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Bifurcation Analysis in a Three-Neuron Artificial Neural Network Model with Distributed Delays

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Abstract In this paper, a three-neuron artificial neural network model with distributed delays is considered. Its dynamics is investigated in term of the linear stability analysis and Hopf bifurcation analysis. By regarding the sum of two delays as a bifurcation parameter and analyzing the associated characteristic equation, we find that Hopf bifurcation occurs when the bifurcation parameter passes through some certain values. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions are derived by using the normal form method and center manifold theory. Finally, computer simulations are given to support the theoretical predictions.

Keywords Artificial neural network · Stability · Hopf bifurcation · Distributed delay · Periodic solution

Mathematics Subject Classification 34K20 · 34C25

1 Introduction

In the last three decades, there has been increasing interest in the dynamical properties of neural networks due to their important applications in many fields such as pattern recognition, classification, optimization, signal and image processing, solving nonlinear algebraic equations, associative memories, cryptography and so on [1]. Many rich mathematical investigations and interesting results on neural networks have been available in the literature (see

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[2–14]). In particular, the Hopf bifurcation behavior is of great interest. In order to obtain a deep and clear understanding of the Hopf bifurcation nature of neural networks, many authors have focused on the studying of simplified neural networks with two, three or four neurons. For instance, Zou et al. [15] considered the stability and Hopf bifurcation of a three-unit neural network with two delays, Shayer and Campbell [16] studied the stability, bifurcation, and multistability in a system of two coupled neurons with multiple time delays, Guo and Huang [17] focused on the linear stability and Hopf bifurcation of a two-neuron network with three delays, Liao et al. [18] analyzed the stability and bifurcation of a tri-neuron model with time delay, Wang and Jian [3] gave a detailed study on the stability and Hopf bifurcation for a four-neuron BAM neural network with distributed delays, Mao and Hu [19] made a detailed discussion on Hopf bifurcation of a four-neuron network with multiple time delays. Majee and Roy [20] dealt with the temporal dynamics of two-neuron continuous network model with time delay. For more work on this aspect, one can see [2,21–39]. In 2008, Gupta et al. [40] considered the following three neurons network with distributed delay

$$\frac{dx_i}{dt} = -px_i(t) + \sum_{j=1}^{3} a_{ij} \tanh\left[\int_{-\infty}^{t} k(t-s)x_j(s)ds\right], \quad i = 1, 2, 3, \tag{1.1}$$

where $p > 0$ is the delay rate of neurons. a_{ij} is the weight of synaptic connections from neuron j to neuron i and k is the delay kernel assumed to satisfy the following conditions:

(i) $k : [0, \infty) \to [0, \infty)$;
(ii) k is piecewise continuous;
(iii) $\int_0^\infty k(s)ds = 1, \int_0^\infty sk(s)ds < \infty.$

The general form of delay kernel k(s) takes the form:

$$k(s) = \beta^{n+1}\frac{s^n e^{\beta s}}{n!}, \quad s \in (0, \infty), \qquad n = 0, 1, 2,$$

where β is a parameter which stands for the rate of decay of the effects of past memories and it is a positive real number. $n = 0$ represents weak kernel, whereas $n = 1$ represents strong kernel.

When $n = 0$, then the delay kernel $k(s)$ reads as

$$k(s) = \beta e^{-\beta s}, s \in (0, \infty).$$

Then system (1.1) takes the form

$$\frac{dx_i}{dt} = -px_i(t) + \sum_{j=1}^{3} a_{ij} \tanh\left[\beta\int_{-\infty}^{t} e^{-\beta(t-s)}x_j(s)ds\right], \quad t > 0, \quad i = 1, 2, 3. \tag{1.2}$$

To make the calculation more tractable, Gupta et al. [40] make the following assumption:

$$p = 1, a_{11} = a_{22} = a_{33} = 0, a_{12} = a_{13} = b, a_{21} = a_{31} = a, a_{23} = a_{32} = 0.$$

Then system (1.2) becomes

$$\begin{cases} \frac{dx}{dt} = -x(t) + b\tanh\left[\beta\int_{-\infty}^{t} e^{-\beta(t-s)}y(s)ds\right] + b\tanh\left[\beta\int_{-\infty}^{t} e^{-\beta(t-s)}z(s)ds\right], \\ \frac{dy}{dt} = -y(t) + a\tanh\left[\beta\int_{-\infty}^{t} e^{-\beta(t-s)}x(s)ds\right], \\ \frac{dz}{dt} = -z(t) + a\tanh\left[\beta\int_{-\infty}^{t} e^{-\beta(t-s)}x(s)ds\right]. \end{cases} \tag{1.3}$$

With the aid of some auxiliary variables, Gupta et al. [40] focused on the asymptotic stability, orbits stability of Hopf bifurcation periodic solution of system (1.3).

We must point out that in real life, there is transmission delay of the signal along the axon of the neuron. Motivated by the viewpoint, we can modify system (1.3) as follows:

$$\begin{cases} \frac{dx}{dt} = -x(t) + b\tanh\left[\beta\int_{-\infty}^{t} e^{-\beta(t-s)}y(s-\tau_2)ds\right] \\ \qquad + b\tanh\left[\beta\int_{-\infty}^{t} e^{-\beta(t-s)}z(s-\tau_2)ds\right], \\ \frac{dy}{dt} = -y(t) + a\tanh\left[\beta\int_{-\infty}^{t} e^{-\beta(t-s)}x(s-\tau_1)ds\right], \\ \frac{dz}{dt} = -z(t) + a\tanh\left[\beta\int_{-\infty}^{t} e^{-\beta(t-s)}x(s-\tau_1)ds\right]. \end{cases} \tag{1.4}$$

Here we wold like to point out that human brain is made up of a large number of cells neurons and their interaction, artificial neural networks are information processing systems which have some common characteristics with biological neural networks. System (1.4) can play a important role in the control of regular dynamical functions such as breathing and hear beating of human.

In this paper, we consider the model (1.4). In order to establish the main results for model (1.4), it is necessary to make the following assumption:

$$\text{(H1)}\ \tau_1 + \tau_2 = \tau.$$

This paper is organized as follows. In Sect. 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are analyzed. In Sect. 3, the direction of Hopf bifurcation and the stability and periods of bifurcating periodic solutions on the center manifold are determined. In Sect. 4, numerical simulations are given to illustrate the validity of the main results. Some main conclusions are drawn in Sect. 5.

