

An Efficient Algorithm for Solving Fuzzy Linear Programming Problems

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Abstract In this article, we consider some well-known approaches for solving fuzzy linear programming (FLP) problems. We present some of the difficulties of these approaches. Then, crisp linear programming problems are suggested for solving FLP problems. A new algorithm is also given. The proposed approach has advantages over the other methods. For example, we can achieve higher membership degrees for objective function and constraints. Moreover, we show that the fuzzy optimal solutions obtained by the proposed approach are efficient enough. Also, we see that unlike the previous methods, our method finds efficient solutions by solving only one crisp linear problem instead of solving two or three crisp problems. Finally some numerical examples are presented to show the efficiency of the given approach over the other approaches.

Keywords Fuzzy linear programming · Membership function · Efficient solution · Crisp linear programming

1 Introduction

In the real world, we deal with the linear and nonlinear programming problems. Some of these problems must be stated in the fuzzy sense, since in this case, decision makers have more flexibility in modeling and solving problems. The definition of a fuzzy linear programming (FLP) problem is not unique and various models of FLP have been suggested by researchers. Tanaka and Asai [16] presented a number of FLP models. In the field of solving FLP problems, the main pioneer researchers are Bellman and Zadeh [4], Tanaka et al. [17], Negoita and Sularia [13], Zimmermann [24,25], Yager [23], etc. In recent decades, many papers and

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books have been published in the field of fuzzy programming, for more information refer to Alex [1], Bector and Chandra [3], Cao [5], Lu et al. [11], Srinivasa Raju and Duckstein [15], Wu [21], Arikan and Gungor [2], Jimenez and Bilbao [9], Wu et al. [22], Ghaznavi et al. [6] Mohanaselvi and Kandasamy [12] and references therein.

All FLP models deal with one of the following cases (see Bellman and Zadeh [4], Tanaka et al. [17], Zimmermann [24,25]):

- (I) Decision maker seeks to achieve an aspiration level for objective function where it is not possible by a nonfuzzy linear programming problem directly.
- (II) The constraints of the problem may be vague. In this case, the inequality constraints can be acceptable by targeted violations. In fact, there is a tolerance for resources in inequality constraints.
- (III) Components of vectors and matrices appearing in the FLP problem can be fuzzy numbers. In this case, the inequality constraints of the problem are interpreted by the ranking of fuzzy number concept.

In this paper, we focus on the models dealing with cases I and II. But in a different point of view the FLP problems are divided to two major classes: symmetric and non-symmetric. The symmetric models introduced by Bellman and Zadeh [4]. In these models, the objective function and constraints of the problem are stated as a fuzzy set. Here, the decision maker considers the confluence of objective function and constraints to achieve an optimal solution (see [25]). Also, the max-min operator is utilized. However, in the non-symmetric models, the objective function is a non-fuzzy or classic function and at least one of the constraints of FLP problem is fuzzy. In this case, two approaches are usually utilized. In the first approach, the problem converts to an equivalent parametric problem (see [18,19]). In the second approach, a membership function may be defined for non-fuzzy objective function and it can be converted to a fuzzy objective function (see [8,9,20,22]). In fact, in the second approach, a non-symmetric problem is converted into a symmetric one.

In defining membership functions for constraints and objective function of an FLP problem, smoothness is very important, since the obtained problem may be nonlinear or nonsmooth. For this, in solving FLP problems, the researchers simplify the membership functions where it can lead to drawbacks or error. In this paper, we revise some drawbacks of the other methods for solving FLP problems. Some of these drawbacks are listed below

- (a) Incorrect application of membership function due to its simplifying.
- (b) When optimal solution is unbounded or the problem is infeasible, the available methods can not solve the problem, correctly.
- (c) Lack of method for finding an optimal solution that has higher membership degree and a good value for the main objective function, simultaneously.

To solve these difficulties, we present modified membership functions for objective function and constraints. At what follows, we propose new crisp linear programming problems for solving FLP problems. We show that optimal solutions of the proposed problems are efficient. We summarize the obtained results in a new algorithm.

The outline of this paper is as follows: in Sect. 2 some well-known approaches for solving FLP problems are reviewed. In Sect. 3 some difficulties of the mentioned methods are shown, by numerical examples. Section 4, contains new models and an algorithm for solving FLP problems. Section 5 is devoted to numerical examples for comparing the suggested algorithm with well-known approaches. Section 6 contains conclusions.

2 Some Well-Known Methods for Solving FLP Problems

In this section, we briefly introduce the Zimmermann [24,25], Werners [20], Guu and Wu [8], Jimenez and Bilbao [9] and Wu et al. [22] approaches for solving FLP problems.

2.1 Zimmermann Approach for Solving Symmetric FLP Problems

Consider the following symmetric FLP problem:

$$\begin{aligned} \text{max } z &= cx \\ \text{subject to } &\begin{cases} A_i x \lesseqgtr b_i, & i = 1, 2, \dots, m, \\ x \in \mathcal{X}, \end{cases} \end{aligned} \tag{1}$$

where $x = (x_1, x_2, \dots, x_n)^T$ is the decision vector, $c = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$, $A_i = (a_{i1}, \dots, a_{in})^T \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $i = 1, 2, \dots, m$ and $\mathcal{X} = \{x \in \mathbb{R}^n : Bx \leq d\}$ with $B \in \mathbb{R}^{l \times n}$ and $d \in \mathbb{R}^l$. In FLP problem (1), the fuzzifier max means that the decision maker wants to reach some aspiration level $b_0 \in \mathbb{R}$ that might not even be definable crisply. Here, \lesseqgtr indicates the fuzzified version of \leq and is described as ‘‘essentially less than or equal to.’’

Assumed that the decision maker proposes an aspiration level b_0 . The FLP problem (1) can be restructured into the following model [25]:

$$\begin{aligned} \text{Find } x \\ \text{subject to } &\begin{cases} cx \gtrsim b_0, \\ A_i x \lesseqgtr b_i, & i = 1, 2, \dots, m, \\ x \in \mathcal{X}. \end{cases} \end{aligned} \tag{2}$$

In the Zimmermann [24,25] approach for FLP problem (1) (or equivalently, problem (2)), we must select a suitable membership function for each of the fuzzy inequalities and then utilize the Max-Min operator (based on the Bellman and Zadeh [4] principle) to identify the fuzzy decision. Assume that $p_0 > 0$ is the allowable tolerance for the cost function and $p_i > 0$, $i = 1, 2, \dots, m$, is the acceptable tolerance for the i th constraint. The membership functions of the objective function and the i th constraint (for $i = 1, 2, \dots, m$) are defined as follows, respectively:

$$\mu_0(cx) = \begin{cases} 1, & cx > b_0, \\ 1 - \frac{b_0 - cx}{p_0}, & b_0 - p_0 \leq cx \leq b_0, \\ 0, & cx < b_0 - p_0, \end{cases} \tag{3}$$

$$\mu_i(A_i x) = \begin{cases} 1, & A_i x < b_i, \\ 1 - \frac{A_i x - b_i}{p_i}, & b_i \leq A_i x \leq b_i + p_i, \\ 0, & A_i x > b_i + p_i, \end{cases} \tag{4}$$

where they are nonlinear (or piecewise linear) and continuous. Using the membership functions (3) and (4), and assumption $\alpha = \min\{\mu_0(cx), \mu_1(A_1x), \dots, \mu_m(A_mx)\}$, we can convert the FLP problem (1) to the following crisp nonlinear programming (CNLP) problem (see for details Zimmermann [25]):

$$\begin{aligned}
 & \text{Max } \alpha \\
 & \text{subject to } \begin{cases} \mu_0(cx) \geq \alpha, \\ \mu_i(A_i x) \geq \alpha, \quad i = 1, 2, \dots, m, \\ 0 \leq \alpha \leq 1, \\ x \in \mathcal{X}. \end{cases} \tag{5}
 \end{aligned}$$

