



Finite-Time Stability for Caputo–Katugampola Fractional-Order Time-Delayed Neural Networks

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Abstract

In this paper, an original scheme is presented, in order to study the finite-time stability of the equilibrium point, and to prove its existence and uniqueness, for Caputo–Katugampola fractional-order neural networks, with time delay. The proposed scheme uses a newly introduced fractional derivative concept in the literature, which is the Caputo–Katugampola fractional derivative. The effectiveness of the theoretical results is shown through simulations for two numerical examples.

Keywords Fractional-order calculus · Neural networks · Finite-time stability · Caputo–Katugampola derivative

1 Introduction

Nowadays, artificial neural networks can be considered as one of the most used and growing techniques in technology. For instance, they are extensively exploited in voice recognition [1], pattern identification [2] and systems control [3]. It is of a great importance to note that, in electronic implementation of neural networks, time delays are very frequent. This is due to various reasons, such as circuit integration and communication delays [4]. Thus, it is very significant to study the stability of delayed neural networks, from both the theoretical aspect and the practical aspect. During the last three decades, several research works have been

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conducted in this context [5–13]. Though, the great majority of these investigations have been based on infinite time intervals.

The finite-time stability definition has been introduced for the first time in [14]. Finite-time stable systems have been proved to have interesting properties, such as disturbance rejection, better robustness and faster convergence [15]. For this reason, several research works have been conducted to study finite-time stability and stabilization for different classes of systems [16–18].

In the last decades, several applications of the fractional calculus in science and engineering have emerged [19, 20]. This fact has considerably stimulated the investigation of fractional-order systems by researchers in the control theory. Indeed, several papers, dealing with this area of research, have been elaborated in the last years, and still, many researchers are working on this field. As examples of the treated queries for fractional-order systems, in the literature, one can cite: model reference control [21], fault reconstruction [22] and finite-time stability analysis [23]. In the last two decades and dealing with artificial neural networks, many researchers have incorporated the fractional calculus in them, see for instance [24–27]. In particular, some remarkable papers have investigated the finite-time stability problem for fractional neural networks [28–31]. It is prominent to indicate that, recently, some interesting papers have been published in relation with finite-time stability for fractional-order neural networks with time delay [32–36]. Note that, in these four works, the well-known definition of Caputo fractional derivative has been used.

In the last years, Katugampola has defined a new fractional derivative concept, called the Caputo–Katugampola derivative. This new concept has been shown to be more general than the classical Caputo one [37, 38]. The Caputo–Katugampola derivative is characterized by two parameters: $\rho > 0$ and $0 < \alpha < 1$. It is noteworthy to indicate that, if $\rho = 1$, this derivative reduces to the classical Caputo one [39]. From a physical point of view, it is being shown in the literature that the Katugampola fractional-order representation is of value. See for instance [40], where it has been insisted that the Caputo–Katugampola derivative is very significant for quantum mechanics.

Motivated by all the above discussions, the authors propose in this paper an original investigation, in which it is question of proving the existence and uniqueness of the equilibrium point, which is finite-time stable, for Caputo–Katugampola fractional-order neural networks, with time delay. In order to demonstrate the existence of a unique equilibrium point, the fixed-point theorem is exploited in this paper. It is of value to note that other research works [41, 42] have used another approach to prove it, which is the topological degree theory. To the best of the authors' knowledge, no analogue study has been done in the literature for the new general class of Katugampola fractional-order systems. To be more precise, some aspects, in the few similar literature papers, have motivated and have inspired the authors to develop the present paper. In the following, the contribution aspect is clarified, and the main advantages of this work, compared to the literature results, are summarized:

- In [32–36], the authors have considered Caputo fractional-order neural networks. The present paper investigates a wider class of systems, since the considered Caputo–Katugampola fractional derivative is more general than the classical Caputo fractional derivative.
- The present paper has another merit, compared to [32, 33, 36], since these three cited works did not investigate the existence and uniqueness problem of the equilibrium point.

Throughout the paper, the authors investigate the fractional-order neural networks with time delay, given by the following representation:

$${}^C D_{t_0}^{\alpha, \rho} x_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t - \tau)) + I_i, i = 1, 2, \dots, n \tag{1}$$

The corresponding vector form is:

$${}^C D_{t_0}^{\alpha, \rho} x(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau)) + I \tag{2}$$

for $t \in [t_0, t_f]$, where ${}^C D_{t_0}^{\alpha, \rho}$ is the Caputo–Katugampola fractional derivative (see Definition 3 in the preliminaries’ section), with the derivation parameters: $0 < \alpha < 1$, and $\rho > 0$. $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$ is the state vector, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \dots, f_n(x_n(t)))^T \in \mathbb{R}^n$ is the neuron activation function, $C = \text{diag}(c_1, c_2, \dots, c_n)$, is the rate, with which the i th neuron resets its potential to the resting state in isolation when disconnection from the networks and the external inputs ($c_i > 0$); $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ represent the connection between the j th neuron and the i th neuron at t and $t - \tau$, respectively, (τ is the nonnegative constant delay), $I = (I_1, I_2, \dots, I_n)^T$ stands for constant inputs.

With the initial conditions:

$$x_i(t_0 + s) = \psi_i(s) \tag{3}$$

$\psi_i(s)$ are continuous functions defined on $[-\tau, 0]$, such that: $\|\psi\| = \sup_{s \in [-\tau, 0]} \sum_{i=1}^n |\psi_i(s)|$.

