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Piecewise Pseudo Almost Periodic Solution for Impulsive Generalised High-Order Hopfield Neural Networks with Leakage Delays

Chaouki Aouiti[1] · **Mohammed Salah M'hamdi**[1] · **Jinde Cao**[2,3] · **Ahmed Alsaedi**[4]

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Abstract Existence of piecewise differentiable pseudo almost-periodic solutions for a class of impulsive high-order Hopfield neural networks with leakage delays are established by employing the fixed point theorem, differential inequality and Lyapunov functionals. The results of this paper are new and they supplement previously known works. Numerical example with graphical illustration is given to illuminate our main results.

Keywords Impulsive high-order Hopfield neural networks · Leakage delays · Piecewise pseudo almost-periodic function · Mixed delays

Mathematics Subject Classification 34C27 · 37B25 · 92C20

✉ Jinde Cao
jdcao@seu.edu.cn

Chaouki Aouiti
chaouki.aouiti@fsb.rnu.tn

Mohammed Salah M'hamdi
mohammedsalah.mhamdi@yahoo.com

Ahmed Alsaedi
aalsaedi@hotmail.com

[1] Department of Mathematics, Faculty of Sciences of Bizerta, University of Carthage, BP W, 7021 Zarzouna, Bizerta, Tunisia

[2] Department of Mathematics, Southeast University, Nanjing 210096, China

[3] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

[4] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia

1 Introduction

In recent years, high-order Hopfield recurrent neural networks have attracted the attention of many scientists : mathematicians, computer scientists, physicists and other. This is due to the fact that high-order neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks [1–10].

Time delays, which is inevitably encountered in the neural networks, is often one of the main sources to cause poor performance, make the dynamic behaviors become more complex, may destabilize the stable equilibria and admit oscillations, bifurcation and chaos [7,14–20]. Therefore, it is of prime importance to consider the delay effects on the stability of neural networks. Recently, Xiao and Meng [6] studied the high-order Hopfield neural networks (HHNNs) with time-varying delays described by

$$x_i'(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(x_j(t-\sigma_{ij}(t)))g_l(x_l(t-\nu_{ij}(t)))$$

In addition, the time delay in the negative feedback terms which is known as leakage have a tendency to destabilize the system [21–24] and have great impact on the dynamical behavior of neural networks. This is to say, it is necessary to consider the effect of leakage delays when studying the stability of state estimation of neural networks.

Moreover, it is known that the existence and the stability of the almost periodic solution play a key role in characterizing the behavior of dynamical system [25–32]. Thus, it is worth while to continue to investigate the existence and stability of almost periodic solution to high-order neutral networks.

The dynamics of evolving processes is usually subjected to suddenly changes such as shocks, harvesting, and natural disasters [11–13]. Often these short-term perturbations are treated as having acted instantaneously or in the form of impulses. High-order recurrent neural networks are often subject to impulsive perturbations that in turn affect the dynamical behaviors. Xu et al. [8] studied the following impulsive high-order Hopfield type neural networks with delays as follows

$$\begin{cases} c_i x_i'(t) = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij}g_j(x_j(t-\tau_j)) \\ \qquad + \sum_{j=1}^{n}\sum_{l=1}^{n} T_{ijl}g_j(x_j(t-\tau_j))g_l(x_l(t-\tau_l)) \\ \qquad + I_i, \quad t \in \mathbb{R},\ t \neq t_k,\ k \in \mathbb{Z} \\ \Delta x_i(t_k) = d_i x_i(t^-) + \sum_{j=1}^{n} W_{ij}h_j(x_j(t^- - \tau_j)) \\ \qquad + \sum_{j=1}^{n}\sum_{l=1}^{n} W_{ijl}h_j(x_j(t^- - \tau_j))g_l(x_l(t^- - \tau_l)) \end{cases}$$

There have been extensive results on the existence and stability of equilibrium points, periodic solutions, almost periodic solutions and anti-periodic solutions for high-order neural networks. However, to the best of our knowledge, there are no published papers considering the piecewise differentiable pseudo almost periodic solutions for impulsive high-order Hopfield neural networks with time-varying delays in the leakage terms.

In this paper, we discuss piecewise differentiable pseudo almost periodic solutions for impulsive high-order Hopfield neural networks (IHOHNNs) with time-varying delays in the leakage terms

$$\begin{cases} x_i'(t) = -c_i(t)x_i(t-\rho(t)) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ \quad + \sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} d_{ij}(u)g_j(x_j(t-u))du \\ \quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(t)g_j(x_j(t-\sigma_{ij}(t)))g_l(x_l(t-\nu_{ij}(t))) \\ \quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \beta_{ijl}(t)\int_0^{\infty} h_{ijl}(u)g_j(x_j(t-u))du \int_0^{\infty} k_{ijl}(u)g_l(x_l(t-u))du \\ \quad + J_i(t), \quad t \in \mathbb{R},\ t \neq t_k,\ k \in \mathbb{Z} \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_k(x_i(t_k)) \end{cases} \tag{1}$$

in which n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the i^{th} unit at the time t, $c_i(t) > 0$ represents the rate with which the i^{th} unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at the time t, $a_{ij}(.)$, $b_{ij}(.)$ and $\alpha_{ijl}(.)$, $\beta_{ijl}(.)$ are respectively the first-order connection weights and the second-order connection weights of the neural network, $0 \le \tau_{ij}(t), \sigma_{ij}(t), \nu_{ij}(t) \le \tau$, $0 \le \rho(t) \le \rho^+$ correspond to the transmission delays, $d_{ij}(.)$, $h_{ijl}(.)$ and $k_{ijl}(.)$ correspond to the transmission delay kernels, $J_i(t)$ denote the external inputs at time t, and g_j is the activation function of signal transmission. The sequence $\{t_k\}$ has no finite accumulation point and $I_k : \mathbb{R}^n \longrightarrow \mathbb{R}$, $k \in \mathbb{Z}$.

The main aim of this article is to establish some sufficient conditions for the existence, uniqueness and exponential stability of piecewise differentiable pseudo almost periodic solutions of Eq. (1).

Throughout this paper, for $i, j, l = 1, 2, \dots, n$, it will be assumed that $a_{ij}, b_{ij}, \alpha_{ijl}, \beta_{ijl}, J_i : \mathbb{R} \longrightarrow \mathbb{R}$ are pseudo almost periodic functions, and let the positive constant $\overline{a}_{ij}, \overline{b}_{ij}, \overline{\alpha}_{ijl}, \overline{\beta}_{ijl}$ and $\overline{J}_i$ such that

$$\overline{a}_{ij} = \sup_{t\in\mathbb{R}} |a_{ij}(t)|, \overline{b}_{ij} = \sup_{t\in\mathbb{R}} |b_{ij}(t)|$$
$$\overline{\alpha}_{ijl} = \sup_{t\in\mathbb{R}} |\alpha_{ijl}(t)|, \overline{\beta}_{ijl} = \sup_{t\in\mathbb{R}} |\beta_{ijl}(t)|, \ \ \overline{J}_i = \sup_{t\in\mathbb{R}} |J_i(t)|.$$

We also assume that the following conditions (H1)–(H6) hold.

(H1) For each $j = \{1, 2, \dots, n\}$, there exist nonnegative constants L_j^g and M_j^g such that

$$g_j(0) = 0, \quad |g_j(u) - g_j(v)| \le L_j^g |u - v|, \ \ and\ |g_j(u)| \le M_j^g, \quad \text{for all } u, v \in \mathbb{R}.$$

(H2) For $i, j, l \in \{1, 2, \dots, n\}$, the delay kernels, $d_{ij}, h_{ijl}, k_{ijl} : [0, \infty) \longrightarrow \mathbb{R}$ are continuous, and there exist nonnegative constants $d_{ij}^+, h_{ijl}^+, k_{ijl}^+, \eta_d, \eta_h, \eta_k$ such that

$$|d_{ij}(u)| \le d_{ij}^+ e^{-\eta_d u}, \ |h_{ijl}(u)| \le h_{ijl}^+ e^{-\eta_h u}, \ |k_{ijl}(u)| \le k_{ijl}^+ e^{-\eta_k u}.$$

(H3) For all $1 \le i \le n$ the functions $t \mapsto c_i(t)$ are almost periodic and

$$0 < c_{i*} = \inf_{t\in\mathbb{R}} \big(c_i(t)\big), \ \ c_i^+ = \sup_{t\in\mathbb{R}} \big(c_i(t)\big).$$

(H4) $I_k \in PAP(\mathbb{Z}, \mathbb{R}^n)$ and there exists a constant L_1 such that $0 < L_1 < 1$ and

$$\| I_k(u) - I_k(v) \| \le L_1 \| u - v \|, \ \ u, v \in \mathbb{R}^n, \ k \in \mathbb{Z}.$$

(H5) Assume that there exist nonnegative constants L, p and q such that

$$\max_{1\le i\le n}\max\left\{\frac{\overline{J}_i}{c_{i*}},\left(1+\frac{c_i^+}{c_{i*}}\right)\overline{J}_i\right\}=L$$

$$p=\max_{1\le i\le n}\max\left\{\left(c_{i*}^{-1}\left[c_i^+\rho^+ +\sum_{j=1}^n\overline{a}_{ij}L_j^g+\sum_{j=1}^n\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g+\sum_{j=1}^n\sum_{l=1}^n\overline{\alpha}_{ijl}L_j^gM_l^g\right.\right.\right.$$
$$+\sum_{j=1}^n\sum_{l=1}^n\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^gM_l^g\Big]+\frac{L_1}{1-e^{-c_{i*}}}\Big),\left(\left(1+\frac{c_i^+}{c_{i*}}\right)\Big[c_i^+\rho^+\right.$$
$$+\sum_{j=1}^n\overline{a}_{ij}L_j^g+\sum_{j=1}^n\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g+\sum_{j=1}^n\sum_{l=1}^n\overline{\alpha}_{ijl}L_j^gM_l^g$$
$$\left.\left.+\sum_{j=1}^n\sum_{l=1}^n\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^gM_l^g\Big]+\frac{c_i^+L_1}{1-e^{-c_{i*}}}\right)\right\}<1,$$

$$q=\max_{1\le i\le n}\max\left\{\left(c_{i*}^{-1}\left[c_i^+\rho^+ +\sum_{j=1}^n\overline{a}_{ij}L_j^g+\sum_{j=1}^n\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g\right.\right.\right.$$
$$+\sum_{j=1}^n\sum_{l=1}^n\overline{\alpha}_{ijl}\left(L_j^gM_l^g+M_j^gL_l^g\right)$$
$$+\sum_{j=1}^n\sum_{l=1}^n\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}\left(L_j^gM_l^g+M_j^gL_l^g\right)\Bigg]+\frac{L_1}{1-e^{-c_{i*}}}\Bigg),\left(\left(1+\frac{c_i^+}{c_{i*}}\right)\Big[c_i^+\rho^+\right.$$
$$+\sum_{j=1}^n\overline{a}_{ij}L_j^g+\sum_{j=1}^n\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g$$
$$+\sum_{j=1}^n\sum_{l=1}^n\overline{\alpha}_{ijl}\left(L_j^gM_l^g+M_j^gL_l^g\right)$$
$$\left.\left.+\sum_{j=1}^n\sum_{l=1}^n\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}\left(L_j^gM_l^g+M_j^gL_l^g\right)\Bigg]+\frac{c_i^+L_1}{1-e^{-c_{i*}}}\right)\right\}<1.$$

(H6) There exist positive constants p_i and q_i, such that: for $t\in[0,\infty)$, $i=1,2,\ldots,n$

$$p_ic_i(t)-q_ic_i^+\rho^+-\sum_{j=1}^np_j|a_{ij}(t)|L_j^g-\sum_{j=1}^np_j|b_{ij}(t)|\frac{d_{ij}}{\eta_d}L_j^g$$
$$-\sum_{j=1}^n\sum_{l=1}^n|\alpha_{ijl}(t)|[p_jL_j^gM_l^g+p_lM_j^gL_l^g]$$
$$-\sum_{j=1}^n\sum_{l=1}^n|\beta_{ijl}(t)|\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}[p_jL_j^gM_l^g+p_lM_j^gL_l^g]>0,$$

$$q_i - p_i c_i(t) - q_i c_i^+ \rho^+ - \sum_{j=1}^{n} p_j |a_{ij}(t)| L_j^g - \sum_{j=1}^{n} p_j |b_{ij}(t)| \frac{d_{ij}}{\eta_d} L_j^g$$
$$- \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_{ijl}(t)| [p_j L_j^g M_l^g + p_l M_j^g L_l^g]$$
$$- \sum_{j=1}^{n} \sum_{l=1}^{n} |\beta_{ijl}(t)| \frac{h_{ijl}^+}{\eta_h} \frac{k_{ijl}^+}{\eta_k} [p_j L_j^g M_l^g + p_l M_j^g L_l^g] > 0,$$

Throughout this paper, we will first recall some basic definitions and lemmas which are used in what follows.

- $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$ stand for the set of natural numbers, integer numbers and real numbers, respectively.
- $C(\mathbb{R}, \mathbb{R}^n)$: the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$.
- $BC(\mathbb{R}, \mathbb{R}^n)$: the set of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. Note that $(BC(\mathbb{R}, \mathbb{R}^n), \| \,.\, \|_\infty)$ is a Banach space where $\| \,.\, \|_\infty$ denotes the sup norm

$$\| f \|_\infty := \max_{1 \le i \le n} \sup_{t \in \mathbb{R}} |f_i(t)|.$$

- Let T be the set consisting of all real sequences $\{t_i\}_{i \in \mathbb{Z}}$ such that $\alpha = \inf_{i \in \mathbb{Z}} (t_{i+1} - t_i) > 0$. It is immediate that this condition implies that $\lim_{i \to +\infty} t_i = +\infty$ and $\lim_{i \to -\infty} t_i = -\infty$.
- $PC(\mathbb{R}, \mathbb{R}^n)$: the space of all piecewise continuous functions $f : \mathbb{R} \to \mathbb{R}^n$ such that
  - $f(.)$ is continuous at t for any $t \notin \{t_i, i \in \mathbb{Z}\}$,
  - $f(t_i^+)$, $f(t_i^-)$ exists and $f(t_i^-) = f(t_i)$ for all $i \in \mathbb{Z}$.
- $PC([-\tau, 0], \mathbb{R}^n) = \big\{ f : [-\tau, 0] \to \mathbb{R}^n | f(t^-) = f(t),$ for $t \in [-\tau, 0]$, $f(t^+)$ exists on $\mathbb{R}$ and $f(t^+) = f(t)$ for all but at most a finite number of points on $[-\tau, 0].\big\}$,
- $PC^1([-\tau, 0], \mathbb{R}^n) = \big\{ f : [-\tau, 0] \to \mathbb{R}^n | f'(t^+)$ and $f'(t^-))$ exist , $f'(t) = f'(t^-)$ for $t \in [-\tau, 0]$, $f'(t^+) = f'(t)$ for all but at most a finite number of pointson $[-\tau, 0].\big\}$,
- $l^\infty(\mathbb{Z}, \mathbb{R}^n) = \big\{ x : \mathbb{Z} \to \mathbb{R}^n : \| x \| = \sup_{n \in \mathbb{Z}} \| x(n) \| < \infty \big\}$.

