Neural Process Lett (2017) 45:95–108
DOI 10.1007/s11063-016-9514-1

Weighted Average Pinning Synchronization for a Class of Coupled Neural Networks with Time-Varying Delays

Qingbo Li[1,2] · **Jin Guo**[1] · **Yuanyuan Wu**[3] · **Chang-Yin Sun**[1]

Published online: 15 March 2016

Abstract This paper deals with the pinning synchronization problem for a class of neural networks with time-varying delay and nonsymmetrical coupling. The weighted average of all the node states is selected as the controlled synchronization state. A pinning feedback controller is proposed, and some sufficient conditions are derived to guarantee the global asymptotical synchronization by use of the Lyapunov–Krasovskii function method. Consequently, the adaptive pinning synchronization is also investigated. Finally, a numerical example is given to show the effectiveness of the main results.

Keywords Coupled neural networks · (Adaptive) pinning synchronization · Time-varying delays · Lyapunov–Krasovskii function

1 Introduction

Over the past few decades, neural networks have received considerable attention for their important applications in many fields of image processing, pattern recognition and associative

✉ Jin Guo
guojin@amss.ac.cn; guojin@ustb.edu.cn

Qingbo Li
qbliyy@163.com

Yuanyuan Wu
wyuanyuan82@163.com

Chang-Yin Sun
cys@ustb.edu.cn

1 School of Automation and Electrical Engineering, University of Science and Technology Beijing, Beijing 100083, People's Republic of China

2 College of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, People's Republic of China

3 College of Electric and Information Engineering, Zhengzhou University of Light Industry, Zhengzhou 450002, People's Republic of China

memories, etc [1–3]. Since the switching speed of information processing and the inherent neuron communication is finite, time delays are always unavoidably existent in neural networks, which may lead to instability or significantly deteriorated performances. Then it is necessary to take into account the time delay in neural networks [4–6].

Also, much concern has been paid to the synchronization of coupled neural networks for its potential applications, such as secure communication, and so on [7–13]. He and Cao [9] discussed the global synchronization of coupled neural networks with one single time-varying delay coupling, and established several sufficient criteria. Song et al. [10] proposed a pinning synchronization control scheme for a class of linearly coupled neural networks. Tang and Wong [11] considered the distributed synchronization problem of coupled neural networks by randomly occurring control method. Employing discontinuous impulsive control scheme, [12] put forward some criteria for the synchronization of coupled networks by impulsive control. Liu and Chen [13] investigated the cluster synchronization problem of the directed networks by intermittent pinning controller. In fact, not all the complex networks can synchronize by themselves. Those, who can not achieve synchronization by themselves, need introducing proper controllers to force them to synchronize. Among some existing control methods, the pinning control is a very important one.

Pinning control strategy applies the local feedback injections to a small fraction of nodes to achieve the global performances of the entire networks, which is effective and practical, especially for the large-size networks. Nowadays, the pinning synchronization of complex networks has been widely investigated, and meanwhile several selection rules of pinned nodes have been proposed in the existing literatures. Wang and Chen [14] concluded that the most-highly connected nodes can be pinned to get the better performance for the undirected networks. Chen et al. [15] showed that even using one single pinned node could control the whole networks as long as the coupling strength was large enough. Song and Cao [16] proposed the pinned nodes selection rule according to the out-degree and in-degree of the nodes, and the synchronization problem was studied for undirected and directed networks. In addition, the pinning control problems of complex networks were investigated, and some sufficient pinning conditions were established in [17–21].

As far as the synchronization state is concerned, most previous studies on synchronization problem of complex networks choose a special solution of the homogeneous systems as the controlled synchronization state [10,16,17,19,20,22,23], usually described by $\dot{s}(t) = f(t, s(t))$. As pointed out in [24,25], the synchronization of the networks not only depends on the topology, but also is heavily related to the dynamic of each node. Then, the contributions and influences of all nodes in synchronization seeking processes of the dynamical networks should be taken into account. In [26], authors considered the average of all the node states $s(t) = \sum_{i=1}^{N} \frac{1}{N} x_i(t)$ to be the synchronization state. For the purpose of practical control strategy, authors in [24,27,28] further investigated the synchronization of complex networks to a desired synchronization state, which was in term of the weighted average of all node states in the networks, that was $s(t) = \sum_{i=1}^{N} \xi_i x_i(t)$ with $\sum_{i=1}^{N} \xi_i = 1$ $(\xi_i > 0)$. In some degree, there research results are important and valuable to the realistic dynamical behaviors of complex works. To the best of authors' knowledge, the synchronization of delayed coupled neural networks in the above sense has not been fully studied by pinning control yet, and it is just the motivation of this paper.

Based on the aforementioned discussions, this paper aims to analyze the synchronization problem for coupled neural networks with the time-varying delay. The synchronization in a different sense is focused on via pinning control and adaptive pinning control respectively. By employing the Lyapunov–Krasovskii (L–K) function theory and an appropriate pinning staggery, the pinning controlled synchronization conditions are derived. Compared with the existing results, the main novelty of this paper lies in two aspects. First, the synchronization state, in terms of the weighted average of all node states, is introduced in coupled neural networks with different time-varying delays. Second, the more simple linear matrix inequalities conditions are established, and then the effectiveness of the proposed method can be checked conveniently by numerical results.

The remainder of this paper is organized as follows. In Sect. 2, the considered coupled neural networks model is formulated and the pinning synchronization problem is introduced. The pinning synchronization conditions are obtained for concerned networks in Sect. 3. Section 4 considers the adaptive pinning synchronization. A numerical example is provided to illustrate the validity of the proposed method in Sect. 5. Some conclusions are made in Sect. 6, together with some potential future study.

Throughout this paper, the superscript ^{T} stands for the transpose of a matrix. $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the n-dimension Euclidean space and set of all $n \times n$ real matrices, respectively. A real symmetric matrix $P > 0$ (≥ 0) denotes P being a positive definite (positive semi-definite) matrix. For a matrix $P \in \mathbb{R}^{n\times n}$, $P^s = \frac{P+P^T}{2}$ is its symmetric part, $\lambda_{max}(P)$ and $\lambda_{min}(P)$ represent its maximum and minimum eigenvalues, respectively. diag$\{\ldots\}$ denotes a block-diagonal matrix, and $\|\cdot\|$ represents the Euclidean norm of a vector and its induced norm of a matrix. I_n stands for an n-dimensional identity matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The symmetric terms in a symmetric matrix are denoted by $*$.

