1 \in \mathcal{B}_{\text{op}_1}$ and $b_2 \in \mathcal{B}_{\text{op}_2}$. The utility of customer i, u_i, depends on consumer i's and operators' choices, as well as on the consumer i's type (i.e., his income / interest class).

$$u_i(d,t) = \left\{ \sum_{b \in \mathcal{B}_{\text{op}_1}} \alpha_{t_i(b)} (\max t_i(b) - P_{\text{op}_1}(b)) d_i(b) \mathbf{1}_{P_{\text{op}_1}(b) \in t_i(b)} + \sum_{b \in \mathcal{B}_{\text{op}_2}} \alpha_{t_i(b)} (\max t_i(b) - P_{\text{op}_2}(b)) d_i(b) \mathbf{1}_{P_{\text{op}_2}(b) \in t_i(b)} \right\} + \left\{ \sum_{b \in \mathcal{B}_{\text{op}_1}} \left(1 - \alpha_{t_i(b)}\right) \max t_i(b) d_i(b) \mathbf{1}_{P_{\text{op}_1}(b) \in t_i(b)} + \sum_{b \in \mathcal{B}_{\text{op}_2}} \left(1 - \alpha_{t_i(b)}\right) \max t_i(b) d_i(b) \mathbf{1}_{P_{\text{op}_2}(b) \in t_i(b)} \right\} \quad (10)$$

The parameter $\alpha_{t_i(b)} \in [0; 1]$ expresses consumer i's price sensitivity depending on his income / interest class for the offer b. For instance, on the class $\mathcal{L}_1$ (low income / interest), $\alpha_{\mathcal{L}_1}$ will be near one; whereas for $\mathcal{L}_3$ (high income / interest), $\alpha_{\mathcal{L}_3}$ will be near zero. Consumer i will not buy the offer b if his maximum admissible price is below the offer b selling price ($P_{\text{op}}(b)$). Besides, he can either buy the offer that he values the most (the one associated with the highest reservation price, i.e., part 2 in the consumer's utility described in Eq. 10), or choose to buy the one guaranteeing him the greatest benefit (part 1 in Eq. 10). If consumer i buys the offer generating the greatest benefit, we speak about cannibalization. Indeed, in such a case, the idea of earning a great benefit seems quite appealing to consumer i and makes him change his initial need to a rather similar, but cheaper one. For example, if the triple-play offer made of mobile / internet / tv proposed by Operator 2 is cheaper than Operator 1's simple offers of wired-phone and internet, even though the consumer does not need to buy a tv access, he will be tempted to buy it since the bundle is far cheaper than the simple offers.

4.2.4 Consumers' subjective beliefs

In the case of two competitive operators, as introduced in Section 2, two attributes of class $\mathcal{A}^1$, i.e., fully comparable: the attribute of trust in Operator 1's brand / loyalty, and the attribute of trust in Operator 2's brand / loyalty. These attributes measure the consumer confidence in each operator brand / loyalty, on each market segment. They contain the consumers' beliefs about whether the operator's cheating probability is high (low confidence), or low (high confidence). As described in Section 2, these attribute values have been estimated for all consumers. On every segment f, the distribution of the consumer trust in Operator 1 brand / loyalty is $\mathcal{T}_{1,f}$ and for the Operator 2, we have: $\mathcal{T}_{2,f}$.

Consumer i's type realization is independent of the others' and consumer i has no prior information about the other consumers' and operators' types; hence $p_i(t_1, t_2, ..., t_{i-1}, t_{i+1}, ..., t_N, t_{\text{op}_1}, t_{\text{op}_2}|t_i) = \prod_{j=1, j\neq i}^{N} p_i(t_j|t_i)\ p_i(t_{\text{op}_1}|t_i)\ p_i(t_{\text{op}_2}|t_i) = \prod_{j=1, j\neq i}^{N} p_i(t_j)\ p_i(t_{\text{op}_1})\ p_i(t_{\text{op}_2})$, where $p_i(t_j) = \frac{1}{3^{|\mathcal{B}|}}$ and $p_i(t_{\text{op}_j}) = \frac{\mathcal{T}_{j,f}(t_{\text{op}_j})}{\sum_{s\in\mathcal{C}} \mathcal{T}_{j,f}(s)}$, where $\mathcal{C}$ is the operators' type space.

4.3 Definition of the operators' game setting

We define the operators' type spaces, action sets, utilities and subjective beliefs to introduce our Bayesian game.

4.3.1 Operators' type spaces

The operator's type space coincides with the cheating probability set. It is the closed interval [0; 1] uniformly discretized[4] according to the stepsize $\frac{1}{10}$; it will be denoted by $\mathcal{C} = \left\{0; \frac{1}{10}; \frac{2}{10}; \frac{3}{10}; ...; 1\right\}$. The operator's cheating probability is *private*, i.e., known only by the operator. Each operator's real cost for every offer is a fraction of the offer selling price. Formally for any offer b belonging to the operator's choice set, we have the relation: $(1 - t_{\text{op}})P_{\text{op}}(b) = c_{\text{op}}(b) \Leftrightarrow 1 - t_{\text{op}} = \frac{c_{\text{op}}(b)}{P_{\text{op}}(b)}$, where $t_{\text{op}}(b) \in \mathcal{C}$ is the operator's type or cheating probability. The cheating probability means that the operator can cheat on his (her) quality, i.e., the offer retail price is not necessarily linked to the operator's investment in the offer quality. For instance, an operator using his (her) brand reputation might try to sell a low quality offer at a high price, pretending that it is an excellent quality product. In fact, the idea of introducing cheating probabilities comes from Akerlof's paper [1] where sellers commercialize good used cars and defective used cars (lemons). The buyer of a car does not know beforehand whether it is a good car or a lemon. In our article, the offer's true costs can be compared with the used car's quality. Besides, cost can be seen as a measure of quality since it represents the operator's investment level in technologies and QoS improvement. As in [1] and [9], the operators have some incentives to cheat about their real costs, i.e., the offer prices do not reveal their true manufacturing costs. Hence, our model deals with information asymmetry, since the operators know more about the offer's true costs than the buyers and, the operators ignore each other's cheating probabilities. As in [1], incentives exist for the seller to pass off a low quality offer to an expensive one sold as being of high quality. However, the introduction of brands might play the role of *guarantees* for the buyer. Since each buyer associates a level of trust with a brand name / loyalty. Indeed, a high cheating probability is not always profitable for the operator: if the operator chooses to invest few funds in the production of his offer and sells it at a high price, then the consumers will be disappointed and by word of mouth effect, no one will trust anymore in the operator's brand name / loyalty. Consequently, the consumers' reservation

[4] The discretization step can be fixed arbitrarily. Here, to fix the ideas, we choose $\frac{1}{10}$.

prices will decrease and the operator will be forced to lower his prices. Then, a compromise has to be found between brand reputation / loyalty preservation and the cheating probability. Due to these guarantees, our market model can be considered as an extension of the well-known lemon market.

The *profit* of the operator $(P_{\text{op}}(b) - (1 - t_{\text{op}})P_{\text{op}}(b))$ is then the difference between the selling price of the offer and this price coefficiented by his (her) type. This latter term represents the hidden offer cost.

The operator's types are independent since they have no prior information about the cost structure of their rival. However, the operators use the publicly known results of the survey (cf. Section 2) to infer some information about the other players' hidden information.

4.3.2 Operators' action sets

The actions of operators are represented by the prices at which they sell their offers. But those prices often do not reveal their real costs, which could include equipment maintenance, advertising / content investment [15], interconnexion contracts, etc. Each operator chooses a price for each of his (her) offers in a *finite discrete* set: $\mathcal{P} \subseteq \mathbb{R}_+$. If the operator selects market share expansion as his (her) marketing strategy, he (she) sells his (her) offers at the same price on each market segment; however, if he (she) prefers a marketing strategy based on multi-level price discrimination or segment targeting, the operator should envisage to sell the same offer at different prices on each market segment. For instance, the operator can propose some rebates on an offer to students (segment 1), or to those who do not have a computer at home and are unfamiliar with internet use (segment 4).