Definition 1 [32]. A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $L(\varepsilon) > 0$ such that every interval of length $L(\varepsilon) > 0$ contains a number τ with the property that $\| f(t + \tau) - f(t) \|_\infty < \varepsilon$, for each $t \in \mathbb{R}$.

The number τ above is called an ε-translation number of f, and the collection of all such functions will be denoted as $AP(\mathbb{R}, \mathbb{R}^n)$.

Definition 2 [32]. A sequence $\{x_n\}$ is called almost periodic if for any $\epsilon > 0$, there exists a relatively dense set of its ϵ-periods, i.e., there exists a natural number $l = l(\epsilon)$, such that for $k \in \mathbb{Z}$, there is at least one number p in $[k, k + l]$, for which inequality $\| x_{n+p} - x_n \| < \epsilon$ holds for all $n \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, \mathbb{R}^n)$, the set of such sequences.

Define

$$PAP_0(\mathbb{Z}, \mathbb{R}^n) := \{ x \in l^\infty(\mathbb{Z}, \mathbb{R}^n) : \lim_{n \longrightarrow \infty} \frac{1}{2n} \sum_{k=-n}^{n} \| x(k) \| = 0 \}.$$

Remark 1 Notice that

1. A sequence vanishing at infinity is a $PAP_0(\mathbb{Z}, \mathbb{R})$ sequence.
2. The sequence $(x(n))_{n\in\mathbb{Z}}$ defined by

$$x(n) = \begin{cases} 1, \ n = 2^k, \\ 0, \ n \neq 2^k, \end{cases}$$

is an example of a $PAP_0(\mathbb{Z}, \mathbb{R})$ sequence which not vanishing at infinity.
3. For $k \in \mathbb{N}$, the sequence $(x(n))_{n\in\mathbb{Z}}$ defined by

$$x(n) = \begin{cases} k, \ n = 2^{k^2}, \\ 0, \ n \neq 2^{k^2}, \end{cases}$$

is an example of an unbounded $PAP_0(\mathbb{Z}, \mathbb{R})$ sequence.

Definition 3 [32]. A sequence $\{x_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, \mathbb{R}^n)$ is called pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, \mathbb{R}^n)$, $x_n^2 \in PAP_0(\mathbb{Z}, \mathbb{R}^n)$. Denote by $PAP(\mathbb{Z}, \mathbb{R}^n)$ the set of such sequences.

For $\{t_i\}_{i\in\mathbb{Z}} \in T$, $\{t_i^j\}$ defined by

$$\{t_i^j = t_{i+j} - t_i\}, \ i, j \in \mathbb{Z}.$$

It is easy to verify that the numbers t_i^j satisfy

$$t_{i+k}^j - t_i^j = t_{i+j}^k - t_i^k, t_i^j - t_i^k = t_{i+k}^{j-k}, \text{ for } i, j, k \in \mathbb{Z}.$$

Definition 4 [34]. A function $f \in PC(\mathbb{R}, \mathbb{R}^n)$ is said to be piecewise almost periodic if the following conditions are fulfilled:

1. $\{t_i^j = t_{i+j} - t_i\}$, $i, j \in \mathbb{Z}$ are equipotentially almost periodic, that is, for any $\epsilon > 0$, there exists a relatively dense set in $\mathbb{R}$ of ϵ-almost periods common for all of the sequences $\{t_i^j\}$.
2. For any $\epsilon > 0$, there exists a positive number $\delta = \delta(\epsilon)$ such that if the points t' and t'' belong to the same interval of continuity of f and $|t' - t''| < \delta$, then $\| f(t') - f(t'') \| < \epsilon$.
3. For any $\epsilon > 0$, there exists a relatively dense set Ω_ϵ in $\mathbb{R}$ such that if $\tau \in \Omega_\epsilon$, then

$$\| f(t + \tau) - f(t) \| < \epsilon$$

for all $t \in \mathbb{R}$ which satisfy the condition $|t - t_i| > \epsilon, \ i \in \mathbb{Z}$.

We denote by $AP_T(\mathbb{R}, \mathbb{R}^n)$ the space of all piecewise almost periodic functions. Obviously, $AP_T(\mathbb{R}, \mathbb{R}^n)$ endowed with the supremum norm is a Banach space. Throughout the rest of this paper, we always assume that $\{t_i^j\}$ are equipotentially almost periodic. Let $UPC(\mathbb{R}, \mathbb{R}^n)$ be the space of all functions $f \in PC(\mathbb{R}, \mathbb{R}^n)$ such that f satisfies the condition (2) in Definition 4.

Define

$$PAP_T^0(\mathbb{R}, \mathbb{R}^n) = \{f \in PC(\mathbb{R}, \mathbb{R}^n), \lim_{r\longrightarrow\infty} \frac{1}{2r} \int_{-r}^{r} \| f(t) \| \, dt = 0\}.$$

Definition 5 [32]. A function $f \in PC(\mathbb{R}, \mathbb{R}^n)$ is said to be piecewise pseudo almost periodic if it can be decomposed $f = g + h$, where $g \in AP_T(\mathbb{R}, \mathbb{R}^n)$ and $h \in PAP_T^0(\mathbb{R}, \mathbb{R}^n)$. Denote by $PAP_T(\mathbb{R}, \mathbb{R}^n)$ the set of all such functions. $PAP_T(\mathbb{R}, \mathbb{R}^n)$ is a Banach space when endowed with the supremum norm.

Remark 2 The functions g and h in Definition 5 are respectively called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f. The decomposition given in Definition 5 is unique. For instance, the function

$$f(t) = \begin{cases} \sin^2(\sqrt{3}t) + \cos^2(\pi t) + \frac{1}{1+t^2}, & t \neq t_k, k \in \mathbb{Z}, \\ \frac{1}{4}\sin(\sqrt{3}t) + \frac{1}{1+t^2}, & t = t_k, k \in \mathbb{Z}, \end{cases}$$

is a piecewise pseudo almost periodic function, where

$$t_k = k + \frac{1}{6}|\sin k - \sin\sqrt{2}k|.$$

Hence, it is easy to see that $f(.)$ is more general than our traditional piecewise almost periodic functions since the ergodic perturbations are introduced.

Lemma 1 *[37]. Suppose that both functions f and its derivative f' are in $PAP(\mathbb{R}, \mathbb{R})$. That is, $f = g + h$ and $f' = g' + h'$, where $g, g' \in AP(\mathbb{R}, \mathbb{R})$ and $h, h' \in PAP_0(\mathbb{R}, \mathbb{R})$. Then the functions g and h are continuous differentiable.*

Remark 3 Let $E = \{f | f, f' \in PAP(\mathbb{R}, \mathbb{R}^n)\}$ equipped with the induced norm defined by

$$\| f \|_E = \max\{\| f \|_\infty, \| f' \|_\infty\}.$$

Follows from [37] that $(PAP(\mathbb{R}, \mathbb{R}^n), \| . \|_E)$ is a Banach space.

The initial conditions associated with (1) are of the form

$$x_i(s) = \varphi_i(s),\ s \in (-\infty, 0],\ i = 1, 2, \ldots, n,$$

where $\varphi(.)$ and $\varphi'(.)$ are real-valued piecewise continuous functions defined on $(-\infty, 0]$.

Lemma 2 [28]. *Let $c_i(.)$ be an almost periodic function on $\mathbb{R}$ and*

$$M[c_i] = \lim_{T\to+\infty} \int_t^{t+T} c_i(s)ds > 0,\ \ i = 1, \ldots, n.$$

Then the linear system

$$x'(t) = diag\Big(-c_1(t), -c_2(t), \ldots, -c_n(t)\Big)x(t) \tag{2}$$

admits an exponential dichotomy on $\mathbb{R}$.

Lemma 3 [36]. *The inhomogeneous linear system*

$$x'(t) = -c(t)x(t) + f(t)$$

has a unique bounded solution for a vector $f \in C(\mathbb{R}, \mathbb{R}^n)$ if and only if the inhomogeneous linear system (31) has exponential dichotomy.

The rest of this paper is organized as follow. The existence and the uniqueness of piecewise differentiable pseudo almost-periodic solutions of Eq. (1) in the suitable convex set are discussed in Sect. 2. Some sufficient conditions on the global exponential stability of piecewise differentiable pseudo almost periodic solutions of Eq. (1) are established in Sect. 3. A numerical example is given in Sect. 4 to illustrate the effectiveness of our results. Finally, we draw conclusion in Sect. 5.

2 Existence of piecewise differentiable pseudo almost periodic solution

In this section, let us recall some lemmas and theorems, which are of importance in proving the main results of this paper.

Lemma 4 [30]. *If* $\varphi \in PAP_T(\mathbb{R}, \mathbb{R}^n)$*, then* $\varphi(. - h) \in PAP_T(\mathbb{R}, \mathbb{R}^n)$.

Lemma 5 [30]. *If* $\varphi, \psi \in PAP_T(\mathbb{R}, \mathbb{R})$*, then* $\varphi \times \psi \in PAP_T(\mathbb{R}, \mathbb{R})$.

Lemma 6 *If* $g \in C(\mathbb{R}, \mathbb{R})$ *a* L^g*−lipschitz function,* $\varphi(.) \in PAP_T(\mathbb{R}, \mathbb{R})$ *and* $\sigma \in C(\mathbb{R}, \mathbb{R})$ *then* $g(\varphi(. - \sigma(.))) \in PAP_T(\mathbb{R}, \mathbb{R})$

Proof We have $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in AP_T(\mathbb{R}, \mathbb{R})$ and $\varphi_2 \in PAP_T^0(\mathbb{R}, \mathbb{R})$. Let

$$\begin{aligned} E(t) &= g(\varphi(t - \sigma(t))) = g(\varphi_1(t - \sigma(t)) \\ &\quad + [g(\varphi_1(t - \sigma(t)) + \varphi_2(t - \sigma(t))) - g(\varphi_1(t - \sigma(t)))] \\ &= E_1(t) + E_2(t). \end{aligned}$$

Firstly, it follows from Theorem 2.11 in [29] that $E_1(.) \in AP_T(\mathbb{R}, \mathbb{R})$. Then, we show that $E_.(t) \in PAP_T^0(\mathbb{R}, \mathbb{R})$ because

$$\begin{aligned} \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} |E_2(t)| dt &= \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} |g(\varphi_1(t - \sigma(t)) + \varphi_2(t - \sigma(t))) \\ &\quad - g(\varphi_1(t - \sigma(t)))| dt \\ &\le \lim_{T \longrightarrow \infty} \frac{L^g}{2T} \int_{-T}^{T} |\varphi_2(t - \sigma(t))| dt = 0. \end{aligned}$$

Thus $E_2(.) \in PAP_T^0(\mathbb{R}, \mathbb{R})$. So, $E(.) \in PAP_T(\mathbb{R}, \mathbb{R})$. The proof is complete. □

Theorem 1 *Under the conditions (H1)-(H2), and for all* $1 \le j \le n$ *,* $x_j(.) \in PAP_T(\mathbb{R}, \mathbb{R})$*, then for all* $1 \le i \le n$*, the function* $\phi_i : t \longmapsto \int_{-\infty}^{t} d_{ij}(t - s) g_j(x_j(s)) ds$ *belongs to* $PAP_T(\mathbb{R}, \mathbb{R})$.

Proof For $x_j(.) \in PAP_T(\mathbb{R}, \mathbb{R})$, it follows from Lemma 6 that $g_j(x_j(.)) \in PAP_T(\mathbb{R}, \mathbb{R})$. Let $g_j(x_j(.)) = u_j(.) + v_j(.)$, where $u_j \in AP_T(\mathbb{R}, \mathbb{R})$ and $v_j \in PAP_T^0(\mathbb{R}, \mathbb{R})$, then

$$\begin{aligned} \phi_i(t) &= \int_{-\infty}^{t} d_{ij}(t - s) g_j(x_j(s)) ds = \int_{-\infty}^{t} d_{ij}(t - s) u_j(s) ds + \int_{-\infty}^{t} d_{ij}(t - s) v_j(s) ds \\ &:= \phi_i^1(t) + \phi_i^2(t). \end{aligned}$$

First, $\phi_i^1(.) \in AP_T(\mathbb{R}, \mathbb{R})$.
It is not difficult to see that $\phi_i^1(.) \in UPC(\mathbb{R}, \mathbb{R})$. Let $t_k < t \le t_{k+1}$.
For $\epsilon > 0$, let Ω_ϵ be a relatively dense set of $\mathbb{R}$ formed by ϵ-periods of u_j. For $\tau \in \Omega_\epsilon$ and $0 < h < \min\{\epsilon, \frac{\alpha}{2}\}$,

$$\begin{aligned} |\phi_i^1(t + \tau) - \phi_i^1(t)| &\le \int_{-\infty}^{t} |d_{ij}(t - s)||u_j(s + \tau) - u_j(s)| ds \\ &\le \sum_{l=-\infty}^{k-1} \int_{t_l + h}^{t_{l+1} - h} |d_{ij}(t - s)||u_j(s + \tau) - u_j(s)| ds \end{aligned}$$

$$+\sum_{l=-\infty}^{k-1}\int_{t_l}^{t_l+h}|d_{ij}(t-s)||u_j(s+\tau)-u_j(s)|ds$$
$$+\sum_{l=-\infty}^{k-1}\int_{t_{l+1}-h}^{t_{l+1}}|d_{ij}(t-s)||u_j(s+\tau)-u_j(s)|ds$$
$$+\int_{t_k}^{t}|d_{ij}(t-s)||u_j(s+\tau)-u_j(s)|ds.$$