2 Problem Formulation

Consider an array of linearly coupled neural networks consisting of N identical nodes in which the dynamics of the ith $(1 \leqslant i \leqslant N)$ node is described by

$$\dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau_1(t))) + J(t) + c_1\sum_{j=1}^{N} G_{1,ij}\Gamma_1 x_j(t) + c_2\sum_{j=1}^{N} G_{2,ij}\Gamma_2 x_j(t-\tau_2(t)), \quad (1)$$

where $x_i(\cdot) = [x_{i1}(\cdot), \ldots, x_{in}(\cdot)]^T \in \mathbb{R}^n$ is the state vector, $C = \text{diag}\{c_1, \ldots, c_n\}$ is a positive diagonal matrix. $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ represent the connection weight matrix and the delayed connection weight matrix, respectively. $f(x_i(\cdot)) = [f_1(x_{i1}(\cdot)), \ldots, f_n(x_{in}(\cdot))]^T \in \mathbb{R}^n$ denotes the neuron activation function. $J(t) = [J_1(t), \ldots, J_n(t)]^T \in \mathbb{R}^n$ is the external input vector. $\tau_1(t)$ and $\tau_2(t)$ are the time-varying delays, which can be constant or bounded and differentiable functions. The positive constants c_1 and c_2 are the strengths for the constant coupling and delayed coupling, respectively. $G_1 = (G_{1,ij})_{N\times N}$ is the constant coupling configuration matrix defined to be diffusive, i.e., $G_{1,ij} \geqslant 0\,(i \neq j)$ and $G_{1,ii} = -\sum_{j=1,j\neq i}^{N} G_{1,ij}$. The delayed coupling configuration matrix $G_2 = (G_{2,ij})_{N\times N}$ is also diffusive with $G_{2,ij} \geqslant 0$ $(i \neq j)$ and $G_{2,ii} = -\sum_{j=1,j\neq i}^{N} G_{2,ij}$. The coupling matrices G_1 and G_2 are nor required to be symmetric or irreducible.

Assumption 2.1 $\tau_1(t)$ and $\tau_2(t)$ are bounded and differentiable with $\bar{\tau}_1 = \sup_{t\geqslant 0}\tau_1(t)$ and $\bar{\tau}_2 = \sup_{t\geqslant 0}\tau_2(t)$. Meanwhile, $\dot{\tau}_1(t) \leqslant \mu_1 < 1$ and $\dot{\tau}_2(t) \leqslant \mu_2 < 1$ for all $t \geqslant 0$ with $\mu_1 > 0$ and $\mu_2 > 0$.

Assumption 2.2 There exists a constant matrix L such that

$$\|f(x) - f(y)\| \leqslant \|L(x - y)\|, \ \forall x, y \in \mathbb{R}. \tag{2}$$

Remark 2.1 Assumptions 2.1 and 2.2 impose restrictions on the time-varying delays and the activation function, and are widely used in the literature [16,20].

The initial conditions for coupled networks (1) are given by

$$x_{ij}(t) = \varphi_{ij}(t) \in C([-\bar{\tau}, 0], R), \ \forall j = 1, \ldots, n, \ i = 1, \ldots, N, \tag{3}$$

where $\bar{\tau} = \max\{\bar{\tau}_1, \bar{\tau}_2\}$, $C([-\bar{\tau}, 0], \mathbb{R})$ denotes the set of continuous functions from $[-\bar{\tau}, 0]$ to $\mathbb{R}$. Since the synchronization problem is considered, we now give the definition.

Definition 2.1 The coupled neural networks (1) is said to be globally asymptotically synchronizable to the goal trajectory $s(t)$ if the following relations

$$\lim_{t \to \infty} \|x_i(t) - s(t)\| = 0, \ i = 1, \ldots, N$$

hold for the initial conditions (3).

We will address the pinning synchronization of (1) by feedback controllers, and the controlled networks model of the ith $(1 \leqslant i \leqslant N)$ node follows

$$\begin{aligned}\dot{x}_i(t) = &-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t - \tau_1(t))) + J(t) + c_1 \sum_{j=1}^{N} G_{1,ij}\Gamma_1 x_j(t) \\ &+ c_2 \sum_{j=1}^{N} G_{2,ij}\Gamma_2 x_j(t - \tau_2(t)) + u_i(t).\end{aligned} \tag{4}$$

This paper aims to design and implement an appropriate pinning controller for (4) such that all the states of the controlled networks can globally asymptotically synchronize to a desired synchronization $s(t)$, which is presented in terms of a weighted average of all the states in the networks. More specifically, that is

$$s(t) = \sum_{k=1}^{N} \xi_k x_k(t), \tag{5}$$

where $x_i(t)$ is the state of the delayed dynamical networks (4), and ξ_k are chosen to satisfy $\sum_{k=1}^{N} \xi_k = 1$ with $\xi_k > 0$, $i = 1, \ldots, N$. Such synchronization strategy takes into account the contributions and confluences of all the nodes of the dynamical networks, compared with a special solution of the homogeneous systems being the synchronization state in most existing literatures. Moreover, the synchronization state $s(t)$ may also be an equilibrium point, a periodic orbit, or a chaotic attractor, among others [24,27].

The following pinning feedback controller is adopted,

$$u_i(t) = -d_i(x_i(t) - s(t)), \ i = 1, \ldots, N \tag{6}$$

where the feedback control gain $d_i > 0$ if the ith node is pinned, otherwise $d_i = 0$. The next section will provide several sufficient conditions for global asymptotical synchronization, where some special cases are also considered in corollaries.