If the operator chooses the same price for each market segment, he gets an action space of the form

$$D_{\text{op}} := \left\{ d_{\text{op}} = \left(P_{\text{op}}(1), P_{\text{op}}(2), ..., P_{\text{op}}(|\mathcal{B}_{\text{op}}|) \right)^t \mid P_{\text{op}}(b) \in \mathcal{P}, \ \forall b \in \mathcal{B}_{\text{op}} \right\},$$

where the $|\mathcal{B}_{\text{op}}|$-dimensional vector d_{op} has b-th coordinate $d_{\text{op}}(b) = P_{\text{op}}(b) \in \mathcal{P}$, which contains the offer b selling price.

Another possibility for the operator is to discriminate between the customer segments and price the offers differently on every market segment, in which case we get

$$D_{\text{op}} = \left\{ d_{\text{op}} = \begin{pmatrix} P_{\text{op}}(1;1) & P_{\text{op}}(1;2) & \dots & P_{\text{op}}(1;F) \\ \vdots & \vdots & \vdots & \vdots \\ P_{\text{op}}(|\mathcal{B}_{\text{op}}|;1) & P_{\text{op}}(|\mathcal{B}_{\text{op}}|;2) & \dots & P_{\text{op}}(|\mathcal{B}_{\text{op}}|;F) \end{pmatrix} \right.$$
$$\left. \text{where } P_{\text{op}}(b;f) \in \mathcal{P}, \ \forall b \in \mathcal{B}_{\text{op}}, \forall f \in \mathcal{F} \right\},$$

$d_{\text{op}}(b, f) = P_{\text{op}}(b; f)$ is the offer b selling price on the market segment f.

4.3.3 Operators' utilities

Three marketing strategies are available for the operators: (i) market share expansion, (ii) segment targeting, (iii) multi-level price discrimination. Hence, his (her) utility can take different forms.

- If he (she) selects a market share expansion strategy, his (her) utility is the sum of the realized profits provided the consumers have bought his (her) offers

$$u_{\text{op}}(d,t) = \sum_{b\in\mathcal{B}_{\text{op}}} \Big(P_{\text{op}}(b) - c_{\text{op}}(b)\Big)\left(\sum_{i=1}^{N} d_i(b)\mathbf{1}_{P_{\text{op}}(b)\in t_i(b)}\right) \tag{11}$$

  where $\mathbf{1}_{P_{\text{op}}(b)\in t_i(b)} = 1$ if the offer b price $P_{\text{op}}(b)$ belongs to the consumer i admissible price interval $t_i(b)$; it is zero otherwise.
- If he (she) prefers a strategy based on market segment targeting, the operator should valuate each segment potential interest for his (her) offers and determine the retail prices which would provide the greatest profit on the most profitable segment

$$u_{\text{op}}(d,t) = \max_{f\in\mathcal{F}} \sum_{b\in\mathcal{B}_{\text{op}}} \Big(P_{\text{op}}(b;\, f) - c_{\text{op}}(b;\, f)\Big)\left(\sum_{i=1}^{N} d_i(b)\mathbf{1}_{i\in f\cap P_{\text{op}}(b;\,f)\in t_i(b)}\right) \tag{12}$$

  where $\mathbf{1}_{i\in f\cap P_{\text{op}}(b;\,f)\in t_i(b)} = 1$ if consumer i belongs to the targeted interval f and the offer b price $(P_{\text{op}}(b;\, f))$ belongs to the consumer i admissible price interval $(t_i(b))$; it vanishes otherwise.
- Finally, if the operator uses multi-level price discrimination, he (she) has to optimize the offer retail prices on each market segment

$$u_{\text{op}}(d,t) = \sum_{f\in\mathcal{F}} \sum_{b\in\mathcal{B}_{\text{op}}} \Big(P_{\text{op}}(b;\, f) - c_{\text{op}}(b;\, f)\Big)\left(\sum_{i=1}^{N} d_i(b)\mathbf{1}_{P_{\text{op}}(b;\,f)\in t_i(b)}\right). \tag{13}$$

We recall that $(1 - t_{\text{op}})P_{\text{op}}(b;\, f) = c_{\text{op}}(b;\, f)$, $\forall b \in \mathcal{B}_{\text{op}}$, $\forall f \in \mathcal{F}$ and $t_{\text{op}} \in \mathcal{C}$.

4.3.4 Operators' subjective beliefs

Operator j's subjective belief about the other actors' types can be defined as $p_{\text{op}_j}(t_1, t_2, ..., t_N, t_{\text{op}_k} | t_{\text{op}_j}) = \prod_{i=1}^{N} p_{\text{op}_j}(t_i | t_{\text{op}_j})\; p_{\text{op}_j}(t_{\text{op}_k} | t_{\text{op}_j})$ since conditionally to his (her) type t_{op_j}, the other operator's cheating probability and the consumers' types are independent from each other.

We start by defining Operator j's subjective belief about his (her) rival (Operator k). The Operator j has no prior information about his (her) rival's type; hence, he (she) uses the consumers' advices about his (her) rival's loyalty, i.e., the estimated densities characterizing the consumers' levels of trust in his (her) rival brand / loyalty on every market segment. Then, Operator j's subjective belief about Operator k's type (cheating probability) is:

$$p_{\mathrm{op}_j}(t_{\mathrm{op}_k}|t_{\mathrm{op}_j}) = p_{\mathrm{op}_j}(t_{\mathrm{op}_k}) \text{ since his (her) subjective belief about the other's type is independent of his (her) own type}$$
$$= \sum_{f \in \mathcal{F}} \frac{\mathcal{T}_{k,f}(t_{\mathrm{op}_k})}{\sum_{s \in \mathcal{C}} \mathcal{T}_{k,f}(s)}.$$

Second, we define Operator j's subjective belief about consumer i's type (i.e., interval of maximum admissible prices): $p_{\mathrm{op}_j}(t_i|t_{\mathrm{op}_j}) = p_{\mathrm{op}_j}(t_i) = \prod_{b \in \mathcal{B}_{\mathrm{op}_j}} p_{\mathrm{op}_j}(t_i(b))$ since Operator j's subjective belief about consumer i's type is independent of his (her) own type and consumer i's type associated to each offer b of the global choice set are independent from each other. Information about the consumers are stored in huge data-bases and the consumer segmentation has been realized off-line, i.e., each operator knows to which segment consumer i belongs. Then, for each offer b of Operator j's choice set, the operator's subjective belief is defined as:

$$p_{\mathrm{op}_j}(t_i(b)) = \begin{cases} 1 - \bar{F}_{b,f_i}(n_1) - \bar{F}_{b,f_i}(0) & \text{if } t_i(b) = \mathcal{L}_1 \\ 1 - \bar{F}_{b,f_i}(n_2) - \bar{F}_{b,f_i}(n_1) & \text{if } t_i(b) = \mathcal{L}_2 \\ 1 - \bar{F}_{b,f_i}(n_3) - \bar{F}_{b,f_i}(n_2) & \text{if } t_i(b) = \mathcal{L}_3 \end{cases}$$

provided consumer i belongs to the market segment $f_i \in \mathcal{F}$.

4.4 Approximation of Bayesian equilibria using simulation

In our model, the utility functions are quite complex, and differ for the various players. Besides, the action and type spaces are quite large.[5] Hence, it is difficult to compute analytically Bayesian equilibria. Tools are available to help solving finite extensive or normal form games [10]. However they are not designed to handle the general or more complex cases. An alternative to provide some solutions might be to introduce simulation in the game to approximate the equilibrium. Indeed, [10], Holenstein uses simulation to approximate the equilibria in an auction game, where the players' types and the action spaces of both players are continuous. Cai and Wurman [5] employ Monte Carlo methods in sequential auctions to sample the type space of the other agents and then solve numerically the resulting complete information game. Since in our model, the equilibria cannot be computed analytically,

[5]The global type space has a cardinal of $3^{N|\mathcal{B}|}\ |\mathcal{C}|^2$; the global action set in the worst case, i.e., if both operators price discriminate between the segments, is of size: $2^{N|\mathcal{B}|}\ |\mathcal{P}|^{F|\mathcal{B}_{\mathrm{op}_1}|}\ |\mathcal{P}|^{F|\mathcal{B}_{\mathrm{op}_2}|}$.

we play a dynamic game between the different actors. Each player responds to the others' actions, using a best response strategy. The idea is to sample the player's type space and then, the other players' strategy profiles being fixed, to determine the randomized strategy which maximizes the player's expected utility. Since the player's randomized action space is quite large, we use Monte-Carlo simulation to maximize his (her) expected utility. The algorithm is described in detail in Section 4.4.1. Our aim is to approach a Bayesian equilibrium. This approach is myopic since it does not consider future consequences of the players' actions, and uses Monte-Carlo methods, which enables us to introduce an underlying Markov chain. Finally, the introduction of a Markov chain enables us to develop convergence proofs in Section 4.4.2.