2 Stability of the Equilibrium and Local Hopf Bifurcations

Let

$$\begin{cases} u(t) = \beta\int_{-\infty}^{t} e^{-\beta(t-s)}x(s-\tau_1)ds, \quad t > 0, \\ v(t) = \beta\int_{-\infty}^{t} e^{-\beta(t-s)}x(s-\tau_1)ds, \quad t > 0, \\ w(t) = \beta\int_{-\infty}^{t} e^{-\beta(t-s)}x(s-\tau_1)ds, \quad t > 0. \end{cases} \tag{2.1}$$

then system (1.4) takes the following equivalent form:

$$\begin{cases} \dfrac{dx}{dt} = -x(t) + b\tanh[v(t)] + b\tanh[w(t)], \\ \dfrac{dy}{dt} = -y(t) + a\tanh[u(t)], \\ \dfrac{dz}{dt} = -z(t) + a\tanh[u(t)], \\ \dfrac{du}{dt} = \beta[x(t-\tau_1) - u(t)], \\ \dfrac{dv}{dt} = \beta[y(t-\tau_2) - v(t)], \\ \dfrac{dw}{dt} = \beta[z(t-\tau_2) - w(t)]. \end{cases} \tag{2.2}$$

For notational and computational simplicity, we can rewrite system (2.2) as follows:

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1(t) + b\tanh[x_5(t)] + b\tanh[x_6(t)], \\ \dfrac{dx_2}{dt} = -x_2(t) + a\tanh[x_4(t)], \\ \dfrac{dx_3}{dt} = -z(t) + a\tanh[x_4(t)], \\ \dfrac{dx_4}{dt} = \beta[x_1(t-\tau_1) - x_4(t)], \\ \dfrac{dx_5}{dt} = \beta[x_2(t-\tau_2) - x_5(t)], \\ \dfrac{dx_6}{dt} = \beta[x_3(t-\tau_2) - x_6(t)]. \end{cases} \tag{2.3}$$

From the paper [40], if the following condition

$$\text{(H2)}\ ab < \frac{1}{2}$$

holds, then (2.3) has a unique equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$. The linear equation of (2.3) at $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$ takes the form:

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1(t) + b\text{sech}^2(x_5^*)x_5(t) + b\text{sech}^2(x_6^*)x_6(t), \\ \dfrac{dx_2}{dt} = -x_2(t) + a\text{sech}^2(x_4^*)x_4(t), \\ \dfrac{dx_3}{dt} = -x_3(t) + a\text{sech}^2(x_4^*)x_4(t), \\ \dfrac{dx_4}{dt} = \beta[x_1(t-\tau_1) - x_4(t)], \\ \dfrac{dx_5}{dt} = \beta[x_2(t-\tau_2) - x_5(t)], \\ \dfrac{dx_6}{dt} = \beta[x_3(t-\tau_2) - x_6(t)]. \end{cases} \tag{2.4}$$

Then the associated characteristic equation of (2.4) is given by

$$\det\begin{pmatrix} \lambda+1 & 0 & 0 & 0 & -b\mathrm{sech}^2(x_5^*) & -b\mathrm{sech}^2(x_6^*) \\ 0 & \lambda+1 & 0 & -b\mathrm{sech}^2(x_4^*) & 0 & 0 \\ 0 & 0 & \lambda+1 & -b\mathrm{sech}^2(x_4^*) & 0 & 0 \\ -\beta e^{-\lambda\tau_1} & 0 & 0 & \lambda+\beta & 0 & 0 \\ 0 & -\beta e^{-\lambda\tau_2} & 0 & 0 & \lambda+\beta & 0 \\ 0 & 0 & -\beta e^{-\lambda\tau_2} & 0 & 0 & \lambda+\beta \end{pmatrix} = 0, \tag{2.5}$$

which leads to the following form:

$$\lambda^6 + r_5\lambda^5 + r_4\lambda^4 + r_3\lambda^3 + r_2\lambda^2 + r_1\lambda + r_0 + (s_2\lambda^2 + s_1\lambda + s_0)e^{-\lambda\tau} = 0, \tag{2.6}$$

where

$$\begin{aligned} r_0 &= \beta^3, r_1 = 3\beta^3 + 3\beta^2, r_2 = 3\beta^3 + 9\beta^2 + 3\beta, \\ r_3 &= \beta^3 + 9\beta^2 + 9\beta + 1, r_4 = 3\beta^2 + 9\beta + 3, r_5 = 3\beta + 3, \\ s_0 &= -ab\beta^3\mathrm{sech}^2(x_6^*)(\mathrm{sech}^2(x_4^*) + \mathrm{sech}^2(x_5^*)), \\ s_1 &= -ab(\beta+1)\beta^2\mathrm{sech}^2(x_6^*)(\mathrm{sech}^2(x_4^*) + \mathrm{sech}^2(x_5^*)), \\ s_2 &= -ab\beta^2\mathrm{sech}^2(x_6^*)(\mathrm{sech}^2(x_4^*) + \mathrm{sech}^2(x_5^*)). \end{aligned}$$

Let $\lambda = i\omega_0$, $\tau = \tau_0$, and substituting this into (2.6), for the sake of simplicity, denote ω_0 and τ_0 by ω and τ, respectively, then (2.6) becomes

$$\begin{aligned} &-\omega^6 + r_5\omega^5 i + r_4\omega^4 - r_3\omega^3 i - r_2\omega^2 + r_1\omega i + r_0 \\ &\quad + (-s_2\omega^2 + s_1\omega i + s_0)(\cos\omega\tau + i\sin\omega\tau) = 0. \end{aligned} \tag{2.7}$$

Separating the real and imaginary parts leads to

$$(s_0 - s_2\omega^2)\cos\omega\tau + s_1\omega\sin\omega\tau = \omega^6 - r_4\omega^4 + r_2\omega^2 - r_0, \tag{2.8}$$

$$s_1\omega\cos\omega\tau - (s_0 - s_2\omega^2)\cos\omega\tau = r_3\omega^3 - r_5\omega^5 - r_1\omega. \tag{2.9}$$

Squaring both sides of (2.8) and (2.9), and adding them up gives

$$(s_0 - s_2\omega^2)^2 + (s_1\omega)^2 = (\omega^6 - r_4\omega^4 + r_2\omega^2 - r_0)^2 + (r_3\omega^3 - r_5\omega^5 - r_1\omega)^2,$$

which is equivalent to

$$\omega^{12} + \theta_5\omega^{10} + \theta_4\omega^8 + \theta_3\omega^6 + \theta_2\omega^4 + \theta_1\omega^2 + \theta_0 = 0, \tag{2.10}$$

where

$$\begin{aligned} \theta_0 &= r_0^2 - s_0^2, \theta_1 = r_1^2 - 2r_0r_2 + 2s_0s_2 - s_1^2, \\ \theta_2 &= r_2^2 - 2r_0r_4 - 2r_1r_3 - s_2^2, \theta_3 = r_3^2 + 2r_1r_5 - 2r_0 - 2r_2r_4, \\ \theta_4 &= r_4^2 + 2r_2 - 2r_3r_5, \theta_5 = r_5^2 - 2r_4. \end{aligned}$$

