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Finite Time Synchronization of Delayed Quaternion Valued Neural Networks with Fractional Order

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Abstract
In this article, the finite time (FT) synchronization problem of fractional order quaternion valued neural networks with time delay is investigated. Without separating the quaternion valued system into two complex valued or four real valued systems, the FT synchronization conditions are derived through using Lyapunov direct method. Furthermore, the setting time is estimated, which is influenced by the order of fractional derivative and control parameters. Finally, numerical simulations are shown to verify the effectiveness of the proposed methods.

Keywords Fractional order quaternion valued neural networks · Finite time synchronization · Time delay · Feedback control

1 Introduction

As is known to all, fractional order calculus can more accurately depict the memory and hereditary properties of several processes, some scholars have combined fractional calculus with neural networks (NN), investigating the dynamical behaviors of fractional order neural networks (FONN), such as dissipativity, periodicity, stability and stabilization, and so on [1–6]. Recently, the synchronization issue of FONN has become an interesting topic because of its extensive applications in secure communication, information science, and biological, etc. In view of the synchronization time, the synchronization problem can be divided into two broad categories: first is the asymptotical synchronization, which shows that synchronization is realized only when the time tends to infinity. Second is the FT synchronization, which means that the systems are synchronized in some FT. In realistic applications, it is more reasonable

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and meaningful to investigate synchronization in FT. For instance, in secure communication, the longer it takes to synchronize, the more likely it is that information will be stolen. At present, many excellent results on FT synchronization of fractional order real valued NN (FORVNN) and fractional order complex valued NN (FOCVNN) have been reported [7–11].

In practical applications concerning multidimensional data, which can not be well handled by neurons of RVNN or CVNN. As of now, quaternion owns one real part and three imaginary parts, which has received lots of attention in numerous fields due to it can deal with perfectly multidimensional data [12,13]. In recent years, combining the advantages of quaternion and FONN, fractional order quaternion valued neural networks (FOQVNN) is established, which can perform relatively well [14,15]. For instance, in color image processing, a color image can be proceed through applying three real valued neurons, however, one QVNN can process a color image through three channels $\imath$, $\jmath$, κ. On the other hand, time delay is inescapable in practical systems and may lead to oscillation, chaos and instability. Therefore, FOQVNN with time delay is a more general model.

Lyapunov method is an important and effective tool to analysis the stability and synchronization for integer order NN. However, it is can not be simply extended to FONN. Achieving stability and synchronization conditions of FONN remains challenging and interesting topics. Nevertheless, so far, investigations about stability analysis and synchronization control for FOQVNN are few [16–18]. In [16], through separating four FORVNN parts, the global Mittag–Leffler stability and synchronization analysis of FOQVNN with linear threshold neurons was discussed. Ref. [17] investigated the global Mittag–Leffler synchronization of FOQVNN with time delay by exploiting direct quaternion method. Authors in [18] introduced novel methods to study Mittag–Leffler synchronization problem of FOQVNN. However, up to now, considering the FT synchronization of FOQVNN with time delay has not been founded, which will make up this gap in this paper.

In the existing literatures, in order to realize FT synchronization of FONN, designing sign function of state variables in controller was effective [10,19]. For example, in [10], FT synchronization of fully FOCVNN was implemented through proposing a complex valued sign function in controller as follow: $u_i(t) = -k_i e_i(t) - \lambda[e_i(t)]|e_i(t)|_2^{\beta}$, where $[e_i(t)] = sign(Re(e_i(t))) + isign(Im(e_i(t))))$. In [19], FT synchronization conditions were obtained by designing the state feedback controllers: $u_p(t) = \xi_p e_p(t) + \varrho_p sign(e_p(t))$ and $u_p(t) = \varpi_p e_p(t) + \lambda_p sign(e_p(t)) + \frac{\sigma_p sign(e_p(t))}{|e_p(t)|}$, where $\xi_p > 0$, $\varrho_p > 0$, $\varpi_p > 0$, $\lambda_p > 0\sigma_p > 0$. It is worth pointing that sign function does not defined in quaternion field. Thereupon, it is difficult and important that how to choose a suitable controller, which is independent of sign function. It is the key problem to derive the FT synchronization.

Propelled by above analysis, the main object in this paper is to discuss the FT synchronization of FOQVNN with time delay by applying Lyapunov direct method to avoid the separation of QVNN. The main advantages are listed below. (1) Due to the sign function does not exist in quaternion field, a new controller without the help of sign function is adopted in the slave system. (2) The FT synchronization of FOQVNN has not be investigated before. (3) The settling time is influenced by the order of fractional derivative and control parameters, which is less conservative.

