




# Finite-Time Synchronization of Memristive Neural Networks with Proportional Delay

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## Abstract

The problem of finite-time synchronization for memristive neural networks (MNNs) with proportional delay is considered. Since proportional delay is unbounded and different from infinite-time distributed delay, the classical finite-time analytical techniques are not applicable anymore. First, a discontinuous state feedback controller is designed such that the delayed MNNs achieve drive-response synchronization in a finite settling time. By using Filippov solution and Lyapunov functional method, sufficient conditions are derived. It is shown that, though the proportional delay is unbounded, complete synchronization can still be realized and the settling time can be explicitly estimated. Second, a special adaptive controller is designed for the finite-time problem in order to reduce the control gains. Finally, numerical simulations are given to verify the effectiveness of the theoretical results.

**Keywords** Memristive · Neural network (NN) · Finite-time synchronization · Proportional delay

## 1 Introduction

Neural networks (NNs) are a class of important model which have significant effect in different filed, such as pattern recognition [1], image processing [2], visual perception [3,4] and so on. Recently, another kind of NN called memristive neural network (MNN) has attracted increasing attention [5–7]. MNN is a memristor-based NN. Memristor, as a contraction of memory resistor, was originally predicted by Chua [8] and realized by HP laboratory in 2008. It is reported that MNNs possesses more computation power and information capacity, which would greatly enhance the applications of NNs for associate memory and information processing. In the literature, there are many results on stabilization, passivity, and synchronization

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of MNNs [6,9–15]. In [11], the Lagrange stability of MNNs with discrete and distributed delays is investigated. Meanwhile, quasi-uniform synchronization, global exponential synchronization, exponential synchronization of MNNs are studied in [10,12,14,16,17].

Time delays are unavoidable in practical systems due to the finite information exchanging between different units, which may cause divergence, instability, or oscillation [9,10,18–23]. Therefore, time delays should be taken into account in studying dynamics of MNNs. Up to now, the dynamical behaviors of MNNs with constant delays [10,12,19,22,24,25], time-varying delays [9,14,18,23,26,27] or distributed delays [11,28] have been intensively studied. However, to the best of our knowledge, seldom authors consider dynamics of MNNs with proportional delay. This motivate us to consider MNNs with proportional delay.

It should be noted that most of existing results concerning synchronization of delayed MNNs are asymptotic, while no published results consider finite-time synchronization of MNNs with proportional delay. Finite-time synchronization means coupled systems achieve synchronization state in a desired time instant called settling time. Compared with asymptotic control, finite-time techniques have better robustness and disturbance rejection [29–33]. In the literature, many papers about finite-time synchronization of coupled systems with delays have been published [13,26,33–35]. In [36], finite-time synchronization of NNs with infinite-time distributed delays was investigated by developing a set of new analytical methods. However, according the results in [36], the settling time cannot be estimated. From practical point of view, finite-time results with unavailable settling time are not convenient for engineering technicians. This motivate us consider finite-time synchronization of MNNs with proportional delay. It is discovered that, although proportional delay is unbounded, the settling time can be explicitly estimated.

It is worth noting that since parameter of MNNs are state-dependent, the parameter of driver and response MNNs are uncertain and cannot be identical all the time when the states of the two MNNs are different. It means that the state-dependent parameters of driver and response MNNs may be mismatched before realizing synchronization. Hence, using the classical analytical techniques and traditional robust analytical techniques of robust synchronization of NNs with matched uncertain parameters [37–40] cannot synchronize MNNs. Therefore, Yang et al. proposed new robust analytical techniques to investigate the synchronization of MNNs in [13]. Unfortunately, the delays of MNNs in [13] are bounded. Since proportional delay is unbounded as time approaches infinity, the analytical techniques proposed in [13] cannot be directly applied to finite-time synchronization of MNNs with proportional delay.

Therefore, how to overcome mismatched parameters and realize stability and synchronization of MNNs has attracted the interest of many authors. In [41], the stability problem is investigated for a class of reaction-diffusion uncertain MNNs with time-varying delays and leakage term. The global asymptotic stability and stabilization of MNNs with communication delays are achieved via event-triggered sampling control in [42]. In [43–45] the synchronization issue of MNNs is studied via different control and analytical methods. For instance, using impulsive control to realize synchronization for delayed memristive based bidirectional associative memory neural networks with random nonlinearities in [43]. And as in [44], the authors discussed exponential synchronization and anti-synchronization of MNNs with time-varying delays via matrix measure strategies. In general, designed controller is required to be simple and reduce the cost of control. Hence, designing suitable controller to achieve finite-time synchronization of MNNs with proportional delay is a challenging work.

Motivated by the above discussions, this paper aims to investigate finite-time synchronization of MNNs with proportional delay. The main contributions of this paper are: (a) By using Filippov solution and differential inclusion [46,47], the drive-response MNNs with

proportional delays are transformed into traditional NNs with unmatched uncertain bounded parameters; (b) New 1-norm-based analytical methods are constructed to deal with the difficult induced by proportional delay; (c) Design a simple discontinuous state controller to overcome mismatched parameters and ensure the synchronization of the considered MNNs with proportional delay in a finite settling time. Moreover, the settling time is theoretically estimated; (d) An adaptive controller is also designed to guarantee finite-time synchronization of drive-response MNNs which can reduce control gains; (e) Without utilizing the finite-time stability theorem in [48], finite-time synchronization criteria are obtained. Results of this paper can easily be extended to finite-time synchronization of classical NNs with proportional delays and parameters uncertainties.

The rest of this paper is organized as follows. In Sect. 2, a model of MNNs with proportional delay is desired. Some necessary definitions and assumptions are also given. Section 3 develops new criteria for finite-time synchronization of the MNNs. In Sect. 4, numerical examples are given to illustrate the effectiveness of our result. Finally, Sect. 5 discusses the conclusions.

**Notations** The notations are quite standard. Throughout this paper,  $\mathbb{R}$  denotes the set of real number,  $\mathbb{R}^n$  is the set of  $n \times 1$  real vector, and  $\mathbb{R}^{n \times m}$  denotes the set of  $n \times m$  matrices;  $D_+(\cdot)$  is the directional derivative in the positive direction; the superscript  $T$  stands for vector or matrix transposition;  $\|\cdot\|_1$  is standard 1-norm of a vector or a matrix.

