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Finite Time Anti-synchronization of Quaternion-Valued Neural Networks with Asynchronous Time-Varying Delays

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Abstract
In this paper, we consider the finite time anti-synchronization (A-SYN) of master-slave coupled quaternion-valued neural networks, where the time-varying delays can be asynchronous and unbounded. Without adopting the general decomposition method, the quaternion-valued state is considered as a whole, which greatly reduces the hassle of further analysis and calculations. The designed controller is delay-free, and the terms with time delay do not need to be bounded globally. Several sufficient conditions for ensuring the finite time A-SYN are obtained under 1-norm and 2-norm respectively. The A-SYN error will be proved to evolve from the initial value to 1 in finite time, and evolve from 1 to 0 also in finite time, hence the finite time A-SYN is proved, which is called two-phases-method. Moreover, adaptive rules for control strengths are also designed to realize the finite time A-SYN. Lastly, a numerical example is presented to demonstrate the correctness and effectiveness of our obtained results.

Keywords Anti-synchronization · Asynchronous · Finite time · Quaternion-valued neural network · Time delay · Unbounded

1 Introduction

In 1843, British mathematician Hamilton introduced quaternion, which was an extension of complex numbers. However, quaternion did not get too much attention or development for quite a long time, where one significant reason was that, unlike complex numbers, quaternion multiplication did not satisfy the commutative law. By the late twentieth century, quaternion ushered in recovery due to its effectiveness in describing spatial rotations. Specifically,

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researchers found that quaternion gave a simple way to encode the rotation information into four numbers, which was more compact and simpler than matrices and Euler angles. Hence, in recent years, quaternion has been widely applied in computer graphics, computer vision, robotics, navigation, and so on.

Neural network [1,2] has become one of the most popular research fields in the past 30 years due to the promising development and wide applications in signal processing, pattern recognition, optimization problems, deep learning, etc. Just as real-valued neural networks (RVNNs) are extended by complex-valued neural networks (CVNNs), QVNNs can also be regarded as an extension of CVNNs, where the neurons' state, activation functions, self-feedback weights, connections weights, and external inputs are all quaternion. Isokawa et al. [3] found that QVNNs performed better on 3D affine transformation task than that with two CVNNs, which illustrated the superiority of QVNNs when dealing with problems related to multi-dimensional information. According to the stringent Cauchy–Riemann–Fueter (CRF) and the generalized Cauchy–Riemann conditions, only constants and linear functions were globally analytical in quaternion domain. Fortunately, researchers have studied this analyticity problem and an alternative condition to CRF (local analyticity condition, LAC) was found [4], which allowed to use standard activation functions, such as tanh function. QVNNs have been successfully applied to various network structures, speech recognition, image processing and classification, and so on [5–7].

Synchronization (SYN) has been a hot topic in network literature for a decade, and many classical results were obtained. However, as a special case of SYN, A-SYN received less attention, which was first observed by Huygens in seventeenth century between two pendulum clocks. When A-SYN occurs, the sum of two correspond state vectors will decrease to zero. A-SYN has been found distinctive applications in communication system, where the security and secrecy of systems can be greatly strengthened by transforming from SYN and A-SYN periodically [8]. Therefore, A-SYN deserves further study in both theory and practice [9,10].

One important factor for network dynamic is time delay, which is inevitable in the real world. There have been many studies on neural network systems with time delays, see [11–26]. For example, Liu and Chen [11] investigated the exponential stability for CVNNs with asynchronous time delays by using the decomposition method, which was a widely used method in the study of CVNNs and QVNNs. Liu et al. [15] decomposed QVNN into two CVNNs. In [20], the pseudo almost periodic SYN of QVNNs with time-varying delays was studied, where the fixed point theorem and Lyapunov functions were applied to ensure the global exponential SYN. Song and Chen [22] focused on the multistability issue for QVNNs with time delays. The decomposition method used in the above papers decouples multi-dimensional state, which simplifies the original problems, but on the other hand, this approach brings more redundancy in calculations and analysis. In fact, if the activation functions satisfy certain characteristics, we can consider the multi-dimensional state as a whole, and analyze it with special calculation arts, for example, Song et al. [24] did not decompose the CVNN, but used the property of $\|u(t)\|$ to analyze the entire complex-valued $u(t)$, which made the proof briefer, and this advantage was more obvious for QVNNs. Zhu and Sun [25] investigated the existence and stability criteria for QVNN with mixed delays by using quaternion-modulus inequality technique. Wei and Cao [26] investigated the SYN of drive-response coupled memristive QVNNs with bounded and differential delay.

It must be stated that, the previous stability/SYN researches are under the concept of asymptotic, exponential, or μ-rate convergence, i.e., the theoretically required time is infinite. Actually, there is another type of convergence: finite/fixed time convergence, and the settling time is dependent/independent on initial values. Finite time stability/SYN is more useful in real applications [27–38]. Since time delay is almost inevitable, there have been many research results on finite/fixed time SYN/A-SYN, which are mainly based on the finite time stability theorem, and two general techniques have been usually applied to deal with time delay. One is designing delay-dependent external controller [31–34], for example (17) in [31], (24) in [32], (22) in [33], (7) in [34]; the other technique is designing delay-free controller but with the boundedness assumption for terms with delay, for example, (H_3) in [31], Assumption 1 in [35], and (H_1) in [36]. Since delay-dependent controller is complex in application, using delay-free controller but without the global boundedness assumption is the trend in recent study of finite time literature, where [37,38] realized this aim, but the time delays were required to be bounded and differentiable. On the other hand, without using finite time stability theorems, recent papers [39–42] investigated the finite time SYN/A-SYN for (inertia) neural networks with delay by developing an integral inequality method.

Recently, a novel method explicitly called two-phases-method (2PM) was proposed in [43], which was inspired by [44,45]. Using 2PM, general frameworks for finite/fixed time stability of delayed system were set up in our works [46–49]. In the first phase, the concerned/measured variable (for example, SYN/A-SYN error) would be proved to evolve from the initial value to 1 in finite time, where the convergence rate is related to the form of the time delay, so this phase can be regarded as a repetition of proving infinite time (including exponential, μ-rate) stability. In the second phase, one can easily enlarge terms with delay by previous obtained boundedness property for system variables, and prove that the concerned/measured variable would evolve from 1 to 0 also in finite time. 2PM has been used to solve finite time A-SYN problem in [43], and can be extended to other hypercomplex-valued neural networks, such as QVNNs in this paper.

To the best of our knowledge, the finite time A-SYN of QVNNs with unbounded asynchronous time delays has not been investigated so far. Motivated by the aforementioned discussions, this paper will concentrate on solving this problem by treating the quaternion state as a whole, and the advantages/contributions of our result can be listed as follows: (1) the finite time A-SYN of QVNNs with delays is realized with just two delay-free controllers. We know one linear negative term cannot realize finite time A-SYN, and three terms can realize A-SYN in [43], in this paper, we prove that only two terms can realize A-SYN, so two can be regarded as the necessary and sufficient terms for finite time A-SYN in this sense; (2) the global boundedness of terms with delay is not required by using 2PM, which is especially important for higher-dimension systems, for example, according to Liouville-theorem, activation functions in CVNN cannot be both bounded and analytic simultaneously; (3) the asynchronous time-varying delays can be unbounded and un-differentiable by using the maximum-value function [11], compared with the bounded and differentiable requirement in [37,38]; (4) the adaptive finite time A-SYN is also realized by designing a suitable adaptive rule and its validity is also strictly proved, which is rarely discussed in mentioned works except for [36].

In Sect. 2, the model description is given, as well as some definitions, assumptions, and lemmas. Sufficient criteria for ensuring (adaptive)finite time A-SYN are derived under 1-norm and 2-norm in Sect. 3. In Sect. 4, a numerical example is presented to show its effectiveness. Finally, Sect. 5 concludes this paper.

2 Model Description

Some notations throughout the whole paper are firstly presented. $\mathbb{R}$ and $\mathbb{H}$ denote the sets of real numbers and quaternions. $\underline{n}$ denotes $\{1, 2, \ldots, n\}$. For any $a = a^R + a^I i + a^J j + a^K k \in$

$\mathbb{H}$, where $i^2 = j^2 = k^2 = -1, ij = k = -ji, jk = i = -kj, ki = j = -ik$, the 1-norm of a is defined as $\|a\|_1 = |a^R| + |a^I| + |a^J| + |a^K|$, and the 2-norm is $\|a\|_2 = \sqrt{a\overline{a}}$, where $\overline{a} = a^R - a^I i - a^J j - a^K k$. For any vector $A = (A_1, A_2, \ldots, A_n) \in \mathbb{R}^{1\times n}$, A^T is its transposition, $A > 0$ means that $A_p > 0$ for any $p \in \underline{n}$.

Next, we present some matrices to show the property of the dot product between two quaternion numbers a and b, where $a = a^R + a^I i + a^J j + a^K k$ and $b = b^R + b^I i + b^J j + b^K k$.

Define a 4-dimensional matrix

$$M = \begin{pmatrix} 1 & i & j & k \\ i & -1 & k & -j \\ j & -k & -1 & i \\ k & j & -i & -1 \end{pmatrix} = M^R + M^I i + M^J j + M^K k, \tag{1}$$

where

$$M^R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad M^I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$M^J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M^K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Definition 1 For any two quaternion numbers $a, b \in \mathbb{H}$, denote

$$\overrightarrow{a} = (a^R, a^I, a^J, a^K)^T, \quad \overrightarrow{b} = (b^R, b^I, b^J, b^K)^T, \tag{2}$$

then

$$ab = \overrightarrow{a}^T M \overrightarrow{b} = \overrightarrow{a}^T M^R \overrightarrow{b} + \overrightarrow{a}^T M^I \overrightarrow{b} i + \overrightarrow{a}^T M^J \overrightarrow{b} j + \overrightarrow{a}^T M^K \overrightarrow{b} k,$$

i.e.,

$$\overrightarrow{ab} = \begin{pmatrix} \overrightarrow{a}^T M^R \overrightarrow{b} \\ \overrightarrow{a}^T M^I \overrightarrow{b} \\ \overrightarrow{a}^T M^J \overrightarrow{b} \\ \overrightarrow{a}^T M^K \overrightarrow{b} \end{pmatrix} = \begin{pmatrix} a^R b^R - a^I b^I - a^J b^J - a^K b^K \\ a^R b^I + a^I b^R + a^J b^K - a^K b^J \\ a^R b^J - a^I b^K + a^J b^R + a^K b^I \\ a^R b^K + a^I b^J - a^J b^I + a^K b^R \end{pmatrix}. \tag{3}$$

Especially, $a\overline{a} = (a^R)^2 + (a^I)^2 + (a^J)^2 + (a^K)^2 = \|a\|_2^2$.

