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Pseudo Almost Periodic Solutions for SICNNs with Leakage Delays and Complex Deviating Arguments

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Abstract This paper is concerned with the existence of pseudo almost periodic solutions for shunting inhibitory cellular neural networks with leakage delays and complex deviating arguments. Applying the contraction mapping fixed point theorem and inequality techniques, some sufficient conditions are presented to ensure the existence of pseudo almost periodic solutions of this model, which are new and supplement some previously known ones.

Keywords Shunting inhibitory cellular neural networks · Pseudo almost periodic solution · Existence · Leakage delay · Complex deviating argument

1 Introduction

In the last two decades, a leakage delay, which is the time delay in the leakage term of the neural networks systems and a considerable factor affecting dynamics for the worse in the systems, is being put to use in the problem of stability for neural networks (see [1–5]). Such time delays in the leakage term are difficult to handle but has great impact on the dynamical behavior of shunting inhibitory cellular neural networks (SICNNs). Therefore, people have paid much attention to the problem on almost periodic solutions and pseudo almost periodic solutions for SICNNs with leakage delays because of its successful applications in variety of areas such as psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing (see [6–8] and the references therein).

On the other hand, over the past several years it has become apparent that functional differential equations with state-dependent delays arise in several areas such as in classical electrodynamics [9], in population models [10], in models of commodity price fluctuations [11], in models of blood cell productions [12–15], and in models of bidirectional associative memory networks (BAMs) and cellular neural networks (CNNs) [16,17]. For more infor-

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mation about BAMs, CNNs and other functional differential equations with state-dependent delays, we refer the reader to [18] and the references therein. It is well known that the CNNs with complex deviating arguments is a special type of state-dependent delay-differential equations. Recently, sufficient conditions for the existence of periodic solutions of continuous-time CNNs and discrete-time CNNs with complex deviating arguments have been obtained in [19,20]. However, to the best of our knowledge, there is no result on the existence of almost periodic solutions and pseudo almost periodic solutions of SICNNs with complex deviating arguments.

Motivated by the above discussions, in this paper, we will consider the existence of pseudo almost periodic solutions for the following SICNNs with leakage delays and complex deviating arguments:

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t-\eta_{ij}(t)) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f(x_{kl}(x_{kl}(t)))x_{ij}(t) + L_{ij}(t), \quad (1.1)$$

where $ij \in J := \{11, \ldots, 1n,\ 21, \ldots, 2n, \ldots,\ m1, \ldots, mn\}$, C_{ij} denotes the cell at the (i, j) position of the lattice. The r-neighborhood $N_r(i, j)$ of C_{ij} is given as

$$N_r(i, j) = \{C_{kl} : \max(|k-i|, |l-j|) \le r, 1 \le k \le m, 1 \le l \le n\},$$

x_{ij} is the activity of the cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, the function $a_{ij}(t) > 0$ represents the passive decay rate of the cell activity, $C_{ij}^{kl}(t)$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and the activity function $f(\cdot)$ is a continuous function representing the output or firing rate of the cell C_{kl}, and $\eta_{ij}(t) \ge 0$ corresponds to the leakage delay.

For convenience, we denote by $\mathbb{R}^\iota(\mathbb{R} = \mathbb{R}^1)$ the set of all $\iota-$dimensional real vectors (real numbers). We will use

$$\{x_{ij}(t)\} = (x_{11}(t), \ldots, x_{1n}(t), \ldots, x_{i1}(t), \ldots, x_{in}(t), \ldots, x_{m1}(t), \ldots, x_{mn}(t)) \in \mathbb{R}^{m\times n}.$$

For any $x(t) = \{x_{ij}(t)\} \in \mathbb{R}^{m\times n}$, we let $|x|$ denote the absolute-value vector given by $|x| = \{|x_{ij}|\}$, and define $\|x(t)\| = \max_{ij\in J}\{|x_{ij}(t)|\}$. A matrix or vector $A \ge 0$ means that all entries of A are greater than or equal to zero. $A > 0$ can be defined similarly. For matrices or vectors A_1 and A_2, $A_1 \ge A_2$ (resp. $A_1 > A_2$) means that $A_1 - A_2 \ge 0$ (resp. $A_1 - A_2 > 0$). Let $BC(\mathbb{R}, \mathbb{R}^{m\times n})$ be the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^{m\times n}$, and $BUC(\mathbb{R}, \mathbb{R}^{m\times n})$ be the set of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^{m\times n}$. Clearly, $(BC(\mathbb{R}, \mathbb{R}^{m\times n}), \|\cdot\|_\infty)$ is a Banach space, where $\|\cdot\|_\infty$ denotes the supremum norm $\|f\|_\infty := \sup_{t\in\mathbb{R}}\|f(t)\|$. Given a bounded and continuous function h defined on $\mathbb{R}$, let h^+ and h^- be defined as

$$h^+ = \sup_{t\in\mathbb{R}}|h(t)|, \quad h^- = \inf_{t\in\mathbb{R}}|h(t)|.$$

We also make the following assumptions.

