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Stochastic Quasi-Synchronization of Delayed Neural Networks: Pinning Impulsive Scheme

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Abstract
This paper studies stochastic quasi-synchronization of delayed neural networks with parameter mismatches and stochastic perturbation mismatch via pinning impulsive control. By pinning selected nodes of stochastic neural network at impulse time, an impulsive control scheme is proposed. Some sufficient conditions are obtained to ensure that the error system can converge to small region in the mean square. Meanwhile, numerical example is provided to illustrate the effectiveness of theoretical results.

Keywords Quasi-synchronization · Stochastic neural networks · Delay · Pinning impulsive control

1 Introduction

In the past few decades, neural networks as an emerging field have been drawing attention from researchers due to their wide application in signal processing, pattern recognition, dynamic optimization, deep learning and so on [1–6]. As an important collective behavior, the synchronization of neural networks is an important topic because of its practical applications in biological systems [7–13].

As is well known, random noises widely exist in the signal transmission of neural networks due to environmental uncertainties, which usually lead to stochastic perturbation and uncertainties of the process for dynamic evolution. Based on the theory of stochastic system [14], a lot of stability and synchronization for neural networks with stochastic perturbations have been obtained [15–17]. In the hardware implementation of neural networks, it is impossible for neurons to respond and communicate simultaneously owing to time-delays. To reduce the negative influence of time-delays, delay-independent method is a powerful tool to check the stability and synchronization of neural networks by constructing Lyapunov–Krasovkii func-

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tional, see [18–20]. On the other hand, in the implementations of neural networks systems, parameter mismatches and stochastic perturbation mismatch are unavoidable. If parameter mismatch and stochastic perturbation mismatch are small, although the stochastic synchronization error can not converge to zero in the mean square with time increasing, but we may show that the error of system is small fluctuations about zero or even a nonzero mean value in the mean square. For background material on parameter mismatch of system, we can find in [21–23].

Pinning control, which is an effectively external control approach, has been widely used for a variety of purposes due to low cost. It is characterized that the controllers are added on only a small fraction of network nodes [24–27]. For example, in [26], the authors proposed a variety of pinning control methods of cluster synchronization in an array of coupled neural networks and proposed a new event-triggered sampled-data transmission strategy. Impulsive control is an energy-saving control due to the instantaneous perturbations input at certain moment, which has been applied efficiently in the fields of engineering, physics, and science as well [28,29]. With the help of the impulsive system theory, a lot of synchronization results of dynamical networks with impulsive input have been obtained [30–33]. It is worth noting that the cost of control can be further reduce by adding the impulsive controllers to a small fraction of networks nodes, which can combine the advantage of pinning control and impulsive control. Recently, some research have been devoted to the synchronization of delayed neural networks with pinning and impulsive controls [34,35]. For instance, in [34], the authors proposed a new pinning impulsive control scheme to investigate the synchronization problem for a class complex networks with time-varying delay.

Motivated by the above discussion, this paper focuses on stochastic quasi-synchronization of delayed neural networks under parameter mismatch and stochastic perturbation mismatch. Although the error of systems will not converge exponentially to zero in the mean square, some effectively sufficient condition are obtained to synchronize the error of systems up to a relatively small bound in the mean square via pinning impulsive control. The contributions in this paper are concluded as follows:

(i) By pinning certain selected nodes of stochastic neural networks at each impulsive time, impulsive control strategy is proposed to achieve stochastic-synchronization.
(ii) By establishing a new lemma of stochastic impulsive system, stochastic-synchronization criteria are derived to guarantee that the nodes of stochastic neural networks can synchronizes desired trajectory to small region in the mean square.
(iii) If the bound of time delay does not exceed the length of impulsive interval, delay-independent method is used to overcome the effects of time delay and impulses by constructing Lyapunov–Krasovkii functional.

The rest of this paper is organized as follows. In Sect. 2, the stochastic neural network is presented and some definitions and lemmas are provided. In addition, a new lemma is established, which plays an important role in the proof of obtained theorems. In Sect. 3, some stochastic quasi-synchronization criteria are obtained via impulsive control technique. In Sect. 4, numerical example is presented to illustrate our results. Finally, some conclusions are given in Sect. 5.

Notations R^n and $R^{n\times m}$ denote n dimensional Euclidean space and the set of $n \times m$ real matrices. The superscript T denotes the transpose. I_n represents the identity matrix with n dimension. $\|\cdot\|$ denotes the Euclidean norm for a vector and a matrix. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent the maximum and minimum eigenvalues of matrix A. $diag\{\cdots\}$ stands for a diagonal matrix. For real symmetric matrices X, the notation $X > 0 (X < 0)$ implies that the matrix X is positive (negative) definite. $\otimes$ represents Kronecker product. Let $\omega(t) =$

$(\omega_1(t), \omega_2(t), \cdots, \omega_n(t))^T$ be an n-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

2 Model Description and Preliminaries

Consider stochastic neural network with delay and N coupled nodes. The dynamics of ith neuron is described by the following form

$$\begin{aligned} dx_i(t) = {} & [C_1 x_i(t) + B_1 f(x_i(t)) + D_1 f(x_i(t-\tau(t))) + \sum_{j=1}^{N} a_{ij}\Gamma x_j(t) + u_i(t)]dt \\ & + h_1(t, x_i(t), x_i(t-\tau(t)))d\omega(t), i = 1, 2, \cdots, N, \end{aligned} \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T$ is the state vector of the i-th neural networks at time t; $C_1 = diag\{c_{11}, c_{12}, \cdots, c_{1n}\}$ denotes the rate with which ith cell resets its potential to the resting state when being isolated from other cells and inputs, $B_1 = (b_{ij}^{(1)})_{n\times n}$, $D_1 = (d_{ij}^{(1)})_{n\times n} \in R^{n\times n}$ are the connection weight matrices; $f(x)$ is the activation function at time t satisfying $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^T$; $h_1(t, x_i(t), x_i(t-\tau(t))) = (h_{11}(x_{i1}(t), x_{i1}(t-\tau(t))), h_{12}(x_{i2}(t), x_{i2}(t-\tau(t))), \cdots, h_{1n}(x_{in}(t), x_{in}(t-\tau(t))))^T$; $\tau(t)$ is transmittal delay, and there exist constant τ and σ such that $0 < \tau(t) \leq \tau$, $\tau^{'}(t) \leq \sigma < 1$; $\Gamma = diag\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ is the inner coupling positive definite matrix between two connected nodes i and j; and a_{ij} is defined as follows: if there is a connection from node j to node $i (j \neq i)$, then $a_{ij} \neq 0$; otherwise, $a_{ij} > 0$.

Let $s(t)$ be the desired trajectory described by the following form:

$$ds(t) = [C_2 s(t) + B_2 f(s(t)) + D_2 f(s(t-\tau(t)))]dt + h_2(t, s(t), s(t-\tau(t)))d\omega(t), \quad (2)$$

where $C_2 = diag\{c_{21}, c_{22}, \cdots, c_{2n}\}$, $B_2 = (b_{ij}^{(2)})_{n\times n}$, $D_2 = (d_{ij}^{(2)})_{n\times n} \in R^{n\times n}$; $h_2(t, x_i(t), x_i(t-\tau(t))) = (h_{21}(x_{i1}(t), x_{i1}(t-\tau(t))), h_{22}(x_{i2}(t), x_{i2}(t-\tau(t))), \cdots, h_{2n}(x_{in}(t), x_{in}(t-\tau(t))))^T$.