## 2 Model Description and Preliminaries

Consider a MNN with proportional delay described as follows:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij}(x_i(t)) f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(x_i(t)) f_j(x_j(qt)) + J_i, \quad (1)$$

where  $i = 1, 2, \dots, n$ ,  $x_i(t) \in \mathbb{R}$  is the voltage of the capacitor  $C_i$  at time  $t > 1$ ,  $c_i > 0$  represents the rate with which the  $i$ th neuron will reset its potential to the resting state,  $f_i(x_i(t))$  denotes the activation function about  $x_i(t)$ ,  $f(x(t)) = (f_1(x(t)), f_2(x(t)), \dots, f_n(x(t)))^T$ ,  $J_i \in \mathbb{R}$  is an external bias on the  $i$ th unit. In particular, the constant  $q$  satisfies  $0 < q < 1$  which is a proportional delay factor, correspondingly,  $qt = t - (1 - q)t$  in which  $(1 - q)t$  is a time-varying continuous function that satisfies  $(1 - q)t \rightarrow +\infty$  as  $t \rightarrow +\infty$ .  $a_{ij}(x_i(t))$ ,  $b_{ij}(x_i(t))$  are the connect non-delayed and time-delayed memristive synaptic connection weight, respectively, which satisfies the following condition:

$$a_{ij}(x_i(t)) = \begin{cases} \hat{a}_{ij}, & \text{if } |x_i(t)| \leq T_i, \\ \check{a}_{ij}, & \text{otherwise,} \end{cases}$$

$$b_{ij}(x_i(t)) = \begin{cases} \hat{b}_{ij}, & \text{if } |x_i(t)| \leq T_i, \\ \check{b}_{ij}, & \text{otherwise,} \end{cases}$$

where  $T_i > 0$  is switching jump,  $\hat{a}_{ij}$ ,  $\check{a}_{ij}$ ,  $\hat{b}_{ij}$ ,  $\check{b}_{ij}$ , ( $i, j = 1, 2, \dots, n$ ) are constants which satisfy  $\hat{a}_{ij} \neq \check{a}_{ij}$  and  $\hat{b}_{ij} \neq \check{b}_{ij}$ . The initial conditions of system (1) are  $x_i(s) = \phi_i$ ,  $s \in [q, 1]$ ,  $i = 1, 2, \dots, n$ .

**Remark 1** The proportional delay exists in many practical systems, for example, in Web quality of service routing decision, proportional delay is usually involved. In general, the proportional factor is denoted by constant  $q$  which satisfies  $0 < q < 1$ , then  $qt = t - (1 - q)t$ , and  $(1 - q)t$  is the proportional delay.

Take system (1) as the drive system and consider controlled response MNN as follows

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^n a_{ij}(y_j(t)) f_j(y_j(t)) + \sum_{j=1}^n b_{ij}(y_j(t)) f_j(y_j(qt)) + J_i + u_i(t), \quad (2)$$

where  $a_{ij}(y_j(t))$  and  $b_{ij}(y_j(t))$  are defined similarly as those given in  $a_{ij}(x_i(t))$  and  $b_{ij}(x_i(t))$ , respectively; and  $u_i(t)$  is the control input. In general, the initial conditions of (2) are different from those given in (1). They can be denoted by  $y_i(s) = \varphi_i, s \in [q, 1], i = 1, 2, \dots, n$ .

Basically, the MNNs (1) and (2) can be considered as traditional NN with state-dependent switching parameters, which is different from the time-dependent switching phenomena in [49]. Our main objective is to establish the relationship of the synchronization control between MNNs and traditional neural networks by transforming the MNNs (1) and (2) into the form of traditional NNs with uncertain parameters. By designing new controllers and using Lyapunov analysis method, some synchronization criteria between (1) and (2) are given.

**Definition 1** (Filippov regularization [46]) The Filippov set-valued map of  $f(x)$  at  $x \in \mathbb{R}^n$  is defined as follows:

$$F(x) = \bigcap_{\delta > 0} \bigcap_{\mu(\Omega)=0} \overline{co}[f(B(x, \delta) \setminus \Omega)],$$

where  $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$ , and  $\mu(\Omega)$  is the Lebesgue measure of the set  $\Omega, \overline{co}[\Omega]$  is the closure of the convex hull of the set  $\Omega$ . Let

$$\begin{aligned} \bar{a}_{ij} &= \max\{\hat{a}_{ij}, \check{a}_{ij}\}, & \bar{b}_{ij} &= \max\{\hat{b}_{ij}, \check{b}_{ij}\}, \\ \underline{a}_{ij} &= \min\{\hat{a}_{ij}, \check{a}_{ij}\}, & \underline{b}_{ij} &= \min\{\hat{b}_{ij}, \check{b}_{ij}\}. \end{aligned}$$

By the measurable selection theorem in [47], there exist measurable functions  $\lambda_{ij}^1(t) \in \overline{co}[\underline{a}_{ij}, \bar{a}_{ij}]$  and  $\lambda_{ij}^2(t) \in \overline{co}[\underline{b}_{ij}, \bar{b}_{ij}]$  such that

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^n \lambda_{ij}^1(t) f_j(x_j(t)) + \sum_{j=1}^n \lambda_{ij}^2(t) f_j(x_j(qt)) + J_i. \quad (3)$$

Similarly as (3), we obtain

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^n \lambda_{ij}^3(t) f_j(y_j(t)) + \sum_{j=1}^n \lambda_{ij}^4(t) f_j(y_j(qt)) + J_i + u_i(t), \quad (4)$$

where  $\lambda_{ij}^3(t) \in \overline{co}[\underline{a}_{ij}, \bar{a}_{ij}]$  and  $\lambda_{ij}^4(t) \in \overline{co}[\underline{b}_{ij}, \bar{b}_{ij}]$  are measurable functions.

**Remark 2** Due to the different initial conditions, when  $|x_i(t)| \leq T_i$  at time  $t$ , the state  $y_i(t)$  in (2) may be  $|y_i(t)| \leq T_i$  or  $|y_i(t)| > T_i$ , which is a common phenomenon for chaotic system because they have the characteristic of initial-value-sensitivity. It means that  $\lambda_{ij}^1(t) = \lambda_{ij}^2(t)$  or  $\lambda_{ij}^3(t) = \lambda_{ij}^4(t)$  can not be guaranteed. Therefore, we will design suitable controller to overcome mismatched parameters in this paper.

The following assumptions are needed.

(H<sub>1</sub>) There exist constants  $p_i$  such that  $|f_i(z)| \leq p_i, \forall z \in \mathbb{R}, i = 1, 2, \dots, n$ .

(H<sub>2</sub>) There exist constants  $l_i, i = 1, 2, \dots, n$ , such that

$$\frac{|f_i(x) - f_i(y)|}{|x - y|} \leq l_i, \quad \forall x, y \in \mathbb{R}, x \neq y.$$

**Definition 2** The MNN (2) is said to be finite-timely synchronized with (1) if, by designing suitable controllers  $u_i(t), i = 1, 2, \dots, n$ , there exist a constant  $t^* > 1$  ( $t^*$  is dependent on the initial values of systems) such that  $\lim_{t \rightarrow t^*} \|y(t) - x(t)\|_1 = 0$  and  $\|y(t) - x(t)\|_1 \equiv 0$  for  $t > t^*$ , where  $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T, x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ .

In order to study the synchronization of MNNs (1) and (2), we only need to consider the synchronization between the systems (3) and (4). Denote the error between  $x_i(t)$  and  $y_i(t)$  by  $z_i(t) = y_i(t) - x_i(t)$ . Then, subtracting (3) from (4) yields

$$\begin{aligned} \dot{z}_i(t) = & -c_i z_i(t) + \sum_{j=1}^n \lambda_{ij}^3(t) g_j(z_j(t)) + \sum_{j=1}^n \lambda_{ij}^4(t) g_j(z_j(qt)) + u_i(t) \\ & + \sum_{j=1}^n (\lambda_{ij}^3(t) - \lambda_{ij}^1(t)) f_j(x_j(t)) + \sum_{j=1}^n (\lambda_{ij}^4(t) - \lambda_{ij}^2(t)) f_j(x_j(qt)), \end{aligned} \tag{5}$$

where  $g_j(z_j(t)) = f_j(y_j(t)) - f_j(x_j(t))$  and  $g_j(z_j(qt)) = f_j(y_j(qt)) - f_j(x_j(qt)), i, j = 1, 2, \dots, n$ .

### 3 Finite-Time Synchronization of the MNNs

In this section, our main objective is to design two different types of controller such that MNN (2) can be synchronized with MNN (1) in finite time, which is equivalent to study finite-time stabilization of the error system (5). Firstly, state feedback controllers are proposed for researching the finite-time synchronization problem. Then adaptive controllers are designed to reduce the control gains. Meanwhile, some sufficient criteria for synchronization between MNNs (2) and (1) in finite time are obtained by mathematical proofs.

State feedback controllers are designed as

$$u_i(t) = -r_i(z_i(t)) - \eta_i \operatorname{sgn}(z_i(t)), \tag{6}$$

where  $i = 1, 2, \dots, n, r_i$  and  $\eta_i$  are positive constants to be determined,  $\operatorname{sgn}(z_i(t))$  stand for sign function.

**Theorem 1** Suppose that Assumptions (H<sub>1</sub>) and (H<sub>2</sub>) are satisfied. The controlled MNN (2) can be synchronized onto MNN (1) under controller (6) in a finite time if the control gains  $r_i$  and  $\eta_i$  in (6) satisfy the following conditions:

$$r_i = -c_i + l_i \sum_{j=1}^n (|\bar{a}|_{ji} + \frac{1}{q} |\bar{b}|_{ji}) \triangleq \Sigma_i, \tag{7}$$

$$\eta_i = \sum_{j=1}^n (|\hat{a}_{ij} - \check{a}_{ij}| + |\hat{b}_{ij} - \check{b}_{ij}|) p_j + \epsilon, \tag{8}$$

where  $i = 1, 2, \dots, n, \epsilon$  is a positive constant,  $|\bar{a}|_{ij} = \max\{|\hat{a}_{ij}|, |\check{a}_{ij}|\}$ , and  $|\bar{b}|_{ij} = \max\{|\hat{b}_{ij}|, |\check{b}_{ij}|\}$ . Moreover, the settling time is estimated as  $\mathcal{T} = \frac{q}{\epsilon} \sum_{i=1}^n [ |z_i(1)| + \frac{1}{q} \sum_{j=1}^n \int_q^1 |\bar{b}|_{ji} l_i |z_i(s)| ds ] + q$ , where  $z_i(s) = \varphi_i - \phi_i, s \in [q, 1], i = 1, 2, \dots, n$ .

**Proof** Consider a Lyapunov functional as follow:

$$V(t) = V_1(t) + V_2(t), \tag{9}$$

where

$$V_1(t) = \sum_{i=1}^n |z_i(t)|,$$

$$V_2(t) = \frac{1}{q} \sum_{i=1}^n \sum_{j=1}^n \int_{qt}^t |\bar{b}|_{ij} l_j |z_i(s)| ds.$$

Differentiating  $V_1(t)$  along the solutions of (5) and considering the controller (6) lead to

$$D_+ V_1(t) = \sum_{i=1}^n \left\{ d_i(t) \left[ \sum_{j=1}^n \lambda_{ij}^3(t) g_j(z_j(t)) + \sum_{j=1}^n \lambda_{ij}^4(t) g_j(z_j(qt)) + \Pi_i(t) - \eta_i d_i(t) \right] - (c_i + r_i) |z_i(t)| \right\}, \tag{10}$$

where  $\Pi_i(t) = \sum_{j=1}^n (\lambda_{ij}^3(t) - \lambda_{ij}^1(t)) f_j(x_j(t)) + \sum_{j=1}^n (\lambda_{ij}^4(t) - \lambda_{ij}^2(t)) f_j(x_j(qt))$ ,  $d_i(t)$  is chosen as follows [24]:

$$d_i(t) = \begin{cases} \text{sgn}(z_i(t)), & \text{if } z_i(t) \neq 0, \\ 0, & \text{if } z_i(t) = 0. \end{cases}$$

By the condition (H<sub>2</sub>), one has

$$d_i(t) \lambda_{ij}^3(t) g_j(z_j(t)) \leq |\lambda_{ij}^3(t) g_j(z_j(t))| \leq |\bar{a}|_{ij} l_j |z_j(t)|, \tag{11}$$

and

$$d_i(t) \lambda_{ij}^4(t) g_j(z_j(qt)) \leq |\lambda_{ij}^4(t) g_j(z_j(qt))| \leq |\bar{b}|_{ij} l_j |z_j(qt)|. \tag{12}$$