Since $u_j \in AP_T(\mathbb{R},\mathbb{R})$, one has $|u_j(t+\tau)-u_j(t)| \le \epsilon$, for all $t \in [t_l+h, t_{l+1}-h]$ and $l \in \mathbb{Z}, l \le k$,

then

$$\begin{aligned}
&\sum_{l=-\infty}^{k-1}\int_{t_l+h}^{t_{l+1}-h}|d_{ij}(t-s)||u_j(s+\tau)-u_j(s)|ds\\
&\le \epsilon\sum_{l=-\infty}^{k-1}\int_{t_l+h}^{t_{l+1}-h}|d_{ij}(t-s)|ds\\
&\le \epsilon d_{ij}^{+}\sum_{l=-\infty}^{k-1}\int_{t_l+h}^{t_{l+1}-h}e^{-\eta_d(t-s)}ds\\
&\le \frac{\epsilon d_{ij}^{+}}{\eta_d}\sum_{l=-\infty}^{k-1}e^{-\eta_d(t-t_{l+1}+h)}\\
&\le \frac{\epsilon d_{ij}^{+}}{\eta_d}\sum_{l=-\infty}^{k-1}e^{-\eta_d\alpha(k-l-1)}\\
&\le \frac{\epsilon d_{ij}^{+}}{\eta_d(1-e^{-\eta_d\alpha})}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{l=-\infty}^{k-1}\int_{t_l}^{t_l+h}|d_{ij}(t-s)||u_j(s+\tau)-u_j(s)|ds\\
&\le 2d_{ij}^{+}|u_j|_\infty\sum_{l=-\infty}^{k-1}\int_{t_l}^{t_l+h}e^{-\eta_d(t-s)}ds\\
&\le 2d_{ij}^{+}|u_j|_\infty\epsilon e^{\eta_d h}\sum_{l=-\infty}^{k-1}e^{-\eta_d(t-t_l)}ds\\
&\le 2d_{ij}^{+}|u_j|_\infty\epsilon e^{\eta_d h}e^{-\eta_d(t-t_k)}\sum_{l=-\infty}^{k-1}e^{-\eta_d\alpha(k-l)}ds\\
&\le \frac{2d_{ij}^{+}|u_j|_\infty\epsilon e^{\frac{(\eta_d\alpha)}{2}}}{1-e^{-\eta_d\alpha}}
\end{aligned}$$

Similarly, one has

$$\sum_{l=-\infty}^{k-1}\int_{t_{l+1}-h}^{t_{l+1}}|d_{ij}(t-s)||u_j(s+\tau)-u_j(s)|ds\le S_1\epsilon,$$

$$\int_{t_k}^{t}|d_{ij}(t-s)||u_j(s+\tau)-u_j(s)|ds\le S_2\epsilon,$$

where S_1, S_2 are some positive constants. Hence, $\phi_i^1(.)\in AP_T(\mathbb{R},\mathbb{R})$.

Second $\phi_i^2(.)\in PAP_T^0(\mathbb{R},\mathbb{R})$.
In fact, for $r>0$, one has

$$\begin{aligned}\lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}|\phi_i^2(t)|dt &= \lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}|\int_{-\infty}^{t}d_{ij}(t-s)v_j(s)ds|dt\\ &= \lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}|\int_{0}^{\infty}d_{ij}(s)v_j(t-s)ds|dt\\ &\le \lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}\big(\int_{0}^{\infty}d_{ij}^{+}e^{-\eta_d s}|v_j(t-s)|ds\big)dt\\ &\le \int_0^\infty d_{ij}^{+}e^{-\eta_d s}\big(\lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}|v_j(t-s)|dt\big)ds.\end{aligned}$$

Since $v_i(.)\in PAP_T^0(\mathbb{R},\mathbb{R})$, it follows that $v_i(.-s)\in PAP_T^0(\mathbb{R},\mathbb{R})$ for each $s\in\mathbb{R}$ by Lemma 4. Using the Lebesgues dominated convergence theorem, we have $\phi_i^2(.)\in PAP_T^0(\mathbb{R},\mathbb{R})$. This completes the proof. □

Similarly, we can obtain:

Corollary 1 *Under the conditions (H1)-(H2), and for all* $1\le j\le n$, $x_j(.)\in PAP_T(\mathbb{R},\mathbb{R})$, *then for all* $1\le i\le n$, *the function* $\phi_i : t\longmapsto \int_{-\infty}^{t}h_{ijl}(t-s)g_j(x_j(s))ds$ *belongs to* $PAP_T(\mathbb{R},\mathbb{R})$.

Corollary 2 *Under the conditions (H1)-(H2), and for all* $1\le l\le n$, $x_j(.)\in PAP_T(\mathbb{R},\mathbb{R})$, *then for all* $1\le i\le n$, *the function* $\phi_i : t\longmapsto \int_{-\infty}^{t}k_{ijl}(t-s)g_l(x_l(s))ds$ *belongs to* $PAP_T(\mathbb{R},\mathbb{R})$.

Lemma 7 *Suppose that assumptions (H1)–(H3) hold. Define the nonlinear operator* $X_\varphi(.)$ *as follows, for each* $\varphi=(\varphi_1,\ldots,\varphi_n)\in PAP_T(\mathbb{R},\mathbb{R}^n)$ *and* $\varphi'=(\varphi_1',\ldots,\varphi_n')\in PAP_T(\mathbb{R},\mathbb{R}^n)$

$$X_\varphi(t)=\begin{pmatrix}\int_{-\infty}^{t}e^{-\int_s^t c_1(u)du}F_1(s)ds\\ \vdots\\ \int_{-\infty}^{t}e^{-\int_s^t c_n(u)du}F_n(s)ds\end{pmatrix}$$

and

$$\begin{aligned}F_i(s)&=c_i(s)\int_{s-\rho(s)}^{s}\varphi_i^{'}(u)du+\sum_{j=1}^{n}a_{ij}(s)g_j(\varphi_j(s-\tau_{ij}(s)))\\ &\quad+\sum_{j=1}^{n}b_{ij}(s)\int_0^\infty d_{ij}(u)g_j(\varphi_j(s-u))du\end{aligned}$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)g_j(\varphi_j(s-\sigma_{ij}(s)))g_l(\varphi_l(s-\nu_{ij}(s)))$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(s)\int_0^{\infty}h_{ijl}(u)g_j(\varphi_j(s-u))du$$
$$\int_0^{\infty}k_{ijl}(u)g_l(\varphi_l(s-u))du+J_i(s),$$

then X_φ *maps* $PAP_T(\mathbb{R},\mathbb{R}^n)$ *into itself.*

Proof First, note that, for all $1\le i\le n$, the function

$$s\mapsto F_i(s)=c_i(s)\int_{s-\rho(s)}^{t}\varphi_i^{'}(u)du+\sum_{j=1}^{n}a_{ij}(s)g_j(\varphi_j(s-\tau_{ij}(s)))$$
$$+\sum_{j=1}^{n}b_{ij}(s)\int_0^{\infty}d_{ij}(u)g_j(\varphi_j(s-u))du$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)g_j(\varphi_j(s-\sigma_{ij}(s)))g_l(\varphi_l(s-\nu_{ij}(s)))$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(s)\int_0^{\infty}h_{ijl}(u)g_j(\varphi_j(s-u))du$$
$$\int_0^{\infty}k_{ijl}(u)g_l(\varphi_l(s-u))du+J_i(s),$$

is in $PAP_T(\mathbb{R},\mathbb{R})$ by using Lemmas 4, 5, 6, Theorem 1, Corollary 1 and 2. Consequently, for all $1\le i\le n$, F_i can be expressed as

$$F_i=F_i^1+F_i^2,$$

where $F_i^1\in AP_T(\mathbb{R},\mathbb{R})$ and $F_i^2\in PAP_T^0(\mathbb{R},\mathbb{R})$. So

$$\Big(X_\varphi(t)\Big)_i=\int_{-\infty}^{t}e^{-\int_s^t c_i(u)du}F_i^1(s)ds+\int_{-\infty}^{t}e^{-\int_s^t c_i(u)du}F_i^2(s)ds$$
$$=G_i^1(t)+G_i^2(t).$$

(i) Firstly, we prove that $G_i^1(.)\in UPC(\mathbb{R},\mathbb{R})$. Let $t^{'},t^{''}\in(t_k,t_{k+1})$, $k\in\mathbb{Z}$, $t^{''}<t^{'}$, then

$$|G_i^1(t^{'})-G_i^1(t^{''})|=|\int_{-\infty}^{t^{'}}e^{-\int_s^{t^{'}}c_i(u)du}F_i^1(s)ds-\int_{-\infty}^{t^{''}}e^{-\int_s^{t^{''}}c_i(u)du}F_i^1(s)ds|$$
$$\le|\int_{-\infty}^{t^{''}}\left[e^{-\int_s^{t^{'}}c_i(u)du}-e^{-\int_s^{t^{''}}c_i(u)du}\right]F_i^1(s)ds|$$
$$+|\int_{t^{''}}^{t^{'}}e^{-\int_s^{t^{'}}c_i(u)du}F_i^1(s)ds|$$
$$\le|e^{-\int_{t^{''}}^{t^{'}}c_i(u)du}-1|\int_{-\infty}^{t^{''}}e^{-\int_s^{t^{''}}c_i(u)du}|F_i^1(s)|ds$$

$$+\int_{t''}^{t'} e^{-\int_s^{t'} c_i(u)du}|F_i^1(s)|ds$$
$$\leq ((t' - t'')c_i^+)\int_{-\infty}^{t''} e^{-(t''-s)c_{i*}}|F_i^1(s)|ds$$
$$+\int_{t''}^{t'} e^{-(t'-s)c_{i*}}|F_i^1(s)|ds,$$

It is easy to see that for any $\epsilon > 0$, there exists

$$0 < \delta < \min\left\{\frac{c_{i*}\epsilon}{2c_i^+ \parallel F_i^1 \parallel}, \frac{\epsilon}{2 \parallel F_i^1 \parallel}\right\}$$

and for a suitable t', t'' satisfying $0 < t' - t'' < \delta$ one has

$$|G_i^1(t') - G_i^1(t'')| \leq \left[(t' - t'')c_i^+ \int_{-\infty}^{t''} e^{-(t''-s)c_{i*}}ds + \int_{t''}^{t'} e^{-(t'-s)c_{i*}}ds\right] \parallel F_i^1 \parallel$$
$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which implies that $G_i^1(.) \in UPC(\mathbb{R}, \mathbb{R})$.
(ii) Secondly, we prove that $G_i^1(.) \in AP_T(\mathbb{R}, \mathbb{R})$. Since $F_i^1 \in AP_T(\mathbb{R}, \mathbb{R})$, for $\epsilon > 0$, there exists a relatively dense set Ω_ϵ such that for $\tau \in \Omega_\epsilon, t \in \mathbb{R}, |t - t_k| > \epsilon, i \in \mathbb{Z}$, then

$$G_i^1(t+\tau) - G_i^1(t) = \int_{-\infty}^{t+\tau} e^{-\int_s^{t+\tau} c_i(u)du} F_i^1(s)ds - \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} F_i^1(s)ds$$
$$= \int_{-\infty}^{t+\tau} e^{-\int_{s-\tau}^{t} c_i(\rho+\tau)d\rho} F_i^1(s)ds - \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} F_i^1(s)ds$$
$$= \int_{-\infty}^{t} e^{-\int_s^t c_i(m+\tau)dm} F_i^1(s+\tau)ds - \int_{-\infty}^{t} e^{-\int_s^t c_i(m+\tau)dm} F_i^1(s)ds$$
$$+\int_{-\infty}^{t} e^{-\int_s^t c_i(m+\tau)dm} F_i^1(s)ds - \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} F_i^1(s)ds$$
$$= \int_{-\infty}^{t} e^{-\int_s^t c_i(u+\tau)du}(F_i^1(s+\tau) - F_i^1(s))ds$$
$$+\int_{-\infty}^{t} (e^{-\int_s^t c_i(u+\tau)du} - e^{-\int_s^t c_i(u)du})F_i^1(s)ds$$

So there exists $\theta \in]0, 1[$ such that

$$|G_i^1(t+\tau) - G_i^1(t)| \leq \parallel F_i^1 \parallel \int_{-\infty}^{t} \left(e^{-\int_s^t c_i(u+\tau)du} - e^{-\int_s^t c_i(u)du}\right) ds$$
$$+\int_{-\infty}^{t} e^{-\int_s^t c_i(u+\tau)du} \left|F_i^1(s+\tau) - F_i^1(s)\right| ds$$
$$\leq \int_{-\infty}^{t} \Big(e^{-\left[\int_s^t c_{ij}(u+\tau)du + \theta(\int_s^t c_i(u)du - \int_s^t c_i(u+\tau)du)\right]}$$

$$
\begin{aligned}
&\int_s^t |c_i(u) - c_i(u+\tau)|du\Big)ds \parallel F_i^1 \parallel \\
&+ \int_{-\infty}^{t} e^{-\int_s^t c_i(u+\tau)du} \left|F_i^1(s+\tau) - F_i^1(s)\right| ds \\
\leq & \int_{-\infty}^{t} \Big(e^{-\left[\int_s^t c_i(u+\tau)du + \theta\left(\int_s^t c_i(u)du - \int_s^t c_i(u+\tau)du\right)\right]} \\
&\int_s^t |c_i(u) - c_i(u+\tau)|du\Big)ds \parallel F_i^1 \parallel \\
&+ \int_{-\infty}^{t} e^{-c_{i*}(t-s)} \left|F_i^1(s+\tau) - F_i^1(s)\right| ds \\
\leq & \int_{-\infty}^{t} \Big\{e^{-c_{i*}(t-s)} e^{-\theta\left(\int_s^t \left|c_i(u) - c_i(u+\tau)\right| du\right)} \\
&\int_s^t |c_i(u) - c_i(u+\tau)|du\Big\}ds \parallel F_i^1 \parallel \\
&+ \int_{-\infty}^{t} e^{-c_{i*}(t-s)} \left|F_i^1(s+\tau) - F_i^1(s)\right| ds \\
\leq & \parallel F_i^1 \parallel \int_{-\infty}^{t} \Big\{e^{-c_{i*}(t-s)} \int_s^t |c_i(u) - c_i(u+\tau)|du\Big\}ds \\
&+ \int_{-\infty}^{t} e^{-c_{i*}(t-s)} \left|F_i^1(s+\tau) - F_i^1(s)\right| ds \\
= & \int_{-\infty}^{t} \Phi_i(t,s)ds + \int_{-\infty}^{t} \Psi_i(t,s)ds
\end{aligned}
$$

where

$$
\Phi_i(t,s) = e^{-c_{i*}(t-s)} \parallel F_i^1 \parallel \int_s^t |c_i(u) - c_i(u+\tau)|du
$$

and

$$
\Psi_i(t,s) = e^{-c_{i*}(t-s)} \left|F_i^1(s+\tau) - F_i^1(s)\right|.
$$

We obtain immediately that, $G_i^1 \in AP_T(\mathbb{R}, \mathbb{R})$.