To facilitate the work and converting the nonlinear membership function to a linear one, it is assumed that $\mu_0(cx) = 1 - \frac{b_0 - cx}{p_0}$ and $\mu_i(A_i x) = 1 - \frac{A_i x - b_i}{p_i}$ for $i = 1, 2, \dots, m$. Therefore, CNLP problem (5) is converted to the following crisp linear programming (CLP) problem:

$$\begin{aligned}
 & \text{Max } \alpha \\
 & \text{subject to } \begin{cases} cx \geq b_0 - (1 - \alpha)p_0, \\ A_i x \leq b_i + (1 - \alpha)p_i, \quad i = 1, 2, \dots, m, \\ 0 \leq \alpha \leq 1, \\ x \in \mathcal{X}. \end{cases} \tag{6}
 \end{aligned}$$

In the Zimmermann approach, if (x^*, α^*) is an optimal solution of the CLP problem (6), then x^* is said to be an optimal solution for the FLP problem (1).

2.2 Werners Approach for Solving Non-symmetric FLP Problems

Consider the following non-symmetric FLP problem with crisp objective function and fuzzy constraints:

$$\begin{aligned}
 & \text{Max } z = cx \\
 & \text{subject to } \begin{cases} A_i x \lesseqgtr b_i, \quad i = 1, 2, \dots, m, \\ x \in \mathcal{X}. \end{cases} \tag{7}
 \end{aligned}$$

Assume that $p_i, i = 1, 2, \dots, m$ is the maximum tolerance of $b_i, i = 1, 2, \dots, m$, as determined by decision maker. Werners [20] suggested a membership function for crisp objective function of FLP problem (7). He proposed solving the following CLP problems:

$$\begin{aligned}
 & \text{Max } z = cx \\
 & \text{subject to } \begin{cases} A_i x \leq b_i, \quad i = 1, 2, \dots, m, \\ x \in \mathcal{X}, \end{cases} \tag{8}
 \end{aligned}$$

and

$$\begin{aligned}
 & \text{Max } z = cx \\
 & \text{subject to } \begin{cases} A_i x \leq b_i + p_i, \quad i = 1, 2, \dots, m, \\ x \in \mathcal{X}. \end{cases} \tag{9}
 \end{aligned}$$

Let z_0 and z_1 be optimal values of problems LPs (8) and (9), respectively. The suggested membership function, for crisp objective function of FLP problem (7), is as follows:

$$\mu_0(cx) = \begin{cases} 1, & cx > z_1, \\ 1 - \frac{z_1 - cx}{z_1 - z_0}, & z_0 \leq cx \leq z_1, \\ 0, & cx < z_0, \end{cases} \tag{10}$$

which is a nonlinear function. Also, the membership functions of the fuzzy constraints are as in relation (4). Now, by applying the Bellman and Zadeh [4] principle and assuming

$\alpha = \min\{\mu_0(cx), \mu_1(A_1x), \dots, \mu_m(A_mx)\}$, the following crisp nonlinear programming (CNLP) problem is proposed for solving the FLP problem (7):

$$\begin{aligned} & \text{Max } \alpha \\ & \text{subject to } \begin{cases} \mu_0(cx) \geq \alpha, \\ \mu_i(A_ix) \geq \alpha, \quad i = 1, 2, \dots, m, \\ 0 \leq \alpha \leq 1 \\ x \in \mathcal{X}. \end{cases} \end{aligned} \tag{11}$$

Here by assuming $\mu_0(cx) = 1 - \frac{z_1 - cx}{z_1 - z_0}$ and $\mu_i(A_ix) = 1 - \frac{A_ix - b_i}{p_i}$, $i = 1, 2, \dots, m$ (to facilitate, but with loss of generality), the CNLP problem (11) will be converted into the following CLP problem:

$$\begin{aligned} & \text{Max } \alpha \\ & \text{subject to } \begin{cases} cx \geq z_1 - (1 - \alpha)(z_1 - z_0), \\ A_ix \leq b_i + (1 - \alpha)p_i, \quad i = 1, 2, \dots, m, \\ 0 \leq \alpha \leq 1, \\ x \in \mathcal{X}. \end{cases} \end{aligned} \tag{12}$$

2.3 Guu and Wu Approach for Solving Non-symmetric FLP Problems

In this subsection, the two phase method suggested by Guu and Wu [8] for FLP problem (7) is presented. Note that there are also some other two phase approaches (see Lee and Li [10] and Guu and Wu [7]). The proposed two phases are as follows:

Phase 1. Solve the following crisp problem

$$\begin{aligned} & \text{Max } \alpha \\ & \text{subject to } \begin{cases} 0 \leq \alpha \leq \mu_i(x) \leq 1, \quad i = 0, 1, \dots, m, \\ x \in \mathcal{X}, \end{cases} \end{aligned} \tag{13}$$

and denote its optimal solution by (x^*, α^*) .

Phase 2. Find the optimal solution $(\alpha_0^{**}, \alpha_1^{**}, \dots, \alpha_m^{**})$ of the following problem:

$$\begin{aligned} & \text{Max } \sum_{i=0}^m \alpha_i \\ & \text{subject to } \begin{cases} \mu_0(cx^*) \leq \alpha_0 \leq \mu_0(cx) \leq 1, \\ \mu_i(A_ix^*) \leq \alpha_i \leq \mu_i(A_ix) \leq 1, \quad i = 1, 2, \dots, m, \\ 0 \leq \alpha_i \leq 1, \quad i = 0, 1, \dots, m, \\ x \in \mathcal{X}. \end{cases} \end{aligned} \tag{14}$$

In this approach, for convenience, it is assumed that $\mu_0(cx) = 1 - \frac{z_1 - cx}{z_1 - z_0}$ and $\mu_i(A_ix) = 1 - \frac{A_ix - b_i}{p_i}$, $i = 1, 2, \dots, m$. Therefore, the CNLP problem (14) is converted to the following CLP problem:

$$\begin{aligned}
 & \text{Max} \quad \sum_{i=0}^m \alpha_i \\
 & \text{subject to} \quad \begin{cases} z_0 + \alpha(z_1 - z_0) \leq cx \leq z_1, \\ b_i \leq A_i x \leq b_i + (1 - \alpha_i)p_i, \quad i = 1, 2, \dots, m, \\ \mu_0(cx^*) \leq \alpha_0, \\ \mu_i(A_i x^*) \leq \alpha_i, \\ 0 \leq \alpha_i \leq 1, \\ x \in \mathcal{X}. \end{cases} \tag{15}
 \end{aligned}$$

Remark 1 In order to simplify, the symbols $\mu_i(x)$ are used uniformly for all $i = 0, 1, \dots, m$. Here $\mu_0(x)$ is $\mu_0(cx)$ and $\mu_i(x), i = 1, 2, \dots, m$ are $\mu_i(A_i x), i = 1, 2, \dots, m$, actually.