2 Preliminaries and Model Description

In order to read the article more convenient, some symbols are illustrated. $\mathcal{R}$ denotes real number, $\mathcal{Q}$ stands for quaternion number. Quaternion y has the following form: $y = y^R +$

$\imath y^I + \jmath y^J + \kappa y^K \in \mathcal{Q}$, where $y^R, y^I, y^J, y^K \in \mathcal{R}$, $\imath, \jmath, \kappa$ are the imaginary units and obey Hamilton rules: $\imath^2 = \jmath^2 = \kappa^2 = -1, \imath\jmath = \kappa = -\jmath\imath, \jmath\kappa = \imath = -\kappa\jmath, \kappa\imath = \jmath = -\imath\kappa$. $\bar{y} = y^R - \imath y^I - \jmath y^J - \kappa y^K$ is the conjugate of y, $|y| = \sqrt{y\bar{y}} = \sqrt{\bar{y}y} = \sqrt{(y^R)^2 + (y^I)^2 + (y^J)^2 + (y^K)^2}$ denotes the module of y.

In this section, some necessary knowledge of Caputo fractional derivative, model of FOQVNN and lemmas are introduced.

Definition 1 [20,21] The fractional integral of function $h(t)$ with order $\alpha > 0$ is defined as:

$$D^{-\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \tag{1}$$

where $t > 0$, $\Gamma(\alpha) = \int_0^\infty \omega^{\alpha-1}e^{-\omega}d\omega$.

Definition 2 [20,21] The Caputo derivative of fractional order $0 < \alpha < 1$ of function $h(t)$ is defined as:

$$D^{\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}h'(\tau)d\tau, \quad t > 0. \tag{2}$$

Considering the FOQVNN with time delay as following form:

$$\begin{aligned} D^{\alpha}q_i(t) = -c_i q_i(t) + \sum_{j=1}^n a_{ij}g_j(q_j(t)) + \sum_{j=1}^n b_{ij}g_j(q_j(t-\tau)) + h_i,\\ i = 1, 2, \ldots, n, \end{aligned} \tag{3}$$

in which $0 < \alpha < 1$, $q_i(t) \in \mathcal{Q}$ is the state vector, $c_i \in R$ denotes the self-feedback coefficient, $a_{ij} \in \mathcal{Q}$, $b_{ij} \in \mathcal{Q}$ stand for the connection weights, $\tau > 0$ is the delay, h_i denotes the external input. $g_j(q_j(t)), g_j(q_j(t-\tau)) : \mathcal{Q} \longrightarrow \mathcal{Q}$ are the neuron activations, which satisfy the following assumption:

Assumption 1 The activation function $g_i(\cdot)$ satisfies the following conditions with constants $l_i > 0$,

$$|g_i(p) - g_i(\tilde{p})| \le l_i|p - \tilde{p}|, \forall p, \quad \tilde{p} \in \mathcal{Q}, \quad i = 1, 2, \ldots, n. \tag{4}$$

The initial condition of system (3) is $q_i(0) = \varsigma_i \in \mathcal{Q}$.

To investigate the FT synchronization of FOQVNN, the model (3) is considered as master system, the corresponding slave model is given by

$$\begin{aligned} D^{\alpha}q_i'(t) = -c_i q_i'(t) + \sum_{j=1}^n a_{ij}g_j(q_j'(t)) + \sum_{j=1}^n b_{ij}g_j(q_j'(t-\tau)) + h_i + u_i(t),\\ i = 1, 2, \ldots, n, \end{aligned} \tag{5}$$

where $q_i'(t)$ denotes the state vector of the system (5), $u_i(t)$ is an external controller. The initial condition of system (5) is $q_i'(0) = \tilde{\varsigma}_i \in \mathcal{Q}$.

Then, the synchronization error is expressed:

$$\begin{aligned} D^{\alpha}e_i(t) = -c_i e_i(t) + \sum_{j=1}^n a_{ij}\left[g_j\left(q_j'(t)\right) - g_j(q_j(t))\right]\\ + \sum_{j=1}^n b_{ij}\left[g_j\left(q_j'(t-\tau)\right) - g_j(q_j(t-\tau))\right] + u_i(t), \end{aligned} \tag{6}$$

where $e_i(t) = q_i'(t) - q_i(t)$ denotes the error vector of system (6). The initial condition of error system (6) is $e_i(0) = \varsigma_i - \tilde{\varsigma}_i \triangleq \sigma_i$.