Define the error signal as $e_i(t) = x_i(t) - s(t)$, $i = 1, 2, \cdots, N$, then we have the following error dynamical system

$$\begin{aligned} de_i(t) = {} & [C_1 e_i(t) + \triangle C s(t) + B_1(f(x_i(t)) - f(s(t))) + \triangle B f(s(t)) \\ & + D_1(f(x_i(t-\tau(t))) - f(s(t-\tau(t)))) + \triangle D f(s(t-\tau(t))) \\ & + \sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + u_i(t)]dt \\ & + [h_1(t, x_i(t), x_i(t-\tau(t))) - h_1(t, s(t), s(t-\tau(t))) \\ & + \triangle h(t, s(t), s(t-\tau(t)))]d\omega(t), \end{aligned} \quad (3)$$

where $\triangle C = C_1 - C_2$, $\triangle B = B_1 - B_2$, $\triangle D = D_1 - D_2$ are parameter mismatch errors and $\triangle h = h_1 - h_2$ is stochastic perturbation mismatch error.

Due to the parameter mismatch and the stochastic perturbation mismatches, the origin $e_i = 0$ is not an equilibrium point of the error system (3), which means that it is impossible to be complete synchronization. However, by pinning impulsive control, stochastic quasi-synchronization with a relatively small error bound can considered.

Let $t_k \geq 0$ be impulsive moments satisfying $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, $\lim\limits_{k\to+\infty} t_k = +\infty$ and $\sup\limits_{k\geq 0}\{\triangle_k\} < +\infty$, where $\triangle_k = t_{k+1} - t_k$. For $t = t_k$, the node errors are arranged in the following two forms

$$(i) \quad E\|e_{i_1}(t_k)\| \geq E\|e_{i_2}(t_k)\| \geq \cdots \geq E\|e_{i_s}(t_k)\| \geq E\|e_{i_{s+1}}(t_k)\| \geq \cdots \geq E\|e_{i_N}(t_k)\|,$$

and

$$(ii) \quad E\|e_{i_1}(t_k)\| \leq E\|e_{i_2}(t_k)\| \leq \cdots \leq E\|e_{i_s}(t_k)\| \leq E\|e_{i_{s+1}}(t_k)\| \leq \cdots \leq E\|e_{i_N}(t_k)\|,$$

where $i_s \in \{1, 2, \cdots, N\}$, $s = 1, 2, \cdots, N$, and $i_u \neq i_v$ for $u \neq v$. Furthermore, if $E\|e_{i_s}(t_k)\| = E\|e_{i_{s+1}}(t_k)\|$, then $i_s < i_{s+1}$. To reach stochastic quasi-synchronization of networks, the pinning impulsive control scheme is used on the nodes. Let $\delta(\cdot)$ be a Dirac function. d_k denotes the impulsive gain. If $-1 < u_k < 1$, the first q nodes are chosen as pinned nodes according to the arrangement (i). If $u_k \geq 1$ or $u_k \leq -1$, then the first q nodes are chosen as the pinning nodes according to the arrangement (ii). The set of pinned nodes can be defined by $\chi(t_k) = \{i_1, i_2, \cdots, i_q\} \subset \{1, 2, \cdots, N\}$ and $\sharp\chi(t_k) = q$. We design the pinning impulsive controller as follows:

$$u_i(t) = \begin{cases} \sum\limits_{k=1}^{+\infty}(v_k - 1)e_i(t)\delta(t - t_k), i \in \chi(t_k), \\ 0, i \in \{1, 2, \cdots, N\}\backslash\chi(t_k). \end{cases} \tag{4}$$

With the help of impulsive control, the error dynamical system can be obtained in the following form:

$$\begin{cases} de_i(t) = [C_1e_i(t) + \triangle Cs(t) + B_1(f(x_i(t)) - f(s(t))) + \triangle Bf(s(t)) \\ \qquad +D_1(f(x_i(t-\tau(t))) - f(s(t-\tau(t)))) + \triangle Df(s(t-\tau(t))) \\ \qquad + \sum\limits_{j=1}^{N} a_{ij}\Gamma e_j(t)]dt + [h_1(t, x_i(t), x_i(t-\tau(t))) - h_1(t, s(t), s(t-\tau(t))) \\ \qquad +\triangle h(t, s(t), s(t-\tau(t)))]d\omega(t), \\ \qquad i = 1, 2, \cdots, N, t \neq t_k, \\ e_i(t_k^+) = v_ke_i(t_k), i \in \chi(t_k), \\ e_i(t_k^+) = e_i(t_k), i \in \{1, 2, \cdots, N\}\backslash\chi(t_k), \end{cases} \tag{5}$$

where $e_i(t_k^+) = \lim\limits_{h\to 0+} e_i(t_k + h), e_i(t_k) = \lim\limits_{h\to 0-} e_i(t_k + h)$ is left-hand continuous at $t = t_k$. The initial condition of $e_i(t)$ is denoted as $e_i(t) = \phi(t) \in PC_{\mathcal{F}_t}([-\tau, 0], R^n)$, where $PC_{\mathcal{F}_t}([-\tau, 0], R^n)$ is the family of all $\mathcal{F}_t$-measurable, $PC([-\tau, 0], R^n)$-value random variable ϕ satisfied $\int_{-\tau}^{0} E[|\phi(\theta)|^2]d\theta < \infty$, $PC([-\tau, 0], R^n)$ is the family of piecewise continuous functions ϕ with the norm $\|\phi\| = \sup\limits_{-\tau\leq\theta\leq 0} \|\phi(\theta)\|$.

Remark 1 Based on the proposed scheme, the norm of synchronization error may vary with impulse time t_k, which implies that the pinned nodes may be different at different t_k. In view of control cost, if $-1 < v_k < 1$, some nodes with large norm value are chosen to be pinned nodes. If $v_k \leq 1$ or $v_k \geq 1$, some nodes with small norm value are considered to be pinned nodes.

Definition 1 Let Φ be a region in the phase space of system (2). The neural networks (1) and (2) are said to be uniformly stochastic quasi-synchronized with error bound $\overline{\theta} > 0$ if there theists a $\tilde{t} \geq 0$ such that for $t \geq \tilde{t}$, $E[\|x_i(0)\|^2]$, $E[\|s(0)\|^2] \in \Phi$, $i = 1, 2, \cdots, N$

$$E[\sum_{i=1}^{N} \|e_i(t)\|^2] = E[\sum_{i=1}^{N} \|x_i(t) - s(t)\|^2] \leq \overline{\theta}.$$

Assumption 1 There exists constant $l > 0$ such that for $\forall x, y \in R^n$

$$\|f(x) - f(y)\| \leq l\|x - y\|.$$

Assumption 2 There exist matrices $M_1 \in R^{n\times n} > 0, M_2 \in R^{n\times n} > 0$ such that for $x, y, u, v \in R^n$

$$\begin{aligned} &trace[(h_1(t, x, y) - h_1(t, u, v))^T h_1(t, x, y) - h_1(t, u, v))] \\ &\leq (x - u)^T M_1(x - u) + (y - v)^T M_2(y - v). \end{aligned}$$

Assumption 3 There exists constant $\rho_1 > 0$ such that

$$\|\triangle C\| + l\|\triangle B\| + l\|\triangle D\| \leq \rho_1.$$

Assumption 4 There exists constants $\rho_2 > 0$, $\rho_3 > 0$ such that

$$trace[\Delta h^T(t, x(t), x(t - \tau(t)))\Delta h(t, x(t), x(t - \tau(t)))] \leq \rho_2^2\|x(t)\|^2 + \rho_3^2\|x(t - \tau(t))\|^2.$$

For the following impulsive stochastic equation with delay:

$$\begin{cases} dx(t) = F(t, x(t), x(t - \tau(t))dt + G(t, x(t), x(t - \tau(t))d\omega(t), t \geq 0, t \neq t_k, \\ \triangle x(t_k) = I_k(t, x(t_k)), \quad k = 1, 2, \cdots, \end{cases} \tag{6}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$, $F : [0, +\infty) \times R^n \times PC([-\tau, 0]; R^n) \to R^n$, $G : [0, +\infty) \times R^n \times PC([-\tau, 0]; R^n) \to R^{n\times n}$, $\triangle x(t_k) = x(t_k^+) - x(t_k)$, $I : [0, +\infty) \times R^n \to R^n$.