Remark 1 This special notation is also used in [13,43], which can greatly reduce the redundancy of calculation and representation in our proof.

Consider the following QVNN model consisting of n neurons and involving asynchronous time-varying delays:

$$\dot{x}_p(t) = -d_p x_p(t) + \sum_{q=1}^{n} a_{pq} f_q(x_q(t)) + \sum_{q=1}^{n} b_{pq} g_q(x_q(t - \tau_{pq}(t))) + I_p, \tag{4}$$

where $x_p \in \mathbb{H}$ is the state of the pth neuron, $p \in \underline{n}$; $d_p \in \mathbb{H}$ is the feedback self-connection weight; $f_p(\cdot)$ and $g_p(\cdot) : \mathbb{H} \to \mathbb{H}$ are quaternion-valued activation functions without and with time delays; $a_{pq}, b_{pq} \in \mathbb{H}$ denote the connection weights without and with time delays;

$\tau_{pq}(t)$ is the asynchronous time-varying delay, and satisfies $0 \leq \tau_{pq}(t) \leq \tau(t)$, where $\tau(t)$ is the upper bound of all $\tau_{pq}(t)$ and $t - \tau(t) \to +\infty$ if $t \to +\infty$; $I_p(t) \in \mathbb{H}$ is time varying and denotes the bounded external input.

Let (4) be the master system, and the slave system is given as follows:

$$\dot{y}_p(t) = -d_p y_p(t) + \sum_{q=1}^{n} a_{pq} f_q(y_q(t)) + \sum_{q=1}^{n} b_{pq} g_q(y_q(t - \tau_{pq}(t))) + I_p + u_p(t), \quad (5)$$

where $p \in \underline{n}$, and parameters in (5) are the same as those in (4), $u_p(t) \in \mathbb{H}$ is the external controller and will be defined later.

Assumption 1 There exist nonnegative real numbers L_ι^f, L_ι^g, H_ι^f, H_ι^g such that

$$\|f_p(x_p) + f_p(y_p)\|_\iota \leq L_\iota^f \|x_p + y_p\|_\iota + H_\iota^f,$$
$$\|g_p(x_p) + g_p(y_p)\|_\iota \leq L_\iota^g \|x_p + y_p\|_\iota + H_\iota^g,$$

where $f_p(\cdot), g_p(\cdot) : \mathbb{H} \to \mathbb{H}$, $x_p, y_p \in \mathbb{H}$, $\iota = 1, 2$, $p \in \underline{n}$.

Remark 2 If $H_\iota^f = 0$, $H_\iota^g = 0$, for any ι, then above conditions can be regarded as the common used Lipschitz condition, thus the above assumption is more general.

Lemma 1 *For any quaternion numbers $a, b \in \mathbb{H}$, the following properties hold:*

$$\text{(i)}.\ \overline{\overline{a}} = a, \quad \text{(ii)}.\ a + \overline{a} = 2a^R, \quad \text{(iii)}.\ \overline{ab} = \overline{b}\,\overline{a}.$$

Proof Results (i) and (ii) are obvious, we just need to prove (iii). From (3),

$$\overrightarrow{\overline{ab}} = \begin{pmatrix} \overrightarrow{a}^T M^R \overrightarrow{b} \\ -\overrightarrow{a}^T M^I \overrightarrow{b} \\ -\overrightarrow{a}^T M^J \overrightarrow{b} \\ -\overrightarrow{a}^T M^K \overrightarrow{b} \end{pmatrix} = \begin{pmatrix} a^R b^R - a^I b^I - a^J b^J - a^K b^K \\ -a^R b^I - a^I b^R - a^J b^K + a^K b^J \\ -a^R b^J + a^I b^K - a^J b^R - a^K b^I \\ -a^R b^K - a^I b^J + a^J b^I - a^K b^R \end{pmatrix}, \quad (6)$$

and

$$\overrightarrow{\overline{b}\,\overline{a}} = \begin{pmatrix} \overrightarrow{\overline{b}}^T M^R \overrightarrow{\overline{a}} \\ \overrightarrow{\overline{b}}^T M^I \overrightarrow{\overline{a}} \\ \overrightarrow{\overline{b}}^T M^J \overrightarrow{\overline{a}} \\ \overrightarrow{\overline{b}}^T M^K \overrightarrow{\overline{a}} \end{pmatrix} = \begin{pmatrix} b^R a^R - (-b^I)(-a^I) - (-b^J)(-a^J) - (-b^K)(-a^K) \\ b^R(-a^I) + (-b^I)a^R + (-b^J)(-a^K) - (-b^K)(-a^J) \\ b^R(-a^J) - (-b^I)(-a^K) + (-b^J)a^R + (-b^K)(-a^I) \\ b^R(-a^K) + (-b^I)(-a^J) - (-b^J)(-a^I) + (-b^K)a^R \end{pmatrix}$$

$$= \begin{pmatrix} a^R b^R - a^I b^I - a^J b^J - a^K b^K \\ -a^R b^I - a^I b^R - a^J b^K + a^K b^J \\ -a^R b^J + a^I b^K - a^J b^R - a^K b^I \\ -a^R b^K - a^I b^J + a^J b^I - a^K b^R \end{pmatrix}.$$

Therefore, $\overrightarrow{\overline{ab}} = \overrightarrow{\overline{b}\,\overline{a}}$, i.e., $\overline{ab} = \overline{b}\,\overline{a}$. □

Definition 2 A sign function for quaternion variables $a = a^R + a^I i + a^J j + a^K k \in \mathbb{H}$ can be defined as

$$\mathbf{sig}(a) \triangleq \mathrm{sign}(a^R) + \mathrm{sign}(a^I)i + \mathrm{sign}(a^J)j + \mathrm{sign}(a^K)k. \quad (7)$$

Similar definition of sign function for complex-variables can be found in [34,37]. This new sign function takes the variable as a whole and will play an important role for the finite time A-SYN analysis, whose properties are presented in the next.

Lemma 2 *For any* $e(t):\mathbb{R}\to\mathbb{H}$,

$$\begin{aligned}
&\text{(i)}.\ \overline{\mathbf{sig}(e(t))}e(t)+\overline{e(t)}\mathbf{sig}(e(t))=2\|e(t)\|_1\geq 2\|e(t)\|_2,\\
&\text{(ii)}.\ \mathbf{sig}(e(t))\overline{\mathbf{sig}(e(t))}=\overline{\mathbf{sig}(e(t))}\mathbf{sig}(e(t))=\|\mathbf{sig}(e(t))\|_1,\\
&\text{(iii)}.\ \frac{d\|e(t)\|_1}{dt}=\frac{1}{2}\left(\overline{\mathbf{sig}(e(t))}\frac{de(t)}{dt}+\frac{d\overline{e(t)}}{dt}\mathbf{sig}(e(t))\right).
\end{aligned}$$

Proof According to Lemma 1, we have

$$\begin{aligned}
\overline{\mathbf{sig}(e(t))}e(t)+\overline{e(t)}\mathbf{sig}(e(t))&=2\overrightarrow{\mathbf{sig}(e(t))}^T M^R\overrightarrow{e(t)}=2\overrightarrow{\mathbf{sig}(e(t))}^T\overrightarrow{e(t)}\\
&=2(|e^R(t)|+|e^I(t)|+|e^J(t)|+|e^K(t)|)=2\|e(t)\|_1\geq 2\|e(t)\|_2,
\end{aligned}$$

where the last inequality is based on norm equivalence property, so (i) is proved.

As for (ii), according to (3), $\overrightarrow{\mathbf{sig}(e(t))\overline{\mathbf{sig}(e(t))}}=(\text{sign}^2(e^R(t))+\text{sign}^2(e^I(t))+\text{sign}^2(e^J(t))+\text{sign}^2(e^K(t)),0,0,0)^T$, therefore, $\mathbf{sig}(e(t))\overline{\mathbf{sig}(e(t))}=|\text{sign}(e^R(t))|+|\text{sign}(e^I(t))|+|\text{sign}(e^J(t))|+|\text{sign}(e^K(t))|=\|\mathbf{sig}(e(t))\|_1$, similar arguments can also be used for $\overline{\mathbf{sig}(e(t))}\mathbf{sig}(e(t))=\|\mathbf{sig}(e(t))\|_1$.

As for (iii), it can be directly obtained by differentiating (i). □

Remark 3 Above results allow us to study the QVNNs without decomposition. We can use 1-norm and 2-norm to quantify the error and convert the norms into the simple quaternion state instead of expanding all dimensions.

Definition 3 For any vector $v=(v_1,v_2,\ldots,v_n)\in\mathbb{R}^{1\times n}$, its ∞-norm is defined as: $\|v\|_\infty=\max_{p\in\underline{n}}|v_p|$. Especially, for any $e(t)=(e_1(t),\ldots,e_n(t))^T$, where $e_p(t)\in\mathbb{H}$, then we can define

$$\|e(t)\|_{\{\iota,\infty\}}=\max_{p\in\underline{n}}\|e_p(s)\|_\iota. \tag{8}$$

Remark 4 ∞-norm has the advantage to deal with asynchronous time delays. A generalized ∞-norm is used to investigate finite time A-SYN for CVNNs [43], which can also be used here, but in order to state our main results more clearly, we adopt the classical ∞-norm, interested readers are encouraged to try by yourself.