3 Sufficient Conditions for Global Asymptotical Synchronization

For convenience, we denote that $\Lambda = \mathrm{diag}\{\xi_1, \ldots, \xi_N\}$, $\Omega_1 = (\Lambda^{\frac{1}{2}} G_1 \Lambda^{-\frac{1}{2}}) \otimes \Gamma_1$ and $\Omega_2 = (\Lambda^{\frac{1}{2}} G_2 \Lambda^{-\frac{1}{2}}) \otimes \Gamma_2$. By defining the synchronization error $e_i(t) = x_i(t) - s(t)$, the error dynamical system is governed as follows

$$\begin{aligned}\dot{e}_i(t) &= -Ce_i(t) + A[f(x_i(t)) - f(s(t))] + B[f(x_i(t-\tau_1(t))) - f(s(t-\tau_1(t)))] + \Pi(t) \\ &\quad + c_1 \sum_{j=1}^{N} G_{1,ij}\Gamma_1 e_j(t) + c_2 \sum_{j=1}^{N} G_{2,ij}\Gamma_2 e_j(t-\tau_2(t)) - d_i e_i(t),\ i = 1, \ldots, N \qquad (7)\end{aligned}$$

where

$$\begin{aligned}\Pi(t) &= A\left[f(s(t)) - \sum_{k=1}^{N} \xi_k f(x_k(t))\right] + B\left[f(s(t-\tau_1(t))) - \sum_{k=1}^{N} \xi_k f(x_k(t-\tau_1(t)))\right] \\ &\quad - c_1 \sum_{k=1}^{N} \xi_k \sum_{j=1}^{N} G_{1,kj}\Gamma_1 e_j(t) - c_2 \sum_{k=1}^{N} \xi_k \sum_{j=1}^{N} G_{2,kj}\Gamma_2 e_j(t-\tau_2(t)) + \sum_{j=1}^{N} \xi_j d_j e_j(t).\end{aligned}$$

Before giving the main results, two lemmas are provided first.

Lemma 3.1 ([29, Schur Complement]). *The linear matrix inequality* $\begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} < 0$ *is equivalent to* $S_{22} < 0$ *and* $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$, *where* $S_{11} = S_{11}^T$, $S_{22} = S_{22}^T$.

Lemma 3.2 ([10]). *For a symmetric matrix* $M \in \mathbb{R}^{N\times N}$ *and a diagonal matrix* $D = \mathrm{diag}\{\bar{d}_1, \ldots, \bar{d}_l, \underbrace{0, \ldots, 0}_{N-l}\}$ *with* $\bar{d}_i > 0, i = 1, \ldots, l$ $(1 \leqslant l \leqslant N)$. *Set* $M - D = \begin{bmatrix} E - \widetilde{D} & F \\ F^T & M_l \end{bmatrix}$, *where* M_l *is the minor matrix of* M *by removing its first* l *row-column pairs,* E *and* F *are matrices with appropriate dimensions,* $\widetilde{D} = \mathrm{diag}\{\bar{d}_1, \ldots, \bar{d}_l\}$. *If* $\bar{d}_i > \lambda_{\max}(E - FM_l^{-1}B^T), i = 1, \ldots, l$, *then* $M - D < 0$ *is equivalent to* $M_l < 0$.

Theorem 3.1 *Under Assumptions 2.1–2.2, the pinning controlled networks (4) with controller (6) can globally asymptotically synchronize to the desired synchronization state* $s(t)$ *given by (5), if there exist positive scalars* ϵ *and* γ_2 *such that the following inequality holds*

$$\begin{bmatrix} I_N \otimes \Phi + c_1\Omega_1^s - D \otimes I_n & \frac{1}{2}c_2\Omega_2 \\ * & -\gamma_2(1-\mu_2)I_{nN} \end{bmatrix} < 0, \qquad (8)$$

where $\Phi = -C + \frac{AA^T}{2} + \frac{BB^T}{2} + \frac{L^TL}{2} + (\gamma_1+\gamma_2)I_n$ *with* $\gamma_1 = \frac{\lambda_1+\epsilon}{1-\mu_1}$ *and* $\lambda_1 = \frac{1}{2}\lambda_{\max}(L^TL)$.

Proof Firstly, a Lyapunov–Krasovskii function is constructed for (7) as follows

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} \xi_i e_i^T(t)e_i(t) + \gamma_1 \sum_{i=1}^{N} \xi_i \int_{t-\tau_1(t)}^{t} e_i^T(s)e_i(s)ds + \gamma_2 \sum_{i=1}^{N} \xi_i \int_{t-\tau_2(t)}^{t} e_i^T(s)e_i(s)ds,$$

where γ_1 and γ_2 are positive scalars to be specialized below. According to Assumptions 2.2 and using Kronecker product technique, the derivative of $V(t)$ along the trajectory of system (4) can be calculated as

$$
\begin{aligned}
\dot{V}(t) = & \sum_{i=1}^{N} \xi_i e_i^T(t)\{-Ce_i(t) \\
& + A[f(x_i(t)) - f(s(t))] + B[f(x_i(t-\tau_1(t))) - f(s(t-\tau_1(t)))] \\
& + \gamma_1 \sum_{i=1}^{N} \xi_i e_i^T(t)e_i(t) - \gamma_1(1-\dot{\tau}_1(t)) \sum_{i=1}^{N} \xi_i e_i^T(t-\tau_1(t))e_i(t-\tau_1(t)) \\
& + \gamma_2 \sum_{i=1}^{N} \xi_i e_i^T(t)e_i(t) - \gamma_2(1-\dot{\tau}_2(t)) \sum_{i=1}^{N} \xi_i e_i^T(t-\tau_2(t))e_i(t-\tau_2(t)) \\
& + \sum_{i=1}^{N} \xi_i e_i^T(t)[c_1 \sum_{j=1}^{N} G_{1,ij}\Gamma_1 e_j(t) + c_2 \sum_{j=1}^{N} G_{2,ij}\Gamma_2 e_j(t-\tau_2(t))] \\
& - \sum_{i=1}^{N} d_i \xi_i e_i^T(t)e_i(t) + \sum_{i=1}^{N} \xi_i e_i^T(t)\Pi(t). \qquad (9)
\end{aligned}
$$