The rest of the paper is organized as follows. In Sect. 2, some useful preliminaries are given. In Sect. 3, the main results of this paper are detailed. Two theorems are demonstrated, in order to prove the existence and uniqueness of the equilibrium point, which is finite-time stable, for Caputo–Katugampola fractional-order neural networks, with time delay. Theorem 1 investigates the case $0 < \alpha \leq \frac{1}{2}$, while Theorem 2 investigates the case $\frac{1}{2} < \alpha < 1$. Finally, in Sect. 4, The effectiveness of the theoretical results is shown through simulations for two numerical examples.

2 Preliminaries

Definition 1 [37] (*Katugampola fractional integral*) Given $\alpha > 0$, $\rho > 0$ and an interval $[a, b]$ of \mathbb{R} , where $0 < a < b$. The Katugampola fractional integral of a function $x \in L^1([a, b])$ is defined by:

$$I_{a^+}^{\alpha, \rho} = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1} u(s)}{(t^\rho - s^\rho)^{1-\alpha}} ds,$$

where Γ is the gamma function.

Definition 2 [38] (*Katugampola fractional derivative*) Given $0 < \alpha < 1$, $\rho > 0$ and an interval $[a, b]$ of \mathbb{R} , where $0 < a < b$. The Katugampola fractional derivative is defined by

$$D_{a^+}^{\alpha, \rho} u(t) = \frac{\rho^\alpha}{\Gamma(1 - \alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1} u(s)}{(t^\rho - s^\rho)^{1-\alpha}} ds,$$

Definition 3 (Caputo–Katugampola fractional derivative) Given $0 < \alpha < 1, \rho > 0$ and an interval $[a, b]$ of \mathbb{R} , where $0 < a < b$. The Caputo–Katugampola fractional derivative is defined by

$$\begin{aligned} {}^C D_{a^+}^{\alpha, \rho} u(t) &= {}^C D_{a^+}^{\alpha, \rho} u(t)[u(t) - u(a)] \\ &= \frac{\rho^\alpha}{\Gamma(1 - \alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1} [u(s) - u(a)]}{(t^\rho - s^\rho)^{1-\alpha}} ds, \end{aligned}$$

Definition 4 [34] The equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ of system (1) is said to be finite time stable with respect to $\{t_0, \delta, \varepsilon, \Theta, \tau\}, 0 < \delta < \varepsilon, \delta, \varepsilon \in \mathbb{R}, \Theta = [t_0, t_0 + t_f]$, such that for any solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ of system (1) with initial conditions (3), if and only if

$$\|\psi - x^*\| < \delta,$$

implies

$$\|x(t) - x^*\| < \varepsilon, \quad \forall t \in \Theta,$$

where

$$\|\psi - x^*\| = \sup_{s \in [-\tau, 0]} \sum_{i=1}^n |\psi_i(s) - x_i^*|, \quad \|x(t) - x^*\| = \sum_{i=1}^n |x_i(t) - x_i^*|.$$

Lemma 1 [43] Let $n \in \mathbb{N}$ and a_1, a_2, \dots, a_n be nonnegative real numbers. Then for $l > 1$;

$$\left(\sum_{i=1}^n a_i \right)^l \leq n^{l-1} \sum_{i=1}^n a_i^l.$$

Lemma 2 (Holder inequality, Cauchy–Schwartz inequality) [44] Let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f_1 \in L^p([a, b]), f_2 \in L^q([a, b])$, then $f_1 f_2 \in L^1([a, b])$ and:

$$\int_a^b |f_1(x) f_2(x)| dx \leq \left(\int_a^b |f_1(x)|^p dx \right)^{\frac{1}{p}} \left(\int_a^b |f_2(x)|^q dx \right)^{\frac{1}{q}}$$

where $L^p([a, b])$ is the Banach space of all Lebesgue measurable functions $f : [a, b] \rightarrow \mathbb{R}$, with $\int_a^b |f(x)|^p dx < \infty$. If $p = q = 2$, then it reduces to the Cauchy–Schwartz inequality:

$$\int_a^b |f_1(x) f_2(x)| dx \leq \left(\int_a^b |f_1(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_a^b |f_2(x)|^2 dx \right)^{\frac{1}{2}}$$

3 Main Results

First, these assumptions are considered:

(H1) The functions f_i are Lipschitz; one can find constants $F_i > 0$ such that

$$\|f_i(x) - f_i(y)\| \leq F_i \|x - y\|$$

for any $x, y \in \mathbb{R}, i = 1, 2, \dots, n$.

(H2) Given a_{ij}, b_{ij}, c_i and F_j , one has:

$$\sum_{i=1}^n \max_{1 \leq j \leq n} \{ |a_{ij}| F_j + |b_{ij}| F_j \} < \min_{1 \leq i \leq n} \{ c_i \}.$$