Definition 3 [22] For any initial values σ_i, the slave system (5) is said to be synchronized in FT with the master system (3), if there exists a time $0 < T(\sigma_i) < \infty$, such that $\lim_{t \to T(\sigma_i)} \|e_i(t)\| = 0$, and $\|e_i(t)\| = 0$ for all $t > T(\sigma_i)$, where $\|\cdot\|$ denotes the Euclidean norm.

Next, we present some lemmas below.

Lemma 1 [23] *Set $x(t) \in \mathcal{Q}$ be a differentiable function. Then, we have:*

$$D^\alpha \overline{x(t)} x(t) \leq D^\alpha \overline{x(t)} x(t) + \overline{x(t)} D^\alpha x(t), \quad t \geq 0 \tag{7}$$

where $0 < \alpha < 1$.

Lemma 2 [24] *Let $\xi, \eta \in \mathcal{Q}$, then:*

$$\bar{\xi}\eta + \bar{\eta}\xi \leq \bar{\xi}\xi + \bar{\eta}\eta, \tag{8}$$

holds.

Lemma 3 [20,21] *Let $h(t) : [0, +\infty) \to \mathcal{Q}$, then one has*

$$D^{-\alpha} D^\alpha h(t) = h(t) - h(0), \quad 0 < \alpha < 1. \tag{9}$$

Lemma 4 [25] *Assume $h(t) \in C^1([0, +\infty), \mathcal{R})$, then*

$$D^\alpha h^k(t) = \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} h^{k-\alpha}(t) D^\alpha h(t), \quad 0 < \alpha < 1, \quad k \in \mathcal{R}. \tag{10}$$

Lemma 5 *Let $V(t)$ be a continuous and positive definite function, which satisfies:*

$$D^\alpha V(t) \leq -\rho V^\beta(t), \quad \rho > 0, \quad 0 \leq \beta < \alpha < 1 \tag{11}$$

then the following statements hold:

(I) *If $\beta = 0$, $V(t) = 0$ for all $t \geq t_1$, where* $t_1 = \left[\frac{V(0)\Gamma(1+\alpha)}{\rho}\right]^{\frac{1}{\alpha}}$.

(II) *If $0 < \beta < \alpha < 1$, $V(t) = 0$ for all $t \geq t_2$, where* $t_2 = \left[\frac{V^{\alpha-\beta}(0)\Gamma(1+\alpha)\Gamma(1-\beta)}{\rho\Gamma(1+\alpha-\beta)}\right]^{\frac{1}{\alpha}}$.

Proof (I) When $\beta = 0$, one can obtain:

$$D^\alpha V(t) \leq -\rho, \tag{12}$$

then there exists a nonnegative function $\eta(t)$ satisfying

$$D^\alpha V(t) + \eta(t) = -\rho. \tag{13}$$

Using Lemma 3, we get:

$$D^{-\alpha} D^\alpha V(t) + D^{-\alpha}\eta(t) = D^{-\alpha}(-\rho), \tag{14}$$

i.e.

$$\begin{aligned} V(t) &= V(0) - D^{-\alpha}\eta(t) + D^{-\alpha}(-\rho) \\ &= V(0) - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\eta(\tau)}{(t-\tau)^{1-\alpha}} d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\rho}{(t-\tau)^{1-\alpha}} d\tau \end{aligned}$$

$$\leq V(0) - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\rho}{(t-\tau)^{1-\alpha}} d\tau$$
$$= V(0) - \frac{\rho t^\alpha}{\Gamma(1+\alpha)}. \tag{15}$$

Note $\Phi(t) = V(0) - \frac{\rho t^\alpha}{\Gamma(1+\alpha)}$. Since $\Phi(t)$ is strictly decreasing, one has $\Phi(t) \leq \Phi(t_1) = 0$ for all $t \geq t_1$, where $t_1 = \left[\frac{V(0)\Gamma(1+\alpha)}{\rho}\right]^{\frac{1}{\alpha}}$. In addition, $V(t) \leq \Phi(t) \leq 0$ and $V(t)$ is nonnegative, one can obtain that $V(t) = 0$ for all $t \geq t_1$.