Now, we turn our attention to G_i^2.

$$
\begin{aligned}
\lim_{r\longrightarrow\infty} \frac{1}{2r} \int_{-r}^{r} |G_i^2(t)|dt &\leq \lim_{r\longrightarrow\infty} \frac{1}{2r} \int_{-r}^{r} \int_{-\infty}^{t} e^{-(t-s)c_{i*}} |F_i^2(s)|dsdt \\
&\leq I_1 + I_2,
\end{aligned}
$$

where

$$I_1 = \lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-r}^{t} e^{-(t-s)c_{i*}} |F_i^2(s)| ds \right) dt$$

and

$$I_2 = \lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-\infty}^{-r} e^{-(t-s)c_{i*}} |F_i^2(s)| ds \right) dt.$$

Pose $m = t - s$, then by Fubini's theorem one has

$$\begin{aligned} I_1 &= \lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-r}^{t} e^{-(t-s)c_{i*}} |F_i^2(s)| ds \right) dt \\ &= \lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{0}^{t+r} e^{-mc_{i*}} |F_i^2(t-m)| dm \right) dt \\ &\leq \lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{0}^{+\infty} e^{-mc_{i*}} |F_i^2(t-m)| dm \right) dt \\ &\leq \int_{0}^{+\infty} e^{-mc_{i*}} \left(\lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} |F_i^2(t-m)| dt \right) dm \end{aligned}$$

since the function $F_i^2(.) \in PAP_T^0(\mathbb{R}, \mathbb{R})$, and by the Lebesgue dominated convergence theorem, we obtain

$$I_1 = 0.$$

On the other hand, notice that $\| F_i^2 \| = \sup_{t \in \mathbb{R}} |F_i^2(t)| < \infty$ then

$$\begin{aligned} I_2 &= \lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-\infty}^{-r} e^{-(t-s)c_{i*}} |F_i^2(s)| ds \right) dt \\ &\leq \lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-\infty}^{-r} e^{sc_{i*}} |F_i^2(s)| ds \int_{-r}^{r} e^{-tc_{i*}} dt \\ &= \lim_{r \longrightarrow \infty} \frac{\| F_i^2 \|}{2rc_{i*}} \int_{-r}^{r} e^{-(r+t)c_{i*}} dt \\ &= 0, \end{aligned}$$

then

$$\lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} | \int_{-\infty}^{t} e^{-\int_s^t c_i(u) du} F_i^2(s) ds | dt = 0.$$

Consequently, the function G_i^2 belongs to $PAP_T^0(\mathbb{R}, \mathbb{R})$. So (X_φ) belongs to $PAP_T(\mathbb{R}, \mathbb{R}^n)$. □

Lemma 8 *Suppose that assumption (H4) hold. For each $\varphi = (\varphi_1, \ldots, \varphi_n) \in PAP_T(\mathbb{R}, \mathbb{R}^n)$, we have*

$$\sum_{t_k < t} e^{-\int_{t_k}^{t} c_i(u) du} I_k(\varphi_i(t_k)) \in PAP_T(\mathbb{R}, \mathbb{R}).$$

Proof We will show that $\sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k(\varphi_i(t_k)) \in PAP_T(\mathbb{R}, \mathbb{R})$. It is not difficult to see that $\sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k(\varphi_i(t_k)) \in UPC(\mathbb{R}, \mathbb{R})$. By Corollary 2.1 (see [30]), $I_k(x_i(t_k)) \in PAP(\mathbb{Z}, \mathbb{R})$, then let $I_k(x_i(t_k)) = I_k^1 + I_k^2$ where $I_k^1 \in AP(\mathbb{Z}, \mathbb{R})$ and $I_k^2 \in PAP_0(\mathbb{Z}, \mathbb{R})$, so

$$\sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k(x_i(t_k)) = \sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k^1 + \sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k^2$$

$$= \Phi_1(t) + \Phi_2(t).$$

Since $\{t_j^k\}$, $k, j \in \mathbb{Z}$ are equipotentially almost periodic, then (see Lemma 3.2, [30]), for any $\epsilon > 0$, there exists relative dense sets of real numbers Ω_ϵ and integers Q_ϵ, such that for $t_k < t \leq t_{k+1}, \tau \in \Omega_\epsilon, q \in Q_\epsilon, |t - t_k| > \epsilon, |t - t_{k+1}| > \epsilon, k \in \mathbb{Z}$, one has

$$t + \tau > t_k + \epsilon + \tau > t_{k+q},$$
$$t_{k+q+1} > t_{k+1} - \epsilon + \tau > t + \tau,$$

that is $t_{k+q} > t + \tau > t_{k+q+1}$; then

$$\begin{aligned}
\| \Phi_1(t+\tau) - \Phi_1(t) \| &= \| \sum_{t_k<t+\tau} e^{-\int_{t_k}^{t+\tau} c_i(u)du} I_k^1 - \sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k^1 \| \\
&\leq \sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} \| I_{k+q}^1 - I_k^1 \| \\
&\leq \epsilon \sum_{t_k<t} e^{-(t-t_k)c_{i*}} \\
&\leq \epsilon \frac{1}{1-e^{-c_{i*}}},
\end{aligned}$$

So, $\Phi_1(.) \in AP_T(\mathbb{R}, \mathbb{R})$.
Next, we show that $\Phi_2(.) \in PAP_T^0(\mathbb{R}, \mathbb{R})$. For a given $k \in \mathbb{Z}$, define the function $\chi(t)$ by

$$\chi(t) = e^{-\int_{t_k}^{t} c_i(u)du} I_k^2, \quad t_k < t \leq t_{k+1},$$

then

$$\lim_{t\to\infty} \| \chi(t) \| = \lim_{t\to\infty} \| e^{-\int_{t_k}^{t} c_i(u)du} I_k^2 \| \leq \lim_{t\to\infty} e^{-(t-t_k)c_{i*}} \sup_{k\in\mathbb{Z}} \| I_k^2 \| = 0,$$

then $\chi \in PAP_T^0(\mathbb{R}, \mathbb{R})$. Define $\chi_n : \mathbb{R} \to \mathbb{R}$ by

$$\chi_n(t) = e^{-\int_{t_{k-n}}^{t} c_i(u)du} I_{k-n}^2, \quad t_k < t \leq t_{k+1}, \; n \in \mathbb{N}.$$

So $\chi_n \in PAP_T^0(\mathbb{R}, \mathbb{R})$. Moreover,

$$\begin{aligned}
\| \chi_n(t) \| &= \| e^{-\int_{t_{k-n}}^{t} c_i(u)du} I_{k-n}^2 \| \\
&\leq e^{-(t-t_{k-n})c_{i*}} \sup_{k\in\mathbb{Z}} \| I_k^2 \| \\
&\leq e^{-(t-t_k)c_{i*}} e^{-c_{i*}\alpha k} \sup_{k\in\mathbb{Z}} \| I_k^2 \| .
\end{aligned}$$

therefore, the series $\Sigma_{n=1}^{\infty}\chi_n$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.2 (see [30]), one has

$$\Phi_2(t) = \sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k^2 = \sum_{n=0}^{\infty} \chi_n \in PAP_T^0(\mathbb{R},\mathbb{R}).$$

So, $\sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k(x_i(t_k)) \in PAP_T(\mathbb{R},\mathbb{R})$. □

Theorem 2 *Suppose that assumptions (H1)–(H4) hold. Define the nonlinear operator Γ as follows, for each $\varphi = (\varphi_1, \ldots, \varphi_n) \in PAP_T(\mathbb{R},\mathbb{R}^n)$, and $\varphi' = (\varphi_1', \ldots, \varphi_n') \in PAP_T(\mathbb{R},\mathbb{R}^n)$,*

$$(\Gamma_\varphi)_i(t) := (X_\varphi)_i(t) + \sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k(\varphi_i(t_k)),$$

then Γ maps $PAP_T(\mathbb{R},\mathbb{R}^n)$ into itself and

$$(\Gamma_\varphi)_i'(t) := F_i(t) - c_i(t)\int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} F_i(s)ds - c_i(t)\sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k(\varphi_i(t_k)),$$

and $\Gamma^{'}$ maps $PAP_T(\mathbb{R},\mathbb{R}^n)$ into itself.

Theorem 3 *Let conditions (H1)–(H5) hold. Then, there exists a unique piecewise differentiable pseudo almost periodic solution of system (1) in the region*

$$B = \Big\{\varphi/\varphi, \varphi' \in PAP_T(\mathbb{R},\mathbb{R}^n), \| \varphi - \varphi_0 \|_E \leq \frac{pL}{1-p}\Big\},$$

where

$$\varphi_0(t) = \Big(\int_{-\infty}^{t} e^{-\int_s^t c_1(u)du} J_1(s)ds, \ldots, \int_{-\infty}^{t} e^{-\int_s^t c_n(u)du} J_n(s)ds\Big)^T.$$

Proof It is easy to see that $B = \Big\{\varphi/\varphi, \varphi' \in PAP_T(\mathbb{R},\mathbb{R}^n), \| \varphi - \varphi_0 \|_E \leq \frac{pL}{1-p}\Big\}$ is a closed convex subset of $PAP_T(\mathbb{R},\mathbb{R}^n)$. According to the definition of the norm of Banach space $PAP_T(\mathbb{R},\mathbb{R}^n)$, we get

$$\begin{aligned} \| \varphi_0 \|_E &= \max\Big\{ \| \varphi_0 \|_\infty, \| \varphi_0' \|_\infty \Big\} \\ &\leq \max_{1\leq i\leq n} \max\Big\{\frac{\overline{J}_i}{c_{i*}}, \Big(1 + \frac{c_i^+}{c_{i*}}\Big)\overline{J}_i\Big\} = L. \end{aligned} \tag{3}$$

Therefore, for $\forall\varphi \in B$, we have

$$\| \varphi \|_E \leq \| \varphi - \varphi_0 \|_E + \| \varphi_0 \|_E \leq \frac{pL}{1-p} + L = \frac{L}{1-p}. \tag{4}$$

In view of (H1), we have

$$|g_j(u)| \leq L_j^g |u|, \text{ for all } u \in \mathbb{R}, j = 1, 2, \ldots, n. \tag{5}$$

Now, we prove that the mapping Γ is a self-mapping from B to B. In fact, for all $\varphi \in B$, by using the estimate just obtained together with (4), (5), Lemmas 2, 3, 7 and 8 we obtain

$$
\begin{aligned}
&\| \Gamma_\varphi - \varphi_0 \|_\infty \\
&= \max_{1\le i\le n} \sup_{t\in\mathbb{R}} \Big\{ \Big| \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} \Big[c_i(s) \int_{s-\rho(s)}^{s} \varphi_i^{'}(m)dm \\
&\quad + \sum_{j=1}^{n} a_{ij}(s) g_j(\varphi_j(s-\tau_{ij}(s))) \\
&\quad + \sum_{j=1}^{n} b_{ij}(s) \int_0^\infty d_{ij}(u) g_j(\varphi_j(s-u))du \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(s) g_j(\varphi_j(s-\sigma_{ij}(s))) g_l(\varphi_l(s-\nu_{ij}(s))) \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \beta_{ijl}(s) \int_0^\infty h_{ijl}(u) g_j(\varphi_j(s-u))du \int_0^\infty k_{ijl}(u) g_l(\varphi_l(s-u))du \Big] ds \\
&\quad + \sum_{t_k<t} e^{-\int_{t_k}^t c_i(u)du} I_k(\varphi_i(t_k)) \Big| \Big\} \\
&\le \max_{1\le i\le n} \sup_{t\in\mathbb{R}} \Big\{ \int_{-\infty}^{t} e^{-c_{i*}(t-s)} \Big[c_i^+ \rho^+ \| \varphi' \|_\infty \\
&\quad + \sum_{j=1}^{n} \overline{a}_{ij} L_j^g \| \varphi \|_\infty + \sum_{j=1}^{n} \overline{b}_{ij} \frac{d_{ij}^+}{\eta_d} L_j^g \| \varphi \|_\infty + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{\alpha}_{ijl} L_j^g M_l^g \| \varphi \|_\infty \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{\beta}_{ijl} \frac{h_{ijl}^+}{\eta_h} \frac{k_{ijl}^+}{\eta_k} L_j^g M_l^g \| \varphi \|_\infty \Big] ds + \sum_{t_k<t} e^{-(t-t_k)c_{i*}} L_1 \| \varphi \|_\infty \Big\} \\
&\le \max_{1\le i\le n} \Big\{ c_{i*}^{-1} \Big[c_i^+ \rho^+ + \sum_{j=1}^{n} \overline{a}_{ij} L_j^g + \sum_{j=1}^{n} \overline{b}_{ij} \frac{d_{ij}^+}{\eta_d} L_j^g + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{\alpha}_{ijl} L_j^g M_l^g \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{\beta}_{ijl} \frac{h_{ijl}^+}{\eta_h} \frac{k_{ijl}^+}{\eta_k} L_j^g M_l^g \Big] + \frac{L_1}{1-e^{-c_{i*}}} \Big\} \| \varphi \|_E
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\| (\Gamma_\varphi - \varphi_0)' \|_\infty \\
&= \max_{1\le i\le n} \sup_{t\in\mathbb{R}} \Big\{ |c_i(t) \int_{t-\rho(t)}^{t} \varphi_i^{'}(m)dm + \sum_{j=1}^{n} a_{ij}(s) g_j(\varphi_j(t-\tau_{ij}(t))) \\
&\quad + \sum_{j=1}^{n} b_{ij}(t) \int_0^\infty d_{ij}(u) g_j(\varphi_j(t-u))du \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(t) g_j(\varphi_j(t-\sigma_{ij}(t))) g_l(\varphi_l(t-\nu_{ij}(t))) \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \beta_{ijl}(t) \int_0^\infty h_{ijl}(u) g_j(\varphi_j(t-u))du \int_0^\infty k_{ijl}(u) g_l(\varphi_l(t-u))du
\end{aligned}
$$