When  $z_i(t) \neq 0$ , it is followed by (H<sub>1</sub>) that, for  $i = 1, 2, \dots, n$ ,

$$d_i(t) \Pi_i(t) \leq \sum_{j=1}^n (|\hat{a}_{ij} - \check{a}_{ij}| + |\hat{b}_{ij} - \check{b}_{ij}|) p_j, \tag{13}$$

and

$$-d_i(t) \eta_i d_i(t) = -\eta_i. \tag{14}$$

When  $z_i(t) = 0$ , for  $i = 1, 2, \dots, n$ , it follows:

$$d_i(t) \Pi_i(t) = -d_i(t) \eta_i d_i(t) = 0. \tag{15}$$

Combining (13)–(15) produces

$$d_i(t) \Pi_i(t) \leq \mu_i \sum_{j=1}^n (|\hat{a}_{ij} - \check{a}_{ij}| + |\hat{b}_{ij} - \check{b}_{ij}|) p_j, \tag{16}$$

and

$$-d_i(t)\eta_i d_i(t) \leq \mu_i \eta_i, \tag{17}$$

where  $\mu_i = 1$  if  $z_i(t) \neq 0$ , otherwise,  $\mu_i = 0$ . Then, it is obtained from (10)–(12), (16), and (17) that

$$D_+ V_1(t) \leq \sum_{i=1}^n \left\{ - (c_i + r_i) |z_i(t)| + \left[ \sum_{j=1}^n |\bar{a}|_{ji} l_j |z_j(t)| + \sum_{j=1}^n |\bar{b}|_{ji} l_j |z_j(qt)| + (\hat{a}_{ij} - \check{a}_{ij}) + |\hat{b}_{ij} - \check{b}_{ij}| p_j \mu_i \right] - \eta_i \mu_i \right\}. \tag{18}$$

The following inequality is derived from  $V_2(t)$ .

$$\dot{V}_2(t) \leq \frac{1}{q} \sum_{i=1}^n \sum_{j=1}^n |\bar{b}|_{ji} l_j |z_j(t)| - \sum_{i=1}^n \sum_{j=1}^n |\bar{b}|_{ji} l_j |z_j(qt)|. \tag{19}$$

The inequalities (9), (18), and (19) yield

$$D_+ V(t) = D_+ V_1(t) + \dot{V}_2(t) \leq \sum_{i=1}^n [(-r_i + \Sigma_i) |z_i(t)| - \epsilon \mu_i]. \tag{20}$$

When  $\|z(t)\|_1 \neq 0$ , the conditions (7) and (8) and the inequality (20) imply

$$D_+ V(t) \leq -\epsilon \sum_{i=1}^n \mu_i. \tag{21}$$

According to (21) and the definition of  $V(t)$  in (9), there exists nonnegative constant  $V^*$  such that

$$\lim_{t \rightarrow +\infty} V(t) = V^* \text{ and } V(t) \equiv V^* \text{ for } t \geq 0. \tag{22}$$

Meanwhile, integrating both sides of the inequality (21) from 1 to  $t$  yields

$$V(t) - V(1) \leq -\epsilon \sum_{i=1}^n \mu_i (t - 1). \tag{23}$$

When  $|z_i(t)| = 0$  at instant  $t^* \in (1, +\infty)$  for  $i = 1, 2, \dots, n$ , the discussion can be proceeded from (25). If  $\|z(t)\|_1 > 0$  for all  $t \in (1, +\infty)$  where  $\|z(t)\|_1 = \sum_{i=1}^n |z_i(t)|$ , there exists at least one  $i_0 \in \{1, 2, \dots, n\}$  such that  $|z_{i_0}(t)| \neq 0$ , then  $-\epsilon \sum_{i=1}^n \mu_i < 0$ . In this case, the inequality (23) means  $\lim_{t \rightarrow +\infty} V(t) = -\infty$ . This contradicts (22), and hence the inequality (22) is not true for  $t \rightarrow +\infty$ . Therefore, there exists  $t^* \in (1, +\infty)$  such that

$$\lim_{t \rightarrow t^*} V(t) = V^* \text{ and } V(t) \equiv V^* \text{ for } t \geq t^*. \tag{24}$$

In the following we prove that

$$\|z(t^*)\|_1 = 0 \text{ and } \|z(t)\|_1 \equiv 0 \text{ for } t \geq t^*. \tag{25}$$

First, we claim  $\|z(t^*)\|_1 = 0$ . Suppose that  $\|z(t^*)\|_1 > 0$ , then there exists a small constant  $\varepsilon > 0$  such that  $\|z(t^*)\|_1 > 0$  for all  $t \in [t^*, t^* + \varepsilon]$ . So there exists at least one  $i_1 \in \{1, 2, \dots, n\}$  such that  $|z_{i_1}(t)| > 0$  for any instant  $t \in [t^*, t^* + \varepsilon]$ . Based on the

previous analysis, it is easy to get that  $\dot{V} \leq -\epsilon < 0$  holds for the instant  $t \in [t^*, t^* + \epsilon]$ . This contradicts (24).

Now, it is necessary to prove  $\|z(t)\|_1 \equiv 0$  for  $\forall t \geq t^*$ . Otherwise, there exists  $t_1 > t^*$  such that  $\|z(t_1)\|_1 > 0$ . Let  $t_s = \sup\{t \in [t^*, t_1] : \|z(t)\|_1 = 0\}$ . We have  $t_s < t_1$ ,  $\|z(t_s)\|_1 = 0$  and  $\|z(t)\|_1 > 0$  for all  $t \in [t_s, t_1]$ . Moreover, there exists  $t_2 \in (t_s, t_1]$  such that  $\|z(t)\|_1$  is monotonously increasing on the interval  $(t_s, t_2]$ . Hence  $V(t)$  is also monotonously increasing on the interval  $(t_s, t_2]$ , i.e.,  $\dot{V}(t) > 0$  for  $t \in (t_s, t_2]$ . On the other hand, because  $\|z(t)\|_1 > 0$  for all  $t \in (t_s, t_2]$ , there exists at least one  $i_2 \in \{1, 2, \dots, n\}$  such that  $|z_{i_2}(t)| > 0$  at any instant  $t \in (t_s, t_2]$ . By the discussion as above, it follows that  $\dot{V}(t) \leq -\epsilon < 0$  holds for the instant  $t \in (t_s, t_2]$ , which is a contradiction with  $\dot{V}(t) > 0$  for  $t \in (t_s, t_2]$ . Hence,  $\|z(t)\|_1 \equiv 0$  for  $\forall t \geq t^*$ . Therefore, (25) holds. According to Definition 1, the synchronization is realized in finite time under the controller (6).