Let $\mathcal{C}_1^2([-\tau, \infty) \times R^n; [0, +\infty))$ be the family of all nonnegative functions $V(t, \phi)$ on $[-\tau, \infty) \times R^n$, V, V_t, V_x, V_{xx} are continuous on $(t_{k-1}, t_k] \times R^n$. For each $V \in \mathcal{C}_1^2([-\tau, \infty) \times R^n; [0, +\infty))$, $\phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\} \in PC_{\mathcal{F}_t}([-\tau, 0]; R^n)$, an operator $\mathcal{L}V : (t_{k-1}, t_k] \times PC_{\mathcal{F}_t}([-\tau, 0]; R^n) \to [0, +\infty)$ associated with Eq. (6) is defined as the following form:

$$\begin{aligned} \mathcal{L}V(t, \phi) = &V_t(t, \phi(0)) + V_x(t, \phi(0))F(t, \phi(0), \phi(-\tau(t)) \\ &+\frac{1}{2}trace[G_1^T(t, \phi(0), \phi)V_{xx}G(t, \phi(0), \phi(-\tau(t)). \end{aligned} \tag{7}$$

Lemma 1 *Assume that* $V \in \mathcal{C}_1^2([-\tau, \infty) \times R^n; R^+)$ *and there exist constants* $d_1 > 0$, $d_2 > 0$, $\eta_k > 0$, $k = 1, 2, \cdots$, $\overline{\mu} \geq 0$, $\hat{\mu} \geq 0$, μ, δ *such that*

(i) $d_1\|x\|^2 \leq V(t, x) \leq d_2\|x\|^2$;
(ii) $E[\mathcal{L}V(t, x(t))] \leq \mu E[V(t, x(t))] + \overline{\mu}E[V(t, x(t - \tau(t)))] + \hat{\mu}$ *for all* $t \in (t_{k-1}, t_k]$;
(iii) $E[V(t_k^+, x(t_k) + I_k(t_k, x(t_k)))] \leq \eta_k E[V(t_k, x(t_k))]$;
(iv) $\ln \eta_k \leq \delta\triangle_{k-1}$, $k = 1, 2, \cdots$;
(v) $\mu + \beta\overline{\mu} + \delta < 0$,

then the zero solution of Eq. (6) *converges exponentially to small region* $\mathcal{K} = \{x(t) \in R^n | E[\|x(t)\|^2] \leq d\}$ *in the mean square with exponent* λ, *where* $\beta = \sup_{1\leq k<+\infty}\{\beta_k\}$, $\beta_k = \max\{e^{\delta\triangle_{k-1}}, e^{-\delta\triangle_{k-1}}\}$, λ *is the unique solution of* $\lambda + \mu + \beta e^{\lambda\tau}\overline{\mu} + \delta = 0$, $d = \frac{-\beta\hat{\mu}}{\mu+\delta+\beta\overline{\mu}}$.

Proof By Itô formula, we can obtain

$$dV(t,x)=\mathcal{L}V(t,x(t))+V_x(t,x(t))G(t,x(t),x(t-\tau(t)))d\omega(t). \tag{8}$$

For $t\in(t_{k-1},t_k]$, we chose $\varepsilon>0$ such that $t+\varepsilon\in(t_{k-1},t_k]$. It follows from integrating the above inequality from t to $t+\varepsilon$ and taking the expectations on both sides of (8) that

$$\begin{aligned}&E[V(t+\epsilon,x(t+\epsilon))]-E[V(t,x(t))])\\&\quad=\int_t^{t+\epsilon}E[\mathcal{L}V(s,x(s))]ds+E\int_t^{t+\epsilon}V_x(s,x(s))G(s,x(s),x(s-\tau(s))d\omega(s).\end{aligned} \tag{9}$$

Let $\varepsilon\to0$, by (ii), it yields that for $t\in(t_{k-1},t_k]$

$$D^+V(t,x(t))=E[\mathcal{L}V(t,x(t))]=\mu E[V(t,x(t))]+\overline{\mu}E[V(t,x(t-\tau(t)))]+\hat{\mu}. \tag{10}$$

Let $V(t)=V(t,x(t))$ and $z(t)=e^{-\mu t}E[V(t)]$. For $t\in(t_{k-1},t_k]$, we have

$$\begin{aligned}D^+z(t)&=e^{-\mu t}D^+E[V(t)]-\mu e^{-\mu t}E[V(t)]\\&=\overline{\mu}e^{-\mu t}E[V(t-\tau(t))]+\hat{\mu}e^{-\mu t}.\end{aligned} \tag{11}$$

By (iii), we have

$$z(t_k^+)=e^{-\mu t_k}E[V(t_k^+)]\le\eta_k z(t_k). \tag{12}$$

For $t\in[0,t_1]$, integrating the inequality (11) from 0 to t, we obtain

$$z(t)=z(0)+\textstyle\int_0^t\overline{\mu}e^{-\mu s}E[V(s-\tau(s))]ds+\int_0^t\hat{\mu}e^{-\mu s}ds, \tag{13}$$

and

$$z(t_1)=z(0)+\textstyle\int_0^{t_1}\overline{\mu}e^{-\mu s}E[V(s-\tau(s))]ds+\int_0^{t_1}\hat{\mu}e^{-\mu s}ds. \tag{14}$$

For $t\in(t_1,t_2]$, by using the same method, we obtain

$$\begin{aligned}z(t)&=z(t_1^+)+\textstyle\int_{t_1}^t\overline{\mu}e^{-\mu s}E[V(s-\tau(s))]ds+\int_{t_1}^t\hat{\mu}e^{-\mu s}ds\\&\le\eta_1\{z(0)+\textstyle\int_0^{t_1}\overline{\mu}e^{-\mu s}E[V(s-\tau(s))]ds+\int_0^{t_1}\hat{\mu}e^{-\mu s}ds\}\\&\quad+\textstyle\int_{t_1}^t\overline{\mu}e^{-\mu s}E[V(s-\tau(s))]ds+\int_{t_1}^t\hat{\mu}e^{-\mu s}ds\\&=\eta_1z(0)+\eta_1\textstyle\int_0^{t_1}\overline{\mu}e^{-\mu s}E[V(s-\tau(s))]ds+\int_{t_1}^t\overline{\mu}e^{-\mu s}E[V(s-\tau(s))]ds\\&\quad+\eta_1\textstyle\int_0^{t_1}\hat{\mu}e^{-\mu s}ds+\int_{t_1}^t\hat{\mu}e^{-\mu s}ds.\end{aligned} \tag{15}$$