Let $z = \theta^2$. Then (2.10) becomes

$$z^6 + \theta_5 z^5 + \theta_4 z^4 + \theta_3 z^3 + \theta_2 z^2 + \theta_1 z + \theta_0 = 0, \tag{2.11}$$

If $\theta_0 < 0$, then (2.11) has at least one positive root. Suppose that Eq. (2.11) has positive roots. Without loss of generality, we assume that (2.11) has six positive roots, denoted by $z_1, z_2, z_3, z_4, z_5, z_6$, Then (2.10) has six positive roots as follows

$$\omega_1 = \sqrt{z_1}, \omega_2 = \sqrt{z_2}, \omega_3 = \sqrt{z_3}, \omega_4 = \sqrt{z_4}, \omega_5 = \sqrt{z_5}, \omega_6 = \sqrt{z_6}.$$

By (2.8) and (2.9), we have

$$\cos \omega_k \tau = \frac{(\omega_k^6 - r_4\omega_k^4 + r_2\omega_k^2 - r_0)(s_0 - s_2\omega_k^2) + (r_3\omega_k^3 - r_5\omega_k^5 - r_1\omega_k)s_1\omega_k}{(s_0 - s_2\omega_k^2)^2 + (s_1\omega_k)^2}. \tag{2.12}$$

Thus, if we denote

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\arccos\left[\frac{(\omega_k^6 - r_4\omega_k^4 + r_2\omega_k^2 - r_0)(s_0 - s_2\omega_k^2) + (r_3\omega_k^3 - r_5\omega_k^5 - r_1\omega_k)s_1\omega_k}{(s_0 - s_2\omega_k^2)^2 + (s_1\omega_k)^2}\right] + 2j\pi\right\}, \tag{2.13}$$

where $k = 1, 2, 3, \dots, 6$ and $j = 0, 1, 2, \dots$. Then $\pm i\omega_k$ are a pair of purely imaginary roots of Eq. (2.6) with $\tau = \tau_k^{(j)}$. Obviously, in view of (2.13), the sequence $\{\tau_k^{(j)}\}_{j=0}^{+\infty}$ is increasing, and

$$\lim_{j\to+\infty} \tau_k^{(j)} = +\infty, \quad k = 1, 2, 3, \dots, 6.$$

Then we can define

$$\tau_0 = \tau_{k0}^{(0)} = \min_{1\le k\le 6}\{\tau_k^{(0)}\}, \qquad \omega_0 = \omega_{k0}. \tag{2.14}$$

Note that when $\tau = 0$, (2.6) becomes

$$\lambda^6 + r_5\lambda^5 + r_4\lambda^4 + r_3\lambda^3 + (r_2 + s_2)\lambda^2 + (r_1 + s_1)\lambda + (r_0 + s_0) = 0. \tag{2.15}$$

All roots of (2.15) have a negative real part if the following well-known Routh-Hurwitz criteria hold.

$$D_1 = r_5 > 0, \tag{2.16}$$

$$D_2 = \det\begin{pmatrix} r_5 & 1 \\ r_3 & r_4 \end{pmatrix} > 0, \tag{2.17}$$

$$D_3 = \det\begin{pmatrix} r_5 & 1 & 0 \\ r_3 & r_4 & r_5 \\ r_1 + s_1 & r_2 + s_2 & r_3 \end{pmatrix} > 0, \tag{2.18}$$

$$D_4 = \begin{pmatrix} r_5 & 1 & 0 & 0 \\ r_3 & r_4 & r_5 & 1 \\ r_1 + s_1 & r_2 + s_2 & r_3 & r_4 \\ 0 & r_0 + s_0 & r_1 + s_1 & r_2 + s_2 \end{pmatrix} \tag{2.19}$$

$$D_5 = \det\begin{pmatrix} r_5 & 1 & 0 & 0 & 0 \\ r_3 & r_4 & r_5 & 1 & 0 \\ r_1 + s_1 & r_2 + s_2 & r_3 & r_4 & r_5 \\ 0 & r_0 + s_0 & r_1 + s_1 & r_2 + s_2 & r_3 \\ 0 & 0 & 0 & r_0 + s_0 & r_1 + s_1 \end{pmatrix} > 0, \tag{2.20}$$

$$D_6 = r_0 + s_0 > 0. \tag{2.21}$$

In order to obtain the main results in this paper, it is necessary to make the following assumptions:

(H3) If (2.16)–(2.21) hold, (2.15) have six roots with negative real parts when $\tau = 0$, (2.1) is stable near the equilibrium.

(H4) $\text{Re}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_0} \neq 0.$

Taking the derivative of λ with respect to τ in (2.6), it is easy to obtain:

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(6\lambda^5 + 5r_5\lambda^4 + 4r_4\lambda^3 + 3r_3\lambda^2 + 2r_2\lambda + r_1)e^{\lambda\tau}}{\lambda(s_2\lambda^2 + s_1\lambda + s_0)} + \frac{2s_2\lambda + s_1}{\lambda(s_2\lambda^2 + s_1\lambda + s_0)} - \frac{\tau}{\lambda}. \tag{2.22}$$

Then

$$\text{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \text{Re}\left\{\frac{(6\lambda^5 + 5r_5\lambda^4 + 4r_4\lambda^3 + 3r_3\lambda^2 + 2r_2\lambda + r_1)e^{\lambda\tau}}{\lambda(s_2\lambda^2 + s_1\lambda + s_0)}\right\}_{\tau=\tau_0} + \left\{\frac{2s_2\lambda + s_1}{\lambda(s_2\lambda^2 + s_1\lambda + s_0)}\right\}_{\tau=\tau_0}. \tag{2.23}$$

Thus

$$\text{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{\Theta_1}{\Theta_2}, \tag{2.24}$$

where

$$\begin{aligned}
\Theta_1 &= s_1\omega_0^2[(6\omega_0^5 - 4r_4\omega_0^3 + 2r_2\omega_0)\sin\omega_0\tau_0 - (5r_4\omega_0^4 - 3r_3\omega_0^2 + r_1)\cos\omega_0\tau_0 - s_1 \\
&\quad + (s_0 - s_2\omega_0^2)[(5r_4\omega_0^4 - 3r_3\omega_0^2 + r_1)\sin\omega_0\tau_0 + (6\omega_0^2 - 4r_4\omega_0^3 + 2r_2\omega_0)\cos\omega_0\tau_0], \\
\Theta_2 &= (s_1\omega_0)^2 + [(s_0 - s_2\omega_0^2)\omega_0]^2.
\end{aligned}$$

In order to investigate the distribution of roots of the transcendental equation (2.6), the following Lemma that is stated in [41] is useful.