Next, one has to estimate the settling time by proving  $V^* = 0$ . If  $V^* > 0$ , then it is obtained from (9) that  $V_1(t^*) > 0$  or  $V_2(t^*) > 0$ . When  $V_1(t^*) > 0$ , there exists at least one  $i_3 \in \{1, 2, \dots, n\}$  such that  $|z_{i_3}(t^*)| > 0$ , then  $\dot{V}(t) \leq -\epsilon \sum_{i=1}^n \mu_i < 0$ , this means that there will exist  $t_3 > t^*$  such that  $V(t_3) > V^*$  contradicts (24). In the other case, when  $V_2(t^*) > 0$ , i.e.,  $\frac{1}{q} \sum_{i=1}^n \sum_{j=1}^n \int_{qt^*}^{t^*} |\bar{b}|_{ji} l_i |z_i(s)| ds > 0$ , there exist  $t_4 \in [qt^*, t^*]$  and a small constant  $\iota > 0$  such that  $\|z(t)\|_1 > 0$  for all  $t \in [t_4 - \iota, t_4 + \iota]$ . Therefore,  $\dot{V}(t) \leq -\epsilon \sum_{i=1}^n \mu_i < 0$ , this also will lead to a contradiction with (24). Hence,  $V^* = 0$ . From the above discussions, one has that, if  $\|z(qt^*)\|_1 = 0$ , then  $\|z(t^*)\|_1 = 0$ . Therefore, when  $V_2(t^*) = 0$  holds,  $V_1(t^*) = 0$  must be true. Integrating both sides of the inequality  $\dot{V}(t) \leq -\epsilon$  when  $\|z(t)\|_1 \neq 0$  from 1 to  $t^*$  yields  $t^* \leq \frac{V(1)}{\epsilon} + 1$ . Meanwhile,  $\|z(\mathcal{T})\|_1 = 0$  and  $\|z(t)\|_1 \equiv 0$  for  $\forall t \geq \mathcal{T}$ , where  $\mathcal{T} = qt^*$ . This completes the proof.  $\square$

**Remark 3** By using the discontinuous state feedback controller and constructing new 1-norm-based Lyapunov–Krasovskii functionals, the controlled MNN (2) with proportional delay is synchronized onto MNN (1) in the estimated settling time. It should be noted that, since  $0 < q < 1$ , the settling time  $\mathcal{T} = qt^* < t^*$  is more accurate which is more convenient for practical application.

**Remark 4** Note that the considered proportional delay is unbounded, which is different from infinite-time distributed delay in [13]. The settling time in [13] cannot be estimated. Because  $qt \rightarrow +\infty$  as  $t \rightarrow +\infty$ , hence, when  $V_2(t) = \frac{1}{q} \sum_{i=1}^n \sum_{j=1}^n \int_{qt}^t |\bar{b}|_{ji} l_i |z_i(s)| ds = 0$ ,  $V_1(t) = 0$  holds.

**Remark 5** The parameter  $\eta_i$  in the controller (6) is used to eliminate proportional delay and mismatch parameters of the systems. Moreover,  $\epsilon$  is a tunable parameter which can adjust the length of the settling time. In general, the larger the  $\epsilon$ , the faster the synchronization will be (as shown in Fig. 2).

It is well known that, compared with state feedback control, adaptive control reduces control gains through adaptive law. Hence, in the following section, adaptive controllers are designed and some criteria are proposed to guarantee the finite-time synchronization between MNN (1) and (2).

Design adaptive controller as follows:

$$\begin{cases} u_i(t) = -r_i(t)(z_i(t)) - \eta_i(t)\text{sgn}(z_i(t)), \\ \dot{r}_i(t) = \varepsilon_i |z_i(t)|, \\ \dot{\eta}_i(t) = \delta_i \mu_i, \end{cases} \tag{26}$$



where  $\mu_i = 1$  if  $z_i(t) \neq 0$ , otherwise,  $\mu_i = 0$ ,  $\varepsilon_i > 0$ ;  $\delta_i > 0, i = 1, 2, \dots, n$ , are constants;  $\text{sgn}(z_i(t))$  stand for sign function.

**Theorem 2** *Suppose that Assumptions (H<sub>1</sub>) and (H<sub>2</sub>) are satisfied. Then, the MNN (2) can be synchronized onto MNN (1) in finite time under adaptive controller (26).*

**Proof** Consider the following Lyapunov–Krasovskii functional:

$$\bar{V}(t) = \bar{V}_1(t) + V_2(t), \tag{27}$$

where

$$\begin{aligned} \bar{V}_1(t) &= \sum_{i=1}^n |z_i(t)| + \sum_{j=1}^n \left\{ \frac{1}{2\varepsilon_i} (r_i(t) - \rho_i)^2 + \frac{1}{2\delta_i} (\eta_i(t) - \xi_i)^2 \right\}, \\ V_2(t) &= \frac{1}{q} \sum_{i=1}^n \sum_{j=1}^n \int_{qt}^t |\bar{b}|_{ji} l_i |z_i(s)| ds, \end{aligned}$$

$\rho_i, \xi_i, i = 1, 2, \dots, n$  are constants to be determined.

Using the same proof procedure as that given in Theorem 1, it is not difficult to find that

$$\begin{aligned} D_+ \bar{V}(t) &\leq \sum_{i=1}^n \left\{ \left[ - (c_i + r_i(t)) + \sum_{j=1}^n (|\bar{a}|_{ji} l_i + \frac{1}{q} |\bar{b}|_{ji} l_i) \right] |z_i(t)| \right. \\ &\quad \left. + \sum_{j=1}^n (|\hat{a}_{ij} - \check{a}_{ij}| + |\hat{b}_{ij} - \check{b}_{ij}|) p_j \mu_i - \eta_i(t) \mu_i \right\} \\ &\quad + \sum_{i=1}^n (r_i(t) - \rho_i) |z_i(t)| + \sum_{i=1}^n (\eta_i(t) - \xi_i) \mu_i \\ &= \sum_{i=1}^n \left[ - c_i + \sum_{j=1}^n (|\bar{a}|_{ji} l_i + \frac{1}{q} |\bar{b}|_{ji} l_i) - \rho_i \right] |z_i(t)| \\ &\quad + \sum_{i=1}^n \left[ \sum_{j=1}^n (|\hat{a}_{ij} - \check{a}_{ij}| + |\hat{b}_{ij} - \check{b}_{ij}|) p_j - \xi_i \right] \mu_i. \tag{28} \end{aligned}$$

Taking  $\rho_i = -c_i + \sum_{j=1}^n (|\bar{a}|_{ji} l_i + \frac{1}{q} |\bar{b}|_{ji} l_i)$  and  $\xi_i = \sum_{j=1}^n (|\hat{a}_{ij} - \check{a}_{ij}| + |\hat{b}_{ij} - \check{b}_{ij}|) p_j - 1$ , it follows from (28) that, when  $\|z(t)\|_1 \neq 0$ ,

$$D_+ \bar{V}(t) \leq - \sum_{i=1}^n \mu_i \leq -1.$$

The rest proof is the same as that given in the proof of Theorem 1. This completes the proof. □

**Remark 6** Compared with the state feedback controller (6), the adaptive controller (26) for Theorem 2 can reduce the control gains. Note that Theorems 1 and 2 are obtained without utilizing the finite-time stability theorem in [44]. In addition, results of this paper can easily

be extended to finite-time synchronization of NNs with proportional delays, parameters uncertainties.