$$- c_i(t)\int_{-\infty}^{t} e^{-\int_s^t c_i(u)du}\Bigg[c_i(s)\int_{s-\rho(s)}^{s}\varphi_i^{'}(m)dm+\sum_{j=1}^{n}a_{ij}(s)g_j(\varphi_j(s-\tau_{ij}(s)))$$

$$+\sum_{j=1}^{n}b_{ij}(s)\int_0^{\infty}d_{ij}(u)g_j(\varphi_j(s-u))du$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)g_j(\varphi_j(s-\sigma_{ij}(s)))g_l(\varphi_l(s-\nu_{ij}(s)))$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(s)\int_0^{\infty}h_{ijl}(u)g_j(\varphi_j(s-u))du\int_0^{\infty}k_{ijl}(u)g_l(\varphi_l(s-u))du\Bigg]ds$$

$$- c_i(t)\sum_{t_k<t}e^{-\int_{t_k}^{t}c_i(u)du}I_k(\varphi_i(t_k))|\Big\}$$

$$\leq \max_{1\leq i\leq n}\sup_{t\in\mathbb{R}}\Bigg\{\Bigg[c_i^+\rho^+ \parallel \varphi' \parallel_\infty+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g \parallel \varphi \parallel_\infty+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g \parallel \varphi \parallel_\infty$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^gM_l^g \parallel \varphi \parallel_\infty+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^gM_l^g \parallel \varphi \parallel_\infty\Bigg]$$

$$+c_i^+\int_{-\infty}^{t}e^{-c_{i*}(t-s)}\Bigg[c_i^+\rho^+ \parallel \varphi' \parallel_\infty+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g \parallel \varphi \parallel_\infty+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g \parallel \varphi \parallel_\infty$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^gM_l^g \parallel \varphi \parallel_\infty+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^gM_l^g \parallel \varphi \parallel_\infty\Bigg]ds$$

$$+c_i^+\sum_{t_k<t}e^{-(t-t_k)c_{i*}}L_1 \parallel \varphi \parallel_\infty\Bigg\}$$

$$\leq \max_{1\leq i\leq n}\Bigg\{\Bigg(1+\frac{c_i^+}{c_{i*}}\Bigg)\big[c_i^+\rho^+ +\sum_{j=1}^{n}\overline{a}_{ij}L_j^g+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^gM_l^g$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^gM_l^g\big]+\frac{c_i^+L_1}{1-e^{-c_{i*}}}\Bigg\} \parallel \varphi \parallel_E$$

where $i=1,2,\ldots,n$. So we can write

$$\parallel \Gamma_\varphi-\varphi_0 \parallel_E = \max\{\parallel \Gamma_\varphi-\varphi_0 \parallel_\infty, \parallel (\Gamma_\varphi-\varphi_0)' \parallel_\infty\}$$

$$\leq \max_{1\leq i\leq n}\max\Bigg\{\Big\{c_{i*}^{-1}\big[c_i^+\rho^+ +\sum_{j=1}^{n}\overline{a}_{ij}L_j^g+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^gM_l^g$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^gM_l^g\big]+\frac{L_1}{1-e^{-c_{i*}}}\Big\},\Big\{\Bigg(1+\frac{c_i^+}{c_{i*}}\Bigg)\big[c_i^+\rho^+$$

$$+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^gM_l^g$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^{+}}{\eta_h}\frac{k_{ijl}^{+}}{\eta_k}L_j^g M_l^g] + \frac{c_i^+ L_1}{1-e^{-c_{i*}}}\Big\}\Big\} \parallel \varphi \parallel_E$$
$$= p \parallel \varphi \parallel_E,$$

where $p < 1$, it implies that $\Gamma_\varphi(.) \in B$. So, the mapping Γ is a self-mapping from B to B.

Next, we prove that the mapping Γ is a contraction mapping of the B. In fact, in view of (H1), for $\forall \phi, \psi \in B$, we have

$$\begin{aligned}
|\big(\Gamma_\phi - \Gamma_\psi\big)_i(t)| = |\int_{-\infty}^{t} & e^{-\int_s^t c_i(u)du}\Big[c_i(s)\int_{s-\rho(s)}^{s}(\phi_i'(m) - \psi'(m))dm \\
&+ \sum_{j=1}^{n} a_{ij}(s)(g_j(\phi_j(s-\tau_{ij}(s))) - g_j(\psi_j(s-\tau_{ij}(s)))) \\
&+ \sum_{j=1}^{n} b_{ij}(s)\int_0^\infty d_{ij}(u)(g_j(\phi_j(s-u)) - g_j(\psi_j(s-u)))du \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)(g_j(\phi_j(s-\sigma_{ij}(s)))g_l(\phi_l(s-\nu_{ij}(s))) \\
&- g_j(\psi_j(s-\sigma_{ij}(s)))g_l(\psi_l(s-\nu_{ij}(s)))) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(s)\Big(\int_0^\infty h_{ijl}(u)g_j(\phi_j(s-u))du\int_0^\infty k_{ijl}(u)g_l(\phi_l(s-u))du \\
&- \int_0^\infty h_{ijl}(u)g_j(\psi_j(s-u))du\int_0^\infty k_{ijl}(u)g_l(\psi_l(s-u))du\Big)\Big]ds \\
&+ \sum_{t_k<t} e^{-\int_{t_k}^t c_i(u)du}(I_k(\varphi_i(t_k)) - I_k(\psi_i(t_k)))| \\
\le \int_{-\infty}^{t} & e^{-(t-s)c_{i*}}\Big[c_i^+\rho^+ \parallel \phi' - \psi' \parallel_\infty + \sum_{j=1}^{n}\overline{a}_{ij}L_j^g \parallel \phi - \psi \parallel_\infty \\
&+ \sum_{j=1}^{n}\overline{b}_{ij}d_{ij}^+ L_j^g \parallel \phi - \psi \parallel_\infty \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}|g_j(\phi_j(s-\sigma_{ij}(s)))g_l(\phi_l(s-\nu_{ij}(s))) \\
&- g_j(\psi_j(s-\sigma_{ij}(s)))g_l(\phi_l(s-\nu_{ij}(s))) \\
&+ g_j(\psi_j(s-\sigma_{ij}(s)))g_l(\phi_l(s-\nu_{ij}(s))) \\
&- g_j(\psi_j(s-\sigma_{ij}(s)))g_l(\psi_l(s-\nu_{ij}(s)))| \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}|\int_0^\infty h_{ijl}(u)g_j(\phi_j(s-u))du\int_0^\infty k_{ijl}(u)g_l(\phi_l(s-u))du \\
&- \int_0^\infty h_{ijl}(u)g_j(\psi_j(s-u))du\int_0^\infty k_{ijl}(u)g_l(\phi_l(s-u))du \\
&+ \int_0^\infty h_{ijl}(u)g_j(\psi_j(s-u))du\int_0^\infty k_{ijl}(u)g_l(\phi_l(s-u))du \\
&- \int_0^\infty h_{ijl}(u)g_j(\psi_j(s-u))du\int_0^\infty k_{ijl}(u)g_l(\psi_l(s-u))du|\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t_k<t} e^{-(t-t_k)c_{i*}} |I_k(\varphi_i(t_k)) - I_k(\psi_i(t_k))| \\
&\leq \int_{-\infty}^{t} e^{-(t-s)c_{i*}} \Bigg[c_i^{+}\rho^{+} \parallel \phi' - \psi' \parallel_\infty + \sum_{j=1}^{n} \overline{a}_{ij} L_j^g \parallel \phi - \psi \parallel_\infty \\
&+ \sum_{j=1}^{n} \overline{b}_{ij} \frac{d_{ij}^{+}}{\eta_d} L_j^g \parallel \phi - \psi \parallel_\infty + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{\alpha}_{ijl} (L_j^g M_l^g + M_j^g L_l^g) \parallel \phi - \psi \parallel_\infty \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{\beta}_{ijl} \frac{h_{ijl}^{+}}{\eta_h} \frac{k_{ijl}^{+}}{\eta_k} (L_j^g M_l^g + M_j^g L_l^g) \parallel \phi - \psi \parallel_\infty \Bigg] ds \\
&+ \sum_{t_k<t} e^{-(t-t_k)c_{i*}} L_1 \parallel \phi - \psi \parallel_\infty \\
&\leq \Bigg\{ c_{i*}^{-1} \Bigg[c_i^{+}\rho^{+} + \sum_{j=1}^{n} \overline{a}_{ij} L_j^g + \sum_{j=1}^{n} \overline{b}_{ij} \frac{d_{ij}^{+}}{\eta_d} L_j^g + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{\alpha}_{ijl} (L_j^g M_l^g + M_j^g L_l^g) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{\beta}_{ijl} \frac{h_{ijl}^{+}}{\eta_h} \frac{k_{ijl}^{+}}{\eta_k} (L_j^g M_l^g + M_j^g L_l^g) \Bigg] + \frac{L_1}{1 - e^{-c_{i*}}} \Bigg\} \parallel \phi - \psi \parallel_E,
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
|\Big(\Gamma_\phi - \Gamma_\psi\Big)_i'(t)| = &\, | \Bigg[c_i(t) \int_{t-\rho(t)}^{t} (\phi_i'(m) - \psi'(m)) dm \\
&+ \sum_{j=1}^{n} a_{ij}(t) (g_j(\phi_j(t - \tau_{ij}(t))) - g_j(\psi_j(t - \tau_{ij}(t)))) \\
&+ \sum_{j=1}^{n} b_{ij}(t) \int_0^\infty d_{ij}(u) (g_j(\phi_j(t-u)) - g_j(\psi_j(t-u))) du \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(t) (g_j(\phi_j(t - \sigma_{ij}(t))) g_l(\phi_l(t - \nu_{ij}(t))) \\
&- g_j(\psi_j(t - \sigma_{ij}(t))) g_l(\psi_l(t - \nu_{ij}(t)))) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} \beta_{ijl}(t) (\int_0^\infty h_{ijl}(u) g_j(\phi_j(t-u)) du \int_0^\infty k_{ijl}(u) g_l(\phi_l(t-u)) du \\
&- \int_0^\infty h_{ijl}(u) g_j(\psi_j(t-u)) du \int_0^\infty k_{ijl}(u) g_l(\psi_l(t-u)) du) \Bigg] \\
&- c_i(t) \int_{-\infty}^{t} e^{-\int_s^t c_i(u) du} \Bigg[c_i(s) \int_{s-\rho(s)}^{s} (\phi_i'(m) - \psi'(m)) dm \\
&+ \sum_{j=1}^{n} a_{ij}(s) (g_j(\phi_j(s - \tau_{ij}(s))) - g_j(\psi_j(s - \tau_{ij}(s)))) \\
&+ \sum_{j=1}^{n} b_{ij}(s) \int_0^\infty d_{ij}(u) (g_j(\phi_j(s-u)) - g_j(\psi_j(s-u))) du \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(s) (g_j(\phi_j(s - \sigma_{ij}(s))) g_l(\phi_l(s - \nu_{ij}(s))) \\
&- g_j(\psi_j(s - \sigma_{ij}(s))) g_l(\psi_l(s - \nu_{ij}(s))))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(s)\bigg(\int_0^{\infty}h_{ijl}(u)g_j(\phi_j(s-u))du\int_0^{\infty}k_{ijl}(u)g_l(\phi_l(s-u))du \\
&-\int_0^{\infty}h_{ijl}(u)g_j(\psi_j(s-u))du\int_0^{\infty}k_{ijl}(u)g_l(\psi_l(s-u))du\bigg)\bigg]ds \\
&-c_i(t)\sum_{t_k<t}e^{-\int_{t_k}^{t}c_i(u)du}(I_k(\varphi_i(t_k))-I_k(\psi_i(t_k)))| \\
\leq&\bigg[c_i^{+}\rho^{+}\parallel\phi'-\psi'\parallel_{\infty}+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g\parallel\phi-\psi\parallel_{\infty}+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^{+}}{\eta_d}L_j^g\parallel\phi-\psi\parallel_{\infty} \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}|g_j(\phi_j(t-\sigma_{ij}(t)))g_l(\phi_l(t-\nu_{ij}(t))) \\
&-g_j(\psi_j(s-\sigma_{ij}(t)))g_l(\phi_l(t-\nu_{ij}(t))) \\
&+g_j(\psi_j(t-\sigma_{ij}(t)))g_l(\phi_l(t-\nu_{ij}(t)))-g_j(\psi_j(t-\sigma_{ij}(t)))g_l(\psi_l(t-\nu_{ij}(t)))| \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}|\int_0^{\infty}h_{ijl}(u)g_j(\phi_j(t-u))du\int_0^{\infty}k_{ijl}(u)g_l(\phi_l(t-u))du \\
&-\int_0^{\infty}h_{ijl}(u)g_j(\psi_j(t-u))du\int_0^{\infty}k_{ijl}(u)g_l(\phi_l(t-u))du \\
&+\int_0^{\infty}h_{ijl}(u)g_j(\psi_j(t-u))du\int_0^{\infty}k_{ijl}(u)g_l(\phi_l(t-u))du \\
&-\int_0^{\infty}h_{ijl}(u)g_j(\psi_j(t-u))du\int_0^{\infty}k_{ijl}(u)g_l(\psi_l(t-u))du|\bigg] \\
&+c_i^{+}\int_{-\infty}^{t}e^{-(t-s)c_{i*}}\bigg[c_i^{+}\rho^{+}\parallel\phi'-\psi'\parallel_{\infty}+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g\parallel\phi-\psi\parallel_{\infty} \\
&+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^{+}}{\eta_d}L_j^g\parallel\phi-\psi\parallel_{\infty}+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}|g_j(\phi_j(s-\sigma_{ij}(s)))g_l(\phi_l(s-\nu_{ij}(s))) \\
&-g_j(\psi_j(s-\sigma_{ij}(s)))g_l(\phi_l(s-\nu_{ij}(s))) \\
&+g_j(\psi_j(s-\sigma_{ij}(s)))g_l(\phi_l(s-\nu_{ij}(s)))-g_j(\psi_j(s-\sigma_{ij}(s)))g_l(\psi_l(s-\nu_{ij}(s)))| \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}|\int_0^{\infty}h_{ijl}(u)g_j(\phi_j(s-u))du\int_0^{\infty}k_{ijl}(u)g_l(\phi_l(s-u))du \\
&-\int_0^{\infty}h_{ijl}(u)g_j(\psi_j(s-u))du\int_0^{\infty}k_{ijl}(u)g_l(\phi_l(s-u))du \\
&+\int_0^{\infty}h_{ijl}(u)g_j(\psi_j(s-u))du\int_0^{\infty}k_{ijl}(u)g_l(\phi_l(s-u))du \\
&-\int_0^{\infty}h_{ijl}(u)g_j(\psi_j(s-u))du\int_0^{\infty}k_{ijl}(u)g_l(\psi_l(s-u))du|\bigg]ds \\
&+c_i^{+}\sum_{t_k<t}e^{-(t-t_k)c_{i*}}|I_k(\varphi_i(t_k))-I_k(\psi_i(t_k))| \\
\leq&\bigg[c_i^{+}\rho^{+}\parallel\phi'-\psi'\parallel_{\infty}+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g\parallel\phi-\psi\parallel_{\infty}+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^{+}}{\eta_d}L_j^g\parallel\phi-\psi\parallel_{\infty} \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^gM_l^g+M_j^gL_l^g)\parallel\phi-\psi\parallel_{\infty}
\end{aligned}
$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^{+}}{\eta_h}\frac{k_{ijl}^{+}}{\eta_k}(L_j^gM_l^g+M_j^gL_l^g)\parallel\phi-\psi\parallel_\infty\Bigg]$$