4.4.1 Description of the Best Response algorithm

We take the point of view of Operator 1, since he is the leader and tries to forecast the two-level game outputs to determine his expected revenue and the best marketing strategy for him. To approximate the Bayesian equilibria, we use his private information and subjective beliefs about the others' types. The algorithm that we introduce, has already been used by Holenstein [10] and Robert [19], and is made of two parts (Algorithm 1 and Algorithm 2). Algorithm 1 samples the players' type spaces. For each player, a type is drawn while the strategies of the others are fixed. The player's type being simulated and the others' strategies remaining fixed, it calls Algorithm 2 to determine the player's best response, i.e., the randomized pricing strategy or purchasing process, maximizing his (her) expected utility. Algorithm 1 is iterated until it reaches a convergence criterion. A temperature parameter is introduced and set to a predefined value: $Temp(0) < +\infty$. It is updated at each iteration of Algorithm 1 according to a decreasing function, e.g., $Temp(t+1) = \exp(-Temp(t))$. It is used in Algorithm 2 to define the Acceptance-Rejection (AR) rule and hence, avoids getting stuck in local extrema defining the player's best response to the others [19, 22]. This rule is issued from the well-known simulated annealing method [8].

- If player i is consumer i;
  the AR rule means that if $\sigma_i^*(.|t_i)$ improves player i's utility, i.e., $U_i^*(\sigma^*|t_i) \geq U_i^*(\sigma^{(k)}|t_i)$ then the Algorithm accepts $\sigma_i^*(.|t_i)$ with probability 1, otherwise it is accepted with probability $\min\left\{1; \exp\left\{-\left[\frac{(U_i^\star(\sigma^{(k)}|t_i) - U_i^\star(\sigma^\star|t_i))}{\text{Temp}(t)}\right]\right\}\right\}$. At time instant t in Algorithm 1, consumer i's best response obtained as output of Algorithm 2 is updated to $\sigma_i^{(t)}(.|t_i) = \sigma_i^{(\text{Iter}_{\max})}(.|t_i)$.
- If player i is Operator i;
  the AR rule means that if $\sigma_{\text{op}_i}^*(.|t_{\text{op}_i})$ improves player i's utility, i.e., $U_{\text{op}_i}^*(\sigma^*|t_{\text{op}_i}) \geq U_{\text{op}_i}^*(\sigma^{(k)}|t_{\text{op}_i})$ then the Algorithm accepts $\sigma_{\text{op}_i}^*(.|t_{\text{op}_i})$ with probability 1, otherwise it is accepted with probability $\min\left\{1; \exp\left\{-\left[\frac{(U_{\text{op}_i}^\star(\sigma^{(k)}|t_{\text{op}_i}) - U_{\text{op}_i}^\star(\sigma^\star|t_{\text{op}_i}))}{\text{Temp}(t)}\right]\right\}\right\}$. At time instant t in Algorithm

1, Operator i's best response obtained as output of Algorithm 2 is updated to $\sigma^{(t)}_{\mathrm{op}_i}(.|t_{\mathrm{op}_i}) = \sigma^{(\mathrm{Iter}_{\max})}_{\mathrm{op}_i}(.|t_{\mathrm{op}_i})$.

Algorithm 1 Bayesian equilibrium approximation

Set time $t = 0$, and intialize the temperature: Temp(0) = 1.
The players' strategies are initialized according to a uniform distribution on the action spaces:
$\sigma^{(0)}_i(d_i|t_i) = \frac{1}{|D_i|},\ \forall i \in \mathcal{N}$ and $\sigma^{(0)}_{\mathrm{op}_i}(d_{\mathrm{op}_i}|t_{\mathrm{op}_i}) = \frac{1}{|D_{\mathrm{op}_i}|},\ \forall i = 1, 2$.
While the norm of the players' conditional strategies changes by more than a fixed constant ε,
for each player i (either consumer or operator),

- if player i is a customer belonging to segment f_i, sample his type according to the multinomial density $t_i(b) \sim \mathcal{M}\Big(1; (1 - \bar{F}_{b,f_i}(n_1) - \bar{F}_{b,f_i}(0)), (1 - \bar{F}_{b,f_i}(n_2) - \bar{F}_{b,f_i}(n_1)), (1 - \bar{F}_{b,f_i}(n_3) - \bar{F}_{b,f_i}(n_2))\Big)$; it means that the probability that i belongs to the category 1 for offer b is $1 - \bar{F}_{b,f_i}(n_1) - \bar{F}_{b,f_i}(0)$, to the category 2, it is $1 - \bar{F}_{b,f_i}(n_2) - \bar{F}_{b,f_i}(n_1)$ and to the category 3: $1 - \bar{F}_{b,f_i}(n_3) - \bar{F}_{b,f_i}(n_2)$.
- If player i is Operator i, then his type is sampled according to the multinomial density $\mathcal{M}\Big(1; \sum_{f\in\mathcal{F}} \frac{\mathcal{T}_{i,f}(0)}{\sum_{s\in\mathcal{C}} \mathcal{T}_{i,f}(s)}, \dots, \sum_{f\in\mathcal{F}} \frac{\mathcal{T}_{i,f}(1)}{\sum_{s\in\mathcal{C}} \mathcal{T}_{i,f}(s)}\Big)$; it means that the probability for Operator i's cheating probability to be $\tilde{s} \in \mathcal{C}$ is $\sum_{f\in\mathcal{F}} \frac{\mathcal{T}_{i,f}(\tilde{s})}{\sum_{s\in\mathcal{C}} \mathcal{T}_{i,f}(s)}$.
- Then, determine the best response given the other players' strategies (via Algorithm 2).
- Update player i's strategy profile.

End
$t = t + 1$, update the temperature which decreases according to a pre-defined law, giving: Temp$(t + 1)$.
End

Algorithm 1 stops when the norm[6] of the players' conditional strategies does not change by more than a fixed constant, i.e.:

$$\begin{aligned} \|\sigma^{(t+1)}_i(.|t_i) - \sigma^{(t)}_i(.|t_i)\| &\leq \varepsilon,\ \forall t_i \in T_i,\ \forall i \in \mathcal{N},\ \varepsilon \\ &\geq 0 \text{ if player } i \text{ coincides with consumer } i \\ \|\sigma^{(t+1)}_{\mathrm{op}_i}(.|t_{\mathrm{op}_i}) - \sigma^{(t)}_{\mathrm{op}_i}(.|t_{\mathrm{op}_i})\| &\leq \varepsilon,\ \forall t_{\mathrm{op}_i} \in T_{\mathrm{op}_i},\ i = 1, 2,\ \varepsilon \\ &\geq 0 \text{ if player } i \text{ is Operator } i \end{aligned}$$

where $\sigma^{(t)}_i(.|t_i) \in \Delta(D_i)$ and $\sigma^{(t)}_{\mathrm{op}_i}(.|t_{\mathrm{op}_i}) \in \Delta(D_{\mathrm{op}})$ are the randomized purchasing strategy and the randomized pricing strategy vectors of consumer i and Operator i, conditionally on his type t_i (resp. t_{op_i}).

[6]We use the L^2-norm defined as $\|x\| = \sum_{i=1}^{n} |x_i|^2$ but other vectorial norms might be used.