(S_1) there exist constants M_f and L^f such that

$$|f(u) - f(v)| \le L^f|u - v|, \quad |f(u)| \le M_f, \quad \text{for all } u, v \in \mathbb{R}.$$

(S_2) there exist positive constants κ and ζ such that

$$0 < \kappa \le L,\ \max_{ij\in J}\left\{\frac{1}{a_{ij}^-}E_{ij},\ \left(1+\frac{a_{ij}^+}{a_{ij}^-}\right)E_{ij}\right\} \le \kappa,\quad \max_{ij\in J}\left\{\frac{1}{a_{ij}^-}F_{ij},\ \left(1+\frac{a_{ij}^+}{a_{ij}^-}\right)F_{ij}\right\} < 1,$$

and

$$\left\{\left[a_{ij}^{+}+a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl\ +}(L^{f}(\kappa+L)+|f(0)|)\right](\kappa+L)+L_{ij}^{+}\right\}\leq\{\zeta\},$$

where

$$E_{ij}=\left[a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl\ +}(L^{f}(\kappa+L)+|f(0)|)\right](\kappa+L),\quad ij\in J,$$

$$F_{ij}=a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl\ +}(M_f+L^{f}(1+\zeta)(\kappa+L)),\quad ij\in J,$$

and

$$L=\max\left\{\max_{ij\in J}\left\{\sup_{t\in R}\left|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}(u)du}L_{ij}(s)\,ds\right|\right\},\right.$$
$$\left.\max_{ij\in J}\left\{\sup_{t\in R}\left|L_{ij}(t)-a_{ij}(t)\int_{-\infty}^{t}e^{-\int_s^t a_{ij}(u)\,du}L_{ij}(s)ds\right|\right\}\right\}$$
$$>0.$$

The paper is organized as follows. Section 2 includes some lemmas and definitions, which can be used to check the existence of pseudo almost periodic solutions of (1.1). In Sect. 3, we present some new sufficient conditions for the existence of the continuously differentiable pseudo almost periodic solution of (1.1). At last, an example is given to illustrate the effectiveness of the obtained results.

2 Preliminary Results

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

Definition 2.1 (*see* [21,22]). Let $u(t)\in BC(\mathbb{R},\mathbb{R}^{m\times n})$. $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon>0$, the set $T(u,\varepsilon)=\{\delta:\|u(t+\delta)-u(t)\|<\varepsilon\ \text{ for all }\ t\in\mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon>0$, it is possible to find a real number $l=l(\varepsilon)>0$ with the property that, for any interval with length $l(\varepsilon)$, there exists a number $\delta=\delta(\varepsilon)$ in this interval such that $\|u(t+\delta)-u(t)\|<\varepsilon$, for all $t\in\mathbb{R}$.

We denote by $AP(\mathbb{R},\mathbb{R}^{m\times n})$ the set of the almost periodic functions from $\mathbb{R}$ to $\mathbb{R}^{m\times n}$. Besides, the concept of pseudo almost periodicity was introduced by C. Zhang in the early nineties of 20 centuries. It is a natural generalization of the classical almost periodicity. Precisely, define the class of functions $PAP_0(\mathbb{R},\mathbb{R}^{m\times n})$ as follows:

$$\left\{g\in BC(\mathbb{R},\mathbb{R}^{m\times n})|\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|\,dt=\mathbf{0}\right\}.$$

A function $\varphi\in BC(\mathbb{R},\mathbb{R}^{m\times n})$ is called pseudo almost periodic if it can be expressed as

$$\varphi=h+g,$$

where $h\in AP(\mathbb{R},\mathbb{R}^{m\times n})$ and $g\in PAP_0(\mathbb{R},\mathbb{R}^{m\times n})$. The collection of such functions will be denoted by $PAP(\mathbb{R},\mathbb{R}^{m\times n})$. The functions h and g in above definition are respectively called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function φ.

It will be assumed that $a_{ij} : \mathbb{R} \to (0 \ +\infty)$ is an almost periodic function, $\eta_{ij} : \mathbb{R} \to [0 \ +\infty)$ and $L_{ij},\ C_{ij}^{kl} : \mathbb{R} \to \mathbb{R}$ are pseudo almost periodic functions, where $ij, kl \in J$. later.

Lemma 2.1 (see [8, Lemma 2.2]). *Let $\mathbf{Z} = \{f | f, f' \in PAP(\mathbb{R}, \mathbb{R}^{m\times n})\}$ be equipped with the induced norm defined by*

$$\|f\|_{\mathbf{Z}} = \max\{\|f\|_\infty, \|f'\|_\infty\} = \max\left\{\sup_{t\in\mathbb{R}} \|f(t)\|, \sup_{t\in\mathbb{R}} \|f'(t)\|\right\}.$$

Then, $\mathbf{Z}$ is a Banach space.