(II) When $0 < \beta < \alpha < 1$, it follows from Lemma 4, one has

$$\begin{aligned} D^\alpha V^{\alpha-\beta}(t) &= \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)} V^{-\beta}(t) D^\alpha V(t) \\ &\leq -\rho \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)} V^{-\beta}(t) V^\beta(t) \\ &= -\rho \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)}. \end{aligned} \tag{16}$$

The rest is the same as the proof of the case (I), taking the α order integration of (16) from 0 to t, we can obtain

$$V^{\alpha-\beta}(t) \leq V^{\alpha-\beta}(0) - \frac{\rho\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)\Gamma(1+\alpha)} t^\alpha. \tag{17}$$

Note $\Psi(t) = V^{\alpha-\beta}(0) - \frac{\rho\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)\Gamma(1+\alpha)} t^\alpha$. Since $\Psi(t)$ is strictly decreasing, one has $\Psi(t) \leq \Psi(t_2) = 0$ for all $t \geq t_2$, where $t_2 = [\frac{V^{\alpha-\beta}(0)\Gamma(1+\alpha)\Gamma(1-\beta)}{\rho\Gamma(1+\alpha-\beta)}]^{\frac{1}{\alpha}}$. In addition, $V(t) \leq \Psi(t) \leq 0$ and $V(t)$ is nonnegative, one can obtain that $V(t) = 0$ for all $t \geq t_2$. □

3 Main Results

In this section, some FT synchronization criteria of FOQVNN are established based on some control schemes.

For realizing the FT synchronization of systems (3) and (5), the controller $u_i(t) (i = 1, 2, \ldots, n)$ is designed as follow:

$$u_i(t) = \begin{cases} -\lambda_i e_i(t) - \omega_i e_i(t-\tau) - \frac{\mu_i e_i(t)}{(\overline{e_i(t)} e_i(t))^\nu}, & e_i(t) \neq 0 \\ 0, & e_i(t) = 0 \end{cases}, \tag{18}$$

where $\lambda_i > 0, \omega_i > 0, \mu_i > 0, 1 - \alpha < \nu < 1$.

Remark 1 Unlike the controllers in [10,19], a novel controller is designed in this paper without choosing sign function, it is because that the sign function does not exist in quaternion domain.

Theorem 1 *Let the Assumption 1 holds, if there exists constants $\delta_i > 0$, λ_i and ω_i such that*

$$\begin{cases} \lambda_i \geq -c_i + \frac{1}{2} \sum\limits_{j=1}^n |a_{ij}|^2 + \frac{1}{2} \sum\limits_{j=1}^n |b_{ij}|^2, \\ \omega_i \geq n l_i^2, \end{cases} \tag{19}$$

then the system (3) and (5) can realize the FT synchronization with $T = \left[\frac{V^{\alpha-1+\nu}(0)\Gamma(1+\alpha)\Gamma(\nu)}{\mu\Gamma(\alpha+\nu)}\right]^{\frac{1}{\alpha}}$.

Proof Choosing the following function

$$V(t)=\sum_{i=1}^{n}\delta_i\overline{e_i(t)}e_i(t). \tag{20}$$

According to Lemma 1,

$$\begin{aligned}
D^{\alpha}V(t)&\leq\sum_{i=1}^{n}\delta_i\left[D^{\alpha}\overline{e_i(t)}e_i(t)+\sum_{i=1}^{n}\overline{e_i(t)}D^{\alpha}e_i(t)\right]\\
&\leq\sum_{i=1}^{n}\delta_i\left\{-c_i\overline{e_i(t)}+\sum_{j=1}^{n}\overline{a_{ij}[g_j(q_j'(t))-g_j(q_j(t))]}\right.\\
&\quad\left.+\sum_{j=1}^{n}\overline{b_{ij}[g_j(q_j'(t-\tau))-g_j(q_j(t-\tau))]}+\overline{u_i(t)}\right\}e_i(t)\\
&\quad+\sum_{i=1}^{n}\delta_i\overline{e_i(t)}\left\{-c_ie_i(t)+\sum_{j=1}^{n}a_{ij}[g_j(q_j'(t))-g_j(q_j(t))]\right.\\
&\quad\left.+\sum_{j=1}^{n}b_{ij}[g_j(q_j'(t-\tau))-g_j(q_j(t-\tau))]+u_i(t)\right\}\\
&=\sum_{i=1}^{n}\delta_i\left\{-c_i\overline{e_i(t)}+\sum_{j=1}^{n}\overline{[g_j(q_j'(t))-g_j(q_j(t))]}\overline{a_{ij}}\right.\\
&\quad\left.+\sum_{j=1}^{n}\overline{[g_j(q_j'(t-\tau))-g_j(q_j(t-\tau))]}\overline{b_{ij}}+\overline{u_i(t)}\right\}e_i(t)\\
&\quad+\sum_{i=1}^{n}\delta_i\overline{e_i(t)}\{-c_ie_i(t)+\sum_{j=1}^{n}a_{ij}[g_j(q_j'(t))-g_j(q_j(t))]\\
&\quad\left.+\sum_{j=1}^{n}b_{ij}[g_j(q_j'(t-\tau))-g_j(q_j(t-\tau))]+u_i(t)\right\}.
\end{aligned} \tag{21}$$