Theorem 1 Assume that (H1) and (H2) hold and $0 < \alpha \leq \frac{1}{2}$. If the following condition is satisfied:

$$\sqrt[3q-1]{NA_2^q [(t_0 + \tau)^\rho - t_0^\rho] \exp \left[t_f^\rho + N \left(\frac{A_1^q}{q} (t_f^\rho - t_0^\rho) + \frac{A_1^q}{q} ((t_f + \delta)^\rho - (t_0 + \delta)^\rho) \right) \right]} < \frac{\varepsilon}{\delta} \tag{4}$$

where $N = \frac{3^{q-1} \rho^{-q\alpha}}{(\Gamma(\alpha))^q} \left(\frac{\Gamma(p(\alpha-1)+1)}{p^{\rho-p+1}} \right)^{\frac{q}{p}}$, $p = 1 + \alpha$ and $q = \frac{\alpha+1}{\alpha}$, $A_1 = \max_{1 \leq i \leq n} (c_i) + \sum_{i=1}^n \max_{1 \leq j \leq n} (|a_{ij}| F_j)$ and $A_2 = \sum_{i=1}^n \max_{1 \leq j \leq n} (|b_{ij}| F_j)$. Then, there exists a unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ of system (1) which is finite time stable with respect to $\{t_0, \delta, \varepsilon, \Theta, \tau\}$, $0 < \delta < \varepsilon, \delta, \varepsilon \in \mathbb{R}, \Theta = [t_0, t_0 + t_f]$.

Proof The first step, is to prove that there exists a unique equilibrium point for the considered class of systems. We consider the function Φ given in [35] by $\Phi(u) = (\Phi_1(u), \Phi_2(u), \dots, \Phi_n(u))^T$ where

$$\Phi_i(u) = \sum_{j=1}^n a_{ij} f_j \left(\frac{u_j}{c_j} \right) + \sum_{j=1}^n b_{ij} f_j \left(\frac{u_j}{c_j} \right) + I_i, \quad i = 1, 2, \dots, n$$

for $u = (u_1, u_2, \dots, u_n)^T$.

Consider two vectors $u = (u_1, u_2, \dots, u_n)^T$ and $v = (v_1, v_2, \dots, v_n)^T$. Then, using assumption (H1) and the same development as [35], we get:

$$\begin{aligned} \|\Phi(u) - \Phi(v)\| &= \sum_{i=1}^n |\Phi_i(u) - \Phi_i(v)| \\ &\leq \frac{\sum_{i=1}^n \max_{1 \leq j \leq n} \{ |a_{ij}| F_j + |b_{ij}| F_j \}}{\min_{1 \leq i \leq n} \{ c_i \}} \sum_{j=1}^n |u_j - v_j| \end{aligned} \tag{5}$$

Hence $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a contraction mapping on \mathbb{R}^n , which means that there exists a unique fixed point $u^* \in \mathbb{R}^n$ satisfying $\Phi(u^*) = u^*$:

$$u_i^* = \sum_{j=1}^n a_{ij} f_j \left(\frac{u_j^*}{c_j} \right) + \sum_{j=1}^n b_{ij} f_j \left(\frac{u_j^*}{c_j} \right) + I_i, \quad i = 1, 2, \dots, n \tag{6}$$

Consider $c_i x_i^* = u_i^*, i = 1, 2, \dots, n$, then

$$- c_i x_i^* + \sum_{j=1}^n a_{ij} f_j (x_j^*) + \sum_{j=1}^n b_{ij} f_j (x_j^*) + I_i = 0, \tag{7}$$

Thus, system (1) has a unique equilibrium point x^* .

Now, the goal is to check the finite time stability of $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$. Define $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ as a solution of system (1). We have

$$\begin{aligned}
 {}^C D_{t_0}^{\alpha, \rho} (x_i(t) - x_i^*) &= -c_i(x_i(t) - x_i^*) + \sum_{j=1}^n a_{ij} \left(f_j(x_j(t)) - f_j(x_j^*) \right) \\
 &\quad + \sum_{j=1}^n b_{ij} \left(f_j(x_j(t - \tau)) - f_j(x_j^*) \right), \quad t \geq t_0,
 \end{aligned}
 \tag{8}$$

The integral equation of (8) is

$$x_i(t) - x_i^* = x_i(t_0) - x_i^* + I_{t_0, t}^{\alpha, \rho} g_i(s), \quad t \geq t_0
 \tag{9}$$

where

$$g_i(s) = -c_i(x_i(s) - x_i^*) + \sum_{j=1}^n a_{ij} \left(f_j(x_j(s)) - f_j(x_j^*) \right) + \sum_{j=1}^n b_{ij} \left(f_j(x_j(s - \tau)) - f_j(x_j^*) \right),
 \tag{10}$$

Then,

$$\begin{aligned}
 \|x(t) - x^*\| &= \sum_{i=1}^n |x_i(t_0) - x_i^* + I_{t_0, t}^{\alpha, \rho} g_i(s)| \\
 &\leq \sum_{i=1}^n |x_i(t_0) - x_i^*| + \left| \sum_{i=1}^n I_{t_0, t}^{\alpha, \rho} g_i(s) \right| \\
 &\leq \|\psi - x^*\| + \left| \sum_{i=1}^n I_{t_0, t}^{\alpha, \rho} g_i(s) \right|, \quad t \geq t_0,
 \end{aligned}
 \tag{11}$$

It follows that,

$$\begin{aligned}
 \left| \sum_{i=1}^n I_{t_0, t}^{\alpha, \rho} g_i(s) \right| &\leq \sum_{i=1}^n c_i I_{t_0, t}^{\alpha, \rho} |x_i(s) - x_i^*| \\
 &\quad + \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| F_j I_{t_0, t}^{\alpha, \rho} |x_j(s) - x_j^*| \\
 &\quad + \sum_{i=1}^n \sum_{j=1}^n |b_{ij}| F_j I_{t_0, t}^{\alpha, \rho} |x_j(s - \tau) - x_j^*|
 \end{aligned}$$