Theorem 1 [8] *Let $(x^{**}, \alpha_0^{**}, \alpha_1^{**}, \dots, \alpha_m^{**})$ be an optimal solution of CNLP problem (14). Then, x^{**} is a fuzzy efficient solution, which means, there is no $x \geq 0$ such that $\mu_i(x) \geq \mu_i(x^{**}), i = 0, 1, \dots, m$ and $\mu_j(x) > \mu_j(x^{**}),$ for some $j \in \{0, 1, \dots, m\}$.*

2.4 Jimenez and Bilbao Approach for Solving Non-symmetric FLP Problems

Jimenez and Bilbao [9] presented an approach for fuzzy (multiple objective) programming problems. Here we explain their method for non-symmetric FLP problems. Consider the FLP problem (7). In Jimenez and Bilbao [9] approach, we have three phases. Phases 1 and 2 are the same as phases 1 and 2 of Guu and Wu [8] approach. Let x^{**} be the optimal solution obtained from phase 2 of Guu and Wu [8] approach and $\mu_0(cx)$ and $\mu_i(A_i x), i = 1, 2, \dots, m$ be defined as relations (10) and (4), respectively. Then, in phase 3 if $\mu_0(cx^{**}) = 1$, the following problem is solved:

$$\begin{aligned}
 & \max \sum_{s=0}^p n_s \\
 & \text{subject to} \quad \begin{cases} cx + n_0 = cx^{**}, \\ A_s x + n_s = A_s x^{**}, \quad s = 1, 2, \dots, p, \\ \mu_r(A_r x) = \mu_r(A_r x^{**}), \quad r = p + 1, p + 2, \dots, m, \\ x \in \mathcal{X}, n_s \geq 0. \end{cases} \tag{16}
 \end{aligned}$$

But, if $\mu_0(cx^{**}) < 1$, then we must solve the following problem:

$$\begin{aligned}
 & \max \sum_{s=1}^p n_s \\
 & \text{subject to} \quad \begin{cases} A_s x + n_s = A_s x^{**}, \quad s = 1, 2, \dots, p, \\ \mu_0(cx) = \mu_0(cx^{**}), \\ \mu_r(A_r x) = \mu_r(A_r x^{**}), \quad r = p + 1, p + 2, \dots, m, \\ x \in \mathcal{X}, n_s \geq 0, \end{cases} \tag{17}
 \end{aligned}$$

where in problems (16) and (17), subscript s refers to membership functions that are equal to 1 and subscript r refers to membership functions that are strictly less than 1. For more details of this approach, we refer to Example 2 of Jimenez and Bilbao [9].

2.5 Wu et al. Approach for Solving Non-symmetric FLP Problems

Wu et al. [22] suggested a new approach for fuzzy (multiple objective) linear programming problems. Consider the FLP problem (7). In Wu et al. [22] approach, the membership functions are suggested as follows:

$$\mu_0(cx) = \begin{cases} \frac{cx-z_0}{z_1-z_0}, & z_0 \leq cx, \\ 0, & cx \leq z_0, \end{cases} \tag{18}$$

$$\mu_i(A_i x) = \begin{cases} 1 - \frac{A_i x - b_i}{p_i}, & A_i x \leq b_i + p_i, \\ 0, & A_i x > b_i + p_i. \end{cases} \tag{19}$$

In this approach, we must solve the following CLP problem in phase 1:

$$\begin{aligned} & \max \lambda \\ \text{subject to } & \begin{cases} \mu_0(cx) \geq \lambda, \\ \mu_i(A_i x) \geq \lambda, \\ x \in \mathcal{X}. \end{cases} \end{aligned} \tag{20}$$

Assume that λ^* is the optimal solution of problem (20), then, in phase 2 the following problem is solved:

$$\begin{aligned} & \max \sum_{i=0}^m s_i \\ \text{subject to } & \begin{cases} \mu_0(cx) - s_0 \geq \lambda^*, \\ \mu_i(A_i x) - s_i \geq \lambda^*, \\ x \in \mathcal{X}, \\ s_i \geq 0, \quad i = 1, 2, \dots, m. \end{cases} \end{aligned} \tag{21}$$

Remark 2 Wu et al. [22] claimed that by redefining the membership functions as relations (18) and (19), we can release the conditions $\mu_i(x) \leq 1, i = 0, 1, \dots, m$ in the Jimenez and Bilbao approach. While, the functions defined in relations (18) and (19) do not satisfy the conditions $0 \leq \mu_i \leq 1, i = 0, 1, \dots, m$ generally. For example, if $A_i x^* = b_i - 1$, for some $i \in \{1, 2, \dots, m\}$, then $\mu_i(A_i x^*) = \frac{p_i + 1}{p_i - i} > 1$ which is not an acceptable membership value.

3 Some Difficulties of the Described Approaches

In the previous section, we briefly described some well-known methods for solving symmetric and non-symmetric FLP problems. However, these methods have difficulties and they can not solve all FLP problems, correctly. Now, in this section, we show some of these drawbacks by numerical examples.

In the Werners [20], Guu and Wu [8], Jimenez and Bilbao [9] and Wu et al. [22] approaches, to define $\mu_0(cx)$ we must solve the LP problems (8) and (9) to find z_0 and z_1 , respectively. However, in some FLP problems it may happen that $z_0 = z_1$. In this case, we can not define μ_0 by relations (10) or (18). The following example shows this difficulty.

Example 1 Consider the following non-symmetric FLP problem:

$$\max \quad x_2$$

$$\text{subject to } \begin{cases} x_1 + x_2 \lesseqgtr 2, \\ x_1 \lesseqgtr 1, \\ x_1, x_2 \geq 0. \end{cases}$$

Let $p_1 = 0$ and $p_2 = 1$. The optimal values of LP problems (8) and (9) are $z_0 = 1$ and $z_1 = 1$, respectively. Therefore, we can not use relations (10) and (18) to define $\mu_0(cx)$.

In Sect. 2, for convenience, it was assumed that $\mu_0(cx) = 1 - \frac{b_0-cx}{p_0}$ (for symmetric FLPs), or $\mu_0(cx) = 1 - \frac{z_1-cx}{z_1-z_0}$ (for non-symmetric FLPs) and $\mu_i(A_i x) = 1 - \frac{A_i x - b_i}{p_i}$ for $i = 1, 2, \dots, m$. However, this may not be generally correct. For example, if x^* is an optimal solution of CLP problem (6) and $cx^* > b_0$ or $A_i x^* < b_i$ for some $i = 1, 2, \dots, m$ then $\mu_0(cx^*) > 1$ or $\mu_i(A_i x^*) > 1$ which contradicts the definition of membership degree.