Definition 4 QVNNs (4) and (5) are said to achieve finite time A-SYN, if there exists a settling time $\mathcal{T}$ depending on (or not) initial functionals, such that

$$\lim_{t\to\mathcal{T}}\|x(t)+y(t)\|_{\{\iota,\infty\}}=0,\quad\text{and}\quad\|x(t)+y(t)\|_{\{\iota,\infty\}}=0,\quad t\geq\mathcal{T},$$

where $x(t)=(x_1(t),\ldots,x_n(t))^T$ and $y(t)=(y_1(t),\ldots,y_n(t))^T$, ι can be 1 or 2.

Remark 5 In fact, finite time SYN can also be defined by replacing $x(t)+y(t)$ by $x(t)-y(t)$. Since A-SYN is more complex and difficult to realize than SYN, so we just discuss finite time A-SYN in this paper, interested readers can deduce finite time SYN by yourself according to our following analysis in the next section.

3 Main Results

Define $e(t) = x(t) + y(t)$ as the A-SYN error between (4) and (5), thus the error system is as follows:

$$
\begin{aligned}
\dot{e}_p(t) = -d_p e_p(t) &+ \sum_{q=1}^{n} a_{pq} \tilde{f}_q(e_q(t)) \\
&+ \sum_{q=1}^{n} b_{pq} \tilde{g}_q(e_q(t - \tau_{pq}(t))) + 2I_p + u_p(t),
\end{aligned} \tag{9}
$$

where $\tilde{f}_q(e_q(t)) = f_q(x_q(t)) + f_q(y_q(t))$, $\tilde{g}_q(e_q(t - \tau_{pq}(t))) = g_q(x_q(t - \tau_{pq}(t))) + g_q(y_q(t - \tau_{pq}(t))$. Initial states of (9) are denoted as $e_p(\theta), \theta \in (-\tau(0), 0]$, $p \in \underline{n}$.

The delay-free controller now can be designed as:

$$
u_p(t) = -\lambda_p e_p(t) - \rho_p \mathbf{sig}(e_p(t)), \tag{10}
$$

where $\lambda_p, \rho_p \in \mathbb{R}$ are positive real constants.

Assumption 2 With the above controller, a necessary condition for A-SYN is that: when $e(t) = 0$, $I_p(t) = 0$, $\tilde{f}_p(e_p(t)) = 0$ and $\tilde{g}_p(e_p(t)) = 0$, $p \in \underline{n}$.

Then we define a special function $\mu(t)$ [50] which is nondecreasing and satisfies the following three properties:

$$
\lim_{t\to+\infty} \mu(t) = +\infty, \quad \lim_{t\to+\infty} \frac{\dot{\mu}(t)}{\mu(t)} = \varsigma, \quad \lim_{t\to+\infty} \frac{\mu(t)}{\mu(t - \tau(t))} = 1 + \eta, \tag{11}
$$

where ς and η are nonnegative constants.

Theorem 1 *Under Assumption 1, 2 and controller (10), the master–slave coupled QVNNs (4) and (5) can achieve finite time A-SYN if*

$$
\lambda_p > \varsigma + |d_p^I| + |d_p^J| + |d_p^K| - d_p^R + \mathcal{A}_{p1} + (1 + \eta)\mathcal{B}_{p1}, \tag{12}
$$

$$
\rho_p > \mathcal{B}_{p1} + \mathcal{C}_{p1} + 2\|I_p\|_1, \tag{13}
$$

where

$$
\mathcal{A}_{p1} = L_1^f \sum_{q=1}^{n} \|a_{pq}\|_1, \quad \mathcal{B}_{p1} = L_1^g \sum_{q=1}^{n} \|b_{pq}\|_1, \quad \mathcal{C}_{p1} = \sum_{q=1}^{n} (H_1^f \|a_{pq}\|_1 + H_1^g \|b_{pq}\|_1). \tag{14}
$$

Proof According to 2PM, the whole process can be analyzed in two phases. From condition (12), there must exist a time $\mathcal{T}_0$ such that

$$
\frac{\dot{\mu}(t)}{\mu(t)} + |d_p^I| + |d_p^J| + |d_p^K| - d_p^R + \mathcal{A}_{p1} + \frac{\mu(t)}{\mu(t - \tau(t))} \mathcal{B}_{p1} - \lambda_p < 0 \tag{15}
$$

holds for all $t \geq \mathcal{T}_0$ and $p \in \underline{n}$.

The following discussions are all from $\mathcal{T}_0$, which is just determined by time delays but not initial values.

Phase 1: We prove that $\sup_{t-\tau(t)\leq s\leq t} \|e(s)\|_{\{1,\infty\}}$ will reach 1 in finite time.

We define a maximum-value function as

$$
\mathcal{M}(t) = \sup_{t-\tau(t)\leq s\leq t} \left(\mu(s)\|e(s)\|_{\{1,\infty\}}\right) = \sup_{t-\tau(t)\leq s\leq t} \left(\mu(s) \max_{p\in\underline{n}} \|e_p(s)\|_1\right). \tag{16}
$$

Obviously, $\mu(t)\|e_p(t)\|_1 \le \mathcal{M}(t)$ holds for any $p \in \underline{n}$, $t \ge \mathcal{T}_0$. Moreover, this fact contains two cases.

(I) If $\mu(t)\max_{p\in\underline{n}}\|e_p(t)\|_1 < \mathcal{M}(t)$, then there must exist a constant $\delta_1 > 0$ such that $\mathcal{M}(s) \le \mathcal{M}(t)$ for any $s \in (t, t+\delta_1)$.

(II) If there exist an index p_1 and a time point $t_1 (t_1 \ge \mathcal{T}_0)$ such that $\mu(t_1)\|e_{p_1}(t_1)\|_1 = \mathcal{M}(t_1)$, then

$$\frac{d(\mu(t)\|e_p(t)\|_1)}{dt}\bigg|_{p=p_1,t=t_1} = \left[\dot{\mu}(t)\|e_p(t)\|_1 + \mu(t)\frac{d\|e_p(t)\|_1}{dt}\right]\bigg|_{p=p_1,t=t_1} \tag{17}$$

From Lemma 2, we have

$$\begin{aligned}
\frac{d\|e_p(t)\|_1}{dt} &= \frac{1}{2}\left(\overline{\mathbf{sig}(e_p(t))}\frac{de_p(t)}{dt} + \frac{d\overline{e_p(t)}}{dt}\mathbf{sig}(e_p(t))\right)\\
&= -\frac{1}{2}\left(\overline{\mathbf{sig}(e_p(t))}d_pe_p(t) + \overline{d_pe_p(t)}\mathbf{sig}(e_p(t))\right)\\
&\quad + \frac{1}{2}\sum_{q=1}^{n}\left(\overline{\mathbf{sig}(e_p(t))}a_{pq}\tilde{f}_q(e_q(t)) + \overline{a_{pq}\tilde{f}_q(e_q(t))}\mathbf{sig}(e_p(t))\right)\\
&\quad + \frac{1}{2}\sum_{q=1}^{n}\left(\overline{\mathbf{sig}(e_p(t))}b_{pq}\tilde{g}_q(e_q(t-\tau_{pq}(t)))\right.\\
&\quad \left.+ \overline{b_{pq}\tilde{g}_q(e_q(t-\tau_{pq}(t)))}\mathbf{sig}(e_p(t))\right)\\
&\quad + \left(\overline{\mathbf{sig}(e_p(t))}I_p + \overline{I_p}\mathbf{sig}(e_p(t))\right) + \frac{1}{2}\left(\overline{\mathbf{sig}(e_p(t))}u_p(t) + \overline{u_p(t)}\mathbf{sig}(e_p(t))\right).
\end{aligned} \tag{18}$$

In the next, we will analyze each term separately in (18) by using (3) and (6).

$$\begin{aligned}
&-\frac{1}{2}\left(\overline{\mathbf{sig}(e_p(t))}d_pe_p(t) + \overline{d_pe_p(t)}\mathbf{sig}(e_p(t))\right)\\
&= -(\overline{\mathbf{sig}(e_p(t))}d_pe_p(t))^R = -\overrightarrow{\overline{\mathbf{sig}(e_p(t))}}^T M^R\overrightarrow{\overline{d_pe_p(t)}}\\
&= -\mathbf{sig}(e_p(t))^R\overrightarrow{d_p}^T M^R\overrightarrow{e_p(t)} - \mathbf{sig}(e_p(t))^I\overrightarrow{d_p}^T M^I\overrightarrow{e_p(t)}\\
&\quad - \mathbf{sig}(e_p(t))^J\overrightarrow{d_p}^T M^J\overrightarrow{e_p(t)} - \mathbf{sig}(e_p(t))^K\overrightarrow{d_p}^T M^K\overrightarrow{e_p(t)}\\
&= -|e_p(t)^R|d_p^R + (e_p(t)^Id_p^I + e_p(t)^Jd_p^J + e_p(t)^Kd_p^K)\mathrm{sign}(e_p(t)^R)\\
&\quad - |e_p(t)^I|d_p^R + (-e_p(t)^Rd_p^I + e_p(t)^Jd_p^K - e_p(t)^Kd_p^J)\mathrm{sign}(e_p(t)^I)\\
&\quad - |e_p(t)^J|d_p^R + (-e_p(t)^Rd_p^J - e_p(t)^Id_p^K + e_p(t)^Kd_p^I)\mathrm{sign}(e_p(t)^J)\\
&\quad - |e_p(t)^K|d_p^R + (-e_p(t)^Rd_p^K + e_p(t)^Id_p^J - e_p(t)^Jd_p^I)\mathrm{sign}(e_p(t)^K)\\
&\le -\|e_p(t)\|_1d_p^R + |e_p(t)^I||d_p^I| + |e_p(t)^J||d_p^J| + |e_p(t)^K||d_p^K|\\
&\quad + |e_p(t)^R||d_p^I| + |e_p(t)^J||d_p^K| + |e_p(t)^K||d_p^J|\\
&\quad + |e_p(t)^R||d_p^J| + |e_p(t)^I||d_p^K| + |e_p(t)^K||d_p^I|\\
&\quad + |e_p(t)^R||d_p^K| + |e_p(t)^I||d_p^J| + |e_p(t)^J||d_p^I|\\
&= (|d_p^I| + |d_p^J| + |d_p^K| - d_p^R)\|e_p(t)\|_1.
\end{aligned} \tag{19}$$