Let

$$B^\zeta = \{\varphi | \varphi \in PAP(\mathbb{R}, \mathbb{R}^{m\times n}), \quad \{|\varphi_{ij}(t_1) - \varphi_{ij}(t_2)|\} \le \{\zeta|t_1 - t_2|\}, \quad \textit{for all } \ t_1, t_2 \in \mathbb{R}\},$$

and

$$B^* = \left\{\varphi \middle| ||\varphi - \varphi_0||_{\mathbf{Z}} \le \kappa, \quad \varphi \in \mathbf{Z}\right\}, \quad \textit{where } \varphi_0 = \left\{\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(w)dw} L_{ij}(s)\, ds\right\}.$$

Lemma 2.2 *B^ζ is a closed subset of $PAP(\mathbb{R}, \mathbb{R}^{m\times n})$.*

Proof Suppose that $\{x_p\}_{p=1}^{+\infty} \subseteq B^\zeta$ satisfies

$$\lim_{p\to+\infty} \|x_p - \varphi\|_\infty = 0. \tag{2.1}$$

Obviously, $\varphi \in PAP(\mathbb{R}, \mathbb{R}^{m\times n})$. We next show that

$$\{|\varphi_{ij}(t_1) - \varphi_{ij}(t_2)|\} \le \{\zeta|t_1 - t_2|\}, \quad \text{for all } \ t_1, t_2 \in \mathbb{R}. \tag{2.2}$$

Let $t_1, t_2 \in \mathbb{R}$, for any $\varepsilon > 0$, from (2.1), we can choose $p > 0$ such that

$$\|x_p - \varphi\|_\infty < \frac{\varepsilon}{2}. \tag{2.3}$$

Since $x_p \in B^\zeta$, we obtain

$$|x_p(t_1) - x_p(t_2)| \le \{\zeta|t_1 - t_2|\},$$

which, together with (2.3), implies that

$$\begin{aligned} |\varphi(t_1) - \varphi(t_2)| &\le |\varphi(t_1) - x_p(t_1)| + |x_p(t_1) - x_p(t_2)| + |x_p(t_2) - \varphi(t_2)| \\ &< \{\varepsilon + \zeta|t_1 - t_2|\}. \end{aligned} \tag{2.4}$$

Letting $\varepsilon \to 0$, (2.4) provides the fact that (2.2) is true. Hence, B^ζ is a closed subset of $PAP(\mathbb{R}, \mathbb{R}^{m\times n})$. This completes the proof of Lemma 2.2. □

Definition 2.2 (see [21,22]). Let $x \in \mathbb{R}^\iota$ and $Q(t)$ be a $\iota \times \iota$ continuous matrix defined on $\mathbb{R}$. The linear system

$$x'(t) = Q(t)x(t) \tag{2.5}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (2.5) satisfying

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\le ke^{-\alpha(t-s)} \qquad \text{for } t \ge s, \\ \|X(t)(I - P)X^{-1}(s)\| &\le ke^{-\alpha(s-t)} \qquad \text{for } t \le s. \end{aligned}$$

Lemma 2.3 (see [21]) *Assume that $Q(t)$ is an almost periodic matrix function and $g(t) \in PAP(\mathbb{R}, \mathbb{R}^{\iota})$. If the linear system (2.5) admits an exponential dichotomy, then pseudo almost periodic system*

$$x'(t) = Q(t)x(t) + g(t) \tag{2.6}$$

has a unique pseudo almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)\, ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)g(s)\, ds. \tag{2.7}$$

Lemma 2.4 (see [21,22])
Let $c_i(t)$ be an almost periodic function on $\mathbb{R}$ and

$$M[c_i] = \lim_{T\to+\infty} \frac{1}{T}\int_{t}^{t+T} c_i(s)\, ds > 0, \quad i = 1, 2, \ldots, \iota.$$

Then the linear system

$$x'(t) = \text{diag}\,(-c_1(t), -c_2(t), \ldots, -c_{\iota}(t))x(t)$$

admits an exponential dichotomy on $\mathbb{R}$.

3 Existence of Pseudo Almost Periodic Solutions

In this section, we establish sufficient conditions on the existence of pseudo almost periodic solutions of (1.1).