Example 2 Consider the following FLP problem:

$$\begin{aligned} & \max \quad x_2 \\ & \text{subject to } \begin{cases} x_1 + x_2 \lesseqgtr 1, \\ x_1 \lesseqgtr 0.5, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Let $p_1 = 1$ and $p_2 = 1$. By solving LPs (8) and (9), we get $z_0 = 1$ and $z_1 = 2$. Using the Werners, Jimenez and Bilbao and Wu et al. methods, we get $x_1^* = 0$ and $x_2^* = 1.5$. Now by $\mu_2(A_2 x^*) = 1 - \frac{A_2 x^* - b_2}{p_2}$ we have $\mu_2(A_2 x^*) = \frac{3}{2} > 1$ which is not an acceptable membership degree.

Even if we want to use the membership functions described in relations (3)–(10), we have to consider all of the constraints to check which of the cases described in the membership functions happens. Thus, we can not find the membership functions directly during the process of solving the related CLP problem.

In the two phase method by Guu and Wu, and the three phase method by Jimenez and Bilbao, in phase 1 and phase 2 we have the constraints $0 \leq \mu_i(x) \leq 1, i = 1, \dots, m$. If we replace $\mu_i(x)$ by $\mu_i(x) = 1 - \frac{A_i x - b_i}{p_i}, i = 1, 2, \dots, m$ we conclude that $b_i \leq A_i x \leq b_i + p_i, i = 1, 2, \dots, m$ [see the example given in Guu and Wu (1999) and Jimenez and Bilbao (2009)]. However, adding the constraints $b_i \leq A_i x \leq b_i + p_i, i = 1, 2, \dots, m$ may cause that problem (13) or problem (14) becomes infeasible. This drawback is shown in the following example:

Example 3 Consider the following non-symmetric FLP-problem:

$$\begin{aligned} & \max \quad z = 2x_1 + x_2 \\ & \text{subject to } \begin{cases} x_1 + x_2 \lesseqgtr 2, \\ 3x_1 + x_2 \lesseqgtr 3, \\ 3x_1 + 2x_2 \lesseqgtr 12, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned} \tag{22}$$

Let $p_1 = p_2 = p_3 = 1$. By solving LPs (8) and (9), we obtain $z_0 = 2.5$ and $z_1 = 3.5$. By substituting $\mu_i(x), i = 0, 1, \dots, m$ as relations (10) and (4), the linear problem in phase 1 is as follows:

$$\begin{aligned} & \max \quad \alpha \\ & \text{subject to} \quad \begin{cases} \alpha \leq 1 - (3.5 - 2x_1 - x_2), \\ \alpha \leq 1 - (x_1 + x_2 - 2), \\ \alpha \leq 1 - (3x_1 + x_2 - 3), \\ \alpha \leq 1 - (3x_1 + 2x_2 - 12), \\ 2 \leq x_1 + x_2, \\ 3 \leq 3x_1 + x_2, \\ 12 \leq 3x_1 + 2x_2, \\ 0 \leq \alpha \leq 1, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

which is an infeasible linear problem.

Even if phase 1 and phase 2 problems are feasible, the obtained solution may not be an optimal solution for the main FLP problem.

Example 4 Consider the following FLP problem:

$$\begin{aligned} & \max \quad 0.1x_1 + 3x_2 \\ & \text{subject to} \quad \begin{cases} x_1 + x_2 \lesseqgtr 4, \\ x_1 + 3x_2 \lesseqgtr 6, \\ x_1 \lesseqgtr 2, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Let $p_1 = 3, p_2 = 2$ and $p_3 = 1$. We solve this problem by Guu and Wu method, Jimenez and Bilbao method and Wu et al. method, respectively.

With phase 1 of Guu and Wu method, the following problem should be solved:

$$\begin{aligned} & \max \quad \alpha \\ & \text{subject to} \quad \begin{cases} 6 + 2\alpha \leq 0.1x_1 + 3x_2 \leq 8, \\ 4 \leq x_1 + x_2 \leq 7 - 3\alpha, \\ 6 \leq x_1 + 3x_2 \leq 8 - 2\alpha, \\ 2 \leq x_1 \leq 3 - \alpha, \\ \alpha, x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Using the simplex method, we can acquire optimal solution $(x_1^*, x_2^*, \alpha^*) = (2.0408, 1.9592, 0.0408)$ and $\mu_0(x^*) = 0.0408, \mu_1(x^*) = 1, \mu_2(x^*) = 0.0408, \mu_3(x^*) = 0.9592$. Moreover, in phase 2 of Guu and Wu method, the following CLP problem is solved:

$$\max \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$$

$$\text{subject to } \begin{cases} 6 + 2\alpha_0 \leq 0.1x_1 + 3x_2 \leq 8, \\ 4 \leq x_1 + x_2 \leq 7 - 3\alpha_1, \\ 6 \leq x_1 + 3x_2 \leq 8 - 2\alpha_2, \\ 2 \leq x_1 \leq 3 - \alpha, \\ 0.0408 \leq \alpha_0 \leq 1, \\ 1 \leq \alpha_1 \leq 1, \\ 0.0408 \leq \alpha_2 \leq 1, \\ 0.9592 \leq \alpha_3 \leq 1, \\ x_1, x_2 \geq 0. \end{cases}$$

Solving this problem we, again, obtain $x^{**} = (2.0408, 1.9592)$, $cx^{**} = 6.0816$, $\mu_0(cx^{**}) = 0.0408$, $\mu_1(A_1x^{**}) = 1$, $\mu_2(A_2x^{**}) = 0.0408$ and $\mu_3(A_3x^{**}) = 0.9592$.

Now, we solve this problem using Jimenez and Bilbao [9] method. Phases 1 and 2 coincide with phases 1 and 2 of Guu and Wu method. In phase 3 (see problem (17)), the following problem should be solved:

$$\begin{aligned} &\max \quad n_1 \\ &\text{subject to } \begin{cases} x_1 + x_2 + n_1 = 4, \\ \frac{0.1x_1 + 3x_2 - 6}{2} = 0.0408, \\ \frac{8 - x_1 - 3x_2}{2} = 0.0408, \\ 3 - x_1 = 0.9592, \\ x_1, x_2, n_1 \geq 0. \end{cases} \end{aligned}$$

Using the simplex method, we obtain the same optimal solutions as those gained by Guu and Wu method.

Finally, we solve this problem by Wu et al. [22] method. Phase 1 of Wu et al. approach (see problem (20)), is as follows:

$$\begin{aligned} &\max \quad \lambda \\ &\text{subject to } \begin{cases} \lambda \leq \frac{0.1x_1 + 3x_2 - 6}{2}, \\ \lambda \leq \frac{7 - x_1 - x_2}{3}, \\ \lambda \leq \frac{8 - x_1 - 3x_2}{2}, \\ \lambda \leq 3 - x_1, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Optimal solution of this problem is $(x_1^*, x_2^*) = (1.7948, 3.6442)$ and $\lambda^* = -2.36$. Using the optimal solutions given in phase 1, Wu et al. formulated phase 2 (see problem (21)) as follows:

$$\begin{aligned} &\max \quad s_0 + s_1 + s_2 + s_3 \\ &\text{subject to } \begin{cases} -2.36 \leq \frac{0.1x_1 + 3x_2 - 6}{2} - s_0, \\ -2.36 \leq \frac{7 - x_1 - x_2}{3} - s_1, \\ -2.36 \leq \frac{8 - x_1 - 3x_2}{2} - s_2, \\ -2.36 \leq 3 - x_1 - s_3, \\ x_1, x_2, s_i \geq 0, i = 0, 1, 2, 3. \end{cases} \end{aligned}$$

Having solved this problem, we obtain $(x_1^{**}, x_2^{**}) = (0, 0.4267)$ and $cx^{**} = 1.2801$. Moreover, we have $(s_0^{**}, s_1^{**}, s_2^{**}, s_3^{**}) = (0.0000, 4.5511, 5.7200, 5.3600)$.