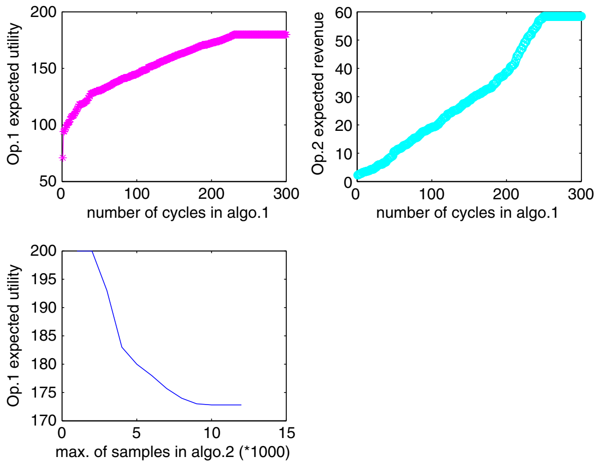

Fig. 3 Convergence of the Bayesian equilibrium approximation algorithm (Algorithm 1) and Operator 1's expected utility as a function of the sample size in Algorithm 2

Algorithm 2 is run for a maximum number of steps called Iter_{max}. It is defined a priori but we check in Fig. 3 that a higher size for the maximum number of samples drawn in Algorithm 2, increases Algorithm 1 convergence to the equilibrium strategies.

4.4.2 Proof of Algorithm 1 convergence

In this proof, we assume that player i denotes indifferently consumer i or Operator i. At time step t in Algorithm 1, the output of Algorithm 2 is kept in memory: $\sigma_i^{(t)}(.|t_i) := \sigma_i^{(\text{Iter}_{\text{max}})}(.|t_i)$.

Player i's state space is made of all the probability distributions on the action space D_i (D_{op} for the operator): $\Delta(D_i)$ (resp. $\Delta(D_{\text{op}})$). To prove the Algorithm 1 convergence, we have to assume that player i's state space is finite: $\mathcal{S}_i \subseteq \Delta(D_i)$ and $|\mathcal{S}_i| < +\infty$ (resp. $\mathcal{S}_{\text{op}} \subseteq \Delta(D_{\text{op}})$ and $|\mathcal{S}_{\text{op}_i}| < +\infty$).

In Algorithm 2 to compute the player's best response, the instrumental density q, is a multidimensional normal density which has been discretized on player i's state space $\mathcal{S}_i$ or $\mathcal{S}_{\text{op}}$ if player i is an operator, and centered in the last accepted randomized-strategy profile for the considered player. q is used in Algorithm 2 to generate a new conditional randomized strategy for player i, and its parameters are updated at each iteration step of Algorithm 2. Consequently, in Algorithm 2, for each player i, conditionally to his type, the others' strategies being fixed, the process is a Markov chain.

Theorem 1 *We suppose that* Iter_{max} *is chosen sufficiently large to guarantee that Algorithm 2 solutions coincide with the players' best responses. Assuming the finite dimensionality of the strategy spaces* $\mathcal{S}_i$ *and* $\mathcal{S}_{op_i}$*, and the irreducibility of the conditional chains for each player, the algorithm converges to a Bayesian equilibrium.*

Algorithm 2 Determination of the best response

Initialize player i's randomized strategy profile: $\sigma_i^{(0)}(.|t_i)$ (or $\sigma_{\text{op}_i}^{(0)}(.|t_{\text{op}_i})$), while the $N+1$ other players' strategies remain unchanged. If $t \geq 2$ in Algorithm 1 then player i's randomized strategy is fixed to the optimal strategy obtained in the last step of Algorithm 1: $\sigma_i^{(0)}(.|t_i) := \sigma_i^{(t-1)}(.|t_i)$ if player i coincides with consumer i and $\sigma_{\text{op}_i}^{(0)}(.|t_{\text{op}_i}) := \sigma_{\text{op}_i}^{(t-1)}(.|t_{\text{op}_i})$ if player i is Operator i. Compute the associated expected utility

- if the player is consumer i

$$U_i^{(0)}(\sigma^{(0)}|t_i) = \sum_{(t_{-i},t_{\text{op}_1},t_{\text{op}_2})\in T_{-i}\times T_{\text{op}_1}\times T_{\text{op}_2}} p_i\big(t_{-i}, t_{\text{op}_1}, t_{\text{op}_2}|t_i\big) \sum_{d\in D} \Big\{ \prod_{j\in\mathcal{N}_{-i}} \sigma_j^{(t)}\big(d_j|t_j\big) \times\sigma_{\text{op}_1}^{(t)}\big(d_{\text{op}_1}|t_{\text{op}_1}\big)\sigma_{\text{op}_2}^{(t)}\big(d_{\text{op}_2}|t_{\text{op}_2}\big)\Big\}\sigma_i^{(0)}(d_i|t_i)\, u_i(d,t),$$

  where, t_i is player i's type, drawn according to the initial marginal distribution, in Algorithm 1.

- if the player is Operator i

$$U_{\text{op}_i}^{(0)}\big(\sigma^{(0)}|t_{\text{op}_i}\big) = \sum_{t\in T_1\times\ldots\times T_N, t_{\text{op}_k}\in T_{\text{op}_k}} p_{\text{op}_i}\big(t_1,\ldots,t_N,t_{\text{op}_k}|t_{\text{op}_i}\big) \times\sum_{d\in D}\Big\{\prod_{j\in\mathcal{N}}\sigma_j^{(t)}\big(d_j|t_j\big)\sigma_{\text{op}_k}^{(t)}\big(d_{\text{op}_k}|t_{\text{op}_k}\big)\Big\}\sigma_{\text{op}_i}^{(0)}\big(d_{\text{op}_i}|t_{\text{op}_i}\big)u_{\text{op}_i}(d,t)$$

  where Operator i's type t_{op_i} has been generated in Algorithm 1.

For k from 0 *to* $\text{Iter}_{\max}$,
sample $z \sim \mathcal{U}_{[0;1]}$,
and sample $\sigma_i^{\star}(.|t_i) \sim q\left(\sigma_i^{\star}(.|t_i) \mid \sigma_i^{(k)}(.|t_i)\right)$ if player i coincides with consumer i and $\sigma_{\text{op}_i}^{\star}(.|t_{\text{op}_i}) \sim q\left(\sigma_{\text{op}_i}^{\star}(.|t_{\text{op}_i}) \mid \sigma_{\text{op}_i}^{(k)}(.|t_{\text{op}_i})\right)$ if player is Operator i.
At each step, compute the expected utility: $U_i^{\star}(\sigma^{\star}|t_i)$ (or $U_{\text{op}_i}^{\star}(\sigma^{*}|t_{\text{op}_i})$) taking into account the other players' strategies which have been updated at the t-th iteration in Algorithm 1. The AR rule is:

- if player i coincides with consumer i
  If $z < \min\left\{1; \exp\left\{-\left[\frac{\left(U_i^{\star}\left(\sigma^{(k)}|t_i\right)-U_i^{\star}(\sigma^{\star}|t_i)\right)}{\text{Temp}(t)}\right]\right\}\right\}$,
  then:
  $\sigma_i^{(k+1)}(.|t_i) = \sigma_i^{\star}(.|t_i)$ and $U_i^{(k+1)}(.|t_i) = U_i^{\star}(\sigma^{\star}|t_i)$.
  Otherwise, $\sigma_i^{(k+1)}(.|t_i) = \sigma_i^{(k)}(.|t_i)$ and $U_i^{(k+1)}(.|t_i) = U_i^{(k)}\big(\sigma^{(k)}|t_i\big)$.
- if player i is Operator i
  If $z < \min\left\{1; \exp\left(-\left\{\frac{\left(U_{\text{op}_i}^{*}\left(\sigma^{(k)}|t_{\text{op}_i}\right)-U_{\text{op}_i}^{*}\left(\sigma^{*}|t_{\text{op}_i}\right)\right)}{\text{Temp}(t)}\right]\right]\right\}$,
  then:
  $\sigma_{\text{op}_i}^{(k+1)}\big(.|t_{\text{op}_i}\big) = \sigma_{\text{op}_i}^{\star}\big(.|t_{\text{op}_i}\big)$ and $U_{\text{op}_i}^{(k+1)}\big(.|t_{\text{op}_i}\big) = U_{\text{op}_i}^{\star}\big(\sigma^{\star}|t_{\text{op}_i}\big)$.
  Otherwise, $\sigma_{\text{op}_i}^{(k+1)}\big(.|t_{\text{op}_i}\big) = \sigma_{\text{op}_i}^{(k)}\big(.|t_{\text{op}_i}\big)$ and $U_{\text{op}_i}^{(k+1)}\big(.|t_{\text{op}_i}\big) = U_{\text{op}_i}^{(k)}\big(\sigma^{(k)}|t_{\text{op}_i}\big)$.
- *End*