By induction, it yields that for $t\in(t_{k-1},t_k]$

$$z(t)\le z(0)\prod_{0\le t_i<t}\eta_i+\overline{\mu}\textstyle\int_0^t\displaystyle\prod_{s\le t_i<t}\eta_ie^{-\mu s}E[V(s-\tau(s))]ds+\hat{\mu}\textstyle\int_0^t\displaystyle\prod_{s\le t_i<t}\eta_ie^{-\mu s}ds, \tag{16}$$

which implies that for $t>0$

$$\begin{aligned}E[V(t)]\le E[V(0)]e^{\mu t}\prod_{0\le t_i<t}\eta_i+\overline{\mu}\int_0^te^{\mu(t-s)}\prod_{s\le t_i<t}\eta_iE[V(s-\tau(s))]ds\\+\hat{\mu}\int_0^te^{\mu(t-s)}\prod_{s\le t_i<t}\eta_ids.\end{aligned} \tag{17}$$

For $t > s$, the impulsive points in $[s, t)$ can be denoted by $t_{i1,i2}, \cdots, t_{ip}$ and t_{i1-1} is the first impulsive point before t_{i1}. If $\delta \geq 0$, by (iv), we have

$$\begin{aligned}\prod_{s\leq t_i<t}\eta_i &= \eta_{i1}\eta_{i2}\cdots\eta_{ip} \leq e^{\delta(t_{i1}-t_{i1-1})}e^{\delta(t_{i2}-t_{i1})}\cdots e^{\delta(t_{ip}-t_{ip-1})}\\ &= e^{\delta(t_{ip}-t_{i1-1})} = e^{\delta(t-s)}e^{\delta(t_{ip}-t)}e^{\delta(s-t_{i1-1})}\\ &\leq e^{\delta(t-s)}e^{\delta(s-t_{i1-1})} \leq \beta e^{\delta(t-s)}.\end{aligned} \tag{18}$$

If $\delta < 0$, by the similar methods, we can conclude that the above inequality holds. It follows that

$$\begin{aligned}E[V(t)] \leq \beta E[V(0)]e^{(\mu+\delta)t} &+ \beta\overline{\mu}\int_0^t e^{(\mu+\delta)(t-s)}E[V(s-\tau(s))]ds\\ &+\beta\hat{\mu}\int_0^t e^{(\mu+\delta)(t-s)}ds.\end{aligned} \tag{19}$$

Let $\varphi(\lambda) = \lambda + \mu + \beta\overline{\mu}e^{\lambda\tau} + \delta$. By (v), we see that $\varphi(0) < 0, \varphi(+\infty) = +\infty$ and $\varphi^{'}(\lambda) = 1 + \beta\overline{\mu}\tau e^{\lambda\tau} > 0$, which means that $\varphi(\lambda) = 0$ has a unique positive solution λ. Next, we can claim that for $t \geq -\tau$

$$E[V(t)] \leq \beta e^{-\lambda t}\sup_{-\tau\leq\varsigma\leq 0}E[V(\varsigma)] + d. \tag{20}$$

Indeed, for $t \in [-\tau, 0]$

$$E[V(t)] \leq \beta\sup_{-\tau\leq\varsigma\leq 0}E[V(\varsigma)] \leq \beta e^{-\lambda t}\sup_{-\tau\leq\varsigma\leq 0}E[V(\varsigma)] + d. \tag{21}$$

Thus we only need to prove that (20) holds for $t > 0$. Otherwise, there exists a $\tilde{t} > 0$ such that

$$\begin{aligned}&E[V(\tilde{t})] > \beta e^{-\lambda\tilde{t}}\sup_{-\tau\leq\varsigma\leq 0}E[V(\varsigma)] + d,\\ &E[V(t)] \leq \beta e^{-\lambda t}\sup_{-\tau\leq\varsigma\leq 0}E[V(\varsigma)] + d, -\tau \leq t < \tilde{t}.\end{aligned} \tag{22}$$

Noting $\varphi(\lambda) = 0$ and (20) (21) yields

$$\begin{aligned}E[V(\tilde{t})] &\leq \beta E[V(0)]e^{(\mu+\delta)\tilde{t}} + \beta\overline{\mu}\int_0^{\tilde{t}} e^{(\mu+\delta)(\tilde{t}-s)}E[V(s-\tau(s))]ds \qquad (22)\\ &\quad + \beta\hat{\mu}\int_0^{\tilde{t}} e^{(\mu+\delta)(\tilde{t}-s)}ds\\ &\leq \beta\sup_{-\tau\leq\varsigma\leq 0}E[V(\varsigma)]e^{(\mu+\delta)\tilde{t}} + \beta^2\overline{\mu}e^{\lambda\tau}\sup_{-\tau\leq\varsigma\leq 0}E[V(\varsigma)]\int_0^{\tilde{t}} e^{(\mu+\delta)(\tilde{t}-s)}e^{-\lambda s}ds\\ &\quad + \beta\overline{\mu}d\int_0^{\tilde{t}} e^{(\mu+\delta)(\tilde{t}-s)}ds + \beta\hat{\mu}\int_0^{\tilde{t}} e^{(\mu+\delta)(\tilde{t}-s)}ds\\ &= \beta e^{-\lambda\tilde{t}}\sup_{-\tau\leq\varsigma\leq 0}E[V(\varsigma)] + d.\end{aligned}$$

□

Lemma 2 [36]. *For any vectors $x, y \in R^n$, there exist constant $\vartheta > 0$, and $\Xi \in R^{n\times n} > 0$ such that*

$$2x^Ty \leq \vartheta x^T\Xi x + \vartheta^{-1}y^T\Xi^{-1}y$$

Lemma 3 ([36] Schur complement). *The linear matrix inequality*

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{pmatrix} < 0$$

is equivalent to

$$U_{22} < 0, \qquad U_{11} - U_{12}U_{22}^{-1}U_{12}^T < 0,$$

where $U_{11} = U_{11}^T$, *and* $U_{22} = U_{22}^T$.

3 Stochastic Quasi-Synchronization in Mean Square

This section devotes to stochastic quasi-synchronization for stochastic neural networks by adding pinning impulsive control.

Theorem 1 *Under Assumption 1–4. Let* $\Theta = \{y \in R^n | E(\|y\|^2) \le \theta\}$ *is the range of system* (2). *If there exist matrices* $P \in R^{n\times n} > 0$, $L_i \in R^{n\times n} > 0$, $i = 1, 2, 3$ *and constants* $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, μ_1, μ_2, ν *such that*

$$\begin{pmatrix} \Xi & \sqrt{\alpha_1} I_N \otimes PB_1 & \sqrt{\alpha_2} I_N \otimes PD_1 & \sqrt{\alpha_3} I_N \otimes P \\ * & -I_N \otimes L_1 & 0 & 0 \\ * & * & -I_N \otimes L_2 & 0 \\ * & * & * & -I_N \otimes L_3 \end{pmatrix} < 0, \tag{23}$$