In Sect. 5, we will obtain the optimal solution $x^{opt} = (0, 2.333)$ with objective value $cx^{opt} = 7$ and higher membership degrees. Therefore, in these examples we see that the methods proposed by Guu and Wu, Jimenez and Bilbao and Wu et al. sometimes, can not solve an FLP problem, correctly (see Table 6).

4 The Suggested Approach

In the previous section, using numerical examples, we presented some drawbacks of the famous approaches for solving FLP problems. Therefore, these methods must be modified. So, in this section, we propose crisp linear programming models for obtaining optimal solutions of FLP problems. Also, an algorithm for finding optimal solutions of FLP problems is recommended which resolves the difficulties described in Sect. 3.

Here we first modify the membership functions (3), (4) and (10), to convert the FLP problems (1) and (7) into an equivalent CLP problem. The proposed membership functions include different cases $A_k x^* < b_k$ for some $k \in \{1, 2, \dots, m\}$, or $b_i \leq A_i x^* \leq b_i + p_i$, for all $i = 1, 2, \dots, m$. Also, we can directly obtain the membership function $\mu_i(A_i x^*)$, for all $i = 1, 2, \dots, m$, by solving a CLP problem. Let $\mathcal{X}_i = \{x \geq 0 : A_i x \leq b_i + p_i\}$. We suggest the following continuous nonnegative membership functions $\mu_i : \mathbb{R} \rightarrow [0, 1], i = 1, \dots, m$, for constraints of the symmetric FLP problem (1) and the non-symmetric FLP problem (7):

$$\mu_i(A_i x) = \text{Minimum}\{1, 1 - \frac{A_i x - b_i}{p_i}\}, \quad x \in \mathcal{X}_i, \quad i = 1, 2, \dots, m. \tag{23}$$

Note that every $x \geq 0$ which $x \notin \mathcal{X}_i$, is infeasible for FLP problem (1) (or FLP problem (7)). So, we disregard the case $A_i x > b_i + p_i$. Hence, the membership function $\mu_i(A_i x), i = 1, 2, \dots, m$ is equal to 1 if $A_i x \leq b_i$, strictly decreasing from 1 to 0 on the interval $(b_i, b_i + p_i)$ and equal to 0 if $A_i x \geq b_i + p_i$.

Also, let $\mathcal{X}_0 = \{x \in \mathcal{X} : cx \geq b_0 - p_0\}$. The proposed membership function for the objective function of symmetric FLP problem (1) is as follows:

$$\mu_0(cx) = \text{Minimum}\{1, 1 - \frac{b_0 - cx}{p_0}\}, \quad x \in \mathcal{X}_0. \tag{24}$$

Moreover, let $\bar{\mathcal{X}}_0 = \{x \in \mathcal{X} : cx \geq z_0\}$. We suggest the following membership function for the objective function of non-symmetric FLP problem (7):

$$\mu_0(cx) = \text{Minimum}\{1, 1 - \frac{z_1 - cx}{z_1 - z_0}\}, \quad x \in \bar{\mathcal{X}}_0. \tag{25}$$

After employing the above membership functions, we propose the following crisp linear programming (CLP) model to solve FLP problems (1) and (7).

$$\begin{aligned} \max \quad & \lambda + \frac{1}{m+1} \sum_{i=0}^m \lambda_i \\ \text{subject to} \quad & \begin{cases} \lambda \leq \lambda_0 \leq \mu_0(cx), & x \in \mathcal{X}_0 \text{ (or } \bar{\mathcal{X}}_0) \\ \lambda \leq \lambda_i \leq \mu_i(A_i x), & x \in \mathcal{X}_i, \quad i = 1, 2, \dots, m, \\ \lambda_i \geq 0, & i = 0, 1, \dots, m, \\ x \in \mathcal{X}. \end{cases} \end{aligned} \tag{26}$$

Remark 3 If the main FLP problem is symmetric, we will use relation (24) for the membership function of the objective function, otherwise, relation (25) will be used for the membership function.

Remark 4 It can be observed that when the problem (26) is solved, the optimal solutions λ_0^{opt} and λ_i^{opt} (for $i = 1, 2, \dots, m$) show the membership degrees of the objective function and the constraints, respectively. Thus, by solving our suggested crisp problem, we can directly achieve membership functions, while in the famous methods described in Sect. 2, we can not see this important property.

Remark 5 The problem (26) is general and can be included different cases, such as, $A_k x^* < b_k$ for some $k \in \{1, 2, \dots, m\}$ or $b_i \leq A_i x^* \leq b_i + p_i$, for all $i = 1, 2, \dots, m$.

Theorem 2 Suppose that $(\lambda^{opt}, x^{opt}, \lambda_0^{opt}, \dots, \lambda_m^{opt})$ is an optimal solution for the problem (26), where the membership functions satisfy relations (23) and (24). Then (λ^{opt}, x^{opt}) is an optimal solution for the CNLP problem (5).

Proof It is trivial that (λ^{opt}, x^{opt}) is a feasible solution for CNLP problem (5). Suppose that (λ^{opt}, x^{opt}) is not an optimal solution for (5). Then, there is a $(\bar{\lambda}, \bar{x})$ such that

$$\bar{\lambda} \leq \mu_0(c\bar{x}), \bar{\lambda} \leq \mu_i(A_i\bar{x}), i = 1, 2, \dots, m, 0 \leq \bar{\lambda} \leq 1, \bar{x} \geq 0.$$

and

$$\bar{\lambda} > \lambda^{opt}. \tag{27}$$

Now, we define

$$\bar{\lambda}_0 = \begin{cases} \frac{\bar{\lambda} + \mu_0(c\bar{x})}{2}, & \lambda_0^{opt} \leq \bar{\lambda} \leq \mu_0(c\bar{x}), \\ \lambda_0^{opt}, & \bar{\lambda} < \lambda_0^{opt} \leq \mu_0(c\bar{x}), \end{cases} \tag{28}$$

and for $i = 1, \dots, m$:

$$\bar{\lambda}_i = \begin{cases} \frac{\bar{\lambda} + \mu_i(A_i\bar{x})}{2}, & \lambda_i^{opt} \leq \bar{\lambda} \leq \mu_i(A_i\bar{x}), \\ \lambda_i^{opt}, & \bar{\lambda} < \lambda_i^{opt} \leq \mu_i(A_i\bar{x}). \end{cases} \tag{29}$$

It is obvious that $(\bar{\lambda}, \bar{x}, \bar{\lambda}_0, \bar{\lambda}_1, \dots, \bar{\lambda}_m)$ is a feasible solution for (26). Moreover, from relations (28) and (29), we have

$$\bar{\lambda}_i \geq \lambda_i^{opt} \text{ for } i = 0, 1, \dots, m. \tag{30}$$

Thus, from relations (27) and (30), $\bar{\lambda} + \frac{1}{m+1} \sum_{i=0}^m \bar{\lambda}_i > \lambda^{opt} + \frac{1}{m+1} \sum_{i=0}^m \lambda_i^{opt}$ which is a contradiction.