Theorem 3.1 *Let (S_1) and (S_2) hold. Then there exists a unique pseudo almost periodic solution of equation (1.1) in the region $\mathbf{B} = B^{\zeta} \bigcap B^{*}$.*

Proof It follows from Lemmas 2.1 and 2.2 that $\mathbf{B}$ is a closed subset of $\mathbf{Z}$. Let $\varphi \in \mathbf{B}$. Obviously, the boundedness of φ' and (S_1) imply that f and φ_{ij} are uniformly continuous functions on $\mathbb{R}$ for $ij \in J$. Set $\widetilde{f}(t, z) = \varphi_{ij}(t - z)(ij \in J)$. By Theorem 5.3 in [21, p. 58] and Definition 5.7 in [21, p. 59], we can obtain that $\widetilde{f} \in PAP(\mathbb{R} \times \Omega)$ and $\widetilde{f}$ is continuous in $z \in K$ and uniformly in $t \in \mathbb{R}$ for all compact subset K of $\Omega \subset \mathbb{R}$. This, together with $\eta_{ij} \in PAP(\mathbb{R}, \mathbb{R})$ and Theorem 5.11 in [21, p. 60], implies that

$$\varphi_{ij}(t - \eta_{ij}(t)) \in PAP(\mathbb{R}, \mathbb{R}),\ ij \in J.$$

From Corollary 5.4 in [21, p. 58], we have

$$f(\varphi_{ij}(\varphi_{ij}(t))) \in PAP(\mathbb{R}, \mathbb{R}),\ ij \in J,$$

which, together with the fact that $\varphi_{ij}(t - \eta_{ij}(t)) \in PAP(\mathbb{R}, \mathbb{R})$, implies

$$a_{ij}(t)\int_{t-\eta_{ij}(t)}^{t} \varphi'_{ij}(s)\, ds = a_{ij}(t)\varphi_{ij}(t) - a_{ij}(t)\varphi_{ij}(t - \eta_{ij}(t)) \in PAP(\mathbb{R}, \mathbb{R}),\ ij \in J,$$

and

$$-\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) f(\varphi_{kl}(\varphi_{kl}(t)))\varphi_{ij}(t) + L_{ij}(t) \in PAP(\mathbb{R}, \mathbb{R}),\ ij \in J.$$

For any $\varphi \in \mathbf{B}$, we consider the pseudo almost periodic solution $x^{\varphi}(t)$ of nonlinear pseudo almost periodic differential equations

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t) + a_{ij}(t)\int_{t-\eta_{ij}(t)}^{t} \varphi'_{ij}(s)\,ds - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) f(\varphi_{kl}(\varphi_{kl}(t)))\varphi_{ij}(t) + L_{ij}(t), \quad ij \in J. \tag{3.1}$$

Then, notice that $M[a_{ij}] > 0, \ \ ij \in J$, it follows from Lemma 2.4 that the linear system

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t), \quad ij \in J, \tag{3.2}$$

admits an exponential dichotomy on $\mathbb{R}$. Thus, by Lemma 2.3, we obtain that the system (3.1) has exactly one pseudo almost periodic solution:

$$\begin{aligned} x^{\varphi}(t) &= \{x_{ij}^{\varphi}(t)\} \\ &= \left\{\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)\,du}\left[a_{ij}(s)\int_{s-\eta_{ij}(s)}^{s} \varphi'_{ij}(u)\,du \right.\right. \\ &\quad \left.\left. - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) f(\varphi_{kl}(\varphi_{kl}(s)))\varphi_{ij}(s) + L_{ij}(s)\right] ds\right\}. \end{aligned} \tag{3.3}$$

Let

$$\Psi_{ij}(t) = a_{ij}(t)\int_{t-\eta_{ij}(t)}^{t} \varphi'_{ij}(s)\,ds - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) f(\varphi_{kl}(\varphi_{kl}(t)))\varphi_{ij}(t) + L_{ij}(t), \ ij \in J.$$

Then, $\{\Psi_{ij}\} \in PAP(\mathbb{R}, \mathbb{R}^{m\times n})$, and Lemmas 2.3 and 2.4 imply that

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t) + \Psi_{ij}(t), \ ij \in J,$$

has exactly one pseudo almost periodic solution

$$\left\{\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)\,du}\Psi_{ij}(s)ds\right\} \in PAP(\mathbb{R}, \mathbb{R}^{m\times n}). \tag{3.4}$$

Then, $\{\Psi_{ij}\} \in PAP(\mathbb{R}, \mathbb{R}^{m\times n})$ and (3.3) imply that

$$(x^{\varphi}(t))' = \{\Psi_{ij}(t) - a_{ij}(t)x^{\varphi}(t)\} \in PAP(\mathbb{R}, \mathbb{R}^{m\times n}). \tag{3.5}$$