Lemma 2.1 [41] *For the transcendental equation*

$$\begin{aligned}
P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\
&\quad + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots \\
&\quad + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0,
\end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *in the open right half plane can change and only if a zero appears on or crosses the imaginary axis.*

Remark 2.1 Lemma 2.1 is a generalization of the Lemma in Cooke and Grossman [42] in which a second order degree exponential polynomial was investigated.

In view of Lemma 2.1, it is easy to obtain the following results:

Theorem 2.2 *If (H1)–(H4) hold, then*

(i) for system (2.3), its equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$ *is asymptotically stable for* $\tau \in [0, \tau_0)$

(ii) system (2.3) undergoes a Hopf bifurcation at the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$ *when* $\tau = \tau_0$*, i.e., system (2.3) has a branch of periodic solutions bifurcating from the equilibrium* $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$ *solution near* $\tau = \tau_0$.

3 Direction and Stability of the Hopf Bifurcation

In the previous section, we have obtained some conditions to ensure that system (2.3) undergoes a single Hopf bifurcation at the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$ when $\tau = \tau_1 + \tau_2$ passes through certain critical values. In this section, we shall study the direction, stability, and the period of bifurcating periodic solutions. The method we used is based on normal form method and the center manifold theory introduced by Hassard et al. [43].

Without loss of generality, we denote the critical value $\tau_i (i = 0, 1, 2, \ldots)$ by $\tilde{\tau} = \tilde{\tau}_1 + \tilde{\tau}_2$ at which system (2.3) undergoes a Hopf bifurcation, where $\tilde{\tau}_1 < \tilde{\tau}_2$ and $\tau = \tilde{\tau} + \mu = (\tilde{\tau}_1 + \mu) + \tilde{\tau}_2$, then $\mu = 0$ is Hopf bifurcation value of system (2.3). We choose the phase space as $C = C([-\tilde{\tau}_2, 0], C^6)$, where for convenience in computation we use C^6 instead of $\mathbb{R}^6$.

Its linear part is given by

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1(t) + b\mathrm{sech}^2(x_5^*)x_5(t) + b\mathrm{sech}^2(x_6^*)x_6(t), \\ \dfrac{dx_2}{dt} = -x_2(t) + a\mathrm{sech}^2(x_4^*)x_4(t), \\ \dfrac{dx_3}{dt} = -x_3(t) + a\mathrm{sech}^2(x_4^*)x_4(t), \\ \dfrac{dx_4}{dt} = \beta[x_1(t-\tau_1) - x_4(t)], \\ \dfrac{dx_5}{dt} = \beta[x_2(t-\tau_2) - x_5(t)], \\ \dfrac{dx_6}{dt} = \beta[x_3(t-\tau_2) - x_6(t)]. \end{cases} \tag{3.1}$$

Its non-linear part is given by

$$f(\mu, x_t) = \begin{pmatrix} c_{11}x_5^2(0) + c_{12}x_5^3(0) + c_{13}x_6^2(0) + c_{14}x_6^3(0) + \text{h.o.t.} \\ c_{21}x_4^2(0) + c_{22}x_4^3(0) + \text{h.o.t.} \\ c_{21}x_4^2(0) + c_{22}x_4^3(0) + \text{h.o.t.} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned} c_{11} &= -2b\mathrm{sech}^2(v^*)\mathrm{th}(v^*), \\ c_{12} &= -b(4\mathrm{sech}^2(v^*)\mathrm{th}^2(v^*) - 2\mathrm{sech}^4(v^*)), \\ c_{13} &= -2b\mathrm{sech}^2(w^*)\mathrm{th}(w^*), \\ c_{14} &= -b(4\mathrm{sech}^2(w^*)\mathrm{th}^2(w^*) - 2\mathrm{sech}^4(w^*)), \\ c_{21} &= -2b\mathrm{sech}^2(u^*)\mathrm{th}(u^*), \\ c_{22} &= -b(4\mathrm{sech}^2(u^*)\mathrm{th}^2(u^*) - 2\mathrm{sech}^4(u^*)) \end{aligned}$$

and

$$\begin{aligned} x_t(\theta) &= (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta), x_{4t}(\theta), x_{5t}(\theta), x_{6t}(\theta))^T \\ &= (x_1(t+\theta), x_2(t+\theta), x_3(t+\theta), x_4(t+\theta), x_5(t+\theta), x_6(t+\theta))^T. \end{aligned}$$

Denote

$$C^k[-\tilde{\tau}_2, 0] = \{\varphi|\varphi : [-\tilde{\tau}_2, 0] \to \mathbb{R}^6,$$
$$\text{each component of } \varphi \text{ has } k \text{ order continuous derivative}\}.$$

For convenience, denote $C[-\tilde{\tau}_2, 0]$ by $C^0[-\tilde{\tau}_2, 0]$.

For $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta), \varphi_4(\theta), \varphi_5(\theta), \varphi_6(\theta))^T \in C([-\tilde{\tau}_2, 0], \mathbb{R}^6)$, define a family of operators

$$L_\mu(\varphi) = B\begin{pmatrix}\varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\\ \varphi_4(0)\\ \varphi_5(0)\\ \varphi_6(0)\end{pmatrix} + B_1\begin{pmatrix}\varphi_1(-\tilde{\tau}_1+\mu)\\ \varphi_2(-\tilde{\tau}_1+\mu)\\ \varphi_3(-\tilde{\tau}_1+\mu)\\ \varphi_4(-\tilde{\tau}_1+\mu)\\ \varphi_5(-\tilde{\tau}_1+\mu)\\ \varphi_6(-\tilde{\tau}_1+\mu)\end{pmatrix} + B_2\begin{pmatrix}\varphi_1(-\tilde{\tau}_2)\\ \varphi_2(-\tilde{\tau}_2)\\ \varphi_3(-\tilde{\tau}_2)\\ \varphi_4(-\tilde{\tau}_2)\\ \varphi_5(-\tilde{\tau}_2)\\ \varphi_6(-\tilde{\tau}_2)\end{pmatrix}, \tag{3.3}$$

where L_μ is a one-parameter family of bounded linear operators in $C([-\tilde{\tau}_2, 0], \mathbb{R}^6) \to \mathbb{R}^6$ and

$$B = \begin{pmatrix}-1 & 0 & 0 & 0 & b\,\mathrm{sech}^2(v^*) & b\,\mathrm{sech}^2(w^*)\\ 0 & -1 & 0 & b\,\mathrm{sech}^2(u^*) & 0 & 0\\ 0 & 0 & -1 & b\,\mathrm{sech}^2(u^*) & 0 & 0\\ 0 & 0 & 0 & -\beta & 0 & 0\\ 0 & 0 & 0 & 0 & -\beta & 0\\ 0 & 0 & 0 & 0 & 0 & -\beta\end{pmatrix},$$

$$B_1 = \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\ \beta&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}, B_2 = \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&\beta&0&0&0&0\\0&0&\beta&0&0&0\end{pmatrix}.$$