According to Assumption 1 and Lemma 2,

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\overline{[g_j(q_j'(t))-g_j(q_j(t))]}\overline{a_{ij}}e_i(t)+\overline{e_i}(t)a_{ij}[g_j(q_j'(t))-g_j(q_j(t))]\right\}\\
&\leq\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{e_i}(t)a_{ij}\overline{a_{ij}}e_i(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{[g_j(q_j'(t))-g_j(q_j(t))]}[g_j(q_j'(t))-g_j(q_j(t))]\\
&\leq\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\overline{e_i(t)}e_i(t)+n\sum_{i=1}^{n}l_i^2\overline{e_i(t)}e_i(t).
\end{aligned} \tag{22}$$

Similarly,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\{\overline{[g_j(q_j'(t-\tau))-g_j(q_j(t-\tau))]}\overline{b_{ij}}e_i(t)+\overline{e_i(t)}b_{ij}[g_j(q_j'(t-\tau))$$
$$-g_j(q_j(t-\tau)))]\}$$
$$\leq\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{e_i(t)}b_{ij}\overline{b_{ij}}e_i(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{[g_j(q_j'(t-\tau))-g_j(q_j(t-\tau))]}[g_j(q_j'(t-\tau))$$
$$-g_j(q_j(t-\tau))]$$
$$\leq\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2\overline{e_i(t)}e_i(t)+n\sum_{i=1}^{n}l_i^2\overline{e_i(t-\tau)}e_i(t-\tau). \tag{23}$$

$$\sum_{i=1}^{n}[\overline{u_i(t)}e_i(t)+\overline{e_i(t)}u_i(t)]$$
$$=\sum_{i=1}^{n}\left[\left(-\lambda_i\overline{e_i(t)}-\omega_i\overline{e_i(t-\tau)}-\frac{\mu_i\overline{e_i(t)}}{(\overline{e_i(t)}e_i(t))^{\nu}}\right)e_i(t)+\overline{e_i}(t)(-\lambda_ie_i(t)\right.$$
$$\left.-\omega_ie_i(t-\tau)-\frac{\mu_ie_i(t)}{(e_i^*(t)e_i(t))^{\nu}})\right]$$
$$=\sum_{i=1}^{n}[-2\lambda_i\overline{e_i(t)}e_i(t)-\omega_i\overline{e_i(t-\tau)}e_i(t)-\omega_i\overline{e_i(t)}e_i(t-\tau)$$
$$-2\mu_i(\overline{e_i(t)}e_i(t))^{1-\nu}]$$
$$\leq\sum_{i=1}^{n}[-2\lambda_i\overline{e_i(t)}e_i(t)-\omega_i\overline{e_i(t)}e_i(t)-\omega_i\overline{e_i}(t-\tau)e_i(t-\tau)$$
$$-2\mu_i(\overline{e_i(t)}e_i(t))^{1-\nu}]. \tag{24}$$

Submitting (22)–(24) into (21), it has:

$$D^{\alpha}V(t)\leq\sum_{i=1}^{n}\delta_i\left\{-2c_i\overline{e_i(t)}e_i(t)+\sum_{j=1}^{n}|a_{ij}|^2\overline{e_i(t)}e_i(t)\right.$$
$$+nl_i^2\overline{e_i(t)}e_i(t)+\sum_{j=1}^{n}|b_{ij}|^2\overline{e_i(t)}e_i(t)+nl_i^2\overline{e_i(t-\tau)}e_i(t-\tau)$$
$$-2\lambda_i\overline{e_i(t)}e_i(t)-\omega_i\overline{e_i(t)}e_i(t)-\omega_i\overline{e_i(t-\tau)}e_i(t-\tau)$$
$$\left.-2\mu_i(\overline{e_i(t)}e_i(t))^{1-\nu}\right\}$$
$$=\sum_{i=1}^{n}\delta_i\left(-2c_i+\sum_{j=1}^{n}|a_{ij}|^2+nl_i^2+\sum_{j=1}^{n}|b_{ij}|^2-2\lambda_i-\omega_i\right)\overline{e_i(t)}e_i(t)$$
$$+\sum_{i=1}^{n}\delta_i(nl_i^2-\omega_i)\overline{e_i(t-\tau)}e_i(t-\tau)$$

$$
\begin{aligned}
&-\sum_{i=1}^{n} 2\delta_i \mu_i (\overline{e_i(t)} e_i(t))^{1-\nu}\} \\
&\leq -2\delta_i \mu_i \sum_{i=1}^{n} (\overline{e_i(t)} e_i(t))^{1-\nu} \\
&\leq -\mu V^{1-\nu}(t),
\end{aligned} \tag{25}
$$

where $\mu = \min_{1 \leq i \leq n} \{2\delta_i \mu_i\}$.