$$\alpha_2^{-1} l^2 L_2 + \lambda_{\max}(P) M_2 < \mu_2 P, \tag{24}$$

$$\ln \xi_k \le \nu \triangle_{k-1}, \tag{25}$$

and

$$\mu_1 + \overline{\nu}\mu_2 + \nu < 0, \tag{26}$$

then the error of system (3) *can converge to small region* $\mathcal{D} = \{(e_1(t), e_2(t), \cdots, e_N(t))^T$ $|E[\sum_{i=1}^{N} \|e_i(t)\|^2)] \le \frac{\overline{d}}{\lambda_{\min}(P)}, e_i(t) \in R^n, i = 1, 2, \cdots, N\}$ *in the mean square with exponent* λ, *where* $\Xi = I_N \otimes (PC + C^T P + \alpha_1^{-1} l^2 L_1 + \lambda_{\max}(P) M_1) + A \otimes P\Gamma + (A \otimes P\Gamma)^T - \mu_1 I_N \otimes P$, $\xi_k = v_k^2 - (v_k - 1)(v_k + 1)\frac{\lambda_{\max}(P)(N-q)}{\lambda_{\min}(P)N}$, $\overline{\nu} = \sup_{1\le k<\infty}\{\nu_k\}$, $\nu_k = \max\{e^{\nu\triangle_{k-1}}, e^{-\nu\triangle_{k-1}}\}$, $\overline{d} = \frac{-\overline{\nu}\theta[\alpha_3^{-1}\lambda_{\max}(L_3)\rho_1^2 + \lambda_{\max}(P)(\rho_2^2+\rho_3^2)]}{\mu_1+\overline{\nu}\mu_2+\nu}$, $\lambda > 0$ *is the unique solution of* $\lambda + \mu_1 + \overline{\nu}e^{\lambda\tau}\mu_2 + \nu = 0$.

Proof Construct a Lyapunov function

$$V(t) = \sum_{i=1}^{N} e_i^T(t) P e_i(t). \tag{27}$$

For $t \in (t_{k-1}, t_k]$, by (7), we have

$$\mathcal{L}V(t) = 2\sum_{i=1}^{N} e_i^T(t) P[C_i e_i(t) + \triangle C s(t) + B_1 F(e_i(t)) + \triangle B f(s(t))$$
$$+ D_1 F(e_i(t - \tau(t))) + \triangle D f(s - \tau(t)) + \sum_{j=1}^{N} a_{ij}\Gamma e_j(t)]$$

$$
+\sum_{i=1}^{N} trace[H^T(t, e_i(t), e_i(t-\tau(t)))PH(t, e_i(t), e_i(t-\tau(t)))]
$$
$$
+\sum_{i=1}^{N} trace[\triangle h^T(t, s(t), s(t-\tau(t)))Ph(t, s(t), s(t-\tau(t)))]. \quad (28)
$$

From Assumption 1 and Lemma 2, there exist $\alpha_1 > 0$, $\alpha_2 > 0$ and $L_1 \in R^{n\times n} > 0$, $L_2 \in R^{n\times n} > 0$ such that

$$
\begin{aligned}
2e_i^T(t)PB_1F(e_i(t)) &\leq \alpha_1 e_i^T(t)PB_1L_1^{-1}B_1^TPe_i(t) + \alpha_1^{-1}F^T(e_i(t))L_1F(e_i(t)) \\
&\leq \alpha_1 e_i^T(t)PB_1L_1^{-1}B_1^TPe_i(t) + \alpha_1^{-1}l^2e_i^T(t)L_1e_i(t), \quad (29)
\end{aligned}
$$

and

$$
\begin{aligned}
2e_i^T(t)PD_1F(e_i(t-\tau(t))) &\leq \alpha_2 e_i^T(t)PD_1L_2^{-1}D_1^TPe_i(t) \\
&\quad +\alpha_2^{-1}F^T(e_i(t-\tau(t)))L_2F(e_i(t-\tau(t))) \\
&\leq \alpha_2 e_i^T(t)PD_1L_2^{-1}D_1^TPe_i(t) \\
&\quad +\alpha_2^{-1}l^2e_i^T(t-\tau(t))L_2e_i(t-\tau(t)). \quad (30)
\end{aligned}
$$

In view of Assumption 2, we can obtain

$$
\begin{aligned}
&trace[H^T(t, e_i(t), e_i(t-\tau(t)))PH(t, e_i(t), e_i(t-\tau(t)))] \\
&\quad \leq \lambda_{\max}(P)[e_i^T(t)M_1e_i(t) + e_i^T(t-\tau(t))M_2e_i(t-\tau(t))]. \quad (31)
\end{aligned}
$$

Let $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$, then

$$
2\sum_{i=1}^{N} e_i^T(t)P\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) = 2e^T(t)(A\bigotimes P\Gamma)e(t). \quad (32)
$$

Noting that the parameter mismatches and stochastic perturbation mismatches satisfy Assumption 3 and Assumption 4, it follows from Lemma 2 that there exist $\alpha_3 > 0$ and $L_3 \in R^{n\times n} > 0$ such that

$$
\begin{aligned}
&2e_i^T(t)P[\triangle Cs(t) + \triangle Bf(s(t)) + \triangle Df(s-\tau(t))] \\
&\quad \leq \alpha_3 e_i^T(t)PL_3^{-1}Pe_i(t) + \alpha_3^{-1}[\triangle Cs(t) + \triangle Bf(s(t)) + \triangle Df(s-\tau(t))]^T \\
&\quad\quad L_3[\triangle Cs(t) + \triangle Bf(s(t)) + \triangle Df(s-\tau(t))]. \quad (33)
\end{aligned}
$$

and

$$
\begin{aligned}
&trace[\triangle h^T(t, s(t), s(t-\tau(t)))P\triangle h(t, s(t), s(t-\tau(t)))] \\
&\quad \leq \lambda_{\max}(P)(\rho_2^2\|s(t)\|^2 + \rho_3^2\|s(t-\tau(t))\|^2). \quad (34)
\end{aligned}
$$

Substituting (29)–(34) into (28) yields

$$
\begin{aligned}
\mathcal{L}V(t) \leq& \sum_{i=1}^{N} e_i^T(t)\Psi_1e_i(t) + 2e^T(t)(A\bigotimes P\Gamma)e(t) + \sum_{i=1}^{N} e_i^T(t-\tau(t))\Psi_2e_i(t-\tau(t)) \\
&+\alpha_3^{-1}[\triangle Cs(t) + \triangle Bf(s(t)) + \triangle Df(s-\tau(t))]^TL_3[\triangle Cs(t) + \triangle Bf(s(t)) \\
&+\triangle Df(s-\tau(t))] + \lambda_{\max}(P)(\rho_2^2\|s(t)\|^2 + \rho_3^2\|s(t-\tau(t))\|^2), \quad (35)
\end{aligned}
$$

where $\Psi_1 = PC + C^T P + \alpha_1 P B_1 L_1^{-1} B_1^T P + \alpha_1^{-1} l^2 L_1 + \alpha_2 P D_1 L_2^{-1} D_1^T P + \lambda_{\max}(P) M_1 + \alpha_3 P L_3^{-1} P$. $\Psi_2 = \alpha_2^{-1} l^2 L_2 + \lambda_{\max}(P) M_2$. By (23) (24) and Lemma 3, we have

$$E[\mathcal{L}V(t)] \leq \mu_1 E[V(t)] + \mu_2 E[V(t-\tau(t))] + [\alpha_3^{-1}\lambda_{\max}(L_3)\rho_1^2 + \lambda_{\max}(P)(\rho_2^2 + \rho_3^2)]\theta. \tag{36}$$