End

Proof of Theorem 1 It can be found in detail in Appendix □

Numerical illustration We start by defining the model's exogeneous parameters. We let $N = 1000$ be the number of consumers, the three category of interest / income upper bounds are: $n_1 = 10$, $n_2 = 30$, $n_3 = 100$. We choose $\alpha_{\mathcal{L}_1} = 0.9$, $\alpha_{\mathcal{L}_2} = 0.5$, and $\alpha_{\mathcal{L}_3} = 0.3$. Operator 1 is integrated and already well-known on the market. He uses multi-level price discrimination as marketing strategy and his cheating probability is $\frac{1}{10}$. Operator 2 is a small new entrant. Her marketing approach is based on segment targeting and her cheating probability is $\frac{7}{10}$.

At the top of Fig. 3, we observe the convergence of both operators' expected utilities conditionally to their types, i.e., $U^*_{\text{op}_1}(\sigma|\frac{1}{10}) = 173.5$ while $U^*_{\text{op}_2}(\sigma|\frac{7}{10}) = 60$. It shows the convergence of the operators' expected utilities conditional to their types, in around 300 iterations of Algorithm 1.

At the bottom of Fig. 3, we have represented Operator 1's expected utility (his type being fixed to $\frac{1}{10}$) as a function of the sample size ($\text{Iter}_{\max}$) used in Algorithm 2, with a fixed number of 300 cycles in Algorithm 1. The convergence rate of Algorithm 1 increases with the number of samples used in Algorithm 2. With $\text{Iter}_{\max} \geq 8000$, the Bayesian equilibrium is reached for 300 iterations in Algorithm 1. Indeed we have more chances to determine the optimal randomized pricing strategy if the random walk describes a larger state space. Besides, the acceptance rate, i.e., the fraction of proposed samples that is accepted in a window of the last $\text{Iter}_{\max}$ samples, at a fixed temperature, is of around 23% for all the players.

To determine the complexity of Algorithm 1, we first note that it contains $N+2$ cycles to cover all the players and for each player, there are $\text{Iter}_{\max} + 1$ iterations to run in Algorithm 2. The problem now, is to determine when Algorithm 1's stopping criterion on the conditional strategies is reached.

For each player i, in each iteration k of Algorithm 2 and for t large enough, the probability that the new strategy vector be accepted converges: $\mathbf{P}[\sigma_i^*(.|t_i) \text{ accepted}] \to \exp\big(-\frac{1}{Temp(t)}\big) = \exp\big(-\frac{1}{\exp(-Temp(t-1))}\big)$, since $U_i(\sigma^{(k)}(.|t_i) - U_i(\sigma^*|t_i)$ is bounded and the temperature updating rule is $Temp(t) = \exp(-Temp(t-1))$. In each cycle of Algorithm 1, a type t_i is drawn for each player i. If we assume that either the algorithm has stopped, or $\sigma_i^*(.|t_i)$ has been rejected; we have the following relation: $\prod_{i=1}^{N+2} \mathbf{P}[|\sigma_i^{\text{Iter}_{\max}}(.|t_i) - \sigma_i^{(t-1)}(.|t_i)| < \varepsilon] > \prod_{i=1}^{N+2} \mathbf{P}[\sigma_i^*(.|t_i) \text{ rejected}]$.

Hence, $\mathbf{P}[\text{Algo.1 stops}] > \big(1 - \exp\big(-\frac{1}{\exp(-Temp(t-1))}\big)\big)^{N+2}$. Practically, if $Temp(t-1) = \log\big(-\log\big(1 - \exp\big(\frac{\log 0.95}{N+2}\big)\big)\big)$ then the probability that Algorithm 1 stops is larger than 0.95. For instance, in our numerical illustration $\log\big(-\log\big(1 - \exp\big(\frac{\log 0.95}{N+2}\big)\big)\big) = 2.3$. Consequently, the algorithm complexity is of the form $O(r(N+2)\text{Iter}_{\max})$ where r is the time needed to decrease the temperature up to $\log\big(-\log\big(1 - \exp\big(\frac{\log 0.95}{N+2}\big)\big)\big)$.

In our numerical illustration, the computational cost, i.e., the average time needed to reach the equilibria, is around 3 hours.

5 First level: computation of the providers' marketing strategies

The second level of the game has been run for each combination of marketing strategies: s_{op_1} for Operator 1 and s_{op_2} for Operator 2. As outputs, we have obtained the optimal randomized strategies $\sigma_{s_{\mathrm{op}_1},s_{\mathrm{op}_2}}$ and the operators's expected utilities conditional to each possible type, i.e. $U_{\mathrm{op}_j}\left(\sigma_{s_{\mathrm{op}_1},s_{\mathrm{op}_2}}|t_{\mathrm{op}_j}\right),\ \forall t_{\mathrm{op}_j}\in\mathcal{C},\ \ j=1,2$.

Now, each operator aims at defining his (her) most profitable marketing strategy between: (i) market share expansion, (ii) segment targeting and (iii) multi-level price discrimination. Operator 1 is integrated, i.e., he is initially more powerful than Operator 2 who is a new entrant on the telecommunication market. The game can then be identified with a Stackelberg one under uncertainties; Operator 1 starts by fixing his marketing strategy, then Operator 2 arrives on the market and has to determine which marketing strategy to choose under uncertainty. Operator 1 can be seen as a *leader*, while Operator 2 is a *follower*.

Once more, we proceed by backward induction.

- First, Operator 2 should solve the following optimization problem for each of her type:

$$\max_{s_{\mathrm{op}_2}\in\{\text{Exp.,Target,Discr.}\}}\sum_{t_{\mathrm{op}_1}\in\mathcal{C}}p_{\mathrm{op}_2}\left(t_{\mathrm{op}_1}|t_{\mathrm{op}_2}\right)\sum_{s_{\mathrm{op}_1}}\sigma_{\mathrm{op}_1}\left(s_{\mathrm{op}_1}|t_{\mathrm{op}_1}\right)U_{\mathrm{op}_2}\left(\sigma_{s_{\mathrm{op}_1},s_{\mathrm{op}_2}}|t_{\mathrm{op}_2}\right) \tag{14}$$

  where $\sigma_{\mathrm{op}_1}(s_{\mathrm{op}_1}|t_{\mathrm{op}_1})\in\{0;1\}$ and $\sum_{s_{\mathrm{op}_1}\in\{\text{Exp.,Target,Discr.}\}}\sigma_{\mathrm{op}_1}(s_{\mathrm{op}_1}|t_{\mathrm{op}_1})=1$, i.e., each operator has to choose a pure marketing strategy.
  As output, Eq. 14 advises Operator 2 about which marketing strategy to use as a function of her type, i.e., we get

$$s^*_{\mathrm{op}_2}:\mathcal{C}\to\left\{\text{Exp., Target, Discr.}\right\}$$
$$t_{\mathrm{op}_2}\mapsto s^*_{\mathrm{op}_2}\left(t_{\mathrm{op}_2}\right)$$

- Then, we assume that Operator 2's marketing strategy ($s^*_{\mathrm{op}_2}$) is fixed.[7] Operator 1, as the leader, should solve

$$\max_{s_{\mathrm{op}_1}\in\{\text{Exp.,Target,Discr.}\}}U_{\mathrm{op}_1}\left(\sigma_{s_{\mathrm{op}_1},s^*_{\mathrm{op}_2}\left(t_{\mathrm{op}_2}\right)}|t_{\mathrm{op}_1}\right) \tag{15}$$

[7] We note that it might be impossible for Operator 1 to infer Operator 2's cheating probability from the sole observation of her marketing strategy. For instance, if $\mathcal{C}=\{0.1;0.5;0.8\}$ and if Operator 2's randomized marketing strategy obtained as output of level 1 is the following: $\sigma_{\mathrm{op}_2}=\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix}$ then if Operator 2 chooses the first marketing strategy, Operator 1 cannot infer her hidden information. However, if $\sigma_{\mathrm{op}_2}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ then Operator 1 should easily guess Operator 2's type.