Noticing that $\sum_{i=1}^{N} \xi_i e_i(t) = 0$, we have $\sum_{i=1}^{N} \xi_i e_i^T(t)\Pi(t) = 0$. Based on Assumption 2.2 and the fact $2x^T y \leqslant x^T x + y^T y, \forall x, y \in \mathbb{R}^n$, one can make the following estimation

$$
\begin{aligned}
& \sum_{i=1}^{N} \xi_i e_i^T(t)A[f(x_i(t)) - f(s(t))] \\
& = \sum_{i=1}^{N} \xi_k e_i^T(t)A[f(e_i(t)+s(t)) - f(s(t))] \\
& \leqslant \sum_{i=1}^{N} \xi_i \left[e_i^T(t)\frac{AA^T}{2}e_i(t) + \frac{1}{2}\|f(e_i(t)+s(t)) - f(s(t))\|^2 \right] \\
& \leqslant \sum_{i=1}^{N} \xi_i e_i^T(t)\frac{AA^T}{2}e_i(t) + \sum_{i=1}^{N} \xi_i e_i^T(t)\frac{L^T L}{2}e_i(t) \\
& = \sum_{i=1}^{N} \xi_i e_i^T(t)\left[\frac{AA^T}{2} + \frac{L^T L}{2}\right]e_i(t). \qquad (10)
\end{aligned}
$$

Similarly, it follows that

$$
\begin{aligned}
& \sum_{i=1}^{N} \xi_i e_i^T(t)B[f(x_i(t-\tau_1(t))) - f(s(t-\tau_1(t)))] \\
& \leqslant \sum_{i=1}^{N} \xi_i e_i^T(t)\frac{BB^T}{2}e_i(t) + \sum_{i=1}^{N} \xi_i e_i^T(t-\tau_1(t))\frac{L^T L}{2}e_i(t-\tau_1(t)). \qquad (11)
\end{aligned}
$$

By denoting $\nu(t) = [\sqrt{\xi_1}e_1^T(t), \ldots, \sqrt{\xi_N}e_N^T(t)]^T$, the transformations can be made as

$$
\sum_{i=1}^{N} \xi_i e_i^T(t)c_1 \sum_{j=1}^{N} G_{1,ij}\Gamma_1 e_j(t)
$$

$$
\begin{aligned}
&= c_1 \sum_{i=1}^{N} \xi_i e_i^T(t) \sum_{j=1}^{N} G_{1,ij} \Gamma_1 e_j(t) = c_1 e^T(t)[(\Lambda G_1) \otimes \Gamma_1] e(t) \\
&= c_1 \nu^T(t)[(\Lambda^{\frac{1}{2}} G_1 \Lambda^{-\frac{1}{2}}) \otimes \Gamma_1] \nu(t) = c_1 \nu^T(t) \Omega_1^s \nu(t),
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{N} \xi_i e_i^T(t) c_2 \sum_{j=1}^{N} G_{2,ij} \Gamma_2 e_j(t - \tau_2(t)) \\
&= c_2 \sum_{i=1}^{N} \xi_i e_i^T(t) \sum_{j=1}^{N} G_{2,ij} \Gamma_2 e_j(t - \tau_2(t)) = c_2 e^T(t)[(\Lambda G_2) \otimes \Gamma_2] e(t - \tau_2(t)) \\
&= c_2 \nu^T(t)[(\Lambda^{\frac{1}{2}} G_2 \Lambda^{-\frac{1}{2}}) \otimes \Gamma_2] \nu(t - \tau_2(t)) = c_2 \nu^T(t) \Omega_2 \nu(t - \tau_2(t)).
\end{aligned} \tag{13}
$$

Submitting the above inequalities (10)–(13) into (9) yields that

$$
\begin{aligned}
\dot{V}(t) \leqslant & \sum_{i=1}^{N} \xi_i e_i^T(t) \Bigg[-C + \frac{AA^T}{2} \\
& + \frac{BB^T}{2} + \frac{L^T L}{2} \Bigg] e_i(t) \\
& + \nu^T(t)[c_1 \Omega_1^s + (\gamma_1 + \gamma_2) I_N \otimes I_n] \nu(t) \\
& + \sum_{i=1}^{N} \xi_i e_i^T(t - \tau_1(t)) \frac{L^T L}{2} e_i(t - \tau_1(t)) - \gamma_1(1 - \mu_1) \nu^T(t - \tau_1(t)) \nu(t - \tau_1(t)) \\
& + c_2 \nu^T(t) \Omega_2 \nu(t - \tau_2(t)) - \gamma_2(1 - \mu_2) \nu^T(t - \tau_2(t)) \nu(t - \tau_2(t)) \\
& - \nu^T(t)(D \otimes I_N) \nu(t) \\
\leqslant & \ \nu^T(t)[I_N \otimes \Phi + c_1 \Omega_1^s - D \otimes I_n] \nu(t) - \gamma_2(1 - \mu_2) \nu^T(t - \tau_2(t)) \nu(t - \tau_2(t)) \\
& + c_2 \nu^T(t) \Omega_2 \nu(t - \tau_2(t)) + [\lambda_1 - \gamma_1(1 - \mu_1)] \nu^T(t - \tau_1(t)) \nu(t - \tau_1(t)).
\end{aligned} \tag{14}
$$

Choose $\gamma_1 = \frac{\lambda_1 + \epsilon}{1 - \mu_1}$ such that $\lambda_1 - \gamma_1(1 - \mu_1) = -\epsilon < 0$. Then, the above inequality (14) degenerates into

$$
\begin{aligned}
\dot{V}(t) \leqslant & \ \nu^T(t)[I_N \otimes \Phi + c_1 \Omega_1^s - D \otimes I_N] \nu(t) + c_2 \nu^T(t) \Omega_2 \nu(t - \tau_2(t)) \\
& \gamma_2(1 - \mu_2) \nu^T(t - \tau_2(t)) \nu(t - \tau_2(t)) \\
= & \ \eta^T(t) \begin{bmatrix} I_N \otimes \Phi + c_1 \Omega_1^s - D \otimes I_n & \frac{1}{2} c_2 \Omega_2 \\ * & -\gamma_2(1 - \mu_2) I_{nN} \end{bmatrix} \eta(t),
\end{aligned} \tag{15}
$$

where $\eta^T(t) = \left[\nu^T(t) \ \nu^T(t - \tau_2(t)) \right]^T$. Furthermore, from the condition (8) in Theorem 3.1 and (15), it follows that $\dot{V}(t) < 0$, which indicates that the error system (7) is globally asymptotically stable, i.e., $\lim_{t \to \infty} e_i(t) = 0$. Therefore, the pinning controlled networks (4) globally asymptotically synchronize to the desired synchronization state $s(t)$ in (5). Now, the proof follows. □