**Remark 7** When the adaptive controllers are considered, the settling time is dependent on the initial values of systems and parameters, the adaptive laws parameters, and synchronization errors. It can be found that the larger of the adaptive laws parameters, for instance,  $\varepsilon_i$  and  $\delta_i$  in (26), the smaller the settling time will be. On the other hand, because the time interval of the increasing of the adaptive laws parameters from initial value to theoretical values is difficult to be estimated, the settling time cannot be explicitly estimated.

### 4 Numerical Examples

In this section, some numerical simulations are given to validate the effectiveness of the above theoretical analysis.

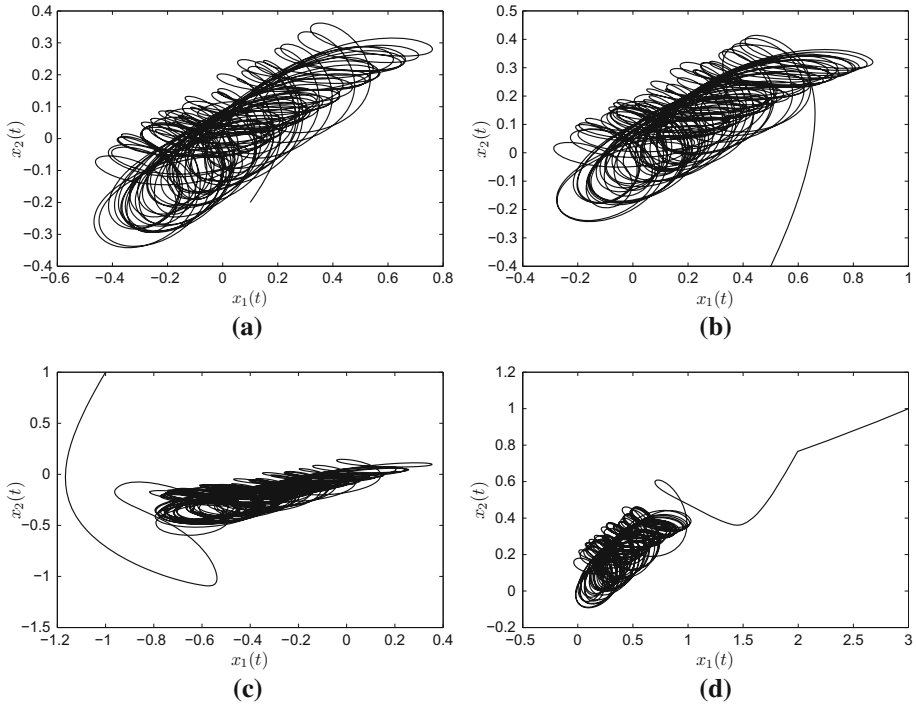
Consider MNN (1) with the following parameters:  $c_1 = c_2 = 1, J_1 = 0.5 \cos(t), J_2 = -0.5 \sin(t), q = 0.5, f_i(x_i) = \tanh(x_i), i = 1, 2$ , and

$$\begin{aligned} a_{11}(x_1) &= \begin{cases} -0.75, & |x_1| \leq 2, \\ -1.5, & |x_1| > 2, \end{cases} & a_{12}(x_1) &= \begin{cases} -2.1, & |x_1| \leq 2, \\ -0.12, & |x_1| > 2, \end{cases} \\ a_{21}(x_2) &= \begin{cases} 1.75, & |x_2| \leq 2, \\ 2.1, & |x_2| > 2, \end{cases} & a_{22}(x_2) &= \begin{cases} -2.85, & |x_2| \leq 2, \\ -3.2, & |x_2| > 2, \end{cases} \\ b_{11}(x_1) &= \begin{cases} 2.7, & |x_1| \leq 2, \\ -1.6, & |x_1| > 2, \end{cases} & b_{12}(x_1) &= \begin{cases} 0.08, & |x_1| \leq 2, \\ -1.11, & |x_1| > 2, \end{cases} \\ b_{21}(x_2) &= \begin{cases} 1.3, & |x_2| \leq 2, \\ 2.2, & |x_2| > 2, \end{cases} & b_{22}(x_2) &= \begin{cases} -1.5, & |x_2| \leq 2, \\ 2.36, & |x_2| > 2. \end{cases} \end{aligned}$$

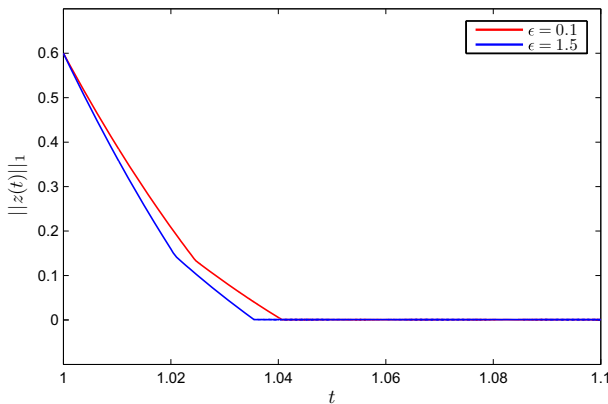
Obviously, Assumptions (H<sub>1</sub>)–(H<sub>2</sub>) are satisfied with  $p_1 = p_2 = 1, l_1 = l_2 = 1$ . The forward Euler numerical scheme in MATLAB is used in the simulations. Figure 1 presents several trajectories of the MNN with different initial values, from which one can see that the trajectories of the MNN with different initial values are different. Hence, Fig. 1 verifies the discussions in Remark 1.

**Example 1** Now we verify Theorem 1. By simple computation, we can obtain that the control gains  $r_1 = 12.4, r_2 = 11.24$ , and  $\eta_1 = 8.22 + \epsilon, \eta_2 = 5.46 + \epsilon$  for any positive constant  $\epsilon$  from (7) and (8), and the drive and response MNNs can be synchronized in finite time by the controller (6). We take the initial values of drive and response systems as  $x(t) = (0.1, -0.2)^T$  and  $y(t) = (0.5, -0.4)^T$ , respectively, where  $t \in [q, 1]$ . The trajectories of the drive and response systems are presented in (a) and (b) of Fig. 1. Figure 2 presents the trajectories of the synchronization error  $\|z(t)\|_1$  under the controller (6) with different values of  $\epsilon$ . Figure 2 also shows that the larger the  $\epsilon$ , the faster the synchronization will be, which demonstrates the discussions in Remark 5. As a matter of fact, the drive and response MNNs are synchronized within the estimation. For instance, when  $\epsilon = 1.5$ , the synchronization is achieved before  $t = 1.1$ , which is within the estimated settling time 2.2534.