So,

$$\begin{aligned}
 \|x(t) - x^*\| &\leq \|\varphi\| + \max_{1 \leq i \leq n} (c_i) I_{t_0, t}^{\alpha, \rho} \|x(s) - x^*\| + \sum_{i=1}^n \max_{1 \leq j \leq n} |a_{ij}| F_j I_{t_0, t}^{\alpha, \rho} \|x(s) - x^*\| \\
 &\quad + \sum_{i=1}^n \max_{1 \leq j \leq n} |b_{ij}| F_j I_{t_0, t}^{\alpha, \rho} \|x(s - \tau) - x^*\|, \quad t \geq t_0 \leq \|\varphi\| \\
 &\quad + A_1 I_{t_0, t}^{\alpha, \rho} \|x(s) - x^*\| + A_2 I_{t_0, t}^{\alpha, \rho} \|x(s - \tau) - x^*\|, \quad t \geq t_0,
 \end{aligned}$$

where $A_1 = \max_{1 \leq i \leq n} (c_i) + \sum_{i=1}^n \max_{1 \leq j \leq n} (|a_{ij}| F_j)$, $A_2 = \sum_{i=1}^n \max_{1 \leq j \leq n} (|b_{ij}| F_j)$ and $\varphi = \psi - x^*$. Let $u(t) = x(t) - x^*$. Using the Holder inequality, one has:

$$\begin{aligned} \|u(t)\| &\leq \|\varphi\| + A_1 \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_0}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \|u(s)\| ds \\ &\quad + A_2 \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_0}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \|u(s - \tau)\| ds \\ &\leq \|\varphi\| + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \left(\int_{t_0}^t \rho e^{ps^\rho} s^{\rho-1} (t^\rho - s^\rho)^{p\alpha-p} ds \right)^{\frac{1}{p}} \\ &\quad \times \left[A_1 \left(\rho \int_{t_0}^t e^{-qs^\rho} s^{\rho-1} \|u(s)\|^q ds \right)^{\frac{1}{q}} + A_2 \left(\rho \int_{t_0}^t e^{-qs^\rho} s^{\rho-1} \|u(s - \tau)\|^q ds \right)^{\frac{1}{q}} \right] \end{aligned}$$

Using the change of variable $\mu = p(t^\rho - s^\rho)$, we get:

$$\int_{t_0}^t \rho e^{ps^\rho} s^{\rho-1} (t^\rho - s^\rho)^{p\alpha-p} ds \leq \frac{e^{pt^\rho}}{p^{\alpha p - p + 1}} \Gamma(p(\alpha - 1) + 1), \quad t \geq t_0,$$

It follows from Lemma 1 (for $n = 3$ and $l = q$) that:

$$\begin{aligned} \left(e^{-t^\rho} \|x(t)\| \right)^q &\leq 3^{q-1} \|\varphi\|^q + N \\ &\quad \left[A_1^q \int_{t_0}^t \rho s^{\rho-1} e^{-qs^\rho} \|x(s)\|^q ds + A_2^q \int_{t_0}^t \rho s^{\rho-1} e^{-qs^\rho} \|x(s - \tau)\|^q ds \right] \end{aligned}$$

Case 1 Let $t \in [t_0, t_0 + \tau]$:

We have

$$\begin{aligned} \left(e^{-t^\rho} \|x(t)\| \right)^q &\leq 3^{q-1} \|\varphi\|^q + N \left[A_2^q \int_{t_0}^t \rho s^{\rho-1} e^{-qs^\rho} \|\varphi(s)\|^q ds + A_1^q \int_{t_0}^t \rho s^{\rho-1} e^{-qs^\rho} \|x(s)\|^q ds \right] \\ &\leq \|\varphi\|^q \left(3^{q-1} + N A_2^q ((t_0 + \tau)^\rho - t_0^\rho) \right) + N A_1^q \int_{t_0}^t \rho s^{\rho-1} e^{-qs^\rho} \|x(s)\|^q ds \end{aligned}$$

Case 2 Let $t > t_0 + \tau$:

We have

$$\begin{aligned} \int_{t_0}^t \rho s^{\rho-1} e^{-qs^\rho} \|x(s - \tau)\|^q ds &= \int_{t_0 - \tau}^{t - \tau} \rho(u + \tau)^{\rho-1} e^{-q(u+\tau)^\rho} \|x(u)\|^q du \\ &= \int_{t_0 - \tau}^{t_0} \rho(u + \tau)^{\rho-1} e^{-q(u+\tau)^\rho} \|x(u)\|^q du \\ &\quad + \int_{t_0}^{t - \tau} \rho(u + \tau)^{\rho-1} e^{-q(u+\tau)^\rho} \|x(u)\|^q du \end{aligned}$$

$$\begin{aligned} &\leq \|\varphi\|^q \int_{t_0-\tau}^{t_0} \rho(u+\tau)^{\rho-1} du + \int_{t_0}^t \rho(u+\tau)^{\rho-1} \left(e^{-u^\rho} (\|x(u)\|) \right)^q du \\ &\leq \|\varphi\|^q \left((t_0+\tau)^\rho - t_0^\rho \right) + \int_{t_0}^t \rho(u+\tau)^{\rho-1} \left(e^{-u^\rho} (\|x(u)\|) \right)^q du \end{aligned}$$