Remark 3.1 The condition (8) in Theorem 3.1 is given in form of linear matrix inequality (LMI) with respect to the invariants ϵ and γ_2, which can be checked by Matlab Toolbox. In addition, the minimization of the estimations of control gains d_i can be conducted. For example, it is assumed that the same control gain d is chosen, and then $D = d\Sigma$ in Theorem

3.1, where $\Sigma = \mathrm{diag}\{\sigma_1, \ldots, \sigma_N\}$ represents the pinning matrix, and $\sigma_i = 1$ when the ith node is pinned, otherwise $\sigma_i = 0$. The minimization of the estimation of control gain d can be formulated as

$$\min_{\epsilon, \gamma_2} \ d, \quad s.t.\ LMI\ (8)\ holds.$$

In fact, the single control gain also can be optimized by similar method.

When there does not exist the uncoupled discrete delay or the coupling delay in networks model, the controlled system (4) will degenerate into the following forms respectively:

$$\begin{aligned} \dot{x}_i(t) &= -C(t)x_i(t) + Af(x_i(t)) + J(t) + c_1 \sum_{j=1}^{N} G_{1,ij}\Gamma_1 x_j(t) \\ &\quad + c_2 \sum_{j=1}^{N} G_{2,ij}\Gamma_2 x_j(t - \tau_2(t)) + u_i(t), \end{aligned} \tag{16}$$

$$\begin{aligned} \dot{x}_i(t) &= -C(t)x_i(t) + Af(x_i(t)) + Bf(x_i(t - \tau_1(t))) + J(t) \\ &\quad + c_1 \sum_{j=1}^{N} G_{1,ij}\Gamma_1 x_j(t) + u_i(t). \end{aligned} \tag{17}$$

Based on Theorem 3.1, we can derive the corresponding results for the pinning synchronization of the above systems (16) and (17). Now, the following corollaries are proposed, and the detailed proofs are omitted.

Corollary 3.1 *Under Assumptions 2.1–2.2, the pinning controlled networks (16) with controller (6) can globally asymptotically synchronize to the desired synchronization state $s(t)$ in (5), if there exists a positive scalar γ_2 such that the following inequality holds*

$$\begin{bmatrix} I_N \otimes \bar{\Phi} + c_1\Omega_1^s - D \otimes I_n & \frac{1}{2}c_2\Omega_2 \\ * & -\gamma_2(1-\mu_2)I_{nN} \end{bmatrix} < 0$$

with $\bar{\Phi} = -C + \frac{AA^T}{2} + \frac{L^T L}{2} + \gamma_2 I_n$.

Corollary 3.2 *Under Assumption 2.1–2.2, the pinning controlled networks (17) with controller (6) can globally asymptotically synchronize to the desired synchronization state $s(t)$ in (5), if there exists a positive scalar ϵ such that the following inequality holds*

$$I_N \otimes \bar{\bar{\Phi}} + c_1\Omega_1^s - D \otimes I_n < 0,$$

where $\bar{\bar{\Phi}} = -C + \frac{AA^T}{2} + \frac{BB^T}{2} + \frac{L^T L}{2} + \gamma_1 I_n$ *with* $\gamma_1 = \frac{\lambda_1 + \epsilon}{1-\mu_1}$ *and* λ_1 *is defined in Theorem 3.1.*

Remark 3.2 By setting $B = 0,\ \tau_1(t) = 0,\ \gamma_1 = 0$ in Theorem 3.1, we can get Corollary 3.1 on the pinning synchronization problem for system (16). Moreover, letting $G_2 = 0,\ \Gamma_2 = 0,\ \gamma_2 = 0$, Corollary 3.2 can be concluded for system (17). Furthermore, choosing $\mu_1 = \mu_2 = 0$ in Theorem 3.1, one can also derive the corresponding results for the constant delays case. With $\xi_i = \frac{1}{N}$ $(1 \leqslant i \leqslant N)$, the achieved results are the ones for the case that the average of all node states is the desired synchronization state.

Corollary 3.3 *Under Assumptions 2.1–2.2, the pinning controlled networks (4) with controller (6) can globally asymptotically synchronize to the desired synchronization state*

$s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t)$, *if there exist positive scalars* ϵ *and* γ_2 *such that the following inequality holds*

$$\begin{bmatrix} I_N \otimes \Phi + c_1(G_1 \otimes \Gamma_1)^s - D \otimes I_n & \frac{1}{2}c_2 G_2 \otimes \Gamma_2 \\ * & -\gamma_2(1-\mu_2)I_{nN} \end{bmatrix} < 0,$$

where Φ *is the same as the one in Theorem 3.1.*

Remark 3.3 Compared with the recent literatures [24] and [27], which focus on the weighted average synchronization problems, the results in this paper have a better application to some extent. First, only the coupling delay is considered in [24], and it is a constant delay. As for the networks in [27], the coupling delay is not involved. Second, from a computational point of view, the criteria in [24,27] are related to some eigenvalues, while our proposed conditions are in form of linear matrix inequalities, which can be checked conveniently by Matlab LMI Toolbox.

4 Adaptive Pinning Synchronization

Based on the above pinning synchronization analysis, we will further study the adaptive pinning synchronization of the networks (4). Without loss of generality, let the nodes $i_1, \ldots, i_l$ be selected as the pinned nodes and rearrange the order of the nodes such that the first l nodes are pinned or controlled. Therefore, it indicates that $d_i = 0$ for $i = l+1, \ldots, N$ in the pinning controller (6). If d_i for $i = 1, \ldots, l$ are designed to be time-variable, the details are as follows

$$u_i(t) = -d_i(t)e_i(t), \quad \dot{d}_i(t) = q_i e_i^T(t)e_i(t), \quad i = 1, \ldots, l, \tag{18}$$

where q_i are positive constants. The controller $u_i(t)$ in (18) is said to be an adaptive pinning controller. In the following content, we will derive some conditions under which the coupled neural networks (4) can achieve the global synchronization to the desired state $s(t)$ in (5) with the adaptive pinning controller (18). To avoid unnecessary duplication with Theorem 3.1, we will omit some details and give a sketch of the proof.