**Example 2** This example considers adaptive control law to verify Theorem 2. Take  $\varepsilon_1 = 0.1, \varepsilon_2 = 0.15, \delta_1 = 0.12, \delta_2 = 0.1$ . By using the adaptive controller (26) with the initial values  $r(t) = (0.3, 0.3)^T$  and  $\eta(t) = (0.2, 0.2)^T$ , and the other parameters are same as those in Example 1, we obtain the time response of the synchronization error shown in Fig. 3. The

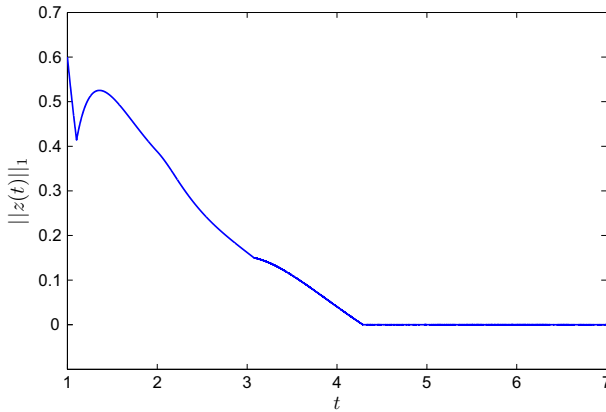


**Fig. 1** Trajectories of the MNN with different initial conditions for  $t \in [q, 1]$ , **a**  $x(t) = (0.1, -0.2)^T$ ; **b**  $x(t) = (0.5, -0.4)^T$ ; **c**  $x(t) = (-1, 1)^T$ ; **d**  $x(t) = (3, 1)^T$

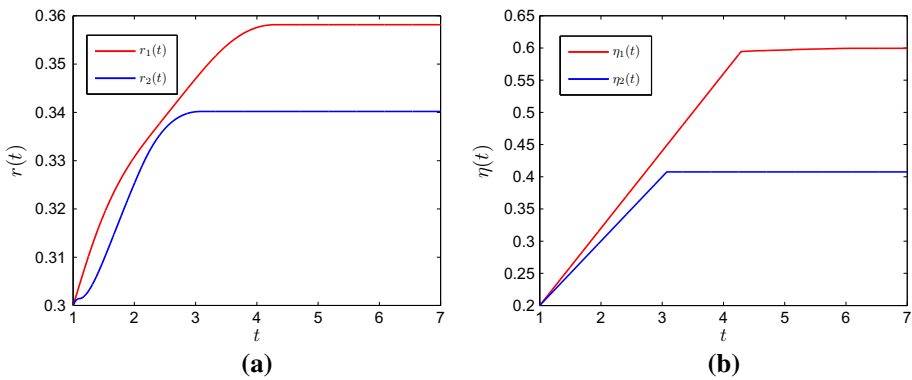


**Fig. 2** Time responses of synchronization error  $\|z(t)\|_1$  under the controller (6) with  $\epsilon = 0.1$  (red) and  $\epsilon = 1.5$  (blue). (Color figure online)

trajectories of the control gains  $r_i(t)$  and  $\eta_i(t)$ ,  $i = 1, 2$  are presented in Fig. 4. Comparing the ultimate control gains with corresponding ones of state feedback control, it is found that the adaptive control gains are smaller, while the synchronization time is larger than that in Fig. 2.



**Fig. 3** Time responses synchronization error  $\|z(t)\|_1$  under the adaptive controller (26)



**Fig. 4** Trajectories of the control gains  $r_1(t)$  and  $r_2(t)$  of the adaptive controller (26) in (a); trajectories of the control gains  $\eta_1(t)$  and  $\eta_2(t)$  of the adaptive controller (26) in (b)

### 5 Conclusions

This paper has addressed the problem of finite-time synchronization of MNNs with proportional delay. A set of powerful state feedback controllers are designed for finite-time synchronization. Then, adaptive controllers are designed to reduce the control gains. Based on the framework of Filippov solution and Lyapunov functional method, sufficient conditions are derived to guarantee the synchronization goal without using existing finite-time stability theorem. Finally, the effectiveness of the theoretical analysis has been validated by numerical simulations.

Note that the settling time is dependent to the initial conditions. When the initial condition is not available, the settling time cannot be estimated. Hence, fixed-time synchronization is our next research topic, where the settling time is not dependent on the initial condition. Moreover, controller without the sign function is also our direction of effort.

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## References

- Lampariello F, Sciandrone M (2001) Efficient training of RBF neural networks for pattern recognition. *IEEE Trans Neural Netw* 12(5):1235–1242
- Ji L, Yi Z, Shang L (2008) An improved pulse coupled neural network for image processing. *Neural Comput Appl* 17(3):255–263
- Grossberg S (1988) Neural networks for visual perception in variable illumination. *Opt News* 14(8):5–10
- Ding X, Cao J, Zhao X (2017) Finite-time stability of fractional-order complex-valued neural networks with time delays. *Neural Process Lett* 46(2):561–580
- Itoh M, Chua LO (2014) Memristor cellular automata and memristor discrete-time cellular neural networks. *Int J Bifurc Chaos* 19(11):3605–3656
- Pershin Y, Ventra MD (2010) Experimental demonstration of associative memory with memristive neural networks. *Neural Netw* 23(7):881–886
- Soudry D, Castro DD, Gal A, Kolodny A, Kvatinsky S (2015) Memristor-based multilayer neural networks with online gradient descent training. *IEEE Trans Neural Netw Learn Syst* 26(10):2408–2421
- Chua LO (1971) Memristor-The missing circuit element. *IEEE Trans Circuit Theory* 18(5):507–519
- Wu A, Zeng Z (2014) Exponential passivity of memristive neural networks with time delays. *Neural Netw* 49(1):11–18
- Guo Z, Yang S, Wang J (2015) Global exponential synchronization of multiple memristive neural networks with time delay via nonlinear coupling. *IEEE Trans Neural Netw Learn Syst* 26(6):1300
- Wu A, Zeng Z (2014) Lagrange stability of memristive neural networks with discrete and distributed delays. *IEEE Trans Neural Netw Learn Syst* 25(4):690
- Yang X, Li C, Huang T, Song Q, Chen X (2017) Quasi-uniform synchronization of fractional-order memristor-based neural networks with delay. *Neurocomputing* 234:205–215
- Yang X, Ho DW (2015) Synchronization of delayed memristive neural networks: robust analysis approach. *IEEE Trans Cybern* 46(12):3377–3387
- Yang X, Cao J, Liang J (2017) Exponential synchronization of memristive neural networks with delays: interval matrix method. *IEEE Trans Neural Netw Learn Syst* 28(8):1878–1888
- Wu A, Zeng Z (2012) Exponential stabilization of memristive neural networks with time delays. *IEEE Trans Neural Netw Learn Syst* 23(12):1919–1929
- Wen S, Zeng Z, Huang T, Zhang Y (2014) Exponential adaptive lag synchronization of memristive neural networks via fuzzy method and applications in pseudorandom number generators. *IEEE Trans Fuzzy Syst* 22(6):1704–1713
- Bao H, Park JH, Cao J (2016) Exponential synchronization of coupled stochastic memristor-based neural networks with time-varying probabilistic delay coupling and impulsive delay. *IEEE Trans Neural Netw Learn Syst* 27(1):190–201
- Rong L, Chen T (2006) New results on the robust stability of Cohen–Grossberg neural networks with delays. *Neural Process Lett* 24(3):193–202
- Yu W, Cao J (2007) Adaptive synchronization and lag synchronization of uncertain dynamical system with time delay based on parameter identification. *Physica A* 375(2):467–482
- Yang X, Song Q, Liang J, He B (2015) Finite-time synchronization of coupled discontinuous neural networks with mixed delays and nonidentical perturbations. *J Frankl Inst* 352(10):4382–4406
- Zhu X, Yang X, Alsaadi F E, Hayat T (2017) Fixed-time synchronization of coupled discontinuous neural networks with nonidentical perturbations. *Neural Process Lett*. <https://doi.org/10.1007/s11063-017-9770-8>
- Shi X, Wang Z, Han L (2017) Finite-time stochastic synchronization of time-delay neural networks with noise disturbance. *Nonlinear Dyn* 88(4):2747–2755
- Wang G, Shen Y (2014) Exponential synchronization of coupled memristive neural networks with time delays. *Neural Comput Appl* 24(6):1421–1430
- Forti M, Nistri P, Papini D (2005) Global exponential stability and global convergence in finite time of delayed neural networks with infinite gain. *IEEE Trans Neural Netw* 16(6):1449–1463
- Bao H, Park JH, Cao J (2015) Adaptive synchronization of fractional-order memristor-based neural networks with time delay. *Nonlinear Dyn* 82(3):1343–1354
- Shi L, Yang X, Li Y, Feng Z (2016) Finite-time synchronization of nonidentical chaotic systems with multiple time-varying delays and bounded perturbations. *Nonlinear Dyn* 83(1–2):75–87