Therefore, according to Lemma 5, systems (3) and (5) can achieve the FT synchronization. Moreover, the setting time is estimated by $T = \left[\frac{V^{\alpha-1+\nu}(0)\Gamma(1+\alpha)\Gamma(\nu)}{\mu\Gamma(\alpha+\nu)}\right]^{\frac{1}{\alpha}}$.

When $\delta_1 = \delta_2 = \cdots = \delta_n = 1$, it can get following conclusion: □

Corollary 1 *If the Assumption 1 holds, the control function $u_i(t)$ satisfies (19), then the system (3) and (5) can realize synchronization in a FT $T = \left[\frac{V^{\alpha-1+\nu}(0)\Gamma(1+\alpha)\Gamma(\nu)}{\tilde{\mu}\Gamma(\alpha+\nu)}\right]^{\frac{1}{\alpha}}$, where $\tilde{\mu} = \min_{1 \leq i \leq n} \{2\mu_i\}$.*

Remark 2 There are some results concerning with FT synchronization of FORVNN and FOCVNN. However, the obtained FT synchronization conditions [7–11] can not be applied to FOQVNN. Furthermore, the main results [7–11] are the special case of this paper.

Remark 3 If the $\tau = 0$, the model was considered in [16,18]. In fact, time-delayed NN models exist in actuality, and time delay could affect the dynamics of NN and cannot be ignored. Here, a typical FOQVNN model is studied.

Remark 4 Compared with the method in [16,18], without dividing the the QVNN into four RVNN in this paper, the FT synchronization of FOQVNN is achieved and some simple criteria are derived by applying quaternion theory. The method is more practical and effective, which can decrease the computational tedious and triviality.

Remark 5 It should be noted that realistic applications require the synchronization is implemented in FT. It is the first time that the FT synchronization of FOQVNN with time delay is discussed, which can be regards as one main novelty of this paper. Thereupon, the obtained results in this paper are different from the existing results in [17].

4 Numerical Simulations

In this section, a numerical example demonstrated the feasibility of the theoretic results.

Consider the following FOQVNN with time delay, which was taken as drive system:

$$
\begin{aligned}
D^{\alpha} q_i(t) &= -c_i q_i(t) + \sum_{j=1}^{n} a_{ij} g_j(q_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(q_j(t-\tau)) + h_i, \\
&\quad i = 1, 2,
\end{aligned} \tag{26}
$$

where $\alpha = 0.95, c_1 = c_2 = 1, q_i(t) = q_i^R(t) + \imath q_i^I(t) + \jmath q_i^J(t) + \kappa q_i^K(t)$ with $q_i^R(t), q_i^I(t), q_i^J(t), q_i^K(t) \in \mathcal{R}$, $g_j(q_j(t)) = tanh(q_i^R(t)) + \imath tanh(q_i^I(t)) + \jmath tanh(q_i^J(t)) + \kappa tanh(q_i^K(t))$, $\tau = 1$, $h_1 = h_2 = 0$. By calculating, we can derive $l_1 = l_2 = 1$.

$$
A = (a_{ij})_{2\times 2} = \begin{pmatrix} 2 + 2\imath + 2\jmath + 2\kappa & -0.1 - 0.1\imath - 0.1\jmath - 0.1\kappa \\ -2 - 3\imath - 2\jmath - 3\kappa & 1 + \imath + \jmath + \kappa \end{pmatrix},
$$

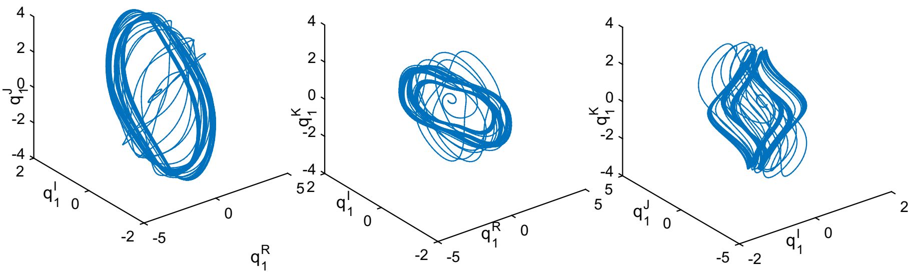

Fig. 1 The chaotic attractors of neurons q_1 of system (26)