Theorem 4.1 *Under Assumptions 2.1–2.2, the coupled neural networks* (4) *can globally asymptotically synchronizes to the desired orbit* $s(t)$ *in* (5) *with the adaptive pinning controller* (18)*, if there exist scalars* $\epsilon > 0$ *and* $\gamma_2 > 0$ *such that the following inequality holds*

$$\begin{bmatrix} I_N \otimes \Phi + c_1\Omega_1^s & \frac{1}{2}c_2\Omega_2 \\ * & -\gamma_2(1-\mu_2)I_{nN} \end{bmatrix}_{nl} < 0, \tag{19}$$

where Φ *is the one in Theorem 3.1.*

Proof Construct a Lyapunov function for coupled networks (4) as follows

$$\bar{V}(t) = V(t) + \sum_{i=1}^{l} \xi_i \frac{1}{2q_i}(d_i(t) - d_i^*)^2,$$

where $q_i > 0$ are defined in (18), and $d_i^* > 0$ are constants to be determined below. Using Kronecker product technique and Assumption 2.1, the time derivative of $\bar{V}(t)$ along the trajectory of system (4) can be computed and estimated as

$$\begin{aligned}\dot{\bar{V}}(t) &= \dot{V}(t) + \sum_{i=1}^{l} \xi_i (d_i(t) - d_i^*) e_i^T(t) e_i(t) \\ &\leqslant \nu^T(t)[I_N \otimes \Phi + c_1 \Omega_1^s - D^* \otimes I_n]\nu(t) - \gamma_2(1-\mu_2)\nu^T(t-\tau_2(t))\nu(t-\tau_2(t)) \\ &\quad + c_2 \nu^T(t)\Omega_2 \nu(t-\tau_2(t)) + [\lambda_1 - \gamma_1(1-\mu_1)]\nu^T(t-\tau_1(t))\nu(t-\tau_1(t)), \end{aligned} \tag{20}$$

where $D^* = \text{diag}\{d_1^*, \ldots, d_l^*, \underbrace{0, \ldots, 0}_{N-l}\}$, $\nu(t)$ and λ_1 are the ones in Theorem 3.1. Letting $\gamma_1 = \frac{\lambda_1 + \epsilon}{1-\mu}$ and from (20), it is known that

$$\dot{\bar{V}}(t) \leqslant \eta^T(t) \begin{bmatrix} I_N \otimes \Phi + c_1 \Omega_1^s - D^* \otimes I_n & \frac{1}{2} c_2 \Omega_2 \\ * & -\gamma_2(1-\mu_2) I_{nN} \end{bmatrix} \eta(t), \tag{21}$$

where $\eta(t)$ has been defined in Theorem 3.1, and $\bar{D} = \text{diag}\{d_1^*, \ldots, d_l^*, \underbrace{0, \ldots, 0}_{2N-l}\}$.

On the other hand, denote the matrix $M = \begin{bmatrix} I_N \otimes \Phi + c_1 \Omega_1^s & \frac{1}{2} c_2 \Omega_2 \\ * & -\gamma_2(1-\mu_2) I_{nN} \end{bmatrix}$, which is symmetric, and let

$$M - \bar{D} \otimes I_n = \begin{bmatrix} \widehat{A} - \widehat{D} & \widehat{B} \\ \widehat{B}^T & (M)_{nl} \end{bmatrix},$$

where $\widehat{D} = \widehat{\bar{D}} \otimes I_n$ with $\widehat{\bar{D}} = \text{diag}\{d_1^*, \ldots, d_l^*\}$, and $\widehat{A}$, $\widehat{B}$ are matrices with appropriate dimensions. According to Lemma 2.1, if the positive constants d_i^* satisfy $d_i^* > \lambda_{\max}(\widehat{A} - \widehat{B}(M)_{nl}^{-1}\widehat{B}^T)$, $(i = 1, \ldots, l)$, then $(M)_{nl} < 0$ can give that $M - \bar{D} \otimes I_n < 0$. Together with (19) and (21), we have $\dot{V}(t) \leqslant 0$ and $\dot{V}(t) = 0$ if and only if $e(t) = e(t - \tau(t)) = 0_{nN}$. Therefore, the pinning controlled coupled neural networks (4) can globally asymptotically synchronize to the trajectory $s(t)$ in (5) by the adaptive pinning controller (18) if the condition (19) holds. This completes the proof. □

Remark 4.1 In the adaptive pinning synchronization design, we can rearrange network nodes in ascending order such that the first l nodes are pinned candidates. Based on the out-degree and in-degree of all the nodes, the pinned nodes can be selected in this paper. The details can be seen in [10] and [16].

5 Numerical Example

We consider the pinning synchronization of coupled neural networks (4) with ten nodes (i.e., $N = 10$), and the relevant parameters are given by [10,30]:

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 3.0 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}.$$

The activation function is descried as $f(x_i(t)) = [\tanh(x_{i1}(t)), \tanh(x_{i2}(t))]^T$. It can be seen from Assumption 2.1 that $L = \text{diag}\{1.0, 1.0\}$. The coupling matrix G_1 is given by (22), which is determined by a directed topology (See Fig. 2 in [10]), and the delayed coupling matrix is set to be $G_2 = 0.5G_1$. The coupling strengths are chosen as $c_1 = 120$, $c_2 = 5$. The weighted average values ξ_i are chosen randomly in (0, 1). Letting $e_i(t) =$