Theorem 3 Suppose that $(\lambda^{opt}, x^{opt}, \lambda_0^{opt}, \dots, \lambda_m^{opt})$ is an optimal solution for the problem (26), where the membership functions satisfy the relations (23) and (25). Then (λ^{opt}, x^{opt}) is an optimal solution for the CNLP problem (11).

Proof The proof is similar to that of Theorem 2.

By Theorems 2 and 3, it can be seen that fuzzy optimal solution of the problem (26) is not worse than the fuzzy optimal solution obtained by Zimmermann approach for FLP problem (1) and Werners approach for FLP problem (7).

In solving an FLP problem, the following two purposes must be considered:

- Finding higher membership degrees for the objective function and the constraints, simultaneously.
- Finding the best value for the main objective function.

Now, we improve the CLP problem (26) and propose a new CLP model for solving a given FLP problem. In this model, we try to meet these requirements. Our suggested crisp linear programming (SCLP) problem for solving FLP problems (1) and (7) is as follows:

$$\begin{aligned}
 & \max \quad \lambda + \frac{1}{m+1} \sum_{i=0}^m \lambda_i + \delta cx \\
 & \text{subject to} \quad \begin{cases} \lambda \leq \lambda_0 \leq \mu_0(cx), & x \in \mathcal{X}_0 \text{ (or } \bar{\mathcal{X}}_0) \\ \lambda \leq \lambda_i \leq \mu_i(A_i x), & x \in \mathcal{X}_i, i = 1, 2, \dots, m, \\ \lambda_i \geq 0, & i = 0, 1, \dots, m, \\ x \in \mathcal{X}, \end{cases} \tag{31}
 \end{aligned}$$

where δ is a sufficiently small positive number (in our suggested algorithm, we will find δ systematically) and the membership functions are determined as problem (26).

We can practically see that the suggested problem (31) gives us a better optimal solution from (6) or (12), since not only we achieve higher membership degrees for constraints and the objective function (see examples in Sect. 5), but also obtain a good value for the objective function.

Now, consider the following multiple objective optimization problem:

$$(MOP) \quad \max_{x \in \mathcal{X}} (\mu_0(x), \mu_1(x), \dots, \mu_m(x), cx). \tag{32}$$

Without loss of generality, we may replace cx with notation $\mu_{m+1}(x)$.

Definition 1 A feasible solution x^* is said to be an efficient solution for the multiple objective optimization problem (32), if there is no other $x \geq 0$ such that $\mu_i(x) \geq \mu_i(x^*)$, for all $i = 0, 1, \dots, m + 1$ and $\mu_j(x) > \mu_j(x^*)$ for some $j \in \{0, 1, \dots, m + 1\}$.

Theorem 4 Suppose that $(\lambda^{opt}, x^{opt}, \lambda_0^{opt}, \dots, \lambda_m^{opt})$ is an optimal solution of the SCLP problem (31). Then, x^{opt} is an efficient solution for the multiple objective optimization problem (32).

Proof Suppose that x^{opt} is not an efficient solution. Then, there exists a feasible solution \bar{x} such that $\mu_i(\bar{x}) \geq \mu_i(x^{opt}), i = 0, 1, \dots, m + 1$ and $\mu_j(\bar{x}) > \mu_j(x^{opt})$ for some $j \in \{0, 1, \dots, m + 1\}$. Now, by assuming $\bar{\lambda}_i = \mu_i(\bar{x}), i = 0, 1, \dots, m$ and $\bar{\lambda} = \min_{i=0,1,\dots,m} \bar{\lambda}_i$, it is obvious that $(\bar{\lambda}, \bar{x}, \bar{\lambda}_0, \dots, \bar{\lambda}_m)$ is feasible for (31). Therefore, we have

$$\lambda^{opt} + \frac{1}{m+1} \sum_{i=0}^m \lambda_i^{opt} \leq \lambda^{opt} + \frac{1}{m+1} \sum_{i=0}^m \mu_i(x^{opt}) < \lambda^{opt} + \frac{1}{m+1} \sum_{i=0}^m \mu_i(\bar{x}). \tag{33}$$

On the other hand, we have:

$$\mu_i(\bar{x}) \geq \mu_i(x^{opt}) \geq \lambda^{opt}, \forall i = 0, 1, \dots, m,$$

and

$$\bar{\lambda} = \min_{i=0,1,\dots,m} \mu_i(\bar{x}) \geq \lambda^{opt}. \tag{34}$$

Therefore, by relations (33) and (34) we have:

$$\lambda^{opt} + \frac{1}{m + 1} \sum_{i=0}^m \lambda_i^{opt} + \delta cx^{opt} < \bar{\lambda} + \frac{1}{m + 1} \sum_{i=0}^m \mu_i(\bar{x}) + \delta c\bar{x},$$

which is in contradiction to optimality of $(\lambda^{opt}, x^{opt}, \lambda_0^{opt}, \dots, \lambda_m^{opt})$. Therefore, x^{opt} is an efficient solution for multiple objective optimization problem (32).

Note that if the CNLP problem (11) corresponding to the FLP problem (7) has only one optimal solution then it is an efficient solution for the following problem:

$$\min_{x \in \mathcal{X}} (\mu_0(x), \mu_1(x), \dots, \mu_m(x)). \tag{35}$$

However, if the CNLP problem (11) corresponding to the FLP problem (7) has multiple optimal solutions, Werners [20] showed that the optimal solutions of the problem (11) are weakly efficient, and at least one of them is efficient.

Theorem 4 shows that, the optimal solutions resulted by this approach are always efficient [for multiple objective problems (32) and (35)] and we can see practically that each membership degree is larger than or equal to the membership degree obtained by the max-min operator. Thus, by our approach we obtain high membership degrees for the objective function and constraints, and provide a better usage of existing resources. Also, we find a good value for the main objective function.

It is important to note that, unlike the methods proposed by Guu and Wu [8], Jimenez and Bilbao [9] and Wu et al. [22], our method finds efficient solutions by solving only one crisp linear problem instead of solving two or three crisp problems. Also, unlike the two phase method, we not only try to achieve high membership degrees, but also find a good value for the main objective function.

Now, for symmetric FLP problem (1), by considering the membership functions given in relations (23) and (24), the suggested CLP problem (31) can be converted to:

$$\begin{aligned} \max \quad & \lambda + \frac{1}{m + 1} \sum_{i=0}^m \lambda_i + \delta cx \\ \text{subject to} \quad & \begin{cases} \lambda \leq \lambda_0, \\ \lambda_0 \leq 1 - \frac{b_0 - cx}{p_0}, \\ \lambda \leq \lambda_i, & i = 1, 2, \dots, m, \\ \lambda_i \leq 1 - \frac{A_i x - b_i}{p_i}, & i = 1, 2, \dots, m, \\ 0 \leq \lambda_i \leq 1, & i = 0, 1, \dots, m, \\ x \in \mathcal{X}_i, & i = 0, 1, \dots, m, \\ x \in \mathcal{X}, \end{cases} \end{aligned} \tag{36}$$

where $\mathcal{X}_0 = \{x \in \mathcal{X} : cx \geq b_0 - p_0\}$ and $\mathcal{X}_i = \{x \geq 0 : A_i x \leq b_i + p_i\}$, $i = 1, 2, \dots, m$.