With the same argument as (19),

$$
\begin{aligned}
&\frac{1}{2}\sum_{q=1}^{n}\Big(\overline{\mathbf{sig}(e_p(t))}a_{pq}\tilde{f}_q(e_q(t)) + \overline{a_{pq}\tilde{f}_q(e_q(t))}\mathbf{sig}(e_p(t))\Big)\\
&= \sum_{q=1}^{n}\Big(\overline{\mathbf{sig}(e_p(t))}a_{pq}\tilde{f}_q(e_q(t))\Big)^R \le \sum_{q=1}^{n}\|a_{pq}\|_1\|\tilde{f}_q(e_q(t))\|_1\\
&\le \sum_{q=1}^{n}\|a_{pq}\|_1(L_1^f\|e_q(t)\|_1 + H_1^f) \qquad (20)\\
&= \frac{\mathcal{M}(t)}{\mu(t)}L_1^f\sum_{q=1}^{n}\|a_{pq}\|_1 + H_1^f\sum_{q=1}^{n}\|a_{pq}\|_1. \qquad (21)
\end{aligned}
$$

Similarly, according to the definition of $\mathcal{M}(t)$ in (16), we have

$$
\begin{aligned}
&\frac{1}{2}\sum_{q=1}^{n}\Big(\overline{\mathbf{sig}(e_p(t))}b_{pq}\tilde{g}_q(e_q(t-\tau_{pq}(t))) + \overline{b_{pq}\tilde{g}_q(e_q(t-\tau_{pq}(t)))}\mathbf{sig}(e_p(t))\Big)\\
&\le \sum_{q=1}^{n}\|b_{pq}\|_1\big(L_1^g\|e_q(t-\tau_{pq}(t))\|_1 + H_1^g\big) \qquad (22)\\
&= \frac{1}{\mu(t)}L_1^g\sum_{q=1}^{n}\|b_{pq}\|_1\frac{\mu(t)}{\mu(t-\tau_{pq}(t))}\mu(t-\tau_{pq}(t))\|e_q(t-\tau_{pq}(t))\|_1 + H_1^g\sum_{q=1}^{n}\|b_{pq}\|_1\\
&\le \frac{\mathcal{M}(t)}{\mu(t)}\frac{\mu(t)}{\mu(t-\tau(t))}L_1^g\sum_{q=1}^{n}\|b_{pq}\|_1 + H_1^g\sum_{q=1}^{n}\|b_{pq}\|_1. \qquad (23)
\end{aligned}
$$

Moreover,

$$
\overline{\mathbf{sig}(e_p(t))}I_p + \overline{I_p}\mathbf{sig}(e_p(t)) \le 2\|I_p\|_1, \qquad (24)
$$

$$
\frac{1}{2}\Big(\lambda_p\big(\overline{\mathbf{sig}(e_p(t))}u_p(t) + \overline{u_p(t)}\mathbf{sig}(e_p(t))\big)\Big) \le -\lambda_p\|e_p(t)\|_1 - \rho_p. \qquad (25)
$$

Therefore, according to (17)–(19), (21), (23)–(25), we have

$$
\begin{aligned}
&\frac{d(\mu(t)\|e_p(t)\|_1)}{dt}\bigg|_{p=p_1,t=t_1}\\
&\le \Big(\frac{\dot{\mu}(t)}{\mu(t)} + |d_p^I| + |d_p^J| + |d_p^K| - d_p^R + \mathcal{A}_{p1} + \frac{\mu(t)}{\mu(t-\tau(t))}\mathcal{B}_{p1} - \lambda_p\Big)\mathcal{M}(t)\\
&\quad + (\mathcal{C}_{p1} + 2\|I_p\|_1 - \rho_p)\mu(t) < 0. \qquad (26)
\end{aligned}
$$

Through the two cases presented above, we have proved that $\mathcal{M}(t)$ is non-increasing for all $t \ge \mathcal{T}_0$, which means that

$$
\mu(t-\tau(t))\sup_{t-\tau(t)\le s\le t}(\|e(s)\|_{\{1,\infty\}}) \le \mathcal{M}(t) \le \mathcal{M}(\mathcal{T}_0),
$$

i.e.,

$$
\sup_{t-\tau(t)\le s\le t}(\|e(s)\|_{\{1,\infty\}}) \le \frac{\mathcal{M}(\mathcal{T}_0)}{\mu(t-\tau(t))}. \qquad (27)
$$

The properties of $\mu(t)$ are given in condition (11), it is easy to see that $\lim_{t\to+\infty}\mu(t-\tau(t))=+\infty$. According to the intermediate value theorem, there exists a time point $\mathcal{T}_1(\mathcal{T}_1\ge\mathcal{T}_0)$ such that $\sup_{t-\tau(t)\le s\le t}(\|e(s)\|_{\{1,\infty\}})\le 1$ holds for all $t\ge\mathcal{T}_1$, hence the proof of phase 1 is completed.

Phase 2: We prove that $\sup_{t-\tau(t)\le s\le t}\|e(s)\|_{\{1,\infty\}}$ will flow to 0 in finite time.

From condition (13), we can pick a small constant ϑ such that

$$0<\vartheta<\rho_p-\mathcal{B}_{p1}-\mathcal{C}_{p1}-2\|I_p\|_1, \tag{28}$$

holds for all $p\in\underline{n}$. We define another function

$$\mathcal{V}(t)=\sup_{t-\tau(t)\le s\le t}(\|e(s)\|_{\{1,\infty\}}+\vartheta s)=\sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_1+\vartheta s). \tag{29}$$

Similar to the proof in phase 1, we will analyze its property in two cases.

(I) If $\max_{p\in\underline{n}}\|e_p(t)\|_1+\vartheta t<\mathcal{V}(t)$, for all $p\in\underline{n}$, then there must exist a constant $\delta_2>0$ such that $\mathcal{V}(s)\le\mathcal{V}(t)$ for any $s\in(t,t+\delta_2)$.

(II) If there exist an index p_2 and a time point $t_2(t_2\ge\mathcal{T}_1)$, such that $\|e_{p_2}(t_2)\|_1+\vartheta t_2=\mathcal{V}(t_2)$, then according to (18)–(20), (22), (24), (25),

$$\begin{aligned}\frac{d}{dt}\Big(\|e_p(t)\|_1+\vartheta t\Big)\Big|_{p=p_2,t=t_2}&=\frac{d\|e_p(t)\|_1}{dt}\Big|_{p=p_2,t=t_2}+\vartheta\\&=(|d_p^I|+|d_p^J|+|d_p^K|-d_p^R+\mathcal{A}_{p1}-\lambda_p)\|e_p(t)\|_1\\&\quad+L_1^g\sum_{q=1}^n\|b_{pq}\|_1\|e_q(t-\tau_{pq}(t))\|_1+\mathcal{C}_{p1}+2\|I_p\|_1-\rho_p+\vartheta \qquad (30)\\&\le\mathcal{B}_{p1}+\mathcal{C}_{p1}+2\|I_p\|_1-\rho_p+\vartheta<0, \qquad (31)\end{aligned}$$

where the inequality from (30) to (31) is based on the result in phase 1, where we have already proved that $\sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_1)\le 1$ holds for all $t\ge\mathcal{T}_1$. As a result, $\|e_q(t-\tau_{pq}(t))\|_1\le 1$ always holds in phase 2.

Hence, $\mathcal{V}(t)$ is a non-increasing function,

$$\|e(t)\|_{\{1,\infty\}}+\vartheta t\le\mathcal{V}(t)\le\mathcal{V}(\mathcal{T}_1)\le 1+\vartheta\mathcal{T}_1,$$

i.e.,

$$\|e(t)\|_{\{1,\infty\}}\le 1-\vartheta(t-\mathcal{T}_1),$$

It is clear that $\|e(t)\|_{\{1,\infty\}}$ converges to 0 as time t increases gradually. We denote $\mathcal{T}_2$ as the first time it reaches 0, where

$$\mathcal{T}_2=\mathcal{T}_1+\frac{1}{\vartheta}, \tag{32}$$

so, QVNNs (4) and (5) will achieve finite time A-SYN no longer than $\mathcal{T}_2$. □

Remark 6 From (12), d_p^R is important for A-SYN, that is to say, if d_p^R is large enough, then λ_p can be chosen as zero, i.e., the term $-\lambda_p e_p(t)$ can be eliminated.

Remark 7 A special case is that Assumption 1 holds with H_1^f and H_1^g being zero, and $I_p=0$, which can be happened in general SYN problem, in this case, $\mathcal{C}_{p1}=\|I_p\|_1=0$, then one can use the following switching-type controller to realize finite time SYN/A-SYN,

$$u_p(t)=\begin{cases}-\lambda_p e_p(t), & \text{if } \sup_{t-\tau(t)\le s\le t}\|e(s)\|_{\{1,\infty\}}>1,\\ -\rho_p\mathbf{sig}(e_p(t)), & \text{otherwise}\end{cases} \tag{33}$$

The time delay in model (9) is said to be asynchronous and time-varying, in fact, we are able to deal with mixed delays, like distributed time delays in [16]. Consider error QVNN described as follows:

$$\dot{e}_p(t) = -d_p e_p(t) + \sum_{q=1}^{n} a_{pq} \tilde{f}_q(e_q(t)) + \sum_{q=1}^{n} b_{pq} \tilde{g}_q(e_q(t - \tau_{pq}(t)))$$
$$+ \sum_{q=1}^{n} \xi_{pq} \int_{t-\tau}^{t} \tilde{\phi}_q(e_q(s))ds + 2I_p + u_p(t), \quad p \in \underline{n}, \tag{34}$$

where $\xi_{pq} \in \mathbb{H}$ is the quaternion-valued distributed connection weight, $\tilde{\phi}_q(e_q(t)) = \phi_q(x_q(t)) + \phi_q(y_q(t))$, and $\phi_q(\cdot) : \mathbb{H} \to \mathbb{H}$ is the quaternion-valued distributed activation function, τ is a constant delay.