So,

$$\begin{aligned} \left(e^{-t^\rho} \|x(t)\| \right)^q &\leq \|\varphi\|^q \left(3^{q-1} + NA_2^q \left((t_0+\tau)^\rho - t_0^\rho \right) \right) \\ &\quad + N \left[\int_{t_0}^t \left(\rho A_1^q s^{\rho-1} + \rho A_2^q (s+\tau)^{\rho-1} \right) \left(e^{-qs^\rho} \|x(s)\| \right)^q ds \right] \end{aligned}$$

The Gronwall inequality on $[t_0; t_f]$, gives:

$$\begin{aligned} \left(e^{-t^\rho} \|x(t)\| \right)^q &\leq \|\varphi\|^q \left(3^{q-1} + NA_2^q \left((t_0+\tau)^\rho - t_0^\rho \right) \right) \\ &\quad \times \exp \left[N \left(A_1^q \left(t^\rho - t_0^\rho \right) + A_2^q \left((t+\tau)^\rho - (t_0+\tau)^\rho \right) \right) \right] \end{aligned}$$

Hence,

$$\begin{aligned} \|x(t)\| &\leq \|\varphi\| \sqrt[q]{3^{q-1} + NA_2^q \left((t_0+\tau)^\rho - t_0^\rho \right)} \\ &\quad \times \exp \left[t^\rho + N \left(\frac{A_1^q}{q} \left(t^\rho - t_0^\rho \right) + \frac{A_2^q}{q} \left((t+\tau)^\rho - (t_0+\tau)^\rho \right) \right) \right], \quad \forall t \in [t_0, t_f] \end{aligned}$$

So, if (4) is satisfied and $\|\varphi\| < \delta$, then $\|x(t)\| < \epsilon$, $\forall t \in [t_0, t_f]$ i.e., system (1) is finite-time stable w.r.t $\{t_0, \delta, \epsilon, \Theta, \tau\}$

Theorem 2 Suppose $\alpha \in (\frac{1}{2}, 1)$ and the fractional-order system (1) satisfies the initial condition $x(t_0 + s) = \varphi(s)$, $-\tau < s < 0$. If the following condition is satisfied:

$$\sqrt{3 + 2MA_2^2 \left((t_0+\tau)^\rho - t_0^\rho \right)} \exp \left(T^\rho + M \left(A_1^2 \left(t_f^\rho - t_0^\rho \right) + A_2^2 \left((t_f+\tau)^\rho - (t_0+\tau)^\rho \right) \right) \right) < \frac{\epsilon}{\delta} \tag{12}$$

where $M = \frac{3\rho-2\alpha}{4^\alpha} \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)^2}$, then (1) is finite-time stable with respect to $\{t_0, \delta, \epsilon, \Theta, \tau\}$, $\delta < \epsilon$.

Proof As the same in Theorem 1, we have the following estimation:

$$\|x(t)\| \leq \|\varphi\| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_0}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} (A_1 \|x(s)\| + A_2 \|x(s-\tau)\|) ds$$

By using the Cauchy-Schwartz inequality, one has

$$\begin{aligned} \|x(t)\| &\leq \|\varphi\| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} A_1 \int_{t_0}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \|x(s)\| ds \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} A_2 \int_{t_0}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \|x(s-\tau)\| ds \end{aligned}$$

$$\leq \|\varphi\| + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \left(\int_{t_0}^t \rho e^{2s^\rho} s^{\rho-1} (t^\rho - s^\rho)^{2\alpha-2} ds \right)^{\frac{1}{2}} \times \left[A_1 \left(\rho \int_{t_0}^t e^{-2s^\rho} s^{\rho-1} \|x(s)\|^2 ds \right)^{\frac{1}{q}} + A_2 \left(\rho \int_{t_0}^t e^{-2s^\rho} s^{\rho-1} \|x(s - \tau)\|^2 ds \right)^{\frac{1}{2}} \right]$$

By using the change of variable $\mu = t^\rho - s^\rho$, we get:

$$\int_{t_0}^t \rho e^{2s^\rho} s^{\rho-1} (t^\rho - s^\rho)^{2\alpha-2} ds \leq 2 \frac{\Gamma(2\alpha - 1)}{4\alpha} e^{2t^\rho}, \quad t \geq t_0,$$

It follows from Lemma 1 (for $n = 3$ and $l = 2$) that:

$$\|x(t)\|^2 \leq 3\|\varphi\|^2 + 2M e^{2t^\rho} \left[A_1^2 \int_{t_0}^t \rho e^{-2s^\rho} s^{\rho-1} \|x(s)\|^2 ds + A_2^2 \int_{t_0}^t \rho e^{-2s^\rho} s^{\rho-1} \|x(s - \tau)\|^2 ds \right],$$

Hence,

$$\left(e^{-t^\rho} \|x(t)\| \right)^2 \leq 3\|\varphi\|^2 + 2M \left[A_1^2 \int_{t_0}^t \rho e^{-2s^\rho} s^{\rho-1} \|x(s)\|^2 ds + A_2^2 \int_{t_0}^t \rho e^{-2s^\rho} s^{\rho-1} \|x(s - \tau)\|^2 ds \right]$$

In the following, there are two cases as $t \in [t_0, t_0 + \tau]$ or $t \in [t_0 + \tau, t_f]$.