Also, for non-symmetric FLP problem (7), by considering the membership functions given in relations (23) and (25), the suggested problem (31) is as follows:

$$\max \quad \lambda + \frac{1}{m + 1} \sum_{i=0}^m \lambda_i + \delta cx$$

$$\text{subject to } \begin{cases} \lambda \leq \lambda_0 \\ \lambda_0 \leq 1 - \frac{z_1 - cx}{z_1 - z_0}, \\ \lambda \leq \lambda_i, & i = 1, 2, \dots, m, \\ \lambda_i \leq 1 - \frac{A_i x - b_i}{p_i}, & i = 1, 2, \dots, m, \\ 0 \leq \lambda_i \leq 1, & i = 0, 1, \dots, m, \\ x \in \mathcal{X}_i, & i = 0, 1, \dots, m, \\ x \in \mathcal{X}, \end{cases} \quad (37)$$

where $\mathcal{X}_0 = \{x \in \mathcal{X} : cx \geq z_0\}$ and $\mathcal{X}_i = \{x \geq 0 : A_i x \leq b_i + p_i\}$, $i = 1, 2, \dots, m$.

The suggested algorithm

Step 1 Determine the FLP problem is symmetric or non-symmetric.

Step 2 2.1 If the FLP problem is symmetric, then take b_0 and $p_i > 0, i = 0, 1, \dots, m$ from decision maker. Also, determine z_0 and z_1 by solving LPs (8) and (9), respectively.

2.2 If the FLP problem is non-symmetric, then determine z_0 and z_1 and take $p_i > 0, i = 1, \dots, m$, from decision maker.

Step 3 For an FLP problem (symmetric or non-symmetric):

3.1 If LP (9) is infeasible, then *STOP*. The main FLP problem is infeasible.

3.2 If $z_0 = z_1$, then *STOP*. Optimal value of the main FLP problem is z_0 and the membership degree of all constraints is equal to one.

3.3 If $z_1 = \infty$, then *STOP*. The main FLP problem is unbounded.

Step 4 Let

$$\delta = \begin{cases} \frac{1}{z_1}, & \text{if } 0 \leq z_0 < z_1, \\ \frac{1}{z_0}, & \text{if } z_0 \leq z_1 \leq 0, \\ 0, & \text{if } z_0 < 0 < z_1. \end{cases}$$

Note that, in this step, we select δ such that $0 \leq \delta cx \leq 1$.

Step 5 For a symmetric FLP, solve the problem (36). Let $(\lambda^{opt}, x^{opt}, \lambda_0^{opt}, \lambda_1^{opt}, \dots, \lambda_m^{opt})$ be the optimal solution of problem (36). Then, x^{opt} is the optimal solution of the main FLP problem and $\lambda_i^{opt}, i = 0, 1, \dots, m$ are the membership degrees of the objective function and the constraints, respectively.

Step 6 For a non-symmetric FLP, solve the problem (37). Let $(\lambda^{opt}, x^{opt}, \lambda_0^{opt}, \lambda_1^{opt}, \dots, \lambda_m^{opt})$ be the optimal solution of problem (37). Then, x^{opt} is the optimal solution of the main FLP problem and $\lambda_i^{opt}, i = 0, 1, \dots, m$ are the membership degrees of the objective function and the constraints, respectively.

5 Numerical Examples

In this section, we will solve some symmetric and non-symmetric FLP problems by our algorithm and compare the results with those of belonging to Zimmermann [24], Werners [20], Guu and Wu [8], Safi et al. [14], Jimenez and Bilbao [9] and Wu et al. [22] approaches. We present the optimal value of the objective function and the membership degrees of the objective function and constraints. The crisp LP problems are solved by using *MATLAB* software.

Table 1 The results of solving Example 5 by using ZM and SA

	x_1^*	x_2^*	$z^* = cx^*$	μ_0	μ_1	μ_2	μ_3	$\sum_{i=0}^3 \mu_i$
ZM	1.2735	3.4823	8.2391	1	1	1	1	4
SA	1	4	9	1	1	1	1	4

Table 2 The results of solving Example 6 by using ZM and suggested algorithm

	x_1^*	x_2^*	$z^* = cx^*$	μ_0	μ_1	μ_2	$\sum_{i=0}^2 \mu_i$
ZM	83.5075	10.1808	93.6883	1	1	1	3
SA	∞	∞	∞	1	1	1	3

In what follows, we use the notations ZM, WM, GWM, Safi, JBM, WEAM, and SA instead of Zimmermann method, Werners method, Guu and Wu method, Safi et al. method, Jimenez and Bilabo method, Wu et al. method and suggested algorithm, respectively.

Example 5 Consider the following symmetric FLP problem:

$$\begin{aligned} & \text{m}\tilde{\text{a}}\text{x} \quad z = x_1 + 2x_2 \\ & \text{subject to} \quad \begin{cases} -x_1 + x_2 \lesssim 3, \\ x_1 + x_2 \lesssim 5, \\ x_1 \lesssim 4, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Let $b_0 = 7, p_0 = 4, p_1 = 3, p_2 = 1,$ and $p_3 = 2.$ Table 1 shows the results of solving this FLP problem by ZM and SA. As it can be seen in Table 1, although the membership degrees of the objective function and the constraints are similar, optimal objective value of the suggested algorithm is better than that of Zimmermann method.

Example 6 Consider a symmetric FLP problem as follows:

$$\begin{aligned} & \text{m}\tilde{\text{a}}\text{x} \quad x_1 + x_2 \\ & \text{subject to} \quad \begin{cases} x_1 + 3x_2 \lesssim 3, \\ 3x_1 + x_2 \lesssim 3, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Let $b_0 = 10, p_0 = 5,$ and $p_1 = p_2 = 1.$ The results of solving this problem by using ZM and SA are as illustrated in Table 2. As it can be seen in this table, this problem is unbounded. Nevertheless, Zimmermann approach can not recognize this subject.

Example 7 Consider the following symmetric FLP problem:

$$\text{m}\tilde{\text{a}}\text{x} \quad z = 2x_1 + 2.2x_2$$

Table 3 The results of solving Example 7 by using ZM, Safi and SA

	x_1^*	x_2^*	cx^*	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	$\sum \mu_i$
ZM	2.0676	0.1824	4.5365	1	1	0.625	1	1	0.625	0.9618	0.8176	1	0.8176	8.847
Safi	1.875	0.375	4.575	1	1	0.625	1	1	0.625	0.833	0.625	1	0.625	8.333
SA	2.25	0	4.5	1	1	0.625	1	1	0.625	1	1	1	1	9.25

$$\text{subject to } \begin{cases} 2x_1 + x_2 \leq 8, \\ x_1 + x_2 \geq 3, \\ -x_1 + x_2 \leq 2, \\ -x_1 + 10x_2 \leq 10, \\ 3x_1 + 3x_2 \leq 6, \\ x_1 - x_2 \geq 2, \\ 2x_1 + 3x_2 \leq 4.5, \\ 3x_1 - x_2 \leq 8, \\ 4x_1 + 7x_2 \leq 9, \\ x_1, x_2 \geq 0. \end{cases}$$

Let $b_0 = 4, p_0 = 1, p_1 = p_2 = p_4 = p_5 = 2, p_3 = p_6 = p_9 = 3$ and $p_7 = p_8 = 1$. Table 3 shows the results of solving this problem by ZM, Safi and SA. The results show that, by our algorithm we achieve higher membership degrees, although the obtained value for the objective function is less than those obtained by Zimmermann and Safi et al. approaches, partially.