$$+c_i^{+}\int_{-\infty}^{t}e^{-(t-s)c_{i*}}\Big[c_i^{+}\rho^{+}\parallel\phi'-\psi'\parallel_\infty$$

$$+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g\parallel\phi-\psi\parallel_\infty+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^{+}}{\eta_d}L_j^g\parallel\phi-\psi\parallel_\infty$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^gM_l^g+M_j^gL_l^g)\parallel\phi-\psi\parallel_\infty$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^{+}}{\eta_h}\frac{k_{ijl}^{+}}{\eta_k}(L_j^gM_l^g+M_j^gL_l^g)\parallel\phi-\psi\parallel_\infty\Bigg]ds$$

$$+c_i^{+}\sum_{t_k<t}e^{-(t-t_k)c_{i*}}L_1\parallel\phi-\psi\parallel_\infty$$

$$\leq\Bigg\{\left(1+\frac{c_i^{+}}{c_{i*}}\right)\Bigg[c_i^{+}\rho^{+}+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^{+}}{\eta_d}L_j^g$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^gM_l^g+M_j^gL_l^g)$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^{+}}{\eta_h}\frac{k_{ijl}^{+}}{\eta_k}(L_j^gM_l^g+M_j^gL_l^g)\Bigg]+\frac{c_i^{+}L_1}{1-e^{-c_{i*}}}\Bigg\}\parallel\phi-\psi\parallel_E,$$

where $i=1,2,\ldots,n$. It follows that

$$\parallel\Gamma_\phi-\Gamma_\psi\parallel_E\leq q\parallel\phi-\psi\parallel_E$$

where

$$q=\max_{1\leq i\leq n}max\Bigg\{\Bigg\{c_{i*}^{-1}\Bigg[c_i^{+}\rho^{+}+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^{+}}{\eta_d}L_j^g+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^gM_l^g+M_j^gL_l^g)$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^{+}}{\eta_h}\frac{k_{ijl}^{+}}{\eta_k}(L_j^gM_l^g+M_j^gL_l^g)\Bigg]+\frac{L_1}{1-e^{-c_{i*}}}\Bigg\},\Bigg\{\left(1+\frac{c_i^{+}}{c_{i*}}\right)$$

$$\Bigg[c_i^{+}\rho^{+}+\sum_{j=1}^{n}\overline{a}_{ij}L_j^g+\sum_{j=1}^{n}\overline{b}_{ij}\frac{d_{ij}^{+}}{\eta_d}L_j^g+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^gM_l^g+M_j^gL_l^g)$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^{+}}{\eta_h}\frac{k_{ijl}^{+}}{\eta_k}(L_j^gM_l^g+M_j^gL_l^g)\Bigg]+\frac{c_i^{+}L_1}{1-e^{-c_{i*}}}\Bigg\}\Bigg\}<1$$

It is clear that the mapping Γ is a contraction. Therefore the mapping Γ possesses a unique fixed point $z^*\in B$, $T(z^*)=z^*$. By (7), z^* satisfies (1). So z^* is a piecewise differentiable pseudo almost periodic solution of system (1) in the region B. The proof is now complete. □

Remark 4 To the best of our knowledge, there have been no results of piecewise pseudo-almost periodic solutions for impulsive high-order Hopfield neural networks with time-varying coefficients, mixed delays and leakage until now. Hence, the obtained results are essentially new and the investigation methods used in this paper can also be applied to study the piecewise pseudo-almost periodic solutions for general Hopfield neural networks and for some other types of neural networks.

3 Exponential stability of piecewise differentiable pseudo almost periodic solution

To study the exponential stability of (1), we need the following lemma and notations. So, for a continuous function $\kappa(.)$, we denote $\overline{\kappa}(t) = \sup_{t-\tau\leq s\leq t} |\kappa(s)|$,

Lemma 9 *Let $\tau \geq 0$ be a given real constant. Assume that $p(t)$ and $q_i(t)(i = 1, 2)$ be continuous functions on $[0, +\infty)$, $k(s)$ be nonnegative function on $[0, +\infty)$ and satisfies that $\int_0^{+\infty} k(s)ds \leq k$ and $\int_0^{+\infty} k(s)e^{\mu s}ds \leq +\infty$ for positive constant μ.*
Moreover, assume that there exist positive constants η and M such that

$$p(t) - q_1(t) - kq_2(t) \geq \eta > 0,\ 0 \leq q_1(t) \leq M,\ 0 \leq q_2(t) \leq M, \forall t \geq 0,$$

then

$$\lambda^* = \inf_{t\geq 0} \left\{\lambda > 0, \lambda - p(t) + q_1(t)e^{\lambda\tau} + q_2(t)\int_0^{+\infty} k(s)e^{\lambda s}ds = 0\right\} > 0.$$

Proof Consider the following equation:

$$G(\lambda) = \lambda - p(t) + q_1(t)e^{\lambda\tau} + q_2(t)\int_0^{+\infty} k(s)e^{\lambda s}ds. \tag{6}$$

Because

$$\begin{aligned} G(0) &= -p(t) + q_1(t) + kq_2(t) < 0, \\ \frac{dG}{d\lambda} &= 1 + q_1(t)\tau e^{\lambda\tau} + q_2(t)\int_0^{+\infty} k(s)se^{\lambda s}ds > 0 \end{aligned}$$

and $G(+\infty) > 0$, we follow that $G(\lambda)$ is a strictly monotone increasing function.
Therefore, for any $t \geq 0$, there is a unique positive $\lambda(t)$ such that

$$\lambda(t) - p(t) + q_1(t)e^{\lambda(t)\tau} + q_2(t)\int_0^{+\infty} k(s)e^{\lambda(t)s}ds = 0.$$

Moreover, λ^* exists and $\lambda^* \geq 0$.
Now, we will prove $\lambda^* > 0$. Suppose this is not true. Pick $\epsilon \in (0, \mu)$ such that $\epsilon < \{\frac{\eta}{3}, \frac{1}{\tau}\ln(1+\frac{\eta}{3M})\}$ and $\int_0^{+\infty} k(s)e^{\epsilon s}ds \leq k+\frac{\eta}{3M}$. Then there exist $t^* > 0$ such that $\lambda^*(t^*) < \epsilon$ and

$$\lambda^*(t^*) - p(t^*) + q_1(t^*)e^{\lambda^*(t^*)\tau} + q_2(t^*)\int_0^{+\infty} k(s)e^{\lambda^*(t^*)s}ds = 0.$$

Now we have

$$\begin{aligned} 0 &= \lambda^*(t^*) - p(t^*) + q_1(t^*)e^{\lambda^*(t^*)\tau} + q_2(t^*)\int_0^{+\infty} k(s)e^{\lambda^*(t^*)s}ds \\ &< \lambda^*(t^*) - p(t^*) + q_1(t^*)e^{\lambda^*(t^*)\tau} + q_2(t^*)\int_0^{+\infty} k(s)e^{\epsilon s}ds \end{aligned}$$

$$
\begin{aligned}
&< \epsilon - p(t^*) + q_1(t^*)\left(1 + \frac{\eta}{3M}\right) + q_2(t^*)\left(k + \frac{\eta}{3M}\right) \\
&< \frac{\eta}{3} - \left(p(t^*) - q_1(t^*) - kq_2(t^*)\right) + \left(q_1(t^*) + q_2(t^*)\right)\frac{\eta}{3M}) + q_2(t^*)\left(k + \frac{\eta}{3M}\right) \\
&< \frac{\eta}{3} - \eta + \frac{2\eta}{3} = 0,
\end{aligned}
$$

which is a contradiction. Hence, $\lambda^* > 0$. The proof of this lemma is completed. □

Then we have

Lemma 10 *Assume that (H1)–(H6) hold and there exist nonnegative vector functions* $(V_1(t), \ldots, V_n(t))^T$ *and* $(W_1(t), \ldots, W_n(t))^T \in PC([-\rho^+, 0], \mathbb{R}^n)$, *where* $V_i(t)$ *is continuous at* $t \neq t_k (k \in \mathbb{N}^*)$, *such that*

$$
\begin{aligned}
D^- V_i(t^-) \leq & -c_i(t)V_i(t^-) + c_i(t)\int_{t^- - \rho(t^-)}^{t^-} W_i(s)ds + \sum_{j=1}^{n} |a_{ij}(t)|L_j^g \overline{V}_j(t^-) \\
& + \sum_{j=1}^{n} |b_{ij}(t)| \int_0^\infty |d_{ij}(u)|L_j^g V_j(t^- - u)du \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n} |\alpha_{ijl}(t)|[L_j^g \overline{V}_j(t^-)M_l^g + M_j^g L_l^g \overline{V}_l(t^-)] \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n} |\beta_{ijl}(t)| \left[\int_0^\infty |h_{ijl}(u)|L_j^g V_j(t^- - u)du \frac{k_{ijl}^+}{\eta_k} M_l^g \right. \\
& \left. + \frac{h_{ijl}^+}{\eta_h} M_j^g \int_0^\infty |k_{ijl}(u)|L_l^g V_l(t^- - u)du \right],
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
W_i(t^+) \leq & \; c_i(t)V_i(t^+) + c_i(t)\int_{t^+ - \rho(t^+)}^{t^+} W_i(s)ds + \sum_{j=1}^{n} |a_{ij}(t)|L_j^g \overline{V}_j(t^+) \\
& + \sum_{j=1}^{n} |b_{ij}(t)| \int_0^\infty |d_{ij}(u)|L_j^g V_j(t^+ - u)du \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n} |\alpha_{ijl}(t)|[L_j^g \overline{V}_j(t^+)M_l^g + M_j^g L_l^g \overline{V}_l(t^+)] \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n} |\beta_{ijl}(t)| \left[\int_0^\infty |h_{ijl}(u)|L_j^g V_j(t^+ - u)du \frac{k_{ijl}^+}{\eta_k} M_l^g \right. \\
& \left. + \frac{h_{ijl}^+}{\eta_h} M_j^g \int_0^\infty |k_{ijl}(u)|L_l^g V_l(t^+ - u)du \right],
\end{aligned} \tag{8}
$$

$$
V_i(t_k^+) \leq L_1 V_i(t^+) \tag{9}
$$

for $t > 0$, $i = 1, 2, \ldots, n$ *and* $k \in \mathbb{N}^*$. *Then for all* $t \geq 0$ *and* $i = 1, 2, \ldots, n$, *there exists a positive constant* M *such that*

$$
V_i(t) \leq M \sum_{l=1}^{n} \max\{\overline{V}_l(0), \overline{W}_l(0)\} e^{-\lambda^* t}, \tag{10}
$$

where λ^* *is defined, respectively, as*

$$\lambda^* = \min\{\lambda_i^*, \widehat{\lambda}_i^* | i = 1, 2, \ldots, n\}, \tag{11}$$

$$\begin{aligned}\lambda_i^* = \inf_{t\geq 0}\Bigg\{ & \lambda(t) > 0,\ \ \lambda(t) - c_i(t) + \frac{q_i}{p_i} c_i^+ \rho^+ e^{\lambda(t)\rho^+} + \sum_{j=1}^{n} \frac{p_j}{p_i} |a_{ij}(t)| L_j^g e^{\lambda(t)\tau} \\ & + \sum_{j=1}^{n} \frac{p_j}{p_i} |b_{ij}(t)| \int_0^\infty |d_{ij}(u)| L_j^g e^{\lambda(t)u} du \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} |\alpha_{ijl}(t)| \left[\frac{p_j}{p_i} L_j^g M_l^g + \frac{p_l}{p_i} M_j^g L_l^g \right] e^{\lambda(t)\tau} \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} |\beta_{ijl}(t)| \left[\frac{p_j}{p_i} \int_0^\infty |h_{ijl}(u)| L_j^g e^{\lambda(t)u} du \frac{k_{ijl}^+}{\eta_k} M_l^g \right. \\ & \left. + \frac{p_l}{p_i} \frac{h_{ijl}^+}{\eta_h} M_j^g \int_0^\infty |k_{ijl}(u)| L_l^g e^{\lambda(t)u} du \right] = 0 \Bigg\} > 0, \end{aligned} \tag{12}$$

$$\begin{aligned}\widehat{\lambda}_i^* = \inf_{t\geq 0}\Bigg\{ & \lambda(t) > 0,\ \ -q_i + c_i(t) p_i + q_i c_i^+ \rho^+ e^{\lambda(t)\rho^+} + \sum_{j=1}^{n} p_j |a_{ij}(t)| L_j^g e^{\lambda(t)\tan} \\ & + \sum_{j=1}^{n} p_j |b_{ij}(t)| \int_0^\infty |d_{ij}(u)| L_j^g e^{\lambda(t)u} du \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} |\alpha_{ijl}(t)| \left[p_j L_j^g M_l^g + p_l M_j^g L_l^g \right] e^{\lambda(t)\tan} \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} |\beta_{ijl}(t)| \left[p_j \int_0^\infty |h_{ijl}(u)| L_j^g e^{\lambda(t)u} du \frac{k_{ijl}^+}{\eta_k} M_l^g \right. \\ & \left. + p_l \frac{h_{ijl}^+}{\eta_h} M_j^g \int_0^\infty |k_{ijl}(u)| L_l^g e^{\lambda(t)u} du \right] = 0 \Bigg\} > 0, \end{aligned} \tag{13}$$