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta, \mu)$ in $[-\tilde{\tau}_2, 0] \to \mathbb{R}^{6^2}$, such that

$$L_\mu(\varphi) = \int_{-\tilde{\tau}_2}^{0} d\eta(\theta, \mu)\varphi(\theta). \tag{3.4}$$

In fact, choosing

$$\eta(\theta, \mu) = \begin{cases} B, & \theta = 0,\\ B_1, & \theta \in [-\tilde{\tau}_1, 0),\\ B_2, & \theta \in [-\tilde{\tau}_2, -\tilde{\tau}_1),\end{cases} \tag{3.5}$$

where $\delta(\theta)$ is Dirac function, then (3.4) is satisfied. For $(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6) \in (C^1[-\tilde{\tau}_2, 0], \mathbb{R}^6)$, define

$$A(\mu)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & -\tilde{\tau}_2 \le \theta < 0,\\ \int_{-\tilde{\tau}_2}^{0} d\eta(s, \mu)\varphi(s), & \theta = 0\end{cases} \tag{3.6}$$

and

$$R\varphi = \begin{cases} 0, & -\tilde{\tau}_2 \le \theta < 0,\\ f(\mu, \varphi), & \theta = 0.\end{cases} \tag{3.7}$$

Then (2.3) is equivalent to the abstract differential equation

$$\dot{u_t} = A(\mu)x_t + R(\mu)x_t, \tag{3.8}$$

where $u = (u_1, u_2, u_3, u_4, u_5, u_6)^T,\ u_t(\theta) = u(t+\theta), \theta \in [-\tilde{\tau}_2, 0]$.

For $\psi \in C([0, \tilde{\tau}_2], (\mathbb{R}^6)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, \tilde{\tau}_2], \\ \int_{-\tilde{\tau}_2}^{0} d\eta^T(t, 0)\psi(-t), & s = 0. \end{cases} \tag{3.9}$$

For $\phi \in C([-\tilde{\tau}_2, 0], \mathbb{R}^6)$ and $\psi \in C([0, \tilde{\tau}_2], (\mathbb{R}^6)^*)$, define the bilinear form

$$\langle \psi, \phi \rangle = \overline{\psi}(0)\phi(0) - \int_{-\tilde{\tau}_2}^{0} \int_{\xi=0}^{\theta} \psi^T(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \tag{3.10}$$

where $\eta(\theta) = \eta(\theta, 0)$. We have the following result on the relation between the operators $A = A(0)$ and A^*.

Lemma 3.1 *$A = A(0)$ and A^* are adjoint operators.*

Proof Let $\phi \in C^1([-\tilde{\tau}_2, 0], \mathbb{R}^6)$ and $\psi \in C^1([0, \tilde{\tau}_2], (\mathbb{R}^6)^*)$. It follows from (3.10) and the definitions of $A = A(0)$ and A^* that

$$\begin{aligned}
\langle \psi(s), A(0)\phi(\theta) \rangle &= \bar{\psi}(0)A(0)\phi(0) - \int_{-\tilde{\tau}_2}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) A(0)\phi(\xi) d\xi \\
&= \bar{\psi}(0) \int_{-\tilde{\tau}_2}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tilde{\tau}_2}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) A(0)\phi(\xi) d\xi \\
&= \bar{\psi}(0) \int_{-\tilde{\tau}_2}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tilde{\tau}_2}^{0} [\bar{\psi}(\xi - \theta) d\eta(\theta)\phi(\xi)]_{\xi=0}^{\theta} \\
&\quad + \int_{-\tilde{\tau}_2}^{0} \int_{\xi=0}^{\theta} \frac{d\bar{\psi}(\xi - \theta)}{d\xi} d\eta(\theta)\phi(\xi) d\xi \\
&= \int_{-\tilde{\tau}_2}^{0} \bar{\psi}(-\theta) d\eta(\theta)\phi(0) - \int_{-\tilde{\tau}_2}^{0} \int_{\xi=0}^{\theta} \left[-\frac{d\bar{\psi}(\xi - \theta)}{d\xi} \right] d\eta(\theta)\phi(\xi) d\xi \\
&= A^*\bar{\psi}(0)\phi(0) - \int_{-\tilde{\tau}_2}^{0} \int_{\xi=0}^{\theta} A^*\bar{\psi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi \\
&= \langle A^*\psi(s), \phi(\theta) \rangle.
\end{aligned}$$

This shows that $A = A(0)$ and A^* are adjoint operators and the proof is complete. □

By the discussions in Sect. 2, we know that $\pm i\omega_0$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively. We have the following result.

Lemma 3.2 *The vector*

$$q(\theta) = (1, a_1, a_2, a_3, a_4, a_5)^T e^{i\omega_0 \theta}, \quad \theta \in [-\tilde{\tau}_2, 0],$$

where

$$a_1 = \frac{a\beta sech^2(u^*)e^{-i\omega_0\tilde{\tau}_1}}{(i\omega_0+1)(\beta+i\omega_0)}, a_2 = \frac{a\beta sech^2(u^*)e^{-i\omega_0\tilde{\tau}_1}}{(i\omega_0+1)(\beta+i\omega_0)},$$
$$a_3 = \frac{\beta e^{-i\omega_0\tilde{\tau}_1}}{\beta+i\omega_0}, a_4 = \frac{a\beta sech^2(u^*)e^{-i\omega_0\tilde{\tau}}}{(i\omega_0+1)(\beta+i\omega_0)^2},$$
$$a_5 = \frac{(i\omega_0+1)^2(\beta+i\omega_0)^2 - a\beta sech^2(u^*)e^{-i\omega_0\tilde{\tau}}}{(i\omega_0+1)(\beta+i\omega_0)^2 bsech^2(w^*)},$$

is the eigenvector of $A(0)$ *corresponding to the eigenvalue* $i\omega_0$*, and*

$$q^*(s) = D(1, a_1^*, a_2^*, a_3^*, a_4^*, a_5^*)e^{i\omega_0 s}, \quad s \in [0, \tilde{\tau}_2],$$