On the other hand, when $t = t_k$, we have

$$\begin{aligned} V(t_k^+) &= \sum_{i=1}^{N} e_i^T(t_k^+) P e_i(t_k^+) \\ &= \sum_{i \in \chi_p(t_k)} e_i^T(t_k^+) P e_i(t_k^+) + \sum_{i \overline{\in} \chi_p(t_k)} e_i^T(t_k^+) P e_i(t_k^+) \\ &= u_k^2 \sum_{i \in \chi(t_k)} e_i^T(t_k) P e_i(t_k) + \sum_{i \overline{\in} \chi(t_k)} e_i^T(t_k) P e_i(t_k) \\ &= u_k^2 \sum_{i=1}^{N} e_i^T(t_k) P e_i(t_k) - (u_k - 1)(u_k + 1) \sum_{i \overline{\in} \chi(t_k)} e_i^T(t_k) P e_i(t_k). \end{aligned} \tag{37}$$

If $-1 < u_k < 1$, in view of the selection of pinning nodes in set $\chi_p(t_k)$, we get

$$\begin{aligned} \frac{1}{N-q} \sum_{i \overline{\in} \chi(t_k)} E[e_i^T(t_k) P e_i(t_k)] &\leq \frac{\lambda_{\max}(P)}{N-q} \sum_{i \overline{\in} \chi(t_k)} E[e_i^T(t_k) e_i(t_k)] \\ &\leq \frac{\lambda_{\max}(P)}{N} \sum_{i=1}^{N} E[e_i^T(t_k) e_i(t_k)] \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P) N} \sum_{i=1}^{N} E[e_i^T(t_k) P e_i(t_k)] \end{aligned} \tag{38}$$

Then we have

$$\begin{aligned} E[V(t_k^+)] \leq{}& v_k^2 \sum_{i=1}^{N} E[e_i^T(t_k) P e_i(t_k)] \\ & -(v_k - 1)(v_k + 1) \sum_{i=1}^{N} \frac{\lambda_{\max}(P)(N-q)}{\lambda_{\min}(P) N} E[e_i^T(t_k) P e_i(t_k)] \\ ={}& \xi_k E[V(t_k)], \end{aligned} \tag{39}$$

where $\xi_k = v_k^2 - (v_k - 1)(v_k + 1)\frac{\lambda_{\max}(P)(N-q)}{\lambda_{\min}(P)N}$. For $v_k \leq -1$ or $v_k \geq 1$, we can conclude that (39) holds by the same method. It follows from (23)–(26) and Lemma 1 that there exists $\bar{l} > 0$ such that

$$E[V(t)] \leq \bar{l} e^{-\lambda t} E\left[\sup_{-\tau \leq \varsigma \leq 0} V(\varsigma)\right] + \bar{d}, t \geq 0 \tag{40}$$

which implies that

$$E\left[\sum_{i=1}^{N} \|e_i(t)\|^2 \leq \frac{\lambda_{\max}(P)\bar{l}}{\lambda_{\min}(P)} e^{-\lambda t} E\left[\sup_{-\tau \leq \varsigma \leq 0} \|e_i(\varsigma)\|^2\right] + \frac{\bar{d}}{\lambda_{\min}(P)}, t \geq 0. \right. \tag{41}$$

Therefore, the error system (3) can converges to small region in the mean square with exponent λ. □

Remark 2 From the proof of Theorem 1, the cluster synchronization criterion is related to ρ_p, η_k and the impulsive interval $t_{k+1} - t_k$. η_k depends on the impulsive gain d_k and the pinned number ρ_p at impulsive time t_k. on the other hand, inequality (10) characterize the relation between η_k and $t_{k+1} - t_k$. Therefore, a suitable pinning impulsive controller can be determined by selecting the value d_k, ρ_p and $t_{k+1} - t_k$.

Remark 3 According to Theorem 1, if $-1 < \upsilon_k < 1$, we can conclude that the number of the pinned nodes is estimated as

$$q > [1 + \frac{(e^{\nu\triangle_{k-1}} - \upsilon_k^2)\lambda_{\min}(P)}{(\upsilon_k - 1)(\upsilon_k + 1)\lambda_{\max}(P)}]N.$$

Correspondingly, if $\upsilon_k \le -1$ or $\upsilon_k \ge 1$, we see that

$$q < [1 + \frac{(e^{\nu\triangle_{k-1}} - \upsilon_k^2)\lambda_{\min}(P)}{(\upsilon_k - 1)(\upsilon_k + 1)\lambda_{\max}(P)}]N.$$

Remark 4 The obtained conditions for stochastic quasi-synchronization are condition (23) (24) (25) and (26) in Theorem 1. To reduce the calculation burden, MATLAB LMI toolbox is used to determine μ_1 and μ_2 by fixing the values of $\alpha_i, i = 1, 2, 3$. If taking $P = L_1 = L_2 = L_3 = I_n, \alpha_1 = \frac{l}{\sqrt{\lambda_{\max}(B_1 B_1^T)}}, \alpha_2 = \frac{l}{\sqrt{\lambda_{\max}(D_1 D_1^T)}}, \alpha_3 = 1, \mu_1 = \lambda_{\max}(C_1 + C_1^T) + 2l\sqrt{\lambda_{\max}(B_1 B_1^T)} + l\sqrt{\lambda_{\max}(D_1 D_1^T)} + \lambda_{\max}[(A \otimes \Gamma) + (A \otimes \Gamma)^T] + \lambda_{\max}(M_1) + 1$, $\mu_2 = l\sqrt{\lambda_{\max}(D_1 D_1^T)} + \lambda_{\max}(M_2)$, $\nu = \sup_{1 \le k < +\infty} \{\frac{\ln[\upsilon_k^2 - (\upsilon_k - 1)(\upsilon_k + 1)\frac{\lambda_{\max}(P)(N-q)}{\lambda_{\min}(P)N}]}{\triangle_{k-1}}\}$, $\overline{d} = \frac{-\overline{\upsilon}\theta(\rho_1^2 + \rho_2^2 + \rho_3^2)}{\mu_1 + \overline{\upsilon}\mu_2 + \nu}$, we can derive the following practical corollary.

Corollary 1 *Under Assumption 1–4. Let $\Theta = \{y \in R^n | E(\|y\|^2) \le \theta\}$ is the range of system (2). If*

$$\mu_1 + \overline{\upsilon}\mu_2 + \nu < 0,$$

then the error of system (3) can converge to small region $\mathcal{D} = \{(e_1(t), e_2(t), \cdots, e_N(t))^T | E[\sum_{i=1}^{N} \|e_i(t)\|^2)] \le \overline{d}, e_i(t) \in R^n, i = 1, 2, \cdots, N\}$ in the mean square.