Example 8 Consider the following symmetric FLP problem:

$$\begin{aligned} &\text{max } x_1 + x_2 \\ &\text{subject to } \begin{cases} -x_1 + x_2 \leq 2, \\ 6x_1 - 2x_2 \leq 30, \\ -x_1 + 10x_2 \geq 10, \\ x_1 + 3x_2 \leq 15, \\ 3x_1 - x_2 \geq 17, \\ 4x_1 + 3x_2 \leq 34, \\ x_1 + 7x_2 \leq 27, \\ x_1 + 2x_2 \geq 12, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Let $b_0 = 3, p_0 = p_1 = p_2 = p_3 = 1, p_4 = p_5 = 2$ and $p_6 = p_7 = p_8 = 1$. Table 4 shows the results after solving this problem by ZM, Safi and SA. The results illustrate that by the suggested algorithm we can achieve a better objective value and higher membership degrees compared with the results of Zimmermann method. In comparison with Safi et al. approach, the membership degrees are higher although the objective value is partially different from Safi et al. algorithm.

Example 9 Consider the non-symmetric FLP problem given in Example 3. In Example 3 we showed that phase 1 of the two phase method, by Guu and Wu [8] and Jimenez and Bilbao

Table 4 The results of solving Example 8 by using ZM, Safi and SA

	x_1^*	x_2^*	cx^*	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	$\sum \mu_i$
ZM	6.0279	2.6839	8.7117	1	1	0.2	1	1	0.2	1	1	0.3955	6.7955
Safi	6.1636	3.0909	9.2545	1	1	0.2	1	0.7819	0.2	1	0.2	1	6.382
SA	6.12	2.96	9.08	1	1	0.2	1	1	0.2	1	1	1	7.4

Table 5 The results of solving Example 10 by using WM, GWM, JBM, WEAM and SA

	x_1^*	x_2^*	x_3^*	x_4^*	cx^*	μ_0	μ_1	μ_2	μ_3	$\sum_{i=0}^3 \mu_i$
WM	4.1892	5.4778	7.833	0	114.6429	0.5	0.5	0.9947	0.5	2.4947
GWM	4.0476	5.6548	7.7976	0	114.6429	0.5	0.5	1	0.5	2.5
JBM	0	10.7142	6.7857	0	114.6429	0.5	0.5	1	0.5	2.5
WEAM	0	10.7142	6.7857	0	114.6429	0.5	0.5	1	0.5	2.5
SA	1.818	8.4417	7.2402	0	114.6429	0.5	0.5	1	0.5	2.5

[9], was infeasible. Therefore, we could not find optimal value of FLP problem (22) by the aforementioned methods. However, we solve the FLP problem (22) by using the suggested algorithm and obtain $(x_1^*, x_2^*) = (0.5, 2)$ and $cx^* = 3$. Also, we have $\mu_0 = \mu_1 = 0.5$, and $\mu_2 = \mu_3 = 1$.

Example 10 Consider the following non-symmetric FLP problem:

$$\begin{aligned} & \max \quad 4x_1 + 5x_2 + 9x_3 + 11x_4 \\ & \text{subject to} \quad \begin{cases} x_1 + x_2 + x_3 + x_4 \lesssim 15, \\ 7x_1 + 5x_2 + 3x_3 + 2x_4 \lesssim 80, \\ 3x_1 + 4.4x_2 + 10x_3 + 15x_4 \lesssim 100, \\ x_1, x_2, x_3 \geq 0. \end{cases} \end{aligned}$$

Let $p_1 = 5, p_2 = 40$ and $p_3 = 30$. By solving LPs (8) and (9), we have $z_0 = 99.2857$ and $z_1 = 130$. Table 5 shows the results of solving this FLP problem by using WM, GWM, WEAM, JBM and SA. The results show that GWM, SA, JBM and WEAM give the same objective value and the same membership degrees. However, in the next example we show that sometimes GWM, JBM and WEAM can not solve an FLP problem, correctly.

Example 11 Consider the non-symmetric FLP problem, solved in Example 4 by using GWM, JBM and WEAM. Now, we solve this problem by the suggested algorithm. The corresponding CLP problem (37), is as follows:

$$\max \quad \lambda + \frac{1}{4}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + \frac{1}{8}(0.1x_1 + 3x_2)$$

Table 6 The results of solving Example 11 by using GWM, JBM, WEAM and SA

	x_1^*	x_2^*	$z^* = cx^*$	μ_0	μ_1	μ_2	μ_3	$\sum_{i=0}^3 \mu_i$
GWM	2.0408	1.9592	6.0816	0.0408	1	0.0408	0.9592	2.5408
JBM	2.0408	1.9592	6.0816	0.0408	1	0.0408	0.9592	2.5408
WEAM	0	0.4267	1.2801	0	2.1919	3.36	3	
SA	0	2.333	7	0.5	1	0.5	1	3

$$\text{subject to } \left\{ \begin{array}{l}
 \lambda \leq \lambda_i, \quad i = 0, 1, 2, 3, \\
 \lambda_0 \leq \frac{0.1x_1 + 3x_2 - 6}{2}, \\
 \lambda_1 \leq 1 - \frac{x_1 + x_2 - 4}{3}, \\
 \lambda_2 \leq 1 - \frac{x_1 + 3x_2 - 6}{2}, \\
 \lambda_3 \leq 3 - x_1, \\
 x_1 + x_2 \leq 7, \\
 x_1 + 3x_2 \leq 8, \\
 x_1 \leq 3, \\
 0 \leq \lambda_i \leq 1, \quad i = 0, 1, 2, 3, \\
 x_1, x_2 \geq 0.
 \end{array} \right.$$

Table 6 shows the results of solving the main FLP problem by using GWM, JBM, WEAM and SA. This table shows that Guu and Wu, Jimenez and Bilbao and Wu et al. approaches can not find the optimal solution, correctly. Moreover, using Wu et al. [22] method, we find membership degrees that are not necessarily between 0 and 1.

6 Conclusions

In this paper, having reviewed some well-known approaches for solving FLP problems, we showed some of their difficulties by numerical examples. We also showed that some of these methods can not solve all the given FLP problems, correctly. Thereafter, we considered a new CLP problem and showed that its optimal solution is also an optimal solution for Zimmermann and Werners approaches. Also, we suggested another new CLP problem and proved that its optimal solutions are efficient. In the proposed model we not only tried to find higher membership degrees, but also attempted to find a good optimal value for the main objective function. Furthermore, we proposed a new algorithm for solving FLP problems and obtained better results than those related to Zimmermann, Werners, Guu and Wu, Safi et al., Jimenez and Bilbao and Wu et al. approaches.

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