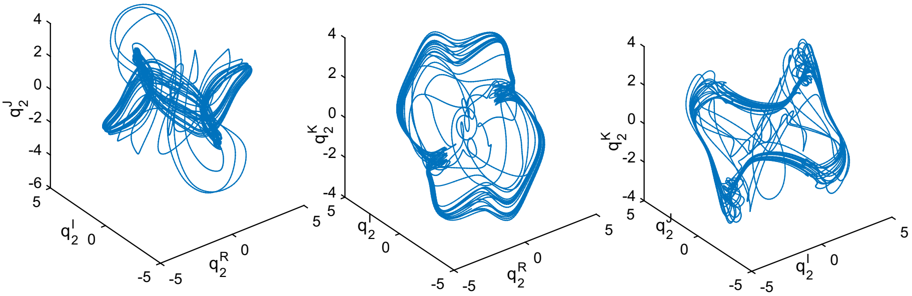

Fig. 2 The chaotic attractors of neurons q_2 of system (26)

$$B = (b_{ij})_{2\times 2} = \begin{pmatrix} -1 - 1.5\imath - 1\jmath - 1.5\kappa & -0.1 - 0.1\imath - 0.1\jmath - 0.1\kappa \\ -0.3 - 0.2\imath - 0.3\jmath - 0.2\kappa & 2 + 2.4\imath + 2.4\jmath + 2.4\kappa \end{pmatrix}.$$

The chaotic attractors of two neurons of system (26) with the initial values $q_{10} = 1 + 2\imath + \jmath + 3\kappa$, $q_{20} = -2 + 2\imath + \jmath + 3\kappa$ are shown in Figs. 1 and 2.

The slave model is given by:

$$D^\alpha q_i'(t) = -c_i q_i'(t) + \sum_{j=1}^{n} a_{ij} g_j('q_j'(t)) + \sum_{j=1}^{n} b_{ij} g_j(q_j'(t-\tau)) + h_i + u_i(t), \quad i = 1, 2, \tag{27}$$

where $q_i'(t) = q_i'^R(t) + \imath q_i'^I(t) + \jmath q_i'^J(t) + \kappa q_i'^K(t)$ with $q_i'^R(t), q_i'^I(t), q_i'^J(t), q_i'^K(t) \in \mathcal{R}$, $u_i(t)$ is the controller, the parameters of model (27) are similar that of model (26). Initial conditions are selected $q_{10} = 0.1 + 0.2\imath + 0.1\jmath + 0.3\kappa$, $q_{20} = -0.2 + 0.2\imath + \jmath + 0.3\kappa$, $q_{10}' = 0.3 + 0.4\imath + \jmath + \kappa$, $q_{20}' = -0.1 + 0.1\imath + 0.1\jmath - 0.1\kappa$. The curves of state variables are shown without controller in Figs. 3 and 4. From the shown results, it can derive that the slave system (27) cannot synchronize with the master system (26).

In controller (18), if we choose $\lambda_1 = 20, \lambda_2 = 15, \omega_1 = \omega_2 = 3$, the condition of Theorem 1 is satisfied, and select $\mu_1 = \mu_2 = 1, \nu = 0.75$, the setting time is $T = 1.1758$. The evolution of synchronization errors are depicted in Fig. 5. The total synchronization error $\| e(t) \|$ is described in Fig. 6.

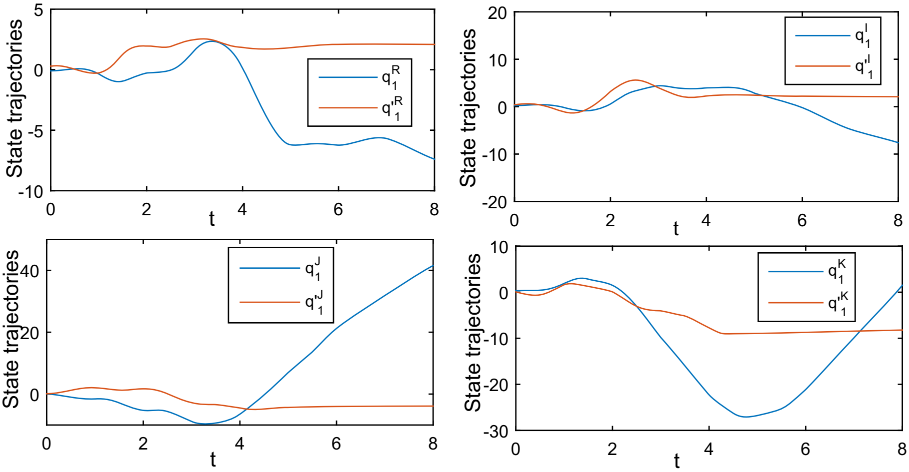

Fig. 3 The state trajectories of $q_1^R, q_1^{\prime R}, q_1^I, q_1^{\prime I}, q_1^J, q_1^{\prime J}, q_1^K, q_1^{\prime K}$ without controller