Theorem 2 *Under Assumption 1–4 and $t_k - t_{k-1} \ge \tau$. Let $\Theta = \{y \in R^n | E(\|y\|^2) \le \theta\}$ is the range of system (2). If there exist matrices $P \in R^{n\times n} > 0$, $L_i \in R^{n\times n} > 0$, $i = 1, 2, 3$ and constants $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, μ_1, μ_2 satisfying (23), (24). Also if there exists a constant $\lambda > 0$ such that for $k = 1, 2 \cdots$*

$$\ln(\xi_k + \frac{\mu_2}{1-\sigma}\triangle_{k-1}) + \rho\triangle_{k-1} \le -\lambda, \tag{42}$$

then the error of system (3) can converge to small region $\mathcal{D} = \{(e_1(t), e_2(t), \cdots, e_N(t))^T | E[\sum_{i=1}^{N} \|e_i(t)\|^2)] \le \frac{\widetilde{b}}{\lambda_{\min}(P)}, e_i(t) \in R^n, i = 1, 2, \cdots, N\}$ in the mean square with exponent $\frac{\lambda}{\triangle}$, where $\rho = \mu_1 + \frac{\mu_2}{1-\sigma}$, $\widetilde{b} = \frac{\varepsilon(e^{-\lambda} - \hat{d})e^{\rho\overline{\triangle}}}{\rho(1 - e^{-\lambda})} + \frac{\varepsilon}{\rho}(e^{\rho\overline{\triangle}} - 1)$, $\varepsilon = [\alpha_3^{-1}\lambda_{\max}(L_3)\rho_1^2 + \lambda_{\max}(P)(\rho_2^2 + \rho_3^2)]\theta$, $\overline{\triangle} = \sup_{k \ge 1}\{\triangle_{k-1}\}$.

Proof Consider a Lyapunov–Krasovskii functional

$$V(t) = V_1(t) + V_2(t), \tag{43}$$

where

$$V_1(t) = \sum_{i=1}^{N} e_i^T(t) P e_i(t), \qquad V_2(t) = \frac{\mu_2}{1-\sigma}\int_{t-\tau(t)}^{t} \sum_{i=1}^{N} e_i^T(s) P e_i(s) ds. \tag{44}$$

Similar to the proof of Theorem 3.1, for $t \in (t_k, t_{k+1}]$, we see that

$$\mathcal{L}V_1(t) \leq \mu_1 V_1(t) + \mu_2 V(t-\tau(t)) + [\alpha_3^{-1}\lambda_{\max}(L_3)\rho_1^2 + \lambda_{\max}(P)(\rho_2^2+\rho_3^2)]\theta. \quad (45)$$

For $t \in (t_k, t_{k+1}]$, it yields

$$\mathcal{L}V_2(t) \leq \tfrac{\mu_2}{1-\sigma}V_1(t) - \mu_2 V_1(t-\tau(t)). \quad (46)$$

Thus

$$E[\mathcal{L}V(t)] \leq (\mu_1 + \tfrac{\mu_2}{1-\sigma})E[V_1(t)] + \varepsilon \leq \rho E[V(t)] + \varepsilon, \quad (47)$$

where $\rho = \mu_1 + \frac{\mu_2}{1-\sigma}$, $\varepsilon = [\alpha_3^{-1}\lambda_{\max}(L_3)\rho_1^2 + \lambda_{\max}(P)(\rho_2^2+\rho_3^2)]\theta$. It follows that for $t \in (t_k, t_{k+1}]$

$$E[V(t)] \leq E[V(t_k^+)]e^{\rho(t-t_k)} + \tfrac{\varepsilon}{\rho}[e^{\rho(t-t_k)} - 1]. \quad (48)$$

When $t = t_k$, according to the proof of Theorem 3.1 and by (48), we can obtain

$$E[V_1(t_k^+)] \leq \xi_k E[V_1(t_k)] \leq \xi_k E[V(t_k)] \leq \xi_k e^{\rho\triangle_{k-1}}E[V(t_{k-1}^+)] + \tfrac{\varepsilon\xi_k}{\rho}[e^{\rho\triangle_{k-1}} - 1]. \quad (49)$$

By (44), there exists a $\bar{t}_k \in (t_{k-1}, t_k]$ such that

$$\begin{aligned} V_2(t_k^+) &= \tfrac{\mu_2}{1-\sigma}\int_{t_k-\tau(t_k)}^{t_k} V_1(s)ds \leq \tfrac{\mu_2}{1-\sigma}\int_{t_{k-1}}^{t_k} V_1(s)ds = \tfrac{\mu_2}{1-\sigma}\triangle_{k-1}V_1(\bar{t}_k) \\ &\leq \tfrac{\mu_2}{1-\sigma}\triangle_{k-1}V(\bar{t}_k). \end{aligned} \quad (50)$$

It follows from (33) and the above inequality that

$$E[V_2(t_k^+)] \leq \tfrac{\mu_2}{1-\sigma}\triangle_{k-1}e^{\rho\triangle_{k-1}}E[V(t_{k-1}^+)] + \tfrac{\mu_2\varepsilon}{(1-\sigma)\rho}\triangle_{k-1}(e^{\rho\triangle_{k-1}} - 1). \quad (51)$$

Submitting (34) and (36) into (28), we have

$$\begin{aligned} E[V(t_k^+)] &\leq \left(\xi_k + \tfrac{\mu_2}{1-\sigma}\triangle_{k-1}\right)e^{\rho\triangle_{k-1}}E[V(t_{k-1}^+)] + \tfrac{\varepsilon}{\rho}\left(\xi_k + \tfrac{\mu_2}{1-\sigma}\triangle_{k-1}\right)(e^{\rho\triangle_{k-1}} - 1) \\ &\leq e^{-\lambda}E[V(t_{k-1}^+)] + \tfrac{\varepsilon}{\rho}(e^{-\lambda} - \overline{b}), \end{aligned} \quad (52)$$

where $\overline{b} = \inf_{k\geq 1}\{\xi_k + \frac{\mu_2}{1-\sigma}\triangle_{k-1}\}$, which yields that

$$E[V(t_k^+)] \leq e^{-\lambda k}E[\sup_{-\tau\leq\varsigma\leq 0} V(\varsigma)] + \tfrac{\varepsilon(e^{-\lambda}-\overline{b})}{\rho(1-e^{-\lambda})}. \quad (53)$$

For $t \in (t_k, t_{k+1}]$, by (26) (33), we see that

$$\begin{aligned} E[V(t)] &\leq e^{\rho(t-t_k)}E[V(t_k^+)] + \tfrac{\varepsilon}{\rho}[e^{\rho(t-t_k)} - 1] \leq e^{\rho\triangle_k}e^{-\lambda k}E[\sup_{-\tau\leq\varsigma\leq 0} V(\varsigma)] + \widetilde{b} \\ &\leq e^{\rho\triangle_k}e^{\frac{-\lambda t_k}{\overline{\triangle}}}E[\sup_{-\tau\leq\varsigma\leq 0} V(\varsigma)] \\ &\quad + \widetilde{b} \leq e^{\rho\triangle_k}e^{\frac{-\lambda(t_k-t_{k+1})}{\overline{\triangle}}}e^{\frac{-\lambda t_{k+1}}{\overline{\triangle}}}E[\sup_{-\tau\leq\varsigma\leq 0} V(\varsigma)] + \widetilde{b} \\ &\leq e^{\rho\overline{\triangle}+\lambda}e^{\frac{-\lambda t_{k+1}}{\overline{\triangle}}}E[\sup_{-\tau\leq\varsigma\leq 0} V(\varsigma)] + \widetilde{b} \leq e^{\rho\overline{\triangle}+\lambda}e^{\frac{-\lambda t}{\overline{\triangle}}}E[\sup_{-\tau\leq\varsigma\leq 0} V(\varsigma)] + \widetilde{b}. \end{aligned} \quad (54)$$

where $\widetilde{b} = \frac{\varepsilon(e^{-\lambda}-\overline{b})e^{\rho\triangle_k}}{\rho(1-e^{-\lambda})} + \frac{\varepsilon}{\rho}(e^{\rho\triangle_k} - 1)$. This completes the proof. □