Corollary 1 *Suppose that Assumptions 1 and 2 holds, and there also exist nonnegative constants L_1^{ϕ} and H_1^{ϕ}, such that $\|\phi_p(x_p) + \phi_p(y_p)\|_1 \leq L_1^{\phi} \|x_p + y_p\|_1 + H_1^{\phi}$ holds. QVNN (34) with delay-free controller (10) will achieve finite time A-SYN if:*

$$\lambda_p > \varsigma + |d_p^I| + |d_p^J| + |d_p^K| - d_p^R + \mathcal{A}'_{p1} + (1+\eta)\mathcal{B}_{p1},$$
$$\rho_p > \mathcal{B}_{p1} + \mathcal{C}'_{p1} + 2\|I_p\|_1.$$

where $\mathcal{A}'_{p1} = A_{p1} L_1^f + \tau L_1^{\phi} \sum_{q=1}^{n} \|\xi_{pq}\|_1$, $\mathcal{C}'_{p1} = \mathcal{C}_{p1} + \tau \sum_{q=1}^{n} \|\xi_{pq}\|_1 (L_1^{\phi} + H_1^{\phi})$.

Theorem 1 is presented under 1-norm, in fact, the result can also be given under 2-norm.

Theorem 2 *Based on Assumption 1, 2, and controller (10), the master–slave coupled QVNNs (4) and (5) can achieve finite time A-SYN if*

$$\lambda_p > \frac{\varsigma}{2} - d_p^R + \mathcal{A}_{p2} + \sqrt{1+\eta}\mathcal{B}_{p2}, \tag{35}$$
$$\rho_p > \mathcal{B}_{p2} + \mathcal{C}_{p2} + 2\|I_p\|_2, \tag{36}$$

hold for all $p \in \underline{n}$, where

$$\mathcal{A}_{p2} = L_2^f \sum_{q=1}^{n} \|a_{pq}\|_2, \quad \mathcal{B}_{p2} = L_2^g \sum_{q=1}^{n} \|b_{pq}\|_2, \quad \mathcal{C}_{p2} = \sum_{q=1}^{n} (H_2^f \|a_{pq}\|_2 + H_2^g \|b_{pq}\|_2).$$

Proof The whole proof is similar to that in Theorem 1 except the deductions related to properties of 1-norm, so some details may be omitted.

Phase 1: According to (35), there must exist a time point $\mathcal{T}_0^{\star}$ such that

$$\frac{\dot{\mu}(t)}{\mu(t)} - 2d_p^R + 2\mathcal{A}_{p2} + 2\mathcal{B}_{p2} \frac{\sqrt{\mu(t)}}{\sqrt{\mu(t-\tau(t))}} - 2\lambda_p < 0. \tag{37}$$

We define

$$\mathcal{M}^{\star}(t) = \sup_{t-\tau(t) \leq s \leq t} \left(\mu(s) \max_{p \in \underline{n}} \|e_p(s)\|_2^2 \right), \qquad t \geq \mathcal{T}_0^{\star}. \tag{38}$$

If there exist an index p and a time point t such that $\mu(t)\|e_p(t)\|_2^2 = \mathcal{M}^{\star}(t)$, then

$$\frac{d(\mu(t)\|e_p(t)\|_2^2)}{dt} = \dot{\mu}(t)\|e_p(t)\|_2^2 + \mu(t)\frac{d\|e_p(t)\|_2^2}{dt}, \tag{39}$$

and

$$
\begin{aligned}
\frac{d\|e_p(t)\|_2^2}{dt} &= \frac{d}{dt} e_p(t)\overline{e_p(t)} \\
&= -\Big(d_p e_p(t)\overline{e_p(t)} + e_p(t)\overline{d_p e_p(t)}\Big) \\
&\quad + \sum_{q=1}^{n}\Big(a_{pq}\tilde{f}_q(e_q(t))\overline{e_p(t)} + e_p(t)\overline{a_{pq}\tilde{f}_q(e_q(t))}\Big) \\
&\quad + \sum_{q=1}^{n}\Big(b_{pq}\tilde{g}_q(e_q(t-\tau_{pq}(t)))\overline{e_p(t)} + e_p(t)\overline{b_{pq}\tilde{g}_q(e_q(t-\tau_{pq}(t)))}\Big) \\
&\quad + 2\Big(I_p\overline{e_p(t)} + e_p(t)\overline{I_p}\Big) - \lambda_p\Big(e_p(t)\overline{e_p(t)} + \overline{e_p(t)}e_p(t)\Big) \\
&\quad - \rho_p\Big(\mathbf{sig}(e_p(t))\overline{e_p(t)} + e_p(t)\overline{\mathbf{sig}(e_p(t))}\Big).
\end{aligned} \tag{40}
$$

For any $p \in \underline{n}$, according to Lemma 1,

$$
\begin{aligned}
d_p e_p(t)\overline{e_p(t)} + e_p(t)\overline{d_p e_p(t)} &= d_p\|e_p(t)\|_2^2 + e_p(t)\overline{e_p(t)}\overline{d_p} \\
&= (d_p + \overline{d_p})\|e_p(t)\|_2^2 = 2d_p^R\|e_p(t)\|_2^2.
\end{aligned} \tag{41}
$$

From Assumption 1, one has

$$
\begin{aligned}
&\sum_{q=1}^{n}\Big(a_{pq}\tilde{f}_q(e_q(t))\overline{e_p(t)} + e_p(t)\overline{a_{pq}\tilde{f}_q(e_q(t))}\Big) \\
&\quad \le 2\sum_{q=1}^{n}\Big(\|a_{pq}\|_2 L_2^f\|e_q(t)\|_2\|e_p(t)\|_2 + \|a_{pq}\|_2 H_2^f\|e_p(t)\|_2\Big) \\
&\quad \le 2\sum_{q=1}^{n}\Big(\|a_{pq}\|_2 L_2^f\|e_p(t)\|_2^2 + \|a_{pq}\|_2 H_2^f\|e_p(t)\|_2\Big)
\end{aligned} \tag{42}
$$

Similarly,

$$
\sum_{q=1}^{n}\Big(b_{pq}\tilde{g}_q(e_q(t-\tau_{pq}(t)))\overline{e_p(t)} + e_p(t)\overline{b_{pq}\tilde{g}_q(e_q(t-\tau_{pq}(t)))}\Big)
$$

$$
\le 2\sum_{q=1}^{n}\Big(\|b_{pq}\|_2 L_2^g\|e_q(t-\tau_{pq}(t))\|_2\|e_p(t)\|_2 + \|b_{pq}\|_2 H_2^g\|e_p(t)\|_2\Big) \tag{43}
$$

$$
\le 2\sum_{q=1}^{n}\Big(\|b_{pq}\|_2 L_2^g\frac{\sqrt{\mu(t)}}{\sqrt{\mu(t-\tau(t))}}\|e_p(t)\|_2^2 + \|b_{pq}\|_2 H_2^g\|e_p(t)\|_2\Big). \tag{44}
$$

Moreover,

$$
I_p\overline{e_p(t)} + e_p(t)\overline{I_p} \le 2\|I_p\|_2\|e_p(t)\|_2, \tag{45}
$$

and from Lemma 2,

$$
\begin{aligned}
&-\lambda_p\Big(e_p(t)\overline{e_p(t)} + \overline{e_p(t)}e_p(t)\Big) - \rho_p\Big(\mathbf{sig}(e_p(t))\overline{e_p(t)} + e_p(t)\overline{\mathbf{sig}(e_p(t))}\Big) \\
&\quad \le -2\lambda_p\|e_p(t)\|_2^2 - 2\rho_p\|e_p(t)\|_2.
\end{aligned} \tag{46}
$$

Substituting (40)–(42) and (44)–(46) into (39), we have

$$\begin{aligned}
&\frac{d(\mu(t)\|e_p(t)\|_2^2)}{dt}\\
&\quad\le \mu(t)\Big\{\Big(\frac{\dot{\mu}(t)}{\mu(t)} - 2d_p^R + 2\mathcal{A}_{p2} + 2\mathcal{B}_{p2}\frac{\sqrt{\mu(t)}}{\sqrt{\mu(t-\tau(t))}} - 2\lambda_p\Big)\|e_p(t)\|_2^2\\
&\qquad + \Big(2\mathcal{C}_{p2} + 4\|I_p\|_2 - 2\rho_p\Big)\|e_p(t)\|_2\Big\} < 0,
\end{aligned}\tag{47}$$

which implies that $\mathcal{M}^\star(t)$ is non-increasing for all $t \ge \mathcal{T}_0^\star$, and

$$\sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2) \le \sqrt{\frac{\mathcal{M}^\star(\mathcal{T}_0^\star)}{\mu(t-\tau(t))}}.$$

According to the increasing property of $\mu(t)$, there must exist a time point $\mathcal{T}_1^\star \ge \mathcal{T}_0^\star$ such that $\sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2) \le 1$ holds for all $t \ge \mathcal{T}_1^\star$.