Now, we define a mapping $T : \mathbf{B} \to PAP(\mathbb{R}, \mathbb{R}^{m\times n})$ by setting

$$(T\varphi)(t) = x^{\varphi}(t), \quad \forall \varphi \in \mathbf{B}.$$

First we show that for any $\varphi \in \mathbf{B}, \ T\varphi = x^{\varphi} \in \mathbf{B}$.
If $\varphi \in \mathbf{B}$, then

$$L = \|\varphi_0\|_{\mathbf{Z}}, \ \ \|\varphi\|_{\mathbf{Z}} \le \|\varphi - \varphi_0\|_{\mathbf{Z}} + \|\varphi_0\|_{\mathbf{Z}} \le \kappa + L. \tag{3.6}$$

Note that

$$
\begin{aligned}
&|(T\varphi)(t) - \varphi_0(t)| \\
&= \left\{ \left| \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)\,du} \left[a_{ij}(s) \int_{s-\eta_{ij}(s)}^{s} \varphi'_{ij}(u)\,du \right.\right.\right. \\
&\quad \left.\left.\left. - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) f(\varphi_{kl}(\varphi_{kl}(s)))\varphi_{ij}(s) \right] ds \right| \right\} \\
&\le \left\{ \int_{-\infty}^{t} e^{-\int_s^t a_{ij}^-\,du} \left[a_{ij}^+ \eta_{ij}^+ \|\varphi\|_{\mathbf{Z}} \right.\right. \\
&\quad \left.\left. + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\ +} (|f(\varphi_{kl}(\varphi_{kl}(s))) - f(0)| + |f(0)|)\|\varphi\|_{\mathbf{Z}} \right] ds \right\} \\
&\le \left\{ \frac{1}{a_{ij}^-} \left[a_{ij}^+ \eta_{ij}^+ + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\ +} (L^f \|\varphi\|_B + |f(0)|) \right] \|\varphi\|_{\mathbf{Z}} \right\} \\
&\le \left\{ \frac{1}{a_{ij}^-} \left[a_{ij}^+ \eta_{ij}^+ + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\ +} (L^f(\kappa + L) + |f(0)|) \right] (\kappa + L) \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&|((T\varphi)(t) - \varphi_0(t))'| \\
&= \left\{ \left| \left[a_{ij}(t) \int_{t-\eta_{ij}(t)}^{t} \varphi'_{ij}(s)\,ds - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) f(\varphi_{kl}(\varphi_{kl}(t)))\varphi_{ij}(t) \right] \right.\right. \\
&\quad - a_{ij}(t) \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)\,du} \left[a_{ij}(s) \int_{s-\eta_{ij}(s)}^{s} \varphi'_{ij}(u)\,du \right. \\
&\quad \left.\left.\left. - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) f(\varphi_{kl}(\varphi_{kl}(s)))\varphi_{ij}(s) \right] ds \right| \right\} \\
&\le \left\{ \left(1 + \frac{a_{ij}^+}{a_{ij}^-} \right) \left[a_{ij}^+ \eta_{ij}^+ + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\ +} (L^f \|\varphi\|_{\mathbf{Z}} + |f(0)|) \right] \|\varphi\|_{\mathbf{Z}} \right\} \\
&\le \left\{ \left(1 + \frac{a_{ij}^+}{a_{ij}^-} \right) \left[a_{ij}^+ \eta_{ij}^+ + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\ +} (L^f(\kappa + L) + |f(0)|) \right] (\kappa + L) \right\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|T\varphi - \varphi_0\|_{\mathbf{Z}} &\le \max_{(i,j)} \left\{ \frac{1}{a_{ij}^-} E_{ij}, \left(1 + \frac{a_{ij}^+}{a_{ij}^-} \right) E_{ij} \right\} \le \kappa, \\
||T\varphi||_\infty &\le ||T\varphi - \varphi_0||_\infty + ||\varphi_0||_\infty \le \kappa + L, \\
|((T\varphi)(t))'| &= \left\{ \left| -a_{ij}(t)((T\varphi)(t))_{ij} + a_{ij}(t) \int_{t-\eta_{ij}(t)}^{t} \varphi'_{ij}(s)\,ds \right.\right. \\
&\quad \left.\left. - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) f(\varphi_{kl}(\varphi_{kl}(t)))\varphi_{ij}(t) + L_{ij}(t) \right| \right\}
\end{aligned}
$$

$$\leq \left\{ \left[a_{ij}^{+} + a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\,+}(L^f(\kappa+L)+|f(0)|) \right](\kappa+L) + L_{ij}^{+} \right\}$$
$$\leq \{\zeta\},\ \forall t \in \mathbb{R}, \tag{3.7}$$

and

$$|(T\varphi)(t_1)-(T\varphi)(t_2)| = \{|(((T\varphi)(t))_{ij})'|_{t=t_1+\Delta(t_2-t_1)}(t_1-t_2)|\} \leq \{\kappa|t_1-t_2|\}, \tag{3.8}$$

where for all $t_1, t_2 \in \mathbb{R}$, $\Delta \in (0,\ 1)$, and $t_1 + \Delta(t_2 - t_1)$ is the mean point in Lagrange's mean value theorem. Thus, (3.7) and (3.8) yield $T\varphi \in \mathbf{B}$. So, the mapping T is a self-mapping from $\mathbf{B}$ to $\mathbf{B}$.