Table 1 Bayesian game outputs, $t_{\text{op}_1} = \frac{1}{4}$

Op.1	Op.2		
	Expansion	Target	Discriminate
Expansion	(80, 180)	(50, 70)	(600, 200)
Target	(5, 180)	(580, 570)	(350, 380)
Discrimate	(8, 400)	(10, 300)	(10, 390)

The optimal marketing strategy for Operator 1 is denoted: $s^*_{\text{op}_1}(t_{\text{op}_1}, t_{\text{op}_2})$.

Practically, for each couple of cheating probabilities $(t_{\text{op}_1}, t_{\text{op}_2})$, once Operator 1 has revealed his marketing strategy $s^*_{\text{op}_1}(t_{\text{op}_1}, t_{\text{op}_2})$, Operator 2 should choose the marketing strategy maximizing her expected utility; she has to consider a system of three equations, one for each available marketing strategy.

1. If Operator 2 selects a market share expansion strategy ($s_{\text{op}_2} = \text{Exp.}$),

$$\sum_{t_{\text{op}_1} \in \mathcal{C}} p_{\text{op}_2}\left(t_{\text{op}_1} | t_{\text{op}_2}\right) U_{\text{op}_2}\left(\sigma_{s^*_{\text{op}_1}, \text{Exp.}} | t_{\text{op}_2}\right) \tag{16}$$

with $s^*_{\text{op}_1} = \arg\max_{s_{\text{op}_1}} U_{\text{op}_1}(\sigma_{s_{\text{op}_1}, \text{Exp}} | t_{\text{op}_1})$;
2. If Operator 2 prefers a multi-level price discrimination strategy ($s_{\text{op}_2} =$ Discr.),

$$\sum_{t_{\text{op}_1} \in \mathcal{C}} p_{\text{op}_2}\left(t_{\text{op}_1} | t_{\text{op}_2}\right) U_{\text{op}_2}\left(\sigma_{s^*_{\text{op}_1}, \text{Discr.}} | t_{\text{op}_2}\right) \tag{17}$$

with $s^*_{\text{op}_1} = \arg\max_{s_{\text{op}_1}} U_{\text{op}_1}\left(\sigma_{s_{\text{op}_1}, \text{Discr}} | t_{\text{op}_1}\right)$;
3. If Operator 2 uses a segment targeting approach ($s_{\text{op}_2} = \text{Target}$),

$$\sum_{t_{\text{op}_1} \in \mathcal{C}} p_{\text{op}_2}\left(t_{\text{op}_1} | t_{\text{op}_2}\right) U_{\text{op}_2}\left(\sigma_{s^*_{\text{op}_1}, \text{Target}} | t_{\text{op}_2}\right) \tag{18}$$

with $s^*_{\text{op}_1} = \arg\max_{s_{\text{op}_1}} U_{\text{op}_1}\left(\sigma_{s_{\text{op}_1}, \text{Target}} | t_{\text{op}_1}\right)$;

Numerical illustrations In a first part, we restrain the cheating probability set to the following one: $\mathcal{C} = \left\{\frac{1}{4}; \frac{1}{2}; \frac{3}{4}; 1\right\}$. Since Operator 2 is new on the market; she is forced to choose $\frac{3}{4}$ as cheating probability; it provides her a guarantee to make profits and to survive against Operator 1. Each operator chooses a unique strategy between the three available marketing strategies: market share expansion, segment targeting or multi-level price discrimination.

Table 2 Bayesian game outputs, $t_{\text{op}_1} = \frac{1}{2}$

Op.1	Op.2		
	Expansion	Target	Discriminate
Expansion	(80, 190)	(50, 5)	(600, 70)
Target	(5, 80)	(580, 10)	(350, 90)
Discrimate	(8, 3)	(10, 200)	(10, 60)

Table 3 Bayesian game outputs, $t_{\text{op}_1} = \frac{3}{4}$

Op.1	Op.2		
	Expansion	Target	Discriminate
Expansion	(80, 90)	(50, 5)	(600, 30)
Target	(5, 70)	(580, 5)	(350, 3)
Discrimate	(8, 3)	(10, 50)	(10, 5)

In Tables 1, 2, 3 and 4 we have computed the operators' expected utilities, Operator 2's type being fixed to $\frac{3}{4}$ while Operator 1's type takes all the possible values in her type space, $\mathcal{C}$. Each element of the payoff matrix contains Operator 1's expected utility and Operator 2's expected utility conditional to her true type.

If we look at the Tables 1, 2, 3 and 4 and, by application of the set of Eqs. 16, 17 and 18, we get the following successive expressions.

- If Operator 2 chooses an Expansion strategy, then Eq. 16 becomes

$$p_{\text{op}_2}\left(\frac{1}{4}\middle|\frac{3}{4}\right)180 + p_{\text{op}_2}\left(\frac{1}{2}\middle|\frac{3}{4}\right)190 + p_{\text{op}_2}\left(\frac{3}{4}\middle|\frac{3}{4}\right)90 + p_{\text{op}_2}\left(1\middle|\frac{3}{4}\right)20;$$

- if Operator 2 selects a multi-level price discrimination strategy, she gets

$$p_{\text{op}_2}\left(\frac{1}{4}\middle|\frac{3}{4}\right)200 + p_{\text{op}_2}\left(\frac{1}{2}\middle|\frac{3}{4}\right)70 + p_{\text{op}_2}\left(\frac{3}{4}\middle|\frac{3}{4}\right)30 + p_{\text{op}_2}\left(1\middle|\frac{3}{4}\right)2;$$

- if segment targeting is selected as marketing strategy for Operator 2, we have

$$p_{\text{op}_2}\left(\frac{1}{4}\middle|\frac{3}{4}\right)570 + p_{\text{op}_2}\left(\frac{1}{2}\middle|\frac{3}{4}\right)10 + p_{\text{op}_2}\left(\frac{3}{4}\middle|\frac{3}{4}\right)5 + p_{\text{op}_2}\left(1\middle|\frac{1}{2}\right)20.$$

Table 4 Bayesian game outputs, $t_{\text{op}_1} = 1$

Op.1	Op.2		
	Expansion	Target	Discriminate
Expansion	(80, 20)	(50, 5)	(600, 2)
Target	(5, 3)	(580, 2)	(350, 2)
Discrimate	(8, 1)	(10, 20)	(10, 3)

Using these set of relations, we infer that the multi-level price discrimination strategy is always worse than the others for Operator 2. Besides, the target strategy is prefered to the expansion one if and only if

$$\begin{aligned} &372x - 162y \geq 67z + 18 \\ &x = p_{\text{op}_2}\left(\frac{1}{4}\middle|\frac{3}{4}\right) \\ &y = p_{\text{op}_2}\left(\frac{1}{2}\middle|\frac{3}{4}\right) \\ &z = p_{\text{op}_2}\left(\frac{3}{4}\middle|\frac{3}{4}\right) \\ &x + y + z \leq 1 \\ &x, y, y \geq 0 \end{aligned}$$

and, in the reverse case, the expansion strategy is better than the target one.

In a second part, we proceed the same way as described in Eqs. 16, 17 and 18. However, Operator 1's cheating probability spans a large number of values in the unit interval [0; 1] (idem for Operator 2). We aim at determining the most profitable marketing strategy to use for Operator 1 in each possible couple of cheating probabilities (i.e., the one maximizing his expected utility conditionally to his type).