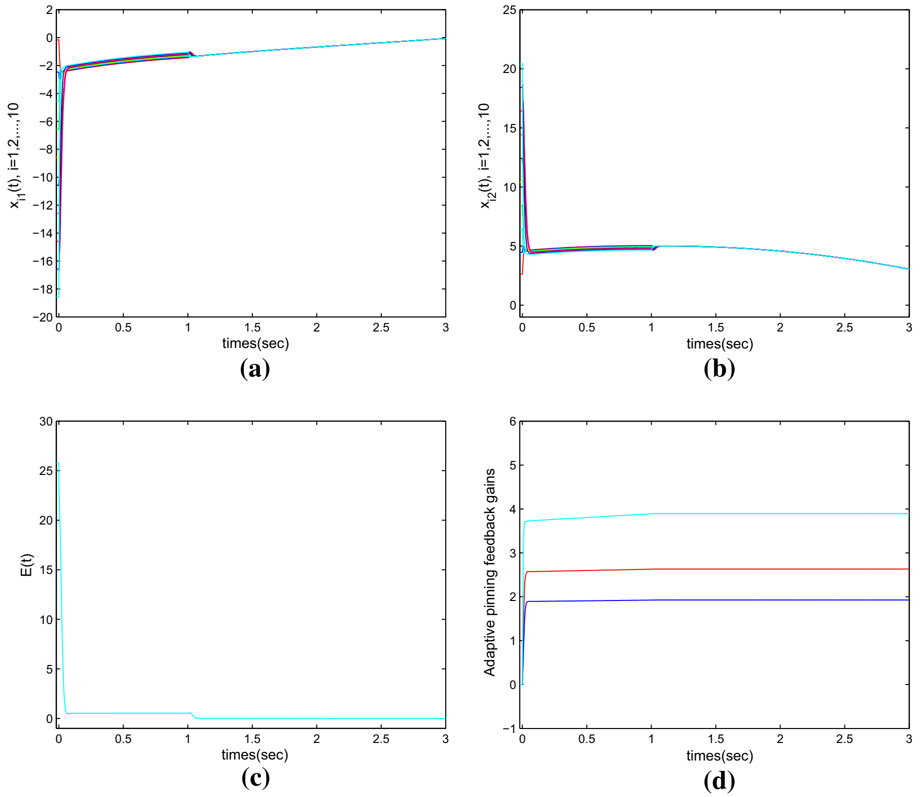

Fig. 1 Constant time delays case. **a** State trajectories, **b** state trajectories, **c** synchronization error, **d** adaptive pinning feedback gains

$x_i(t) - s(t)$ with $s(t) = \sum_{i=1}^{10} \xi_i x_i(t)$, and the synchronization error is measured by $E(t) = \sqrt{\frac{1}{10}\sum_{i=1}^{10} e_i^T(t) e_i(t)}$.

$$G_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \tag{22}$$

First, we set the time delays are constant scalars $\tau_1(t) = \tau_2(t) = 1.0$ with $\mu_1 = \mu_2 = 0$, and the inner connecting matrices are defined as $\Gamma_1 = \Gamma_2 = I_n$. For the case, the pinning synchronization to the isolate node is studied in [10]. Now, we deal with the pinning synchronization to the weighted average of all node states. Employing Theorem 4.1 and pinning the nodes 3, 4, 9, we can check the synchronization of the concerned system to $s(t)$ by LMI

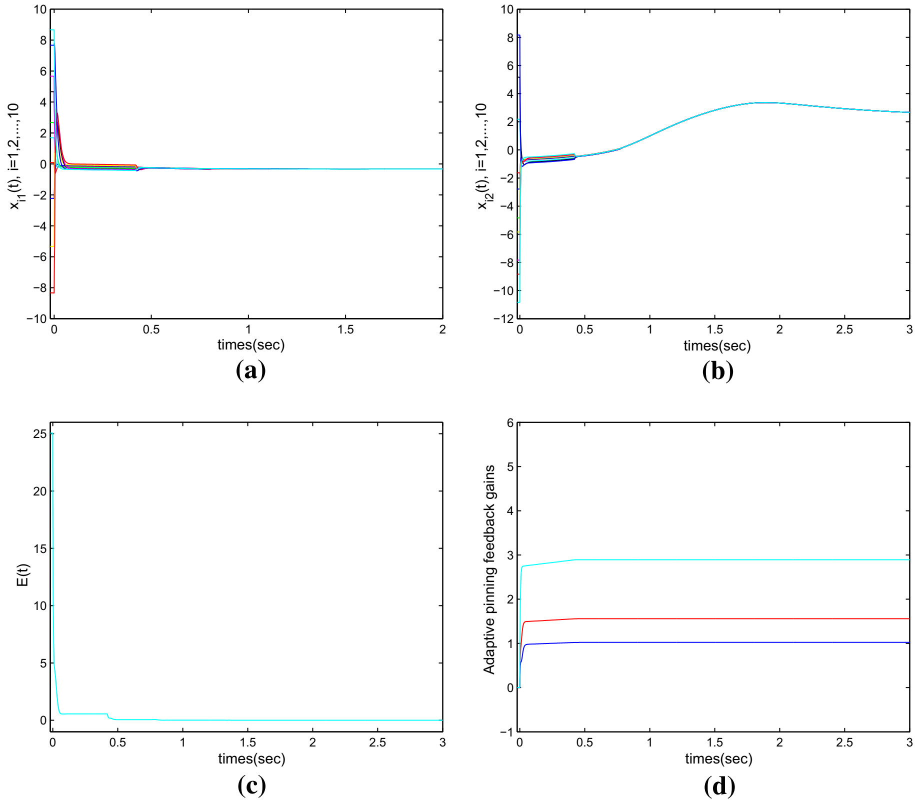

Fig. 2 Time-varying delays case. **a** State trajectories, **b** state trajectories, **c** synchronization error, **d** adaptive pinning feedback gains

Matlab toolbox with controller (18), and obtain that the feasible solutions are $\epsilon = 1.5894$ and $\gamma_2 = 24.4658$. The state trajectories, the synchronization error $E(t)$ and the adaptive feedback gains are illustrated in Fig. 1. It shows that the coupled neural networks with parameters in (22) can synchronize to the state $s(t)$ under the adaptive pinning controller (18).