Phase 2: Based on (36), we can choose a small constant $\vartheta^\star$ such that

$$0 < \vartheta^\star < \rho_p - \mathcal{B}_{p2} - \mathcal{C}_{p2} - 2\|I_p\|_2 \tag{48}$$

holds for all $p \in \underline{n}$. Then we define

$$\mathcal{V}^\star(t) = \sup_{t-\tau(t)\le s\le t}(\|e(s)\|_{\{2,\infty\}} + \vartheta^\star s) = \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2 + \vartheta^\star s). \tag{49}$$

If there exist an index p and a time point $t \ge \mathcal{T}_1^\star$ such that $\|e_p(t)\|_2 + \vartheta^\star t = \mathcal{V}^\star(t)$, then from inequalities (40)–(43), (45), and (46), we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\|e_p(t)\|_2 + \vartheta^\star t\Big) = \frac{d}{dt}\Big((e_p(t)\overline{e_p(t)})^{\frac{1}{2}}\Big) + \vartheta^\star\\
&= \frac{1}{2}(e_p(t)\overline{e_p(t)})^{-\frac{1}{2}}\frac{d}{dt}e_p(t)\overline{e_p(t)} + \vartheta^\star\\
&\le (e_p(t)\overline{e_p(t)})^{-\frac{1}{2}}\Big\{(-d_p^R + \mathcal{A}_{p2} - \lambda_p)\|e_p(t)\|_2^2\\
&\quad + (\sum_{q=1}^{n}\|b_{pq}\|_2 L_2^g\|e_q(t-\tau_{pq}(t))\|_2 + \mathcal{C}_{p2} + 2\|I_p\|_2 - \rho_p)\|e_p(t)\|_2\Big\} + \vartheta^\star \qquad (50)\\
&\le \|e_p(t)\|_2^{-1}\Big((\mathcal{B}_{p2} + \mathcal{C}_{p2} + 2\|I_p\|_2 - \rho_p)\|e_p(t)\|_2\Big) + \vartheta^\star \qquad (51)\\
&= \mathcal{B}_{p2} + \mathcal{C}_{p2} + 2\|I_p\|_2 - \rho_p + \vartheta^\star < 0,
\end{aligned}$$

where the reasons from (50) to (51) are condition (35) and the fact proved in phase 1 that $\sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2) \le 1$.

Hence, $\mathcal{V}^\star(t)$ is proved to be a non-increasing function, so

$$\|e(t)\|_{\{2,\infty\}} + \vartheta^\star t \le \mathcal{V}^\star(t) \le \mathcal{V}^\star(\mathcal{T}_1^\star) \le 1 + \vartheta^\star\mathcal{T}_1^\star,$$

i.e., $\|e(t)\|_{\{2,\infty\}} \le 1 - \vartheta^\star(t - \mathcal{T}_1^\star)$, if we denote $\mathcal{T}_2^\star$ as

$$\mathcal{T}_2^\star = \mathcal{T}_1^\star + \frac{1}{\vartheta^\star}, \tag{52}$$

then $\|e(t)\|_{\{2,\infty\}}$ will be 0 after $\mathcal{T}_2^\star$, so finite time A-SYN is finally realized. □

Remark 8 From above discussions, we have considered 1-norm and 2-norm, in fact, we can also consider the ∞-norm for quaternions, but this norm would need the decomposition technique used in our previous paper [43], which violates the idea that we will treat quaternions as a whole, so we do not consider ∞-norm in this paper, interested readers can consider this norm as [43].

In fact, we can improve our previous theoretical results by applying the adaptive technique on the control strengths, since adaptive technique is especially powerful in circumstances with unknown parameters.

At first, we consider 1-norm, then delay-free controller with adaptive control strengths can be designed as:

$$u_p(t) = -\lambda_p(t)e_p(t) - \rho_p(t)\mathbf{sig}(e_p(t)), \tag{53}$$

with

$$\dot{\lambda}_p(t) = \begin{cases} \omega_{p1}\mu(t)\|e_p(t)\|_1, & \text{if } \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) \ge 1 \\ \omega_{p1}\|e_p(t)\|_1, & \text{if } 0 < \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) < 1 \\ 0, & \text{if } \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) = 0 \end{cases} \tag{54}$$

and

$$\dot{\rho}_p(t) = \begin{cases} \omega_{p2}\mu(t), & \text{if } \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) \ge 1 \\ \omega_{p2}, & \text{if } 0 < \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) < 1 \\ 0, & \text{if } \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) = 0 \end{cases} \tag{55}$$

where $\omega_{p1}, \omega_{p2} \in \mathbb{R}$ are positive real constants, and $\lambda_p(0) = \rho_p(0) = 0$, $p \in \underline{n}$.

Theorem 3 *Under Assumption 1, 2, and controller (53), QVNNs (4) and (5) can achieve finite time A-SYN under adaptive rules (54) and (55).*

Proof 2PM will also be used.

Phase 1: Define

$$\begin{aligned}&\mathcal{M}^\bullet(t)\\ &= \sup_{t-\tau(t)\le s\le t}\left\{\mu(s)\max_{p\in\underline{n}}\left(\|e_p(s)\|_1 + \frac{1}{2\omega_{p1}}(\lambda_p(s)-\lambda_p^\bullet)^2 + \frac{1}{2\omega_{p2}}(\rho_p(s)-\rho_p^\bullet)^2\right)\right\},\end{aligned}$$

where $\lambda_p^\bullet$ and $\rho_p^\bullet$ are sufficiently large enough constants satisfying

$$\lambda_p^\bullet > \varsigma + |d_p^I| + |d_p^J| + |d_p^K| - d_p^R + \mathcal{A}_{p1} + (1+\eta)\mathcal{B}_{p1}, \quad \rho_p^\bullet > \mathcal{B}_{p1} + \mathcal{C}_{p1} + 2\|I_p\|_1.$$

(I) If $\mu(t)\max_{p\in\underline{n}}(\|e_p(t)\|_1 + \frac{1}{2\omega_{p1}}(\lambda_p(t)-\lambda_p^\bullet)^2 + \frac{1}{2\omega_{p2}}(\rho_p(t)-\rho_p^\bullet)^2)) < \mathcal{M}^\bullet(t)$, then there must exist a constant $\delta_3 > 0$ such that $\mathcal{M}^\bullet(s) \le \mathcal{M}^\bullet(t)$, $s \in (t, t+\delta_3)$.

(II) If there exist an index p_3 and a time point $t_3 (t_3 \ge \mathcal{T}_0)$ such that

$$\mu(t_3)\Big(\|e_{p_3}(t_3)\|_1 + \frac{1}{2\omega_{p1}}(\lambda_p(t_3)-\lambda_p^\bullet)^2 + \frac{1}{2\omega_{p2}}(\rho_p(t)-\rho_p^\bullet)^2\Big) = \mathcal{M}^\bullet(t_3).$$

Since $\lambda_p(t)$ and $\rho_p(t)$ are non-decreasing, $(\lambda_p(t)-\lambda_p^\bullet)^2$ and $(\rho_p(t)-\rho_p^\bullet)^2$ would be non-increasing, then $\mu(t_3)\|e_{p_3}(t_3)\|_1$ is the maximum value for $\mu(s)\|e_p(s)\|_1$, $s \in [t_3-\tau(t_3), t_3]$. According to (26),

$$\frac{d}{dt}\Big(\mu(t)\|e_p(t)\|_1 + \frac{1}{2\omega_{p1}}(\lambda_p(t)-\lambda_p^\bullet)^2 + \frac{1}{2\omega_{p2}}(\rho_p(t)-\rho_p^\bullet)^2\Big)\Big|_{p=p_3, t=t_3}$$

$$
\begin{aligned}
&= \dot{\mu}(t)\|e_p(t)\|_1 + \mu(t)\frac{d\|e_p(t)\|_1}{dt} + (\lambda_p(t) - \lambda_p^\bullet)\mu(t)\|e_p(t)\|_1 + (\rho_p(t) - \rho_p^\bullet)\mu(t) \\
&\leq \Big(\frac{\dot{\mu}(t)}{\mu(t)} + |d_p^I| + |d_p^J| + |d_p^K| - d_p^R + \mathcal{A}_{p1} + \frac{\mu(t)}{\mu(t-\tau(t))}\mathcal{B}_{p1} - \lambda_p^\bullet\Big)\mu(t)\|e_p(t)\|_1 \\
&\quad + (\mathcal{C}_{p1} + 2\|I_p\|_1 - \rho_p^\bullet)\mu(t) < 0.
\end{aligned}
$$

Therefore, $\mathcal{M}^\bullet(t)$ is non-increasing for all $t \geq \mathcal{T}_0$, there exists a time point $\mathcal{T}_1^\bullet$ such that $\sup_{t-\tau(t)\leq s\leq t}(\|e(s)\|_{\{1,\infty\}}) \leq 1$ holds for all $t \geq \mathcal{T}_1^\bullet$.

Phase 2: Define

$$
\begin{aligned}
&\mathcal{V}^\bullet(t) \\
&\quad = \sup_{t-\tau(t)\leq s\leq t} \Big(\|e(s)\|_{\{1,\infty\}} + \vartheta^\bullet s + \frac{1}{2\omega_{p1}}(\lambda_p(s) - \lambda_p^\bullet)^2 + \frac{1}{2\omega_{p2}}(\rho_p(s) - \rho_p^\bullet)^2\Big),
\end{aligned}
$$

where $0 < \vartheta^\bullet < \rho_p^\bullet - \mathcal{B}_{p1} - \mathcal{C}_{p1} - 2\|I_p\|_1$, $p \in \underline{n}$.