Practically, the cheating probability usually depends on the operator size / power in the relation Operator 1-Operator 2: if the cheating probability is small, the operator has a big structure, i.e., he is powerful; if the cheating probability is of middle-value, the operator is not too powerful in the relation Operator 1-Operator 2 (he (she) does not cheat too much because he (she) already has some money issued from other business areas, for instance tv access, beverage or airline tickets sellings; but needs to expand his market share / increase his (her) revenue); finally if the cheating probability is high, the operator needs to make big profits as soon as possible, i.e., it may be a small new entrant, or a virtual network operator lacking funds.

Using Fig. 4, we infer that

- when both operators are small or equally powerful, they would rather use a market share expansion strategy to extend rapidly their market shares and try to *survive* on the market;
- when the considered operator is big, he (she) systematically uses a multi-level price discrimination strategy. Indeed, since the other operator does not represent a threat, he (she) tries to *add value* to his (her) brand by differentiating between the segments;
- when the considered operator is of middle-size while the other is a threat for him (her), he (she) would rather use a segment targeting strategy. To survive on the market, the operator wants to differentiate from his (her) rival by *capturing market niches*.

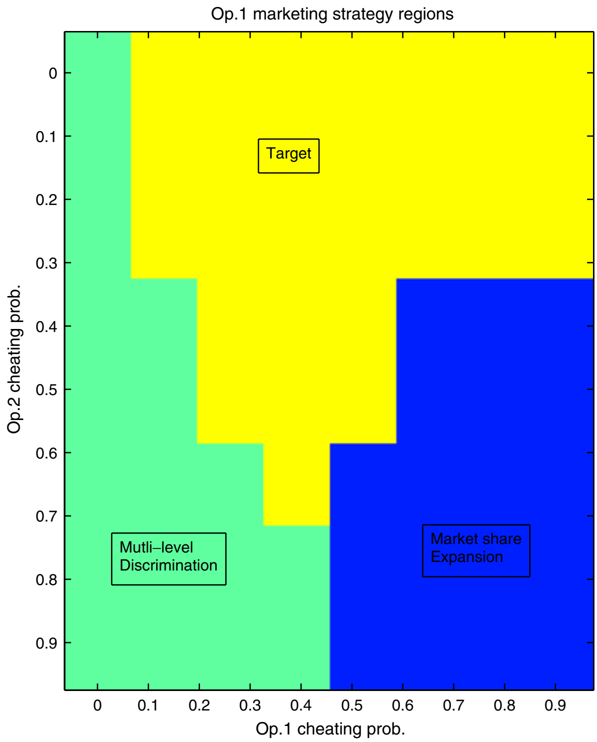

Fig. 4 Most profitable marketing strategy for Operator 1 as functions of both operators' cheating probabilities

6 Conclusions

In this paper, we have presented a way to model customer preferences using discrete choice theory for telecommunication operators willing to propose offers and in particular, bundles, on the market. Using information taken from a panel of customers, we have developed in Section 2 and in [14] an approach based on data augmentation and Bayesian networks to help an operator to price optimally his offers, in order to maximize his revenue. But this study was in the case where no competition was carefully modeled among operators. For this reason, we have studied in this part the case where operators propose substitutable offers and compete for customers. Customers participate into the game by selecting the provider the most relevant to them, using the previously studied preference model. A two-level game is introduced. Due to the uncertainties involved, the second level is modeled as a Bayesian game, and the resulting equilibrium analyzed. The first level of the game enables the operator to identify the marketing strategy maximizing his (her) expected revenue under uncertainties about his (her) rivals' behaviours and private information.

The paper might be extended by introducing alliances between the operators, e.g., one of the operators might lack network facilities and become a virtual operator on the network of his rival. Then, some contracts could be introduced between the operators and the utilities modified accordingly. Another difficulty might be to define a dynamic model dealing with consumer preference evolution and learning the rival's private information by observing his (her) strategy and revenue evolution. To tackle such approaches, stochastic programming has been employed by Audestadt et al. [2], or Stackelberg game theory (cf. Leleno and Sherali [16]) with partial feedback (cf. Cesa-Bianchi and Lugosi [6]).

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Appendix

Proof of Theorem 1 The proof is inspired from [22], who proves the convergence of a simulated annealing algorithm for multiobjective optimization problems. Here, we do not use a classical simulated annealing algorithm and the incomplete information framework adds some difficulties to the convergence proof. Player i indifferently refers to consumer i or Operator i.

(1) *Definition of Algorithm 2's underlying Markov chain transition probabilities.*

Let $\mathcal{B}^\star$ be the set of Bayesian equilibria associated with our game. We consider player i. At the temperature Temp(t), we suppose that the type $t_i \in T_i$ has been drawn. In Algorithm 2, conditionally to the type t_i, we choose an Acceptance-Rejection (AR) rule similar to simulated annealing algorithms (cf. [19]):

$$A_i(s_1, s_2) \;:=\; \min\left\{1; \exp\left(-\left[\frac{U_i^\star(s_1|t_i) - U_i^\star(s_2|t_i)}{\text{Temp}^{(t)}}\right]\right)\right\}\;, \;\forall s_1,\; s_2 \in \mathcal{S}_i. \quad (19)$$

Let a^+, be the positive part of the real number a:

$$a^+ \;=\; \begin{cases} a, & \text{if } a > 0,\\ 0, & \text{otherwise.} \end{cases}$$

Using this definition, Eq. 19 can be simplified to give

$$A_i(s_1, s_2) \;:=\; \exp\left\{-\left[\frac{U_i^\star(s_1|t_i) - U_i^\star(s_2|t_i)}{\text{Temp}^{(t)}}\right]^+\right\}\;, \; s_1,\; s_2 \in \mathcal{S}_i. \quad (20)$$

At the iteration k, the probability associated with the simulation of the probability vector s_2, starting at s_1, and conditionally to the type t_i for player i, is $q(.|s_1)$. It is a normal density centered in s_1 with covariance matrix Σ. Then, the generating matrix G_i associated with the random walk generating distribution q, is of form

$$G_i(s_1, s_2) \;=\; \frac{\exp\left(-\frac{1}{2}(s_2 - s_1)^T \Sigma^{-1}(s_2 - s_1)\right)}{\sum_{s\in\mathcal{S}_i} \exp\left(-\frac{1}{2}(s - s_1)^T \Sigma^{-1}(s - s_1)\right)}. \quad (21)$$

Note that $G_i(s_1, s_2) = G_i(s_2, s_1)$, since the strategies are drawn according to a normal density symetrically distributed around the origin.

Using the acceptance probability defined in Eq. 19 conditionally to the type t_i and the generating matrix definition in Eq. 21, we get the expression of the transition probability from the state s_1 to the state s_2, for player i

$$P_i(s_1, s_2) = \begin{cases} G_i(s_1, s_2)\ A_i(s_1, s_2)\ , \text{ if } s_2 \neq s_1, \\ 1 - \sum_{s \in \mathcal{S}_i - \{s_1\}} P_i(s_1, s). \end{cases} \tag{22}$$

Let $P_i(t_i)$, be the matrix containing the transition probabilities issued from the algorithm, which generates the Markov chain $\{X_i^{(k)}(t_i)\}_k$, for player i, conditionally to his type t_i, at the temperature $\text{Temp}(t)$. Then

$$X_i^{(k)} = \left\{ X_i^{(k)}(t_i) \in \mathcal{S}_i \mid t_i \in T_i \right\},\ i \in \mathcal{N} \cup \text{Op}_1 \cup \text{Op}_2$$

contains the set of Markov chains, generated for each player, conditionally to every type.

(2) *Identification of a stationary distribution for the underlying Markov chain.*

Recall that if the distribution π satisfies (cf. [8]),

$$\pi(s_1)\ P(s_1, s_2) = \pi(s_2)\ P(s_2, s_1)\ ,\ \forall s_1, s_2 \in \mathcal{S}_i, \tag{23}$$

then it is stationary. Assume that G_i generates an irreducible Markov chain for each player i conditionally to his type t_i. Then, the associated matrix G_i, will be called *irreducible*. Since $P_i(s_1, s_1) > 0$, the associated Markov chain is irreducible and aperiodic. Then, there exists a unique stationary distribution. Hence, if we determine an invariant distribution for the probability transition matrix, it is the limit distribution towards which the chain converges.