Second, we show that T is a contract operator.

In fact, in view of (3.3), (3.5), (S_1) and (S_2), for $\varphi, \psi \in \mathbf{B}$, we have

$$\begin{aligned}
&|(T\varphi)(t)-(T\psi)(t)| \\
&= \{|((T\varphi)(t)-(T\psi)(t))_{ij}|\} \\
&= \left\{ \left| \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)\,du} \left[a_{ij}(s)\int_{s-\eta_{ij}(s)}^{s} (\varphi_{ij}'(u)-\psi_{ij}'(u))\,du \right.\right.\right. \\
&\quad \left.\left.\left. - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)(f(\varphi_{kl}(\varphi_{kl}(s)))\varphi_{ij}(s) - f(\psi_{kl}(\psi_{kl}(s)))\psi_{ij}(s)) \right] ds \right| \right\} \\
&\leq \left\{ \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)\,du} \left[a_{ij}^{+}\eta_{ij}^{+}\|\varphi-\psi\|_{\mathbf{Z}} \right.\right. \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\,+}(|f(\varphi_{kl}(\varphi_{kl}(s)))||\varphi_{ij}(s)-\psi_{ij}(s))| \\
&\quad \left.\left. + |f(\varphi_{kl}(\varphi_{kl}(s))) - f(\psi_{kl}(\psi_{kl}(s)))||\psi_{ij}(s))|) \right] ds \right\} \\
&\leq \left\{ \frac{1}{a_{ij}^{-}} \left[a_{ij}^{+}\eta_{ij}^{+}\|\varphi-\psi\|_{\mathbf{Z}} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\,+}(M_f\|\varphi-\psi\|_{\mathbf{Z}} \right.\right. \\
&\quad \left.\left. + L^f(|\varphi_{kl}(\varphi_{kl}(s))-\psi_{kl}(\varphi_{kl}(s))| + |\psi_{kl}(\varphi_{kl}(s))-\psi_{kl}(\psi_{kl}(s))|)\|\psi\|_{\mathbf{Z}}) \right] \right\} \\
&= \left\{ \frac{1}{a_{ij}^{-}} \left[a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\,+}(M_f + L^f(1+\zeta)\|\psi\|_{\mathbf{Z}}) \right] \|\varphi-\psi\|_{\mathbf{Z}} \right\} \\
&\leq \left\{ \frac{1}{a_{ij}^{-}} \left[a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\,+}(M_f + L^f(1+\zeta)(\kappa+L)) \right] \|\varphi-\psi\|_{\mathbf{Z}} \right\},
\end{aligned}$$

and

$$\begin{aligned}
&|((T\varphi)(t))' - ((T\psi)(t))'| \\
&= \{|(((T\varphi)(t))' - ((T\psi)(t))')_{ij}|\} \\
&= \left\{ \left| \left[a_{ij}(t)\int_{t-\eta_{ij}(t)}^{t} (\varphi_{ij}'(s)-\psi_{ij}'(s))\,ds \right.\right.\right. \\
&\quad \left. - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)(f(\varphi_{kl}(\varphi_{kl}(t)))\varphi_{ij}(t) - f(\psi_{kl}(\psi_{kl}(t)))\psi_{ij}(t) \right]
\end{aligned}$$

$$
\begin{aligned}
&- a_{ij}(t)\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)\,du}\Big[a_{ij}(s)\int_{s-\eta_{ij}(s)}^{s}(\varphi_{ij}'(u)-\psi_{ij}'(u))\,du \\
&- \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)(f(\varphi_{kl}(\varphi_{kl}(s)))\varphi_{ij}(s) - f(\psi_{kl}(\psi_{kl}(s))\psi_{ij}(s)\Big]ds\Big|\Big\} \\
&\le \Big\{\Big(1+\frac{a_{ij}^{+}}{a_{ij}^{-}}\Big)\Big[a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\ +}(M_f + L^f(1+\zeta)(\kappa+L))\Big]\|\varphi-\psi\|_{\mathbf{Z}}\Big\},
\end{aligned}
$$

which yields

$$
\|T\varphi - T\psi\|_{\mathbf{Z}} \le \max_{(i,j)}\Big\{\frac{1}{a_{ij}^{-}}F_{ij},\ \Big(1+\frac{a_{ij}^{+}}{a_{ij}^{-}}\Big)F_{ij}\Big\}\|\varphi-\psi\|_{\mathbf{Z}}.
$$

It follows from $\max_{ij\in J}\{\frac{1}{a_{ij}^{-}}F_{ij},\ \Big(1+\frac{a_{ij}^{+}}{a_{ij}^{-}}\Big)F_{ij}\} < 1$ that the mapping $T : \mathbf{B} \longrightarrow \mathbf{B}$ is a contraction mapping. Therefore, we obtain that the mapping T possesses a unique fixed point

$$
x^* = \{x_{ij}^*(t)\} \in \mathbf{Z}, \quad Tx^* = x^*.
$$

By (3.1) and (3.3), x^* satisfies (3.1). So (1.1) has at least one continuously differentiable pseudo almost periodic solution x^* in the region $\mathbf{B}$. The proof of Theorem 3.1 is now completed. □

4 Example

In this section, we give an example to demonstrate the results obtained in previous sections.