where

$$a_1^* = \frac{b\beta sech^2(u^*)e^{-i\omega_0\tilde{\tau}_2}}{(i\omega_0+1)(\beta+i\omega_0)}, a_2^* = \frac{b\beta sech^2(w^*)e^{-i\omega_0\tilde{\tau}_2}}{(i\omega_0+1)(\beta+i\omega_0)},$$
$$a_3^* = \frac{i\omega_0+1}{\beta e^{-i\omega_0\tilde{\tau}_1}}, a_4^* = \frac{bsech^2(u^*)}{i\omega_0+\beta},$$
$$a_5^* = \frac{bsech^2(w^*)}{i\omega_0+\beta},$$

is the eigenvector of A^* *corresponding to the eigenvalue* $-i\omega_0$*, moreover,* $\langle q^*(s), q(\theta)\rangle = 1$*, where*

$$D = \frac{1}{1+\sum_{i=1}^{5}\bar{a}_i a_i^* + a_3^*\beta e^{i\omega_0\tilde{\tau}_1} + \bar{a}_1 a_4^*\beta e^{i\omega_0\tilde{\tau}_2} + \bar{a}_2 a_5^*\beta e^{i\omega_0\tilde{\tau}_2}}.$$

Proof Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$ and $q^*(s)$ be the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0$, namely, $A(0)q(\theta) = i\omega_0 q(\theta)$ and $A^*q(s) = -i\omega_0 q^*(s)$. From the definitions of $A(0)$ and A^*, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^*q(s) = -\frac{dq^*(s)}{ds}$. Thus, $q(\theta) = q(0)e^{i\omega_0\theta}$ and $q^*(s) = q(0)e^{i\omega_0 s}$. In addition,

$$\begin{aligned}\int_{-\tilde{\tau}_2}^{0} d\eta(\theta)q(\theta) &= \begin{pmatrix} -1 & 0 & 0 & 0 & b\text{sech}^2(v^*) & b\text{sech}^2(w^*) \\ 0 & -1 & 0 & b\text{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & -1 & b\text{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta \end{pmatrix} q(0) \\ &+ \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ \beta&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} q(-\tilde{\tau}_1) + \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&\beta&0&0&0&0 \\ 0&0&\beta&0&0&0 \end{pmatrix} q(-\tilde{\tau}_2) \\ &= A(0)q(0) = i\omega_0 q(0). \end{aligned} \tag{3.11}$$

That is

$$\begin{pmatrix} -1 + a_4\text{sech}^2(v^*) + a_5\text{sech}^2(w^*) \\ -a_1 + a_3\text{sech}^2(u^*) \\ -a_2 + a_3\text{sech}^2(u^*) \\ \beta e^{i\omega_0\tilde{\tau}_1} - a_3\beta \\ \beta a_1 e^{i\omega_0\tilde{\tau}_2} - a_4\beta \\ \beta a_2 e^{i\omega_0\tilde{\tau}_2} - a_5\beta \end{pmatrix} = \begin{pmatrix} i\omega_0 \\ ia_1\omega_0 \\ ia_2\omega_0 \\ ia_3\omega_0 \\ ia_4\omega_0 \\ ia_5\omega_0 \end{pmatrix}. \tag{3.12}$$

Therefore, we can easily obtain

$$\begin{aligned} a_1 &= \frac{a\beta\text{sech}^2(u^*)e^{-i\omega_0\tilde{\tau}_1}}{(i\omega_0+1)(\beta+i\omega_0)}, \qquad a_2 = \frac{a\beta\text{sech}^2(u^*)e^{-i\omega_0\tilde{\tau}_1}}{(i\omega_0+1)(\beta+i\omega_0)}, \\ a_3 &= \frac{\beta e^{-i\omega_0\tilde{\tau}_1}}{\beta+i\omega_0}, a_4 = \frac{a\beta\text{sech}^2(u^*)e^{-i\omega_0\tilde{\tau}}}{(i\omega_0+1)(\beta+i\omega_0)^2}, \\ a_5 &= \frac{(i\omega_0+1)^2(\beta+i\omega_0)^2 - a\beta\text{sech}^2(u^*)e^{-i\omega_0\tilde{\tau}}}{(i\omega_0+1)(\beta+i\omega_0)^2 b\text{sech}^2(w^*)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{-\tilde{\tau}_2}^{0} q^*(-t)d\eta(t) &= \begin{pmatrix} -1 & 0 & 0 & 0 & b\text{sech}^2(v^*) & b\text{sech}^2(w^*) \\ 0 & -1 & 0 & b\text{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & -1 & b\text{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta \end{pmatrix}^T q^*(0) \\ &+ \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ \beta&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}^T q^*(-\tilde{\tau}_1) + \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&\beta&0&0&0&0 \\ 0&0&\beta&0&0&0 \end{pmatrix}^T q(-\tilde{\tau}_2) \\ &= A^*q^*(0) = -i\omega_0 q^*(0). \end{aligned} \tag{3.13}$$

Namely,

$$\begin{pmatrix} -1 + \beta a_3^* e^{i\omega_0\tilde{\tau}_1} \\ -a_1^* + \beta a_3^* e^{i\omega_0\tilde{\tau}_2} \\ -a_2^* + \beta a_5^* e^{i\omega_0\tilde{\tau}_2} \\ a_1^* a\text{sech}^2(u^*) + a_2^* a\text{sech}^2(u^*) - \beta a_3^* \\ b\text{sech}^2(u^*) - \beta a_4^* \\ b\text{sech}^2(w^*) - \beta a_5^* \end{pmatrix} = \begin{pmatrix} 1 \\ ia_1^*\omega_0 \\ ia_2^*\omega_0 \\ ia_3^*\omega_0 \\ ia_4^*\omega_0 \\ ia_5^*\omega_0 \end{pmatrix}. \tag{3.14}$$

Therefore, we can easily obtain

$$a_1^* = \frac{b\beta \mathrm{sech}^2(u^*)e^{-i\omega_0\tilde{\tau}_2}}{(i\omega_0+1)(\beta+i\omega_0)}, a_2^* = \frac{b\beta \mathrm{sech}^2(w^*)e^{-i\omega_0\tilde{\tau}_2}}{(i\omega_0+1)(\beta+i\omega_0)},$$
$$a_3^* = \frac{i\omega_0+1}{\beta e^{-i\omega_0\tilde{\tau}_1}}, a_4^* = \frac{b\mathrm{sech}^2(u^*)}{i\omega_0+\beta},$$
$$a_5^* = \frac{b\mathrm{sech}^2(w^*)}{i\omega_0+\beta}.$$