Similarly, with the same arguments, if there exist an index p_4 and a time point $t_4 (t_4 \geq \mathcal{T}_1^\bullet)$, such that $\|e_{p_4}(t_4)\|_1 + \vartheta^\bullet t_4 + \frac{1}{2\omega_{p1}}(\lambda_p(t_4) - \lambda_p^\bullet)^2 + \frac{1}{2\omega_{p2}}(\rho_p(t_4) - \rho_p^\bullet)^2 = \mathcal{V}^\bullet(t_4)$, then

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|e_p(t)\|_1 + \vartheta^\bullet t + \frac{1}{2\omega_{p1}}(\lambda_p(t) - \lambda_p^\bullet)^2 + \frac{1}{2\omega_{p2}}(\rho_p(t) - \rho_p^\bullet)^2\Big)\Big|_{p=p_4, t=t_4} \\
&\quad \leq (|d_p^I| + |d_p^J| + |d_p^K| - d_p^R + \mathcal{A}_{p1} - \lambda_p^\bullet)\|e_p(t)\|_1 + (\mathcal{B}_{p1} + \mathcal{C}_{p1} + 2\|I_p\|_1 - \rho_p^\bullet + \vartheta^\bullet) \\
&\quad < 0.
\end{aligned}
$$

Hence, $\mathcal{V}^\bullet(t)$ is a non-increasing function, i.e.,

$$
\|e(t)\|_{\{1,\infty\}} + \vartheta^\bullet t \leq \mathcal{V}^\bullet(t) \leq \mathcal{V}^\bullet(\mathcal{T}_1^\bullet) \leq 1 + \vartheta^\bullet \mathcal{T}_1^\bullet + \max_{p\in\underline{n}}\Big(\frac{(\lambda_p^\bullet)^2}{2\omega_{p1}} + \frac{(\rho_p^\bullet)^2}{2\omega_{p2}}\Big),
$$

so, if we define

$$
\mathcal{T}_2^\bullet = \mathcal{T}_1^\bullet + \frac{1}{\vartheta^\bullet}\Big[1 + \max_{p\in\underline{n}}\Big(\frac{(\lambda_p^\bullet)^2}{2\omega_{p1}} + \frac{(\rho_p^\bullet)^2}{2\omega_{p2}}\Big)\Big],
$$

then QVNNs (4) and (5) will achieve finite time A-SYN no longer than $\mathcal{T}_2^\bullet$. □

Remark 9 From definitions of adaptive rules (54), (55), and the proof process, we can design non-segment adaptive rules, like

$$
\dot{\lambda}_p(t) = \begin{cases} \omega_{p1}\mu(t)\|e_p(t)\|_1, & \text{if } \sup_{t-\tau(t)\leq s\leq t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) > 0 \\ 0, & \text{if } \sup_{t-\tau(t)\leq s\leq t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) = 0 \end{cases} \tag{56}
$$

and

$$
\dot{\rho}_p(t) = \begin{cases} \omega_{p2}\mu(t), & \text{if } \sup_{t-\tau(t)\leq s\leq t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) > 0 \\ 0, & \text{if } \sup_{t-\tau(t)\leq s\leq t}(\max_{p\in\underline{n}}\|e_p(s)\|_1) = 0 \end{cases} \tag{57}
$$

The only requirement is that $\mu(t) \geq 1$, and this condition is easy to be satisfied. For example, if the time delay is bounded, then $\mu(t)$ can be chosen as $e^{\alpha t}$, $\alpha > 0$, obviously, for this $\mu(t)$, it is larger than 1. One advantage of these new adaptive rules is that there are no further judges between $\sup_{t-\tau(t)\leq s\leq t}(\max_{p\in\underline{n}}\|e_p(s)\|_1)$ and 1. Of course, the disadvantage is to result in larger values of $\lambda_p(t)$ and $\rho_p(t)$.

We can also present the corresponding result with adaptive rules for 2-norm.

Theorem 4 *Under Assumption 1, 2, and controller (53), QVNNs (4) and (5) can achieve finite time A-SYN under adaptive rules*

$$\dot{\lambda}_p(t) = \begin{cases} \omega_{p1}\mu(t)\|e_p(t)\|_2^2, & \text{if } \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2) \ge 1 \\ \omega_{p1}\|e_p(t)\|_2, & \text{if } 0 < \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2) < 1 \\ 0, & \text{if } \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2) = 0 \end{cases} \quad (58)$$

and

$$\dot{\rho}_p(t) = \begin{cases} \omega_{p2}\mu(t)\|e_p(t)\|_2, & \text{if } \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2) \ge 1 \\ \omega_{p2}, & \text{if } 0 < \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2) < 1 \\ 0, & \text{if } \sup_{t-\tau(t)\le s\le t}(\max_{p\in\underline{n}}\|e_p(s)\|_2) = 0 \end{cases} \quad (59)$$

where $\omega_{p1}, \omega_{p2} \in \mathbb{R}$ are positive real constants, and $\lambda_p(0) = \rho_p(0) = 0$, $p \in \underline{n}$.

Proof **Phase 1:** Define

$$\begin{aligned}&\mathcal{M}^\circ(t)\\ &= \sup_{t-\tau(t)\le s\le t}\left\{\mu(s)\max_{p\in\underline{n}}\left(\|e_p(s)\|_2^2 + \frac{1}{\omega_{p1}}(\lambda_p(s)-\lambda_p^\circ)^2 + \frac{1}{\omega_{p2}}(\rho_p(s)-\rho_p^\circ)^2\right)\right\},\end{aligned}$$

where λ_p° and ρ_p° are sufficiently large enough constants satisfying

$$\lambda_p^\circ > \frac{\varsigma}{2} - d_p^R + \mathcal{A}_{p2} + \sqrt{1+\eta}\mathcal{B}_{p2}, \quad \rho_p^\circ > \mathcal{B}_{p2} + \mathcal{C}_{p2} + 2\|I_p\|_2.$$

According to (47),

$$\begin{aligned}&\frac{d}{dt}\left(\mu(t)\|e_p(t)\|_2^2 + \frac{1}{\omega_{p1}}(\lambda_p(t)-\lambda_p^\circ)^2 + \frac{1}{\omega_{p2}}(\rho_p(t)-\rho_p^\circ)^2\right)\\ &\quad\le \mu(t)\Bigg\{\left(\frac{\dot{\mu}(t)}{\mu(t)} - 2d_p^R + 2\mathcal{A}_{p2} + 2\mathcal{B}_{p2}\sqrt{\mu(t)}\sqrt{\mu(t-\tau(t))} - 2\lambda_p^\circ\right)\|e_p(t)\|_2^2\\ &\qquad + \left(2\mathcal{C}_{p2} + 4\|I_p\|_2 - 2\rho_p^\circ\right)\|e_p(t)\|_2\Bigg\} < 0,\end{aligned}$$

Therefore, $\mathcal{M}^\circ(t)$ is non-increasing for all $t \ge \mathcal{T}_0^\star$, there exists a time point $\mathcal{T}_1^\circ$ such that $\sup_{t-\tau(t)\le s\le t}(\|e(s)\|_{\{2,\infty\}}) \le 1$ holds for all $t \ge \mathcal{T}_1^\circ$.

Phase 2: Define

$$\begin{aligned}&\mathcal{V}^\circ(t)\\ &= \sup_{t-\tau(t)\le s\le t}\left(\|e(s)\|_{\{2,\infty\}} + \vartheta^\circ s + \frac{1}{2\omega_{p1}}(\lambda_p(s)-\lambda_p^\circ)^2 + \frac{1}{2\omega_{p2}}(\rho_p(s)-\rho_p^\circ)^2\right),\end{aligned}$$

where $0 < \vartheta^\circ < \rho_p^\circ - \mathcal{B}_{p2} - \mathcal{C}_{p2} - 2\|I_p\|_2$, $p \in \underline{n}$. Then, according to (50) and (51),

$$\begin{aligned}&\frac{d}{dt}\left(\|e_p(t)\|_2 + \vartheta^\circ t + \frac{1}{2\omega_{p1}}(\lambda_p(t)-\lambda_p^\circ)^2 + \frac{1}{2\omega_{p2}}(\rho_p(t)-\rho_p^\circ)^2\right)\\ &\quad\le (-d_p^R + \mathcal{A}_{p2} - \lambda_p^\circ)\|e_p(t)\|_2 + (\mathcal{B}_{p2} + \mathcal{C}_{p2} + 2\|I_p\|_2 - \rho_p^\circ + \vartheta^\circ) < 0.\end{aligned}$$

Hence, $\mathcal{V}^\circ(t)$ is a non-increasing function, and if we define

$$\mathcal{T}_2^\circ = \mathcal{T}_1^\circ + \frac{1}{\vartheta^\circ}\left[1 + \max_{p\in\underline{n}}\left(\frac{(\lambda_p^\circ)^2}{2\omega_{p1}} + \frac{(\rho_p^\circ)^2}{2\omega_{p2}}\right)\right],$$

then QVNNs (4) and (5) will achieve finite time A-SYN no longer than $\mathcal{T}_2^\circ$. □

4 Numerical Example

Consider a two-neuron master-slave coupled QVNN described as follows:

$$\begin{cases} \dot{x}_1(t) = -d_1 x_1(t) + a_{11} f_1(x_1(t)) + a_{12} f_2(x_2(t)) \\ \qquad\quad +b_{11} g_1(x_1(t-\tau_{11}(t))) + b_{12} g_2(x_2(t-\tau_{12}(t))) + I_1, \\ \dot{x}_2(t) = -d_2 x_2(t) + a_{21} f_1(x_1(t)) + a_{22} f_2(x_2(t)) \\ \qquad\quad +b_{21} g_1(x_1(t-\tau_{21}(t))) + b_{22} g_2(x_2(t-\tau_{22}(t))) + I_2, \\ \dot{y}_1(t) = -d_1 y_1(t) + a_{11} f_1(y_1(t)) + a_{12} f_2(y_2(t)) \\ \qquad\quad +b_{11} g_1(y_1(t-\tau_{11}(t))) + b_{12} g_2(y_2(t-\tau_{12}(t))) + I_1 + u_1, \\ \dot{y}_2(t) = -d_2 y_2(t) + a_{21} f_1(y_1(t)) + a_{22} f_2(y_2(t)) \\ \qquad\quad +b_{21} g_1(y_1(t-\tau_{21}(t))) + b_{22} g_2(y_2(t-\tau_{22}(t))) + I_2 + u_2, \end{cases} \tag{60}$$

where $d_1 = 0.0077 + 0.1120i + 0.2911j + 0.5029k$, $d_2 = 0.1128 + 0.5858i + 0.5528j + 0.0276k$, $I_1 = -0.4340 - 0.5493 - 0.3374 - 0.0935k$, $I_2 = 0.4748 + 0.0198i - 0.235j + 0.811k$, $a_{11} = -0.0223 - 0.7751i + 0.2771j - 0.0028k$, $a_{12} = -0.1470 - 0.4188i + 0.2667j + 0.9555k$, $a_{21} = 0.9428 + 0.4864i + 0.1884j + 0.1357k$, $a_{22} = -0.8475 + 0.1227i + 0.8616j - 0.8128k$, $b_{11} = 0.3235 - 0.0524i - 0.0488j + 0.9193k$, $b_{12} = 0.5955 + 0.8244i - 0.4134j + 0.0083k$, $b_{21} = 0.2055 - 0.2875i + 0.3420j - 0.8218k$, $b_{22} = 0.1816 - 0.7977i - 0.8968j + 0.5368k$.