For the case that the time delays are different time-varying functions $\tau_1(t) = 0.7 - 0.3\cos(t)$ and $\tau_2(t) = 0.5 - 0.2\sin(t)$ and resorting to Matlab LMI toolbox, solving the inequality (19) for setting $\mu_1 = 0.3$ and $\mu_2 = 0.2$ leads to some feasible solutions $\epsilon = 1.9892$ and $\gamma_2 = 21.8983$. The corresponding simulation results are given in Fig. 2.

Remark 5.1 Since the weighted average synchronization problem is concerned, the results in [10,30] may not be employed here. In addition, for the case that $\tau_1(t)$ and $\tau_2(t)$ are different time-varying functions, the methods in [24,27] may not work.

6 Concluding Remarks

This paper has investigated the problem of pinning synchronization for a class of coupled neural networks with time-varying delays. The Lyapunov–Krasovskii function method is developed to derive some sufficient criteria, which can guarantee that the pinning controlled networks globally asymptotically synchronize to the desired state that is in terms of the weight average of all node states in networks. Further study can focus on some advanced analysis technique to obtain the less conservative results, the synchronization of switched networks, and other control methods for complex networks.

Acknowledgements This research was supported in part by the Projects of International Cooperation and Exchanges NSFC (61520106009), in part by the National Natural Science Foundation of China (61403027), in part by the Fundamental Research Funds for the Central Universities (FRF-TP-15-071A1), and in part by the Science Foundation of China Postdoctoral under Grant 2014M561543.

References

1. Narendra K, Parthasarathy K (1990) Identification and control of dynamical systems using neural networks. IEEE Trans Neural Netw 1(1):4–27
2. Bishop C (1995) Neural networks for pattern recognition. Oxford University Press, New York
3. Watta PB, Wang K, Hassoun MH (1997) Recurrent neural nets as dynamical boolean systems with application to associative memory. IEEE Trans Neural Netw 8(5):1268–1280
4. Arunkumar A, Sakthivel R, Mathiyalagan K (2015) Robust reliable H_∞ control for stochastic neural networks with randomly occurring delays. Neurocomputing 149:1524–1534
5. Sakthivel R, Vadivel P, Mathiyalagan K, Arunkumar A, Sivachitra M (2015) Design of state estimation for bidirectional associative memory neural networks with leakage delays. Inf Sci 296:263–274
6. Wu YY, Cao JD, Alofic A, AL-Mazrooeic A, Elaiwc A (2015) Finite-time boundedness and stabilization of uncertain switched neural networks with time-varying delay. Neural Netw 69:135–143
7. Cao J, Li P, Wang W (2006) Global synchronization in arrays of delayed neural networks with constant and delayed coupling. Phys Lett A 353:318–325
8. Lu J, Ho DWC, Liu M (2007) Globally exponential synchronization in an array of asymmetric coupled neural networks. Phys Lett A 369:444–451
9. He W, Cao JD (2009) Global synchronization in arrays of coupled networks with one single time-varying delay coupling. Phys Lett A 373(31):2682–2694
10. Song Q, Cao JD, Liu F (2012) Pinning synchronization of linearly coupled delayed neural networks. Math Comput Simul 86:39–51
11. Tang Y, Wong WK (2013) Distributed synchronization of coupled neural networks via randomly occurring control. IEEE Trans Neural Netw Learn Syst 24(3):43–447
12. Li P, Cao J, Wang Z (2007) Robust impulsive synchronization of coupled delayed neural networks with uncertainties. Phys A 373:261–272
13. Liu X, Chen T (2011) Cluster synchronization in directed networks via intermittent pinning control. IEEE Trans Neural Netw 22(7):1009–1020
14. Wang X, Chen G (2002) Pinning control of scale-free dynamical networks. Phys A 310:521–531
15. Chen T, Liu X, Lu W (2007) Pinning complex networks by a single controller. IEEE Trans Circuits Syst I(54):1317–1326
16. Song Q, Cao J (2010) On pinning synchronization of directed and undirected complex dynamical networks. IEEE Trans Circuits Syst I(57):672–680
17. Yu W, Chen G, Lü J (2009) On pinning synchronization of complex dynamical networks. Automatica 45(2):429–435
18. Zhou J, Lu J (2008) Pinning adaptive synchronization of a general complex dynamical network. Automatica 44(4):996–1003
19. Zheng C, Cao J (2014) Robust synchronization of coupled neural networks with mixed delays and uncertain parameters by intermittent pinning control. Neurocomputing 141:153–159
20. Li L, Cao J (2011) Cluster synchronization in an array of coupled stochastic delayed neural networks via pinning control. Neurocomputing 74:846–856
21. Sorrentino F, Bernardo M, Garafalo F, Chen G (2007) Controllability of complex networks via pinning. Phys Rev E 75 (4), Article ID: 046103, 6 p
22. Wang Q, Yu C, Gao H (2014) Synchronization of identical linear dynamic systems subject to input saturation. Syst Control Lett 64:107–113
23. Wang Q, Yu C, Gao H (2014) Semiglobal synchronization of multiple generic linear agents with input saturation. Int J Robust Nonlinear Control 24(18):3239–3254

24. Zhou J, Wu Q, Xiang L, Cai S, Liu Z (2011) Impulsive synchronization seeking in general complex delayed dynamical networks. Nonlinear Anal 5:513–524
25. Cai S, Zhou J, Xiang L, Liu Z (2008) Robust impulsive synchronization of complex delayed dynamical networks. Phys Lett A 372:4990–4995
26. Zhou J, Chen T (2006) Synchronization in general complex delayed dynamical networks. IEEE Trans Circuits Syst I(53):733–744
27. Wu Q, Zhang H (2015) Drive network to a desired orbit by pinning control. Kybernetika 51(1):150–172
28. Cai ZM, Zhou PP, Liu ZR (2015) Synchronization analysis of hybrid-coupled delayed dynamical networks with impulsive effects: a unified synchronization criterion. J Frankl Inst 352:2065–2089
29. Boyd SP, El Ghaoui L, Feron E, Balakrishnan V (1994) Linear matrix inequalities in system and control theory, vol 15. Society for industrial and applied mathematics (SIAM), Philadelphia
30. Lu W, Chen T (2004) Synchronization of coupled connected neural networks with delays. IEEE Trans Circuits Syst I 51(12):2491–2503