Proof By the similar analysis in Lemma 9, we can deduce that λ_i^* and $\widehat{\lambda}_i^*$ exist uniquely and $\lambda_i^* > 0, \widehat{\lambda}_i^* > 0$. Consequently, $\lambda^* > 0$.
Choose a positive constant θ such that

$$\min\{p_i, q_i | i = 1, 2, \ldots, n\}\theta > 1.$$

Let

$$\begin{aligned} \Phi_i(t) &= \max\left\{ \frac{1}{p_i} V_i(t), \frac{1}{q_i} W_i(t) \right\}, i = 1, 2, \ldots, n, \\ \Psi(t) &= \theta \sum_{l=1}^{n} \max\{\overline{V}_l(0), \overline{W}_l(0)\} e^{-\lambda^* t}. \end{aligned} \tag{14}$$

Then for all $t \in (-\infty, 0]$ and $\gamma > 1$, we have

$$\gamma\Psi(t) = \gamma\theta \sum_{l=1}^{n} \max\{\overline{V}_l(0), \overline{W}_l(0)\} e^{-\lambda^* t} > \Phi_i(t). \tag{15}$$

Then

$$\Phi_i(t) < \gamma\Psi(t), \ \ t \in [0, \infty), \ i = 1, 2, \dots, n. \tag{16}$$

For the sake of contradiction, assume that there exist $i \in \{1, 2, \dots, n\}$ and $\bar{t} > 0$ such that

$$\Phi_i(\bar{t}^+) \geq \gamma\Psi(\bar{t}), \Phi_j(t) < \gamma\Psi(t), \ \ for \ \ t \in [0, \bar{t}), j \in \{1, 2, \dots, n\}. \tag{17}$$

The we have the following
(I) $(1/p_i)V_i(\bar{t}^+) \geq \gamma\Psi(\bar{t})$ then we have the following subcases.
(i) $\bar{t} \neq t_k, t_k \in\in \mathbb{N}^*$. So $V_i(t)$ is continuous at $\bar{t}$. By 17, we have

$$\frac{1}{p_i}V_i(\bar{t}) = \gamma\Psi(\bar{t}), \ \ \frac{1}{p_i}D^-V_i(\bar{t}) > \gamma\Psi'(\bar{t}) \tag{18}$$

From (H6), (17) and the definition of λ^*, we have

$$\begin{aligned}
\frac{1}{p_i}D^-V_i(\bar{t}) - \gamma\Psi'(\bar{t}) \leq & -c_i(t)\gamma\Psi(\bar{t}) \\
& + \frac{q_i}{p_i}c_i(\bar{t})\int_{\bar{t}-\rho(\bar{t})}^{\bar{t}} \gamma\Psi(s)ds + \sum_{j=1}^{n}\frac{p_j}{p_i}|a_{ij}(t)|L_j^g\gamma\Psi(\bar{t}-\tau) \\
& + \sum_{j=1}^{n}\frac{p_j}{p_i}|b_{ij}(\bar{t})|\int_0^\infty |d_{ij}(u)|L_j^g\gamma\Psi(\bar{t}-u)du \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n}|\alpha_{ijl}(\bar{t})|\left[\frac{p_j}{p_i}L_j^gM_l^g + \frac{p_l}{p_i}M_j^gL_l^g\right]\gamma\Psi(\bar{t}-\tau) \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n}|\beta_{ijl}(\bar{t})|\left[\frac{p_j}{p_i}\int_0^\infty h_{ijl}(u)|L_j^g\gamma\Psi(\bar{t}-u)du\frac{k_{ijl}^+}{\eta_k}M_l^g\right. \\
& \left. + \frac{p_l}{p_i}\frac{h_{ijl}^+}{\eta_h}M_j^g\int_0^\infty k_{ijl}(u)|L_l^g\gamma\Psi(\bar{t}-u)du\right] + \lambda^*\gamma\Psi(\bar{t}) \\
\leq & \ \gamma\Psi(\bar{t})(\lambda^* - c_i(t) + \frac{q_i}{p_i}c_i^+\rho^+e^{\lambda^*\rho^+} + \sum_{j=1}^{n}\frac{p_j}{p_i}|a_{ij}(t)|L_j^ge^{\lambda^*\tau} \\
& + \sum_{j=1}^{n}\frac{p_j}{p_i}|b_{ij}(\bar{t})|\int_0^\infty |d_{ij}(u)|L_j^ge^{\lambda^*u}du \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n}|\alpha_{ijl}(\bar{t})|\left[\frac{p_j}{p_i}L_j^gM_l^g + \frac{p_l}{p_i}M_j^gL_l^g\right]e^{\lambda^*\tau} \\
& + \sum_{j=1}^{n}\sum_{l=1}^{n}|\beta_{ijl}(\bar{t})|\left[\frac{p_j}{p_i}\int_0^\infty |h_{ijl}(u)|L_j^ge^{\lambda^*u}du\frac{k_{ijl}^+}{\eta_k}M_l^g\right. \\
& \left. + \frac{p_l}{p_i}\frac{h_{ijl}^+}{\eta_h}M_j^g\int_0^\infty |k_{ijl}(u)|L_l^ge^{\lambda^*u}du\right]) < 0,
\end{aligned} \tag{19}$$

which is a contradiction with (18).

(ii) There exists $k_0 \in \mathbb{N}^*$ such that $\bar{t} = t_k$. By (17), we have

$$\frac{1}{p_i} V_i(\bar{t}) \le \gamma \Psi(\bar{t}) \le \frac{1}{p_i} V_i(\bar{t}^+). \tag{20}$$

Noting $\frac{1}{p_i} V_i(\bar{t}^-) \ne \frac{1}{p_i} V_i(\bar{t}^+)$, we have $\frac{1}{p_i} V_i(\bar{t}^-) < \gamma \Psi(\bar{t})$ or $\gamma \Psi(\bar{t}) < \frac{1}{p_i} V_i(\bar{t}^+)$. Without loss of generality, we assume that $\gamma \Psi(\bar{t}) < \frac{1}{p_i} V_i(\bar{t}^+)$. from (9) and (20) we get that

$$\gamma \Psi(\bar{t}) < \frac{1}{p_i} V_i(\bar{t}^+) \le \gamma L_1 \Psi(\bar{t}). \tag{21}$$

Simplifying (21), we obtain $L_1 > 1$, which contradict that $L_1 < 1$.
If (I) does not hold, then
(II)

$$\frac{1}{q_i} W_i(\bar{t}^+) \ge \gamma \Psi(\bar{t}), \quad \frac{1}{q_j} W_j(t) < \gamma \Psi(t), \quad \frac{1}{p_j} W_j(t) \ge \gamma \Psi(t), \\ \text{for } t \in [0, \bar{t}), \ \ j \in \{1, 2, \ldots, n\}. \tag{22}$$

Then from (8) and (H6) we have

$$\begin{aligned}
0 \le & -W_i(\bar{t}^+) + c_i(t) V_i(\bar{t}^+) + c_i(t^+) \int_{\bar{t}^+ - \rho(t^+)}^{t^+} W_i(s) ds + \sum_{j=1}^n |a_{ij}(t^+)| L_j^g \overline{V}_j(\bar{t}^+) \\
& + \sum_{j=1}^n |b_{ij}(t^+)| \int_0^\infty |d_{ij}(u)| L_j^g V_j(\bar{t}^+ - u) du \\
& + \sum_{j=1}^n \sum_{l=1}^n |\alpha_{ijl}(t^+)| \left[L_j^g \overline{V}_j(\bar{t}^+) M_l^g + M_j^g L_l^g \overline{V}_l(\bar{t}^-) \right] \\
& + \sum_{j=1}^n \sum_{l=1}^n |\beta_{ijl}(t^+)| \left[\int_0^\infty |h_{ijl}(u)| L_j^g V_j(\bar{t}^+ - u) du \frac{k_{ijl}^+}{\eta_k} M_l^g \right. \\
& \left. + \frac{h_{ijl}^+}{\eta_h} M_j^g \int_0^\infty |k_{ijl}(u)| L_l^g V_l(\bar{t}^+ - u) du \right] \\
\le & \gamma \Psi(\bar{t}) (-q_i + c_i(t) p_i + q_i c_i^+ \rho^+ e^{\lambda^* \rho^+} + \sum_{j=1}^n p_j |a_{ij}(t^+)| L_j^g e^{\lambda^* \tau} \\
& + \sum_{j=1}^n p_j |b_{ij}(t^+)| \int_0^\infty |d_{ij}(u)| L_j^g e^{\lambda^* u} du \\
& + \sum_{j=1}^n \sum_{l=1}^n |\alpha_{ijl}(t^+)| \left[p_j L_j^g M_l^g + p_l M_j^g L_l^g \right] e^{\lambda^* \tan} \\
& + \sum_{j=1}^n \sum_{l=1}^n |\beta_{ijl}(t^+)| \left[p_j \int_0^\infty |h_{ijl}(u)| L_j^g e^{\lambda^* u} du \frac{k_{ijl}^+}{\eta_k} M_l^g \right. \\
& \left. + p_l \frac{h_{ijl}^+}{\eta_h} M_j^g \int_0^\infty |k_{ijl}(u)| L_l^g e^{\lambda^* u} du \right]) < 0
\end{aligned} \tag{23}$$

which is a contradiction. From (I) and (II), (16) holds. Letting $\gamma \to 1^+$ in (16), we have

$$\Phi_i(t) \le \gamma \Psi(t), \ \ t \in [0, \infty), \ i = 1, 2, \ldots, n. \tag{24}$$

So $\frac{1}{p_i} V_i(t) \le \Psi(t)$ for all $t \in [0, \infty), \ i = 1, 2, \ldots, n$. Let $\widetilde{L} = \max_{1 \le i \le n} \{p_i \theta\}$ then for $t \ge 0$ and $i = 1, 2, \ldots, n$, we have

$$V_i(t) \le \widetilde{L} \sum_{l=1}^{n} \max\{\overline{V}_l(0), \overline{W}_l(0)\} e^{-\lambda^* t}, \tag{25}$$

The proof is complete. □

Theorem 4 *Assume that (H1)-(H6) hold, then the unique piecewise differentiable pseudo almost periodic solution of system (1) is globally exponentially stable.*

Proof It follows from Theorem 3 that system (1) has at least one piecewise differentiable pseudo almost periodic solution $x^*(t) = (x_1^*(t), \ldots, x_n^*(t))^T \in \mathbb{B}$ with initial value $\phi^*(t)$. Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ be an arbitrary solution of system (1) with initial value $\phi(t)$. Let $V_i(t) = |x_i(t) - x_i^*(t)|$, $W_i(t) = |x_i'(t) - x_i^{*'}(t)|$ for $t \in \mathbb{R}^+$, and $i = 1 \ldots n$, Then,

$$\begin{aligned}
D^- V_i(t^-) \le & -c_i(t) V_i(t^-) + c_i(t) \int_{t^- - \rho(t^-)}^{t^-} W_i(s) ds + \sum_{j=1}^{n} |a_{ij}(t)| L_j^g \overline{V}_j(t^-) \\
& + \sum_{j=1}^{n} |b_{ij}(t)| \int_0^\infty |d_{ij}(u)| L_j^g V_j(t^- - u) du \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_{ijl}(t)| [L_j^g \overline{V}_j(t^-) M_l^g + M_j^g L_l^g \overline{V}_l(t^-)] \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} |\beta_{ijl}(t)| \Bigg[\int_0^\infty |h_{ijl}(u)| L_j^g V_j(t^- - u) du \frac{k_{ijl}^+}{\eta_k} M_l^g \\
& + \frac{h_{ijl}^+}{\eta_h} M_j^g \int_0^\infty |k_{ijl}(u)| L_l^g V_l(t^- - u) du \Bigg],
\end{aligned} \tag{26}$$

$$\begin{aligned}
W_i(t^+) \le & \ c_i(t) V_i(t^+) + c_i(t) \int_{t^+ - \rho(t^+)}^{t^+} W_i(s) ds + \sum_{j=1}^{n} |a_{ij}(t^+)| L_j^g \overline{V}_j(t^+) \\
& + \sum_{j=1}^{n} |b_{ij}(t)| \int_0^\infty |d_{ij}(u)| L_j^g V_j(t^+ - u) du \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_{ijl}(t)| [L_j^g \overline{V}_j(t^+) M_l^g + M_j^g L_l^g \overline{V}_l(t^-)] \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} |\beta_{ijl}(t)| \Bigg[\int_0^\infty |h_{ijl}(u)| L_j^g V_j(t^+ - u) du \frac{k_{ijl}^+}{\eta_k} M_l^g \\
& + \frac{h_{ijl}^+}{\eta_h} M_j^g \int_0^\infty |k_{ijl}(u)| L_l^g V_l(t^+ - u) du \Bigg],
\end{aligned} \tag{27}$$