27. Bao H, Cao J, Kurths J, Alsaadi A, Ahmad B (2018)  $H_\infty$  state estimation of stochastic memristor-based neural networks with time-varying delays. *Neural Netw* 99:79–91
28. Wang X, Li C, Huang T, Chen L (2015) Dual-stage impulsive control for synchronization of memristive chaotic neural networks with discrete and continuously distributed delays. *Neurocomputing* 149:621–628
29. Yang X, Wu Z, Cao J (2013) Finite-time synchronization of complex networks with nonidentical discontinuous nodes. *Nonlinear Dyn* 73(4):2313–2327
30. Zhou C, Zhang W, Yang X (2017) Finite-time synchronization of complex-valued neural networks with mixed delays and uncertain perturbations. *Neural Process Lett* 46:271–291
31. Wang L, Liu Y, Wu Z, Alsaadi FE (2018) Strategy optimization for static games based on STP method. *Appl Math Comput* 316:390–399
32. Tong L, Liu Y, Lou J, Lu J, Alsaadi FE (2018) Static output feedback set stabilization for context-sensitive probabilistic Boolean control networks. *Appl Math Comput* 332:263–275
33. Yang X, Ho DWC, Lu J, Song Q (2015) Finite-time cluster synchronization of T-S fuzzy complex networks with discontinuous subsystems and random coupling delays. *IEEE Trans Fuzzy Syst* 23(6):2302–2316
34. Wu H, Li R, Zhang X, Yao R (2015) Adaptive finite-time complete periodic synchronization of memristive neural networks with time delays. *Neural Process Lett* 42(3):563–583
35. Wu H, Zhang X, Li R, Yao R (2015) Finite-time synchronization of chaotic neural networks with mixed time-varying delays and stochastic disturbance. *Memet Comput* 7(3):231–240
36. Yang X (2014) Can neural networks with arbitrary delays be finite-timely synchronized? *Neurocomputing* 143(16):275–281
37. Yan JJ, Hung ML, Chiang TY, Yang YS (2006) Robust synchronization of chaotic systems via adaptive sliding mode control. *Phys Lett A* 356(3):220–225
38. Liang J, Wang Z, Liu Y, Liu X (2008) Robust synchronization of an array of coupled stochastic discrete-time delayed neural networks. *IEEE Trans Neural Netw* 19(11):1910–1921
39. Tang Y, Gao H, Kurths J (2014) Distributed robust synchronization of dynamical networks with stochastic coupling. *IEEE Trans Circuits Syst-I Reg Pap* 61(5):1508–1519
40. Yang M, Wang YW, Xiao JW, Wang HO (2010) Robust synchronization of impulsively-coupled complex switched networks with parametric uncertainties and time-varying delays. *Nonlinear Anal B Real World Appl* 11(4):3008–3020
41. Li R, Cao J (2016) Stability analysis of reaction-diffusion uncertain memristive neural networks with time-varying delays and leakage term. *Appl Math Comput* 278:54–69
42. Zhang R, Zeng D, Zhong S, Yu Y (2017) Event-triggered sampling control for stability and stabilization of memristive neural networks with communication delays. *Appl Math Comput* 310:57–74
43. Mathiyalagan K, Ju HP, Sakthivel R (2015) Synchronization for delayed memristive BAM neural networks using impulsive control with random nonlinearities. *Appl Math Comput* 259:967–979
44. Bao H, Ju HP, Cao J (2015) Matrix measure strategies for exponential synchronization and anti-synchronization of memristor-based neural networks with time-varying delays. *Appl Math Comput* 270:543–556
45. Wang W, Li L, Peng H, Kurths J, Xiao J, Yang Y (2016) Finite-time anti-synchronization control of memristive neural networks with stochastic perturbations. *Neural Process Lett* 43(1):49–63
46. Arscott FM (1988) *Differential equations with discontinuous righthand sides*. Kluwer, Dordrecht
47. Aubin JP, Cellina A (1986) *Differential inclusions: set-valued maps and viability theory*. *Acta Appl Math* 6(2):215–217
48. Bhat SP, Bernstein DS (2000) *Finite-time stability of continuous autonomous systems*, vol 38. Society for Industrial and Applied Mathematics, Philadelphia
49. Qin J, Yu C, Gao H (2014) Coordination for linear multiagent systems with dynamic interaction topology in the leader-following framework. *IEEE Trans Ind Electron* 61(5):2412–2422