Case 1 $t \in [t_0, t_0 + \tau]$:

We have

$$\begin{aligned} \left(e^{-t^\rho} \|x(t)\| \right)^2 &\leq 3\|\varphi\|^2 + 2M \left[A_2^2 \int_{t_0}^t \rho s^{\rho-1} e^{-2s^\rho} \|\varphi(s)\|^2 ds + A_1^2 \int_{t_0}^t \rho s^{\rho-1} e^{-2s^\rho} \|x(s)\|^2 ds \right] \\ &\leq \varphi^2 (3 + 2MA_2^2((t_0 + \tau)^\rho - t_0^\rho)) + 2MA_1^2 \int_{t_0}^t \rho s^{\rho-1} e^{-2s^\rho} x(s)^2 ds \end{aligned}$$

Case 2 Let $t > t_0 + \tau$

We have

$$\begin{aligned} \int_{t_0}^t \rho s^{\rho-1} e^{-2s^\rho} \|x(s - \tau)\|^2 ds &= \int_{t_0 - \tau}^{t - \tau} \rho(u + \tau)^{\rho-1} e^{-2(u+\tau)^\rho} \|x(u)\|^2 du \\ &= \int_{t_0 - \tau}^{t_0} \rho(u + \tau)^{\rho-1} e^{-2(u+\tau)^\rho} \|x(u)\|^2 du \\ &\quad + \int_{t_0}^{t - \tau} \rho(u + \tau)^{\rho-1} e^{-2(u+\tau)^\rho} \|x(u)\|^2 du \end{aligned}$$

Table 1 Finite time stability of Example 1, at $\delta = 0.31$

α	ρ	t_f	$\ x(t)\ $
0.5	0.5	0.3216	1.2752
	1	0.7967	1.1567
	1.5	0.9036	1.0475
	2	0.9585	0.9804
0.9	0.5	1.0846	1.8720
	1	1.2567	1.4019
	1.5	1.1873	1.1244
	2	1.1424	0.9592

Table 2 Finite time stability of Example 2, at $\delta = 0.31$

α	ρ	t_f	$\ x(t)\ $
0.5	0.5	0.0151	0.6575
	1	0.1583	0.5154
	1.5	0.3432	0.5173
	2	0.5005	0.5037
0.9	0.5	0.1556	0.9620
	1	1.1719	1.2667
	1.5	1.1748	1.0236
	2	1.1389	0.8457

$$\begin{aligned} &\leq \|\varphi\|^2 \int_{t_0-\tau}^{t_0} \rho(u+\tau)^{\rho-1} du + \int_{t_0}^t \rho(u+\tau)^{\rho-1} \left(e^{-u\rho} (\|x(u)\|) \right)^2 du \\ &\leq \|\varphi\|^2 ((t_0+\tau)^\rho - t_0^\rho) + \int_{t_0}^t \rho(u+\tau)^{\rho-1} \left(e^{-u\rho} (\|x(u)\|) \right)^2 du \end{aligned}$$

So,

$$\begin{aligned} \left(e^{-t\rho} \|x(t)\| \right)^2 &\leq \|\varphi\|^2 (3 + 2MA_2^2((t_0+\tau)^\rho - t_0^\rho)) \\ &\quad + 2M \left[\int_{t_0}^t (\rho A_1^2 s^{\rho-1} + \rho A_2^2 (s+\tau)^{\rho-1}) \left(e^{-2s\rho} \|x(s)\| \right)^2 ds \right] \end{aligned}$$

The Gronwall inequality on $[t_0; t_f]$, gives:

$$\begin{aligned} \left(e^{-t\rho} \|x(t)\| \right)^2 &\leq \|\varphi\|^2 (3 + 2MA_2^2((t_0+\tau)^\rho - t_0^\rho)) \\ &\quad \times \exp[2M(A_1^2(t^\rho - t_0^\rho) + A_2^2(((t+\tau)^\rho - (t_0+\tau)^\rho)))] \end{aligned}$$

Hence,

$$\begin{aligned} \|x(t)\| &\leq \|\varphi\| \sqrt{3 + 2MA_2^2((t_0+\tau)^\rho - t_0^\rho)} \\ &\quad \times \exp[t^\rho + 2M(A_1^2(t^\rho - t_0^\rho) + A_2^2(((t+\tau)^\rho - (t_0+\tau)^\rho)))] , \quad \forall t \in [t_0, t_f] \end{aligned}$$