Example 4.1 Consider the following SICNNs with leakage delays and complex deviating arguments:

$$
\begin{aligned}
\frac{dx_{ij}}{dt} &= -a_{ij}(t)x_{ij}(t-\eta_{ij}(t)) \\
&\quad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(x_{kl}(t)))x_{ij}(t) + L_{ij}(t), \quad i,j = 1,2,3. \qquad (4.1)
\end{aligned}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} = \begin{bmatrix} 1 & 1 & 3\\ 3 & 1 & 3\\ 3 & 1 & 3\end{bmatrix},\quad \begin{bmatrix} C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.1\\ 0.2 & 0 & 0.2\\ 0.1 & 0.2 & 0.1\end{bmatrix}, \qquad (4.2)
$$

$$
\begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13}\\ \eta_{21} & \eta_{22} & \eta_{23}\\ \eta_{31} & \eta_{32} & \eta_{33}\end{bmatrix} = 0.01\begin{bmatrix} \sin^2\sqrt{3}t + \frac{0.1}{1+t^2} & \cos^2\sqrt{3}t + \frac{0.1}{1+t^2} & \sin^2 2t + \frac{0.1}{1+t^2}\\ \cos^2\sqrt{5}t + \frac{0.1}{1+t^2} & \sin^2\sqrt{5}t + \frac{0.1}{1+t^2} & \cos^2 2t + \frac{0.1}{1+t^2}\\ \sin^2 2t + \frac{0.1}{1+t^2} & \cos^2 3t + \frac{0.1}{1+t^2} & \sin^2\sqrt{2}t + \frac{0.1}{1+t^2}\end{bmatrix}, \qquad (4.3)
$$

$$
\begin{bmatrix} L_{11} & L_{12} & L_{13}\\ L_{21} & L_{22} & L_{23}\\ L_{31} & L_{32} & L_{33}\end{bmatrix} = \frac{3}{5}\begin{bmatrix} 0.7 + 0.24\sin^2\sqrt{2}t - \frac{1}{1+t^2} & 0.41 + 0.5\cos^2 t & 1\\ 0.61 + 0.2\cos^2 t - \frac{1}{1+t^2} & 0.67 + 0.2\sin^2 t & 1\\ 0.59 + 0.4\cos^4 t - \frac{1}{1+t^2} & 0.5 + 0.41\sin^2 t & 1\end{bmatrix}. \qquad (4.4)
$$

Set

$$f(x) = \frac{1}{50}(|x-1| - |x+1|), \quad \kappa = 0.5, \quad \zeta = 4, \quad r = 1.$$

Clearly,

$$M_f = 0.04, \ \ L_f = 0.04, \sum_{C_{kl}\in N_1(3,3)} C_{33}^{kl} = 0.5,$$

$$\sum_{C_{kl}\in N_1(1,1)} C_{11}^{kl} = 0.5, \quad \sum_{C_{kl}\in N_1(1,2)} C_{12}^{kl} = 0.8, \quad \sum_{C_{kl}\in N_1(1,3)} C_{13}^{kl} = 0.5, \quad \sum_{C_{kl}\in N_1(2,1)} C_{21}^{kl} = 0.8,$$

$$\sum_{C_{kl}\in N_1(2,2)} C_{22}^{kl} = 1.2, \quad \sum_{C_{kl}\in N_1(2,3)} C_{23}^{kl} = 0.8, \quad \sum_{C_{kl}\in N_1(3,1)} C_{31}^{kl} = 0.5, \quad \sum_{C_{kl}\in N_1(3,2)} C_{32}^{kl} = 0.8,$$

where $ij \in J = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}$. Then,

$$L = \max\left\{\max_{ij\in J}\left\{\sup_{t\in R}\left|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} L_{ij}(s)\,ds\right|\right\},\right.$$

$$\left.\max_{ij\in J}\left\{\sup_{t\in R}\left|L_{ij}(t) - a_{ij}(t)\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} L_{ij}(s)\,ds\right|\right\}\right\} = 0.6 > 0,$$