By (9) and (H5) we have

$$V_i(t_k^+) \leq L_1 V_i(t^+), \quad with \;\; L_1 < 1. \tag{28}$$

By (26)–(28), (H1)–(H6) and Lemma 10, there exists a positive constant M such that

$$V_i(t) \leq M \sum_{l=1}^{n} \max\{\overline{V}_l(0), \overline{W}_l(0)\} e^{-\lambda^* t}, \tag{29}$$

where λ^* is defined in (11). □

4 Application

Example 1 Consider the following impulsive High-order Hopfield Neural Networks with time-varying coefficients, continuously distributed delays and leakage:

$$\begin{cases} x_i'(t) = -c_i(t)x_i(t-\rho(t)) + \sum_{j=1}^{2} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ \qquad + \sum_{j=1}^{2} b_{ij}(t)\int_0^\infty d_{ij}(u)g_j(x_j(t-u))du \\ \qquad + \sum_{j=1}^{2}\sum_{l=1}^{2} \alpha_{ijl}(t)g_j(x_j(t-\sigma_{ij}(t)))g_l(x_l(t-\nu_{ij}(t))) \\ \qquad + \sum_{j=1}^{2}\sum_{l=1}^{2} \beta_{ijl}(t)\int_0^\infty h_{ijl}(u)g_j(x_j(t-u))du \int_0^\infty k_{ijl}(u)g_l(x_l(t-u))du \\ \qquad + I_i(t), \quad t \in \mathbb{R}, \; t \neq 2k, \; k \in \mathbb{Z} \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_k(x_i(t_k)) \end{cases} \tag{30}$$

where

$$c(t) = \begin{pmatrix} 4 + \sin^2(t) \\ 4 + \cos^2(t) \end{pmatrix} \Rightarrow c_{1*} = c_{2*} = 4,$$

for all $t \in \mathbb{R}$

$$\begin{aligned} g_1(t) &= g_2(t) = \sin t \Rightarrow L_1^g = L_2^g = M_1^g = M_2^g = 1, \\ \tau_{ij}(t) &= \sigma_{ij}(t) = \nu_{ij}(t) = \rho(t) = \frac{1}{80}|\cos t|, \text{ for } i, j \in \{1, 2\} \\ d_{ij}(t) &= h_{ijl}(t) = k_{ijl}(t) = e^{-t} \Rightarrow \frac{d_{ij}^+}{\eta_d} = \frac{h_{ijl}^+}{\eta_h} \\ &= \frac{k_{ijl}^+}{\eta_k} = 1, \quad \text{for } i, j, l \in \{1, 2\} \end{aligned}$$

$$\begin{pmatrix} \Delta x_1(2k) \\ \Delta x_2(2k) \end{pmatrix} = \begin{pmatrix} -\frac{1}{80}x_1(2k) + \frac{1}{80}\sin(x_1(2k)) + \frac{1}{20} \\ -\frac{1}{80}x_2(2k) + \frac{1}{80}\cos(x_2(2k)) + \frac{1}{30} \end{pmatrix} \Rightarrow L = \frac{1}{80}.$$

$$a(t) = \begin{pmatrix} 0.02\sin t & 0.03\cos t \\ 0.03\cos t + \frac{0.01}{1+t^2} & 0.01\cos\sqrt{2}t \end{pmatrix} \Rightarrow \overline{a} = \begin{pmatrix} 0.02 & 0.03 \\ 0.04 & 0.01 \end{pmatrix},$$

$$b(t) = \begin{pmatrix} 0.02\cos t + \frac{0.01}{1+t^2} & 0.02\sin t \\ 0.01\cos\sqrt{2}t + \frac{0.01}{1+t^2} & 0.02\sin t + \frac{0.01}{1+t^2} \end{pmatrix} \Rightarrow \overline{b} = \begin{pmatrix} 0.03 & 0.02 \\ 0.02 & 0.03 \end{pmatrix},$$

$$(\alpha_{1jl}(t))_{1\le j,l\le 2} = \begin{pmatrix} 0 & 0.04\cos t + \frac{0.01}{1+t^2} \\ 0 & 0 \end{pmatrix} \Rightarrow (\overline{\alpha}_{1jl})_{1\le j,l\le 2} = \begin{pmatrix} 0 & 0.05 \\ 0 & 0 \end{pmatrix},$$

$$(\alpha_{2jl}(t))_{1\le j,l\le 2} = \begin{pmatrix} 0 & 0.06\sin t + \frac{0.01}{1+t^2} \\ 0 & 0 \end{pmatrix} \Rightarrow (\overline{\alpha}_{2jl})_{1\le j,l\le 2} = \begin{pmatrix} 0 & 0.07 \\ 0 & 0 \end{pmatrix},$$

$$(\beta_{1jl}(t))_{1\le j,l\le 2} = \begin{pmatrix} 0 & 0.05\sin t + \frac{0.01}{1+t^2} \\ 0 & 0 \end{pmatrix} \Rightarrow (\overline{\beta}_{1jl})_{1\le j,l\le 2} = \begin{pmatrix} 0 & 0.06 \\ 0 & 0 \end{pmatrix},$$

$$(\beta_{2jl}(t))_{1\le j,l\le 2} = \begin{pmatrix} 0 & 0.4\cos t + \frac{0.01}{1+t^2} \\ 0 & 0 \end{pmatrix} \Rightarrow (\overline{\beta}_{2jl})_{1\le j,l\le 2} = \begin{pmatrix} 0 & 0.05 \\ 0 & 0 \end{pmatrix},$$

$$I(t) = \begin{pmatrix} 0.9\sin\sqrt{3}t + \frac{0.1}{1+t^2} \\ 0.8\cos t \end{pmatrix} \Rightarrow \overline{I} = \begin{pmatrix} 0.9 \\ 0.8 \end{pmatrix}.$$

Then

$$\max_{1\le i\le n} \max\left\{ \frac{\overline{J}_i}{c_{i*}}, \left(1 + \frac{c_i^+}{c_{i*}}\right)\overline{J}_i \right\} = 2.0250 = L$$

$$p = \max_{1\le i\le 2} \max\left\{ \left\{ c_{i*}^{-1}\left[c_i^+\rho^+ + \sum_{j=1}^{2}\overline{a}_{ij}L_j^g + \sum_{j=1}^{2}\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\alpha}_{ijl}L_j^gM_l^g \right.\right.\right.$$
$$\left. + \sum_{j=1}^{2}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^gM_l^g\right] + \frac{L_1}{1-e^{-c_{i*}}}\Big\}, \left\{ \left(1 + \frac{c_i^+}{c_{i*}}\right)\left[c_i^+\rho^+ + \sum_{j=1}^{2}\overline{a}_{ij}L_j^g \right.\right.$$
$$+ \sum_{j=1}^{2}\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\alpha}_{ijl}L_j^gM_l^g$$
$$\left.\left.\left. + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^gM_l^g\right] + \frac{c_i^+L_1}{1-e^{-c_{i*}}}\right\}\right\}$$
$$= 0.6712 < 1.$$

$$q = \max_{1\le i\le 2} max\left\{\left\{ c_{i*}^{-1}\left[c_i^+\rho^+ + \sum_{j=1}^{2}\overline{a}_{ij}L_j^g + \sum_{j=1}^{2}\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g \right.\right.\right.$$
$$+ \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\alpha}_{ijl}(L_j^gM_l^g + M_j^gL_l^g)$$
$$\left.\left. + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^gM_l^g + M_j^gL_l^g)\right] + \frac{L_1}{1-e^{-c_{i*}}}\right\},$$
$$\left\{\left(1 + \frac{c_i^+}{c_{i*}}\right)\left[c_i^+\rho^+ + \sum_{j=1}^{2}\overline{a}_{ij}L_j^g + \sum_{j=1}^{2}\overline{b}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^g + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\alpha}_{ijl}(L_j^gM_l^g + M_j^gL_l^g) \right.\right.$$
$$\left.\left.\left. + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^gM_l^g + M_j^gL_l^g)\right] + \frac{c_i^+L_1}{1-e^{-c_{i*}}}\right\}\right\}$$
$$= 0.9412 < 1.$$

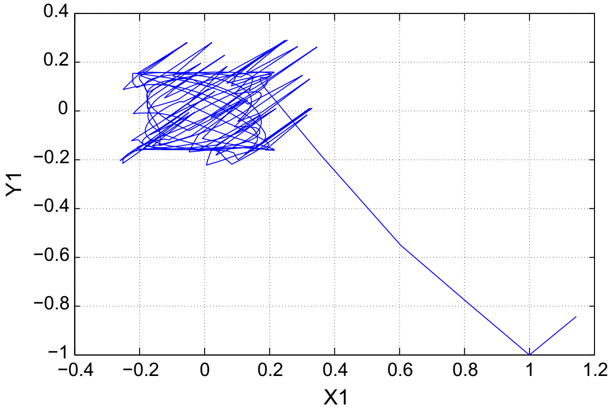

Fig. 1 The orbit of X1 − X2 for the system

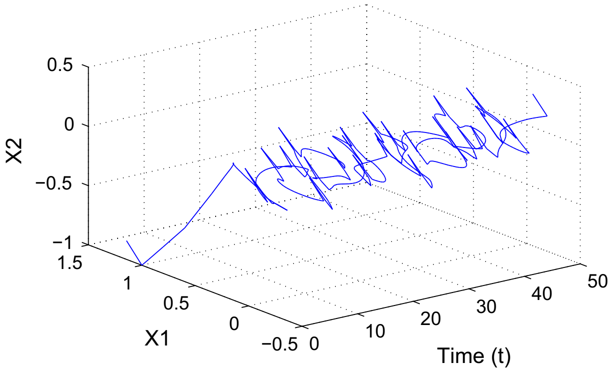

Fig. 2 The phase system for the system

Let $p_1 = p_2 = 1$ and $q_1 = q_2 = 10$, from the above assumption, the (H6) is satisfied. Therefore, all conditions from Theorems 3 and 4 are satisfied, then the delayed impulsive high order Hopfield neural network with leakage of Example1 has a unique piecewise differentiable pseudo almost-periodic solution (Figs. 1, 2, 3).

Example1 without leakage:

$$\begin{cases} x_i'(t) = -c_i(t)x_i(t) + \sum_{j=1}^{2} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ \qquad + \sum_{j=1}^{2} b_{ij}(t)\int_0^\infty d_{ij}(u)g_j(x_j(t-u))du \\ \qquad + \sum_{j=1}^{2}\sum_{l=1}^{2} \alpha_{ijl}(t)g_j(x_j(t-\sigma_{ij}(t)))g_l(x_l(t-\nu_{ij}(t))) \\ \qquad + \sum_{j=1}^{2}\sum_{l=1}^{2} \beta_{ijl}(t)\int_0^\infty h_{ijl}(u)g_j(x_j(t-u))du \int_0^\infty k_{ijl}(u)g_l(x_l(t-u))du \\ \qquad + I_i(t), \quad t \in \mathbb{R},\ t \neq 2k,\ k \in \mathbb{Z} \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_k(x_i(t_k)) \end{cases} \tag{31}$$

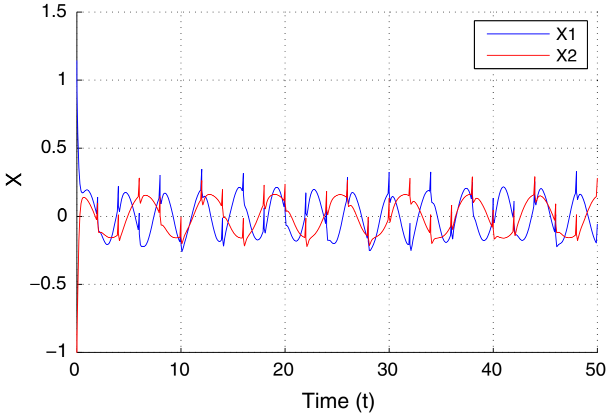

Fig. 3 Transient response of state variables X1 and X2 for the system

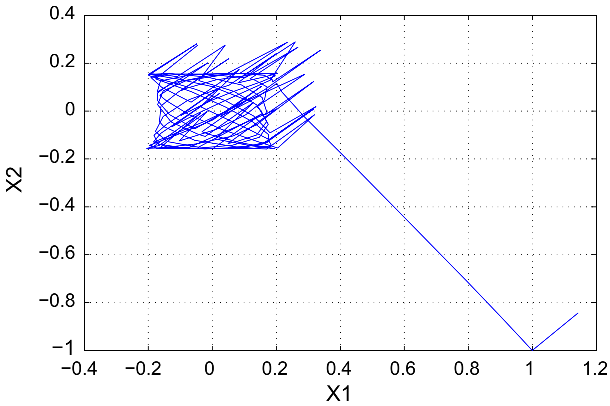

Fig. 4 The orbit of X1–X2 for the system without leakage

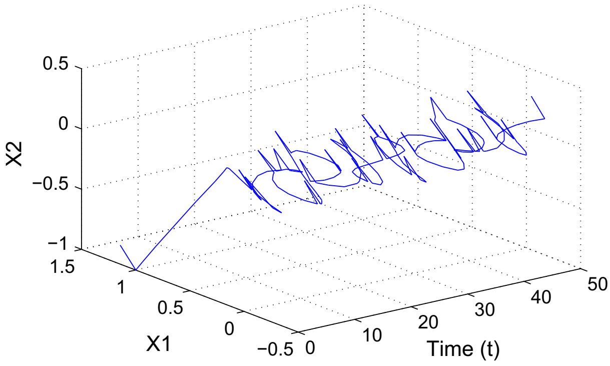

Fig. 5 The phase system for the system without leakage

The example considered here is a special case of ***Example 1***. Simulation results are depicted in Figure 4, Figure 5 and Figure 6. If the leakage delays are not small enough they can have a destabilizing influence on the system leading to delay induced instability and oscillations and even periodicity.

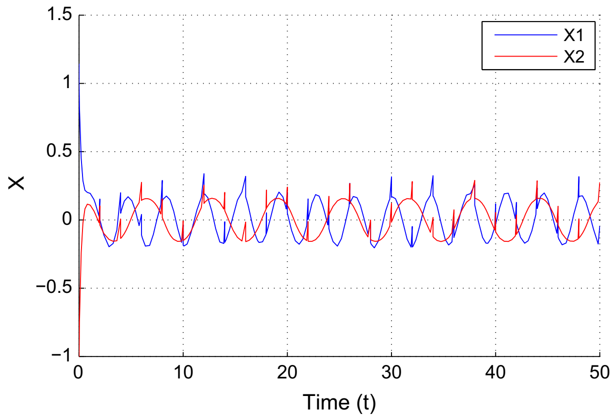

Fig. 6 Transient response of state variables X1 and X2 for the system without leakage

5 Conclusion

A class of impulsive high order neural networks described with mixed delays is considered. By means of fixed point theorem, Lyapunov functional method and differential inequality techniques, criteria on existence and global exponential stability of piecewise differentiable pseudo almost periodic solution for model (1) are derived. Many adjustable parameters are introduced in criteria to provide flexibility for the design and analysis of the system.The results of this paper are new and they supplement previously known results. An example is given to illustrate the results.

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