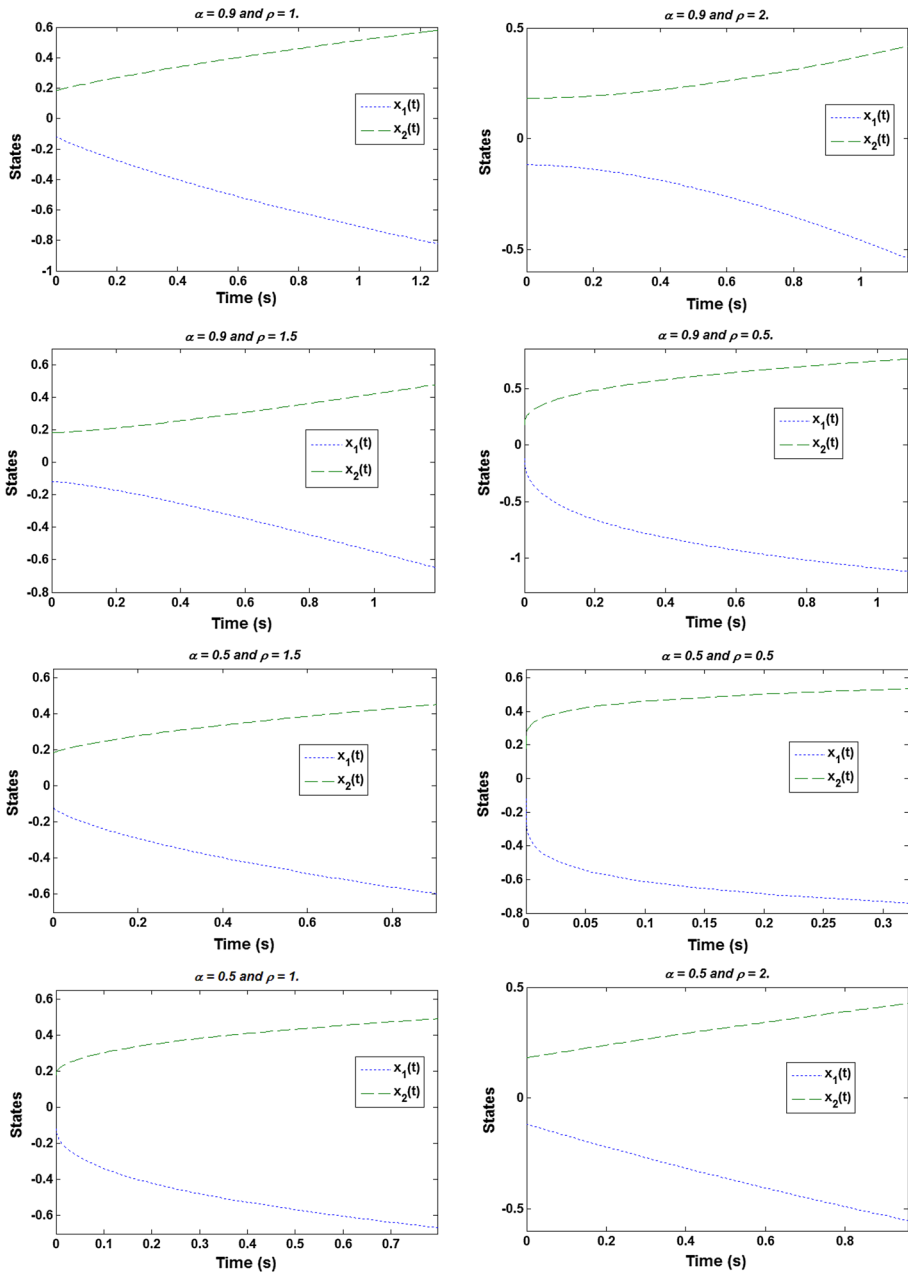


Fig. 1 States evolution for Example 1 (different cases of ρ and α)

So, if (12) is satisfied and $\varphi < \delta$, then $x(t) < \varepsilon, \forall t \in [t_0, t_f]$ i.e., system (1) is finite-time stable w.r.t $\{t_0, \delta, \varepsilon, \Theta, \tau\}$.

Remark 1 If $\rho = 1$, then system (1) will be reduced to a fractional-order neural networks system, under the Caputo derivative definition. For that case of Caputo derivative, the study