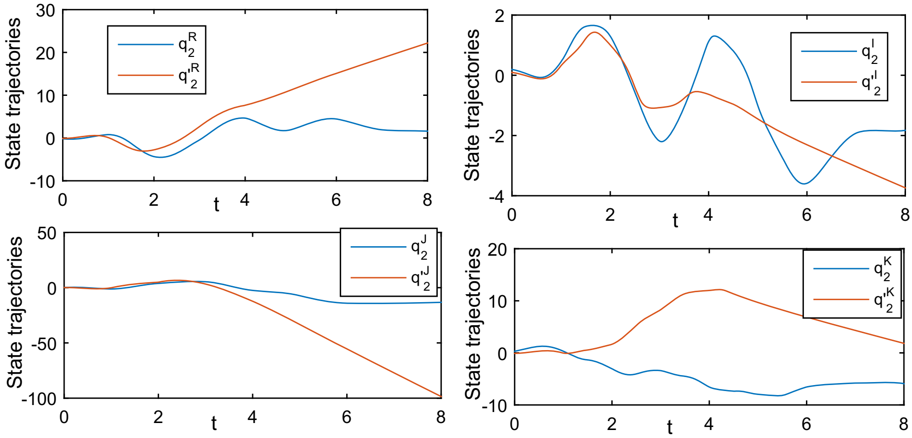

Fig. 4 The state trajectories of $q_2^R, q_2^{\prime R}, q_2^I, q_2^{\prime I}, q_2^J, q_2^{\prime J}, q_1^K, q_2^{\prime K}$ without controller

5 Conclusions

This paper main concerns with the FT synchronization of FOQVNN with time delay based on Lyapunov direct method. By making use of the suitable controller without the help of sign function, some synchronization sufficient criteria are given. Moreover, the setting time is estimated. The correction and feasibility of the proposed methods are verified by a numerical example. It is different from the decomposition approaches in the existing literatures, the model is considered as a entirety. Furthermore, the proposed controller is novelty, the obtained results are less conservative in this paper. However, the FT synchronization time depends on initial values, the fixed time synchronization of FOQVNN with time delay will be investigated without separation method in future work.

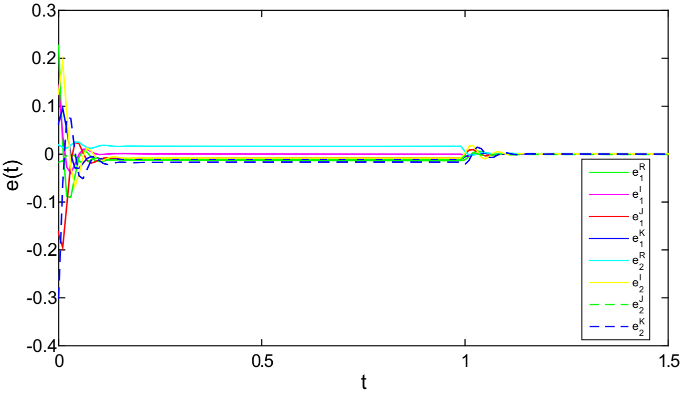

Fig. 5 The synchronization error $e_1^R, e_2^R, e_1^I, e_2^I, e_1^J, e_2^J, e_1^K, e_2^K$ under controller (18)

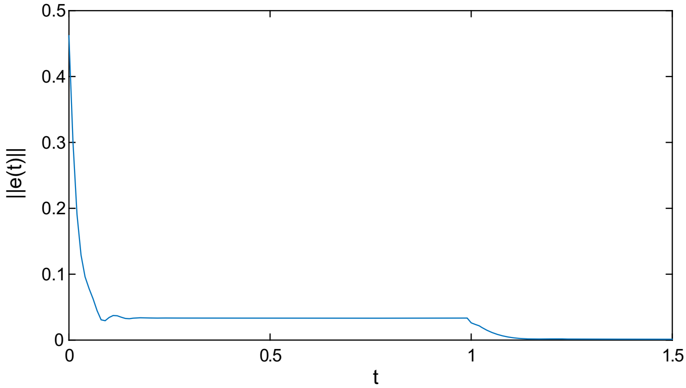

Fig. 6 The synchronization error norm $\| e(t) \|$ under controller (18)

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