In the sequel, we shall verify that $\langle q^*(s), q(\theta)\rangle = 1$. In fact, from (3.10), we have $\langle q^*(s), q(\theta)\rangle$

$$\begin{aligned}
&= \bar{D}(1, \bar{a}_1^*, \bar{a}_2^*, \bar{a}_3^*, \bar{a}_4, \bar{a}_5^*)(1, a_1, a_2, a_3, a_4, a_5)^T \\
&\quad - \int_{-\tilde{\tau}_2}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{a}_1^*, \bar{a}_2^*, \bar{a}_3^*, \bar{a}_4, \bar{a}_5^*)e^{-i\omega_0\tau_0(\xi-\theta)}d\eta(\theta)(1, a_1, a_2, a_3, a_4, a_5)^T e^{i\omega_0\xi}d\xi \\
&= \bar{D}\left[1+\sum_{i=1}^{5}a_i\bar{a}_i^* - \int_{-\tilde{\tau}_2}^{0}(1, \bar{a}_1^*, \bar{a}_2^*, \bar{a}_3^*, \bar{a}_4, \bar{a}_5^*)\theta e^{i\omega_0\theta}d\eta(\theta)(1, a_1, a_2, a_3, a_4, a_5)^T\right] \\
&= \bar{D}\left\{1+\sum_{i=1}^{5}a_i\bar{a}_i^* + (1, \bar{a}_1^*, \bar{a}_2^*, \bar{a}_3^*, \bar{a}_4, \bar{a}_5^*)\Lambda(1, a_1, a_2, a_3, a_4, a_5)^T\right\} \\
&= \bar{D}\left[1+\sum_{i=1}^{5}a_i\bar{a}_i^* + \bar{a}_3^*\beta e^{-i\omega_0\tilde{\tau}_1} + a_1\bar{a}_4^*\beta e^{-i\omega_0\tilde{\tau}_2} + a_2\bar{a}_5^*\beta e^{-i\omega_0\tilde{\tau}_2}\right] = 1.
\end{aligned}$$

where

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta e^{-i\omega_0\tilde{\tau}_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta e^{-i\omega_0\tilde{\tau}_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta e^{-i\omega_0\tilde{\tau}_1} & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

Next, we use the same notations as those in Hassard, Kazarinoff and Wan [43], and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of Eq.(2.3) when $\mu = 0$.

Define

$$z(t) = \langle q^*, x_t\rangle, \;\; W(t,\theta) = x_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\} \tag{3.16}$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \tag{3.17}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \tag{3.18}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if x_t is real, we consider only real solutions. For solutions $x_t \in C_0$ of (2.3),

$$
\begin{aligned}
\dot{z}(t) &= \langle q^*(s), \dot{x}_t\rangle = \langle q^*(s), A(0)x_t + R(0)x_t\rangle \\
&= \langle q^*(s), A(0)x_t\rangle + \langle q^*(s), R(0)x_t\rangle \\
&= \langle A^* q^*(s), x_t\rangle + \bar{q}^*(0)R(0)x_t \\
&\quad - \int_{-\bar{\tau}_2}^{0}\int_{\xi=0}^{\theta} \bar{q}^*(\xi-\theta)d\eta(\theta)A(0)R(0)x_t(\xi)d\xi \\
&= \langle i\omega_0 q^*(s), x_t\rangle + \bar{q}^*(0)f(0, x_t(\theta)) \\
&\overset{\text{def}}{=} i\omega_0 z(t) + \bar{q}^*(0)f_0(z(t), \bar{z}(t)).
\end{aligned} \tag{3.19}
$$

That is

$$
\dot{z}(t) = i\omega_0\tau_0 z + g(z, \bar{z}), \tag{3.20}
$$

where

$$
g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.21}
$$

Hence, we have

$$
\begin{aligned}
g(z, \bar{z}) &= \bar{q}^*(0)f_0(z, \bar{z}) = f(0, u_t) \\
&= \tau_0\bar{D}(1, \bar{a}_1^*, \bar{a}_2^*, \bar{a}_3^*, \bar{a}_4^*, \bar{a}_5^*) \\
&\quad \times (f_1(0, x_t), f_2(0, x_t), f_3(0, x_t), f_4(0, x_t), f_5(0, x_t), f_6(0, x_t))^T,
\end{aligned} \tag{3.22}
$$

where

$$
\begin{aligned}
f_1(0, x_t) &= c_{11}x_{5t}^2(0) + c_{12}x_{5t}^3(0) + c_{13}x_{6t}^2(0) + c_{14}x_{6t}^3(0) + \text{h.o.t.}, \\
f_2(0, x_t) &= c_{21}x_{4t}^2(0) + c_{22}x_{4t}^3(0) + \text{h.o.t.}, \\
f_3(0, x_t) &= c_{21}x_{4t}^2(0) + c_{22}x_{4t}^3(0) + \text{h.o.t.}, \\
f_4(0, x_t) &= f_5(0, x_t) = f_6(0, x_t) = 0.
\end{aligned}
$$

Noticing that

$$
x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta), x_{4t}(\theta), x_{5t}(\theta), x_{6t}(\theta))^T = W(t, \theta) + zq(\theta) + \bar{z}\bar{q}
$$

and

$$
q(\theta) = (1, a_1, a_2, a_3, a_4, a_5)^T e^{i\omega_0\theta},
$$

we have

$$
\begin{aligned}
x_{4t}(0) &= a_3 z + \bar{a}_3\bar{z} + W_{20}^{(4)}(0)\frac{z^2}{2} + W_{11}^{(4)}(0)z\bar{z} + W_{02}^{(4)}(0)\frac{\bar{z}^2}{2} + \cdots, \\
x_{5t}(0) &= a_4 z + \bar{a}_4\bar{z} + W_{20}^{(5)}(0)\frac{z^2}{2} + W_{11}^{(5)}(0)z\bar{z} + W_{02}^{(5)}(0)\frac{\bar{z}^2}{2} + \cdots, \\
x_{6t}(0) &= a_5 z + \bar{a}_5\bar{z} + W_{20}^{(6)}(0)\frac{z^2}{2} + W_{11}^{(6)}(0)z\bar{z} + W_{02}^{(6)}(0)\frac{\bar{z}^2}{2} + \cdots.
\end{aligned}
$$