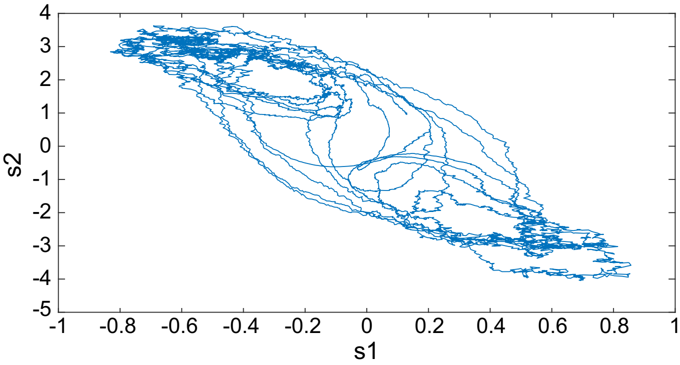

Fig. 1 The state variables $s(t)$ with initial value (0.2, 0.5)

4 Numerical Simulations

In this section, a numerical example is given to demonstrate our results. Consider the following neural networks with stochastic perturbation as the leader:

$$ds(t) = [C_2 s(t) + B_2 f(s(t)) + D_2 f(s(t-1))]dt + [h_2(t, s(t), s(t-1))]d\omega(t), \quad (55)$$

where $s(t) = (s_1(t), s_2(t))^T$, $f(s(t)) = (f_1(s_1(t)), f_2(s_2(t)))^T$, $f_1(s_1(t)) = \arctan s_1(t)$, $f_2(s_2(t)) = \arctan s_2(t)$,

$$C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, B_2 = \begin{pmatrix} 2 & -0.1 \\ -5 & 1.5 \end{pmatrix}, D_2 = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -1 \end{pmatrix},$$

$$h_2(t, s(t), s(t-1)) = \begin{pmatrix} 0.1s_1(t) & 0 \\ 0 & 0.03s_2(t) \end{pmatrix} + \begin{pmatrix} -0.1s_1(t) & 0 \\ 0 & -0.05s_2(t) \end{pmatrix}$$

Figure 1 depicts the trajectory of $(s_1(t), s_2(t))$ with value (0.2, 0.5). This is a chaotic attractor with stochastic perturbation and the range $\Theta = \{s \in R^2 | E[\|s\|] \le 16\}$.

We assume the response neural networks is the following form:

$$\begin{aligned} dx_i(t) = [&C_1 x_i(t) + B_1 f(x_i(t)) + D_1 f(x_i(t-1)) + \sum_{j=1}^{4} a_{ij}\Gamma x_i(t) + u_i(t)]dt \\ &+[h_1(t, x_i(t), x_i(t-1))]d\omega(t), \quad i = 1, 2, 3, 4, \end{aligned} \quad (55)$$

where $\Gamma = diag\{1.2, 1.5\}$

$$C_1 = \begin{pmatrix} -1.002 & 0 \\ 0 & -1.003 \end{pmatrix}, B_1 = \begin{pmatrix} 2.001 & -0.102 \\ -4.99 & 1.502 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} -1.502 & -0.09 \\ -0.203 & -1.002 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 0.6 & 0.8 & 0.6 \\ 0.5 & -3 & 0.5 & 2 \\ 1 & 0.2 & -2.5 & 1.3 \\ 0.8 & 0 & 1.2 & -2 \end{pmatrix},$$

$$h_1(t, x_i(t), x_i(t-1)) = \begin{pmatrix} -0.15x_{i1}(t) & 0 \\ 0 & 0.04x_{i2}(t) \end{pmatrix} + \begin{pmatrix} -0.12x_{i1}(t) & 0 \\ 0 & -0.04x_{i2}(t) \end{pmatrix}.$$

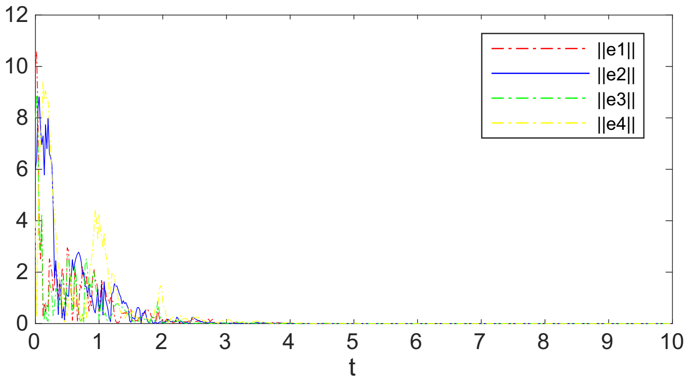

Fig. 2 The errors $\|e_i\|, i = 1, 2, 3, 4$ of synchronization with initial value (0.2, 0.5)

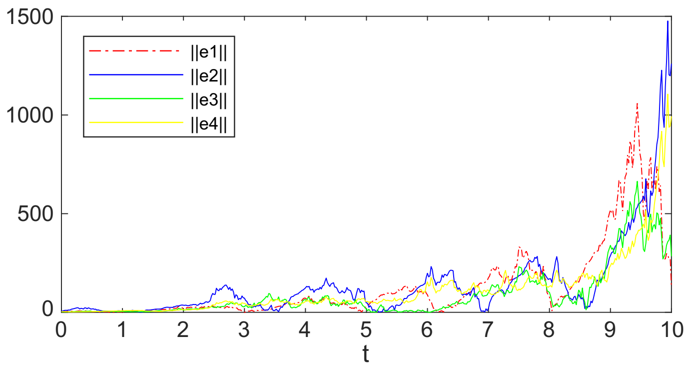

Fig. 3 The errors $\|e_i\|, i = 1, 2, 3, 4$ of synchronization without pinning impulsive input

We design the pinning impulsive input with pinned nodes $q = 2$, $t_k = 0.02k$, $v_k = 0.25$. By simple calculation, we conclude that $\mu_1 + \overline{\nu}\mu_2 + \nu \approx -13.8816 < 0$. By Corollary 1, we can estimate the convergence region $\mathcal{D} = \{(e_1, e_2, e_3, e_4)|E[\sum_{i=1}^{4} \|e_i\|^2] \leq 0.098\}$. Figure 2 depicts the errors $\|e_1\|$ and $\|e_2\|$ of synchronization with initial value $(10, -5)$. Figure 2 depicts the errors $\|e_i\|$, $(i = 1, 2, 3, 4)$ of synchronization with initial value (0.2, 0.5). Figure 3 depicts the errors $\|e_i\|$, $(i = 1, 2, 3, 4)$ of synchronization without pinning impulsive input.

Remark 5 It is necessary to select suitable nodes when applying pinning impulsive control scheme. In [24,25], random nodes can be selected to control. However, since the expectation of synchronization error $e_i(t)$ may be different at impulsive times $t = t_k$, so the pinned nodes are not invariant. Figure 2 implies that our pinning algorithm is more general than ones in [24,25,35]. Figure 3 shows that pinning impulsive input plays an important role in stochastic quasi-synchronization of delayed networks.

5 Conclusions

In this paper, stochastic quasi-synchronization is studied in a leader-follower delayed neural networks by using pinning impulsive control scheme. First, by pinning selected nodes of stochastic neural networks and establishing a new lemma of stochastic impulsive system, a general criterion is obtained to ensure stochastic quasi-synchronization between the leader and the followers with two different topologies. Finally, an example is provided to illustrate the effectiveness of the obtained results.

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