The activation functions are defined as:

$$f_1(x_q) = 0.5\tanh(x_q^R) + 0.5i\tanh(x_q^I) + 0.5j\tanh(x_q^J) + 0.5k\tanh(x_q^K),$$

$$f_2(x_q) = 0.25\tanh(x_q^R) + 0.25i\tanh(x_q^I) + 0.25j\tanh(x_q^J) + 0.25k\tanh(x_q^K),$$

$$g_1(x_q) = 0.4\tanh(x_q^R) + 0.4i\tanh(x_q^I) + 0.4j\tanh(x_q^J) + 0.4k\tanh(x_q^K),$$

$$g_2(x_q) = 0.2\tanh(x_q^R) + 0.2i\tanh(x_q^I) + 0.2j\tanh(x_q^J) + 0.2k\tanh(x_q^K),$$

According to Assumption 1, we get that $L_1^f = 0.5$, $H_1^f = 0$, $L_1^g = 0.4$, $H_1^g = 0$.

The time delays are assumed to be unbounded, asynchronous, and time-varying,

$$\tau_{11}(t) = 0.4t, \quad \tau_{12}(t) = 0.5t, \quad \tau_{21}(t) = 0.5t, \quad \tau_{22}(t) = 0.4t.$$

Define A-SYN error as $\|e_p(t)\|_1 = \|x_p(t) + y_p(t)\|_1$, $p = 1, 2$, when there are no controllers, i.e., $u_1 = u_2 = 0$, Fig. 1 shows that A-SYN cannot be achieved.

From Theorem 1, we can design a controller based on 1-norm, and we choose $\mu(t) = t^{0.6}$, according to conditions (11)–(13), we have

$$\varsigma = 0, \quad \eta = 0.5157, \quad \lambda_1 > 4.2623, \quad \rho_1 > 1.2742, \ \lambda_2 > 5.7189, \quad \rho_2 > 1.6279.$$

Therefore, the controller can be designed as:

$$\begin{cases} u_1(t) = -5e_1(t) - 1.5\mathbf{sig}(e_1(t)), \\ u_2(t) = -6e_2(t) - 2\mathbf{sig}(e_2(t)). \end{cases} \tag{61}$$

Figure 2 shows the trajectories of system (60) under controller (61), which implies that finite time A-SYN has been achieved.

Next, we apply adaptive technique to realize finite time A-SYN, where the controller is

$$u_p(t) = -\lambda_p(t)e_p(t) - \rho_p(t)\mathbf{sig}(e_p(t)), \quad p = 1, 2, \tag{62}$$

where adaptive rules for $\lambda_p(t)$ and $\rho_p(t)$ are define in (54) and (55) with coefficients $\omega_{11} = \omega_{12} = 0.2$ and $\omega_{21} = \omega_{22} = 0.4$. The dynamics of A-SYN errors can be found in Fig. 3, and dynamics of control strengths can be found in Fig. 4.

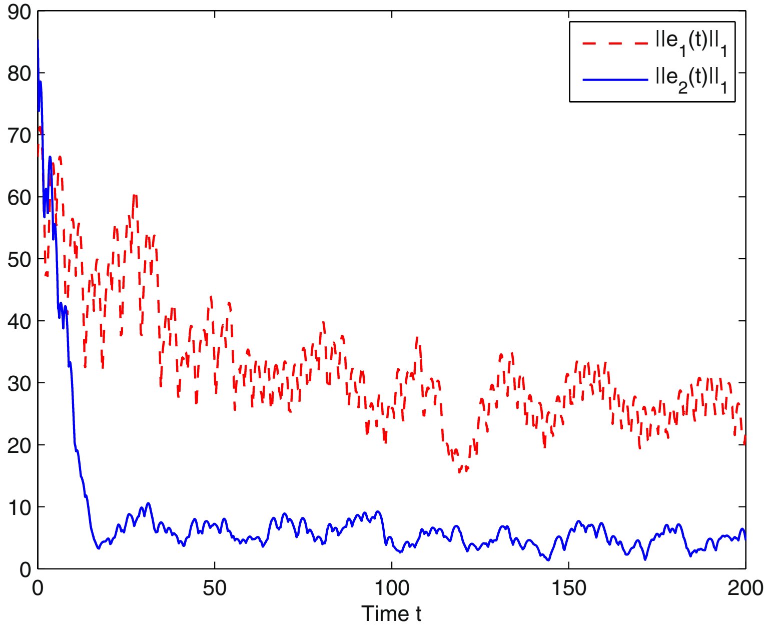

Fig. 1 Error trajectories of system (60) without control

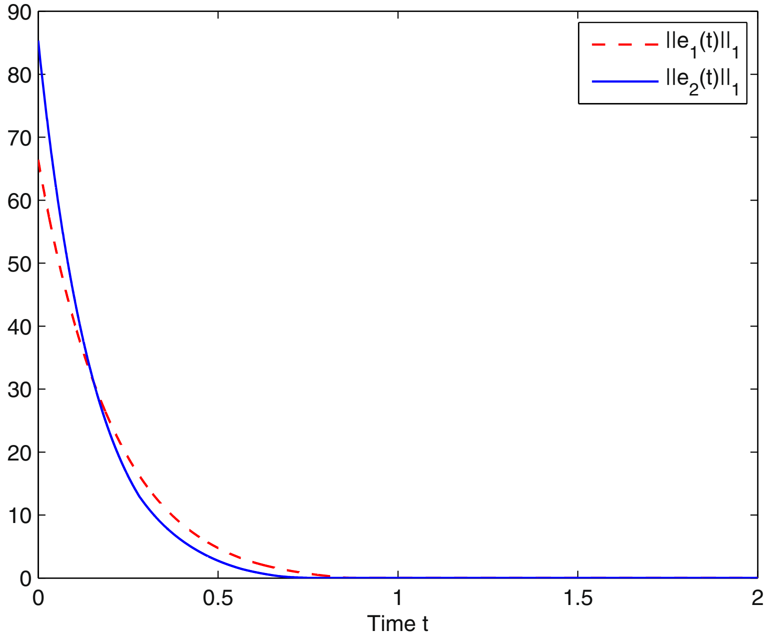

Fig. 2 Error trajectories of system (60) under controller (61)

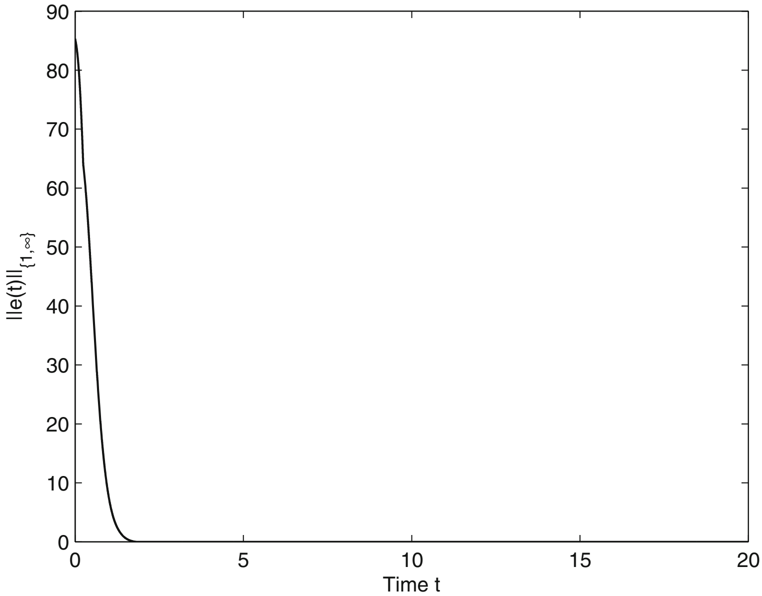

Fig. 3 Error trajectories of system (60) under controller (62)

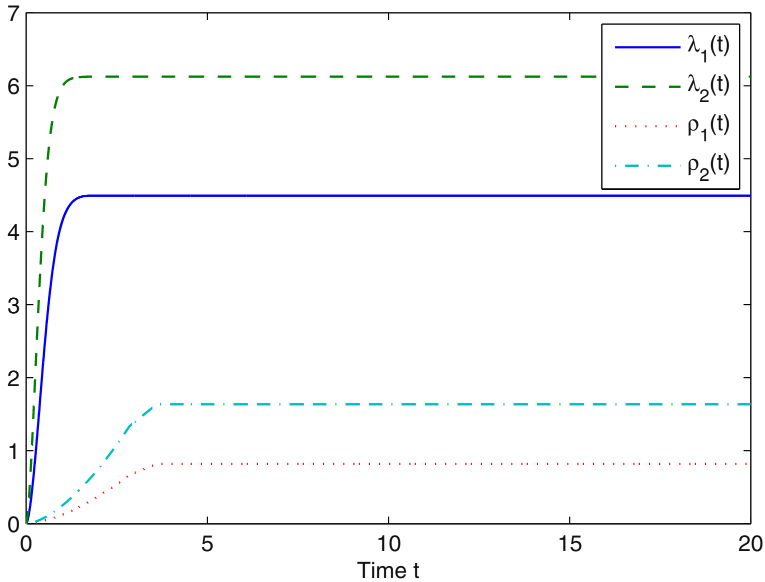

Fig. 4 Dynamics of adaptive control strengths of system (60) under controller (62)

5 Conclusion

We study the finite time A-SYN problem for QVNNs with asynchronous time-varying delays. With the help of the quaternion sign function and its special properties, we treat the QVNN's state as a whole instead of using decomposition method. The error is quantified with 1-norm and 2-norm, respectively. For each norm, 2PM is used to derive the sufficient conditions under simple delay-free controllers for ensuring finite time A-SYN. Moreover, adaptive rules for control strengths are also designed to realize finite time A-SYN. Finally, we present a numerical example to show the effectiveness of our obtained criteria.

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