Recall also that Boltzmann's distribution (cf. [8]) is a probability distribution on the action strategy space which puts most of the weights on the states (here, the randomized vectors), maximizing the player's objective function. The Boltzmann's distribution for player i, conditionally to his type t_i, is defined as follows:

$$\beta_i(s_1) = \frac{\exp\left[\frac{U_i(s_1|t_i)}{\text{Temp}(t)}\right]}{\sum_{s \in \mathcal{S}_i} \exp\left[\frac{U_i(s|t_i)}{\text{Temp}(t)}\right]}\ ,\ \forall s_1 \in \mathcal{S}_i. \tag{24}$$

We have already seen, using weight symmetry that, $G_i(s_1, s_2) = G_i(s_2, s_1)$, $\forall t_i \in T_i$, $\forall i \in \mathcal{N} \cup \text{Op}_1 \cup \text{Op}_2$. Consequently, to prove that the Boltzmann's distribution is invariant for our transition matrix, i.e., $\beta_i(s_1)\ P_i(s_1, s_2) = \beta_i(s_2)\ P_i(s_2, s_1)$, it is sufficient to show that $\beta_i(s_1)\ A_i(s_1, s_2) = \beta_i(s_2)\ A_i(s_2, s_1)$. We indeed have

$$\begin{aligned}
\beta_i(s_1)\ A_i(s_1, s_2) &= \frac{\exp\left[\frac{U_i(s_1|t_i)}{\mathrm{Temp}(t)}\right]}{\sum_{s\in\mathcal{S}_i}\exp\left[-\frac{U_i(s|t_i)}{\mathrm{Temp}(t)}\right]}\ \exp\left[-\left(\frac{U_i(s_1|t_1)-U_i(s_2|t_1)}{\mathrm{Temp}(t)}\right)\right]^+,\\
&= \frac{\exp\left[\frac{U_i(s_2|t_i)}{\mathrm{Temp}(t)}\right]}{\sum_{s\in\mathcal{S}_i}\exp\left[\frac{U_i(s|t_i)}{\mathrm{Temp}(t)}\right]}\ \exp\left[\frac{U_i(s_1|t_i)-U_i(s_2|t_i)}{\mathrm{Temp}(t)}\right]\\
&\quad\times\exp\left[-\left(\frac{U_i(s_1|t_i)-U_i(s_2|t_i)}{\mathrm{Temp}(t)}\right)^+\right].
\end{aligned} \tag{25}$$

But, each real number $a \in \mathbb{R}$ can be written under the form, $a = a^+ - a^-$ with $a^+,\ a^- \in \mathbb{R}^+$. It gives us $a^+ = a + a^-$. But $a^- = (-a)^+$. Hence, we infer that $a^+ = a + (-a)^+$.

By application of this identity in the Eq. 25, we get

$$\begin{aligned}
\exp\left[-\left(\frac{U_i(s_1|t_i)-U_i(s_2|t_i)}{\mathrm{Temp}(t)}\right)^+\right] = \exp\Bigg[-\Bigg\{\left[\frac{U_i(s_1|t_i)-U_i(s_2|t_i)}{\mathrm{Temp}(t)}\right]\\
+\left[\frac{U_i(s_2|t_i)-U_i(s_1|t_i)}{\mathrm{Temp}(t)}\right]^+\Bigg\}\Bigg].
\end{aligned}$$

Using simplifications, it gives:

$$\beta_i(s_1)\ A_i(s_1, s_2)\ =\ \beta_i(s_2)\ A_i(s_2, s_1).$$

Then, β_i is a stationary distribution for the Markov chain associated with the transition matrix $P_i(t_i)$. For every player i, conditionally to each type $t_i \in T_i$, since we assume that G_i is irreducible, the Markov chain issued from our algorithm converges towards the stationary distribution, β_i.

(3) *Proof that Algorithm* 1 *converges to a Bayesian equilibrium and that it is unique under irreducibility of the generating matrix* G_i.

Now, we want to check that asymptotically β_i is a Bayesian equilibrium, i.e., that the algorithm converges.

Indeed, by definition [22], we say that the algorithm converges with probability one if and only if:

$$\lim_{\text{Temp}\to^{+}0} \mathbf{P}\Big[\big\{X^{(k)} \in \mathcal{B}^{\star}\big\}\Big] = 1. \tag{26}$$

We note $\mathcal{BR}_i^{(t)}$, the set of player i's best responses, conditionally to his type t_i, at the time instant t, at the temperature Temp(t). This set contains the probability vectors maximizing player i's expected utility, conditionally to his type t_i, the strategies of the other players being fixed. We note $U_i^{\star}(t_i)$, player i's expected utility conditionally to his type t_i, for every element of the best response space, $\mathcal{BR}_i^{(t)}$. Using these definitions, Boltzmann's measure at the time instant t, becomes:

$$\beta_i(s_1) = \frac{\exp\left[\frac{U_i(s_1|t_i)}{\text{Temp}^{(t)}}\right]}{\sum_{s\in\mathcal{S}_i}\exp\left[\frac{U_i(s|t_i)}{\text{Temp}^{(t)}}\right]}, \tag{27}$$

$$= \frac{\exp\left[\frac{-U_i^{\star}(t_i)+U_i(s_1|t_i)}{\text{Temp}(t)}\right]}{\sum_{s\in\mathcal{S}_i}\exp\left[\frac{-U_i^{\star}(t_i)+U_i(s|t_i)}{\text{Temp}(t)}\right]}\left(\mathbf{1}_{s_1\in\mathcal{BR}_i^{(t)}} + \mathbf{1}_{s_1\in\mathcal{S}_i-\mathcal{BR}_i^{(t)}}\right), \tag{28}$$

$$= \frac{1}{\sum_{s\in\mathcal{S}_i}\exp\left[\frac{-U_i^{\star}(t_i)+U_i(s|t_i)}{\text{Temp}(t)}\right]}\mathbf{1}_{s_1\in\mathcal{BR}_i^{(t)}}$$

$$+\frac{\exp\left[\frac{-U_i^{\star}(t_i)+U_i(s_1|t_i)}{\text{Temp}^{(t)}}\right]}{\sum_{s\in\mathcal{S}_i}\exp\left[\frac{-U_i^{\star}(t_i)+U_i(s|t_i)}{\text{Temp}(t)}\right]}\mathbf{1}_{s_1\in\mathcal{S}_i-\mathcal{BR}_i}. \tag{29}$$

As the temperature decreases towards 0, the second term vanishes, since $U_i(s_1|t_i) \leq U_i^{\star}(t_i)$. Under this hypothesis, we have:

$$\beta_i(s_1) \to \frac{1}{|\mathcal{BR}_i^{(t)}|}\mathbf{1}_{s_1\in\mathcal{BR}_i^{(t)}}.$$

Hence, Algorithm 1 converges.

In the case of a finite dimensional Bayesian game, there always exists a Bayesian equilibrium (cf. [17]). Provided $\mathcal{B}^{\star} \subseteq \times_{i\in\mathcal{N}_g}\mathcal{S}_i$, for t large enough, $\mathcal{BR}^{(t)}$, the set of the best responses obtained using our algorithm, is either empty, either identifiable with a Bayesian equilibrium provided it converges (cf. [10]). Consequently:

$$\lim_{\text{Temp}^{(t)}\to 0}\mathbf{P}\left[X^{(k)} \in \mathcal{B}^{\star}\right] \geq \lim_{\text{Temp}^{(t)}\to 0}\mathbf{P}\left[X^{(k)} \in \mathcal{BR}^{(t)}\right] = 1. \tag{30}$$

Besides, since the Markov chain is supposed irreducible and aperiodic, there exists a unique stationary distribution, which means that the Bayesian equilibrium once reached, is unique. □

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