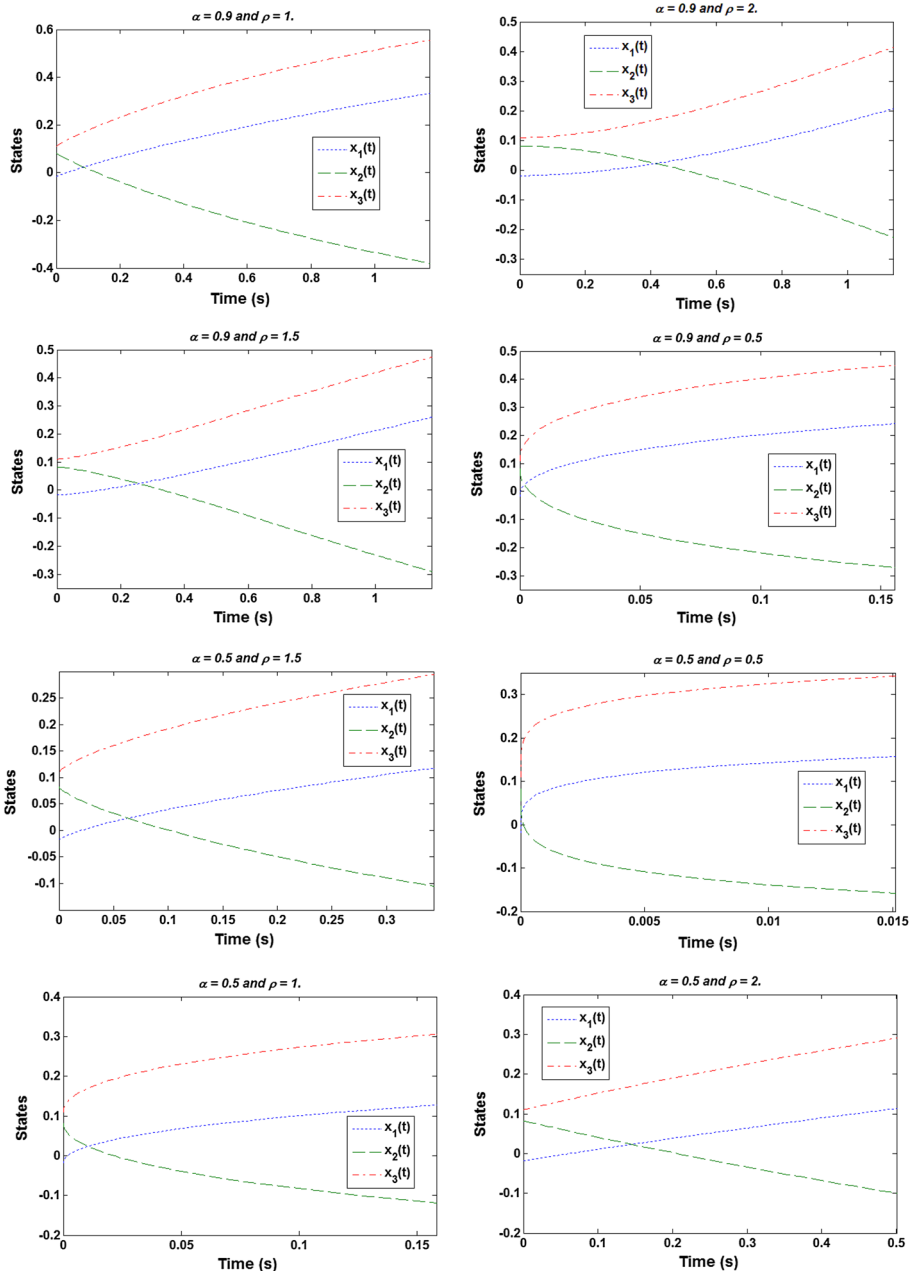


Fig. 2 States evolution for Example 2 (different cases of ρ and α)

on the existence of a unique equilibrium point and finite-time stability for system (1) has been given in [34, 35]. To the best of the authors' knowledge, this is the first time that the problem of finite-time stability for Caputo–Katugampola fractional-order neural networks, with time-delay, is investigated.

Remark 2 The proofs of Theorems 1 and 2 have been based on the ones of the authors’ sister paper [23]. It is not possible to demonstrate these to theorems at the same time, using $0 < \alpha \leq 1$. That is why the authors have divided the analysis into two theorems.

4 Numerical Examples

In this section, two expository examples with their numerical results will be given in order to clarify the validity of the theoretical results which obtained in the previous sections.

Example 1 Let us consider the Caputo–Katugampola fractional-order neural networks system, with time delay:

$${}^C D_{t_0}^{\alpha, \rho} x_i(t) = -c_i x_i(t) + \sum_{j=1}^2 a_{ij} f_j(x_j(t)) + \sum_{j=1}^2 b_{ij} f_j(x_j(t - \tau)) + I_i, i = 1, 2$$

where $\tau = 0.5, f_j(x_j(t)) = \tanh(x_j(t)), j = 1, 2$ and:

$$C = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad A = \begin{pmatrix} 0.01 & 0.02 \\ 0.03 & -0.01 \end{pmatrix}, \quad B = \begin{pmatrix} 0.03 & -0.01 \\ -0.01 & 0.03 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} -0.7 \\ 0.4 \end{pmatrix}.$$

Example 2 Let us consider the Caputo–Katugampola fractional-order neural networks system, with time delay:

$${}^C D_{t_0}^{\alpha, \rho} x_i(t) = -c_i x_i(t) + \sum_{j=1}^3 a_{ij} f_j(x_j(t)) + \sum_{j=1}^3 b_{ij} f_j(x_j(t - \tau)) + I_i, i = 1, 2, 3$$

where $\tau = 0.2, f_j(x_j(t)) = \frac{1}{2}(|x + 1| - |x - 1|), j = 1, 2, 3$ and

$$C = \begin{pmatrix} 0.45 & 0 & 0 \\ 0 & 0.45 & 0 \\ 0 & 0 & 0.65 \end{pmatrix}, \quad A = \begin{pmatrix} 0.05 & 0.01 & 0.02 \\ 0.05 & 0.08 & 0.03 \\ 0.03 & 0.08 & 0.02 \end{pmatrix}, \quad B = \begin{pmatrix} 0.01 & 0.03 & 0.04 \\ 0.02 & 0.05 & 0.02 \\ 0.06 & 0.02 & 0.05 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 0.35 \\ -0.5 \\ 0.6 \end{pmatrix}.$$

Clearly, the function f in both examples satisfies the assumption (H1). Also, the hypothesis (H2) is satisfied for $F_j = 1, j = 1, 2$. In Examples 1 and 2, let us assume that $\delta = 0.31, \varepsilon = 2$. According to the inequalities (4) and (12) with various values of ρ and α , we can compute the estimated finite t_f , of the finite-time stability of both examples as shown in Tables 1 and 2. Moreover, it is obvious from the obtained results in Tables 1 and 2 that the norm of the approximated solutions does not override the value of ε . Figures 1 and 2, show the numerical simulations with various values of ρ and α . From the obtained results in the tables and all figures, we can indicate that our results coincided with the theoretical one and the finite-time stability of the proposed systems.

5 Conclusion

In this research paper, fractional-order neural networks with time-delay have been investigated. An advantageous and newly introduced fractional-order derivative concept in the literature, has been exploited: the Caputo–Katugampola fractional derivative. The main purpose of the paper has been to demonstrate the existence of a unique equilibrium point, and to prove its finite-time stability, for the general considered class of fractional neural networks.

In order to further show the effectiveness of the used methodology, two simulation examples have been given and analyzed.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflicts of interest.

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