$$E_{ij} = \left[a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\,+}(L^f(\kappa+L) + |f(0)|)\right](\kappa+L) \le 0.095, \quad ij \in J,$$

$$F_{ij} = a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\,+}(M_f + L^f(1+\zeta)(\kappa+L)) \le 0.289, \quad ij \in J,$$

$$0.5 = \kappa \le L = 0.6, \quad \max_{(i,j)}\left\{\frac{1}{a_{ij}^{-}}E_{ij}, \left(1 + \frac{a_{ij}^{+}}{a_{ij}^{-}}\right)E_{ij}\right\} \approx 0.23 \le \kappa,$$

$$\max_{(i,j)}\left\{\frac{1}{a_{ij}^{-}}F_{ij}, \left(1 + \frac{a_{ij}^{+}}{a_{ij}^{-}}\right)F_{ij}\right\} \approx 0.59 < 1,$$

and

$$\left\{\left[a_{ij}^{+} + a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl\,+}(L^f(\kappa+L) + |f(0)|)\right](\kappa+L) + L_{ij}^{+}\right\} \le \{3.8\} \le \{\zeta\}.$$

It follows that system (4.1) satisfies all the conditions in Theorem 3.1 Hence, system (4.1) has exactly one pseudo almost periodic solution in the region $\mathbf{B} = B^{\zeta} \bigcap B^{*}$.

Example 4.2 Assume that (4.2) and (4.4) hold. Consider the following SICNNs with leakage delays and complex deviating arguments:

$$\frac{dx_{ij}}{dt} = -a_{ij}(t)x_{ij}(t - \eta_{ij}(t)) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(x_{kl}(t)))x_{ij}(t) + L_{ij}(t), \quad i, j = 1, 2 \tag{4.5}$$

where

$$\begin{bmatrix}\eta_{11} & \eta_{12}\\ \eta_{21} & \eta_{22}\end{bmatrix} = \frac{1}{20}\begin{bmatrix}0.1\sin t + 0.1 + \frac{0.2}{1+t^2} & 0.1\cos t + 0.1 + \frac{0.2}{1+t^2}\\ 0.1\cos t + 0.1 + \frac{0.2}{1+t^2} & 0.1\sin t + 0.1 + \frac{0.2}{1+t^2}\end{bmatrix},$$

and

$$f(x) = \begin{cases} \frac{1}{50}\tanh x, & x \leq 0, \\ \frac{1}{50}(|x-1| - |x+1|), & x > 0. \end{cases} \tag{4.6}$$

Recently, $0.1\sin t + 0.1$, $0.1\cos t + 0.1$ and (4.6) have been considered as leakage delays and the activation function to simulate the practical neural network models in [29,30] and [31], respectively. This means that system (4.5) is a practical neural network model. It is straightforward to check that all assumptions needed in Theorem 3.1 are satisfied. Therefore, system (4.5) has exactly one pseudo almost periodic solution in the region $\mathbf{B} = B^{\zeta} \bigcap B^{*}$.

Remark 4.1 For all we know, there is no research on the existence of pseudo almost periodic solutions to SICNNs with complex deviating arguments. We also mention that all results in the references [6–8,15–20,23,24] cannot be directly applied to imply the existence of pseudo almost periodic solutions to (4.1) and (4.2). Here we present a novel proof to establish some criteria to guarantee the existence of pseudo almost periodic solutions for SICNNs with leakage delays and complex deviating arguments. This implies that the result of this paper is essentially new.

Remark 4.2 Most recently, a typical time delay called as Leakage (or "forgetting") delay may exist in the negative feedback terms of the neural network system and it has a great impact on the dynamic behaviors of delayed neural networks (See Gopalsamy [4] and Liu [5]). Moreover, as pointed out by Ait Dads and Ezzinbi in [25], it would be of great interest to study the dynamics of pseudo almost periodic systems with time delays. It is well known that the SICNNs (1.1) with complex deviating arguments is a special type of state-dependent delay differential equations and plays an important role in applications (see [18]). In this paper, the existence of pseudo almost periodic solutions on a class of shunting inhibitory cellular neural networks with leakage delays and complex deviating arguments has been studied for the first time.

5 Conclusion

In this paper, a class of shunting inhibitory cellular neural networks systems with leakage delays and complex deviating arguments have been studied. Under some appropriate conditions, the existence of pseudo almost periodic solutions for this model has been established by using the exponential dichotomy theory, contraction mapping fixed point theorem and inequality analysis technique. Our results can be applied for some practical problems concerning neural networks. Moreover, an example is given to illustrate the effectiveness of our new results. In the real world, fuzzy theory is considered as a more suitable method for the sake of taking vagueness into consideration (see [26–28]). Whether or not our results and method in this paper are available for the existence of pseudo almost periodic solutions the fuzzy SICNNs models, it is an interesting problem and we leave it as our work in the future.

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