From (3.21) and (3.22), we can obtain the expression of $g(z, \bar{z})$ as follows

$$
\begin{aligned}
g(z,\bar{z}) = {} & (c_{11}a_4^2 + c_{13}a_5^2 + \bar{a_1^*}c_{21}a_3^2 + \bar{a_2^*}c_{21}a_3^2)z^2 \\
& + (2c_{11}|a_4|^2 + 2c_{13}|a_5|^2 + 2\bar{a_1^*}c_{21}|a_3|^2 + 2\bar{a_2^*}c_{21}|a_3|^2)z\bar{z} \\
& + (c_{11}\bar{a}_4^2 + c_{13}\bar{a}_5^2 + \bar{a_1^*}c_{21}\bar{a}_3^2 + \bar{a_2^*}c_{21}\bar{a}_3^2)\bar{z}^2 \\
& + \Big\{c_{11}\left(2a_4W_{11}^{(5)}(0) + W_{20}^{(5)}(0)\bar{a}_4\right) + 3c_{12}a_4^2\bar{a}_4 \\
& + c_{13}\left(2a_5W_{11}^{(6)}(0) + W_{20}^{(6)}(0)\bar{a}_5\right) + 3c_{14}a_5^2\bar{a}_5 + (a_1^* + a_2^*) \\
& \times \Big[c_{21}\left(2a_3W_{11}^{(4)}(0) + W_{20}^{(4)}(0)\bar{a}_3\right) + 3c_{22}a_3^2\bar{a}_3\Big]\Big\}z^2\bar{z} + \text{h.o.t.}.
\end{aligned}
$$

It is easy to obtain

$$
\begin{aligned}
g_{20} &= 2\bar{D}(c_{11}a_4^2 + c_{13}a_5^2 + \bar{a_1^*}c_{21}a_3^2 + \bar{a_2^*}c_{21}a_3^2) \\
g_{11} &= 2\bar{D}(c_{11}|a_4|^2 + c_{13}|a_5|^2 + \bar{a_1^*}c_{21}|a_3|^2 + \bar{a_2^*}c_{21}|a_3|^2), \\
g_{02} &= 2\bar{D}\Big\{c_{11}\left(2a_4W_{11}^{(5)}(0) + W_{20}^{(5)}(0)\bar{a}_4\right) + 3c_{12}a_4^2\bar{a}_4 \\
& \quad + c_{13}\left(2a_5W_{11}^{(6)}(0) + W_{20}^{(6)}(0)\bar{a}_5\right) + 3c_{14}a_5^2\bar{a}_5 + (a_1^* + a_2^*) \\
& \quad \times \Big[c_{21}\left(2a_3W_{11}^{(4)}(0) + W_{20}^{(4)}(0)\bar{a}_3\right) + 3c_{22}a_3^2\bar{a}_3\Big]\Big\}.
\end{aligned}
$$

Since

$$W_{11}^{(4)}(0),\ W_{11}^{(5)}(0),\ W_{11}^{(6)}(0),\ W_{20}^{(4)}(0),\ W_{20}^{(5)}(0),\ W_{20}^{(6)}(0)$$

in g_{21}, we still need to compute them. In view of (3.8) and (3.9), we have

$$
\begin{aligned}
W' &= \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(\theta)\}, & -\tilde{\tau}_2 \le \theta < 0, \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(\theta)\} + f_0, & \theta = 0. \end{cases} \\
&\overset{\text{def}}{=} AW + H(z,\bar{z},\theta),
\end{aligned}
\tag{3.23}
$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.24}$$

Comparing the coefficients, we obtain

$$(A - 2i\omega_0)W_{20} = -H_{20}(\theta), \tag{3.25}$$

$$AW_{11}(\theta) = -H_{11}(\theta), \tag{3.26}$$

For $\theta \in [-\tilde{\tau}_2, 0)$,

$$H(z,\bar{z},\theta) = -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta). \tag{3.27}$$

Comparing the coefficients of (3.24) with (3.27) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{3.28}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{3.29}$$

From (3.25), (3.28) and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2i\omega_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta). \tag{3.30}$$

Noting that $q(\theta) = q(0)e^{i\omega_0\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0} q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta}, \tag{3.31}$$

where E_1 is a constant vector. Similarly, from (3.26), (3.29) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g_{11}}\bar{q}(\theta), \tag{3.32}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0} q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{11}}{\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_2. \tag{3.33}$$

where E_2 is a constant vector.

In what follows, we shall seek appropriate $E_1,\ E_2$ in (3.31), (3.33), respectively. It follows from the definition of A and (3.28), (3.29) that

$$\int_{-\tilde{\tau}_2}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0 W_{20}(0) - H_{20}(0) \tag{3.34}$$

and

$$\int_{-\tilde{\tau}_2}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.35}$$

where $\eta(\theta) = \eta(0, \theta)$. It follows from (3.25) that

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + 2(H_1, H_2, H_3, H_4, H_5, H_6)^T, \tag{3.36}$$

where

$$H_1 = c_{11}a_4^2 + c_{13}a_5^2,\ H_2 = c_{21}a_3^2,\ H_3 = c_{21}a_3^2,\ H_4 = H_4 = H_6 = 0.$$

From (3.26), we have

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + \tau_0(P_1, P_2, P_3, P_4, P_5, P_6)^T, \tag{3.37}$$

where

$$P_1 = c_{11}|a_4|^2 + c_{13}|a_5|^2,\ P_2 = c_{21}|a_3|^2,\ P_3 = c_{21}|a_3|^2,\ P_4 = P_5 = P_6 = 0.$$

Noting that

$$\left(i\omega_0 I - \int_{-\tilde{\tau}_2}^{0} e^{i\omega_0\theta} d\eta(\theta)\right) q(0) = 0, \tag{3.38}$$

$$\left(-i\omega_0 I - \int_{-\tilde{\tau}_2}^{0} e^{-i\omega_0\theta} d\eta(\theta)\right) \bar{q}(0) = (H_1, H_2, H_3, H_4, H_5, H_6)^T \tag{3.39}$$

and substituting (3.31) and (3.36) into (3.34), we have

$$\left(2i\omega_0 I - \int_{-\tilde{\tau}_2}^{0} e^{2i\omega_0\theta} d\eta(\theta)\right) E_1 = (H_1, H_2, H_3, H_4, H_5, H_6)^T. \tag{3.40}$$

That is

$$\begin{pmatrix} 2i\omega_0+1 & 0 & 0 & 0 & -b\text{sech}^2(v^*) & -b\text{sech}^2(w^*) \\ 0 & 2i\omega_0+1 & 0 & -b\text{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & 2i\omega_0+1 & -b\text{sech}^2(u^*) & 0 & 0 \\ -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 & 0 \\ 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 \\ 0 & 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta \end{pmatrix}$$
$$\times(E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_1^{(4)}, E_1^{(5)}, E_1^{(6)})^T = 2(H_1, H_2, H_3, 0, 0, 0)^T. \tag{3.41}$$

Hence,

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1},\ E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},\ E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1},\ E_1^{(4)} = \frac{\Delta_{14}}{\Delta_1},\ E_1^{(5)} = \frac{\Delta_{15}}{\Delta_1},\ E_1^{(6)} = \frac{\Delta_{16}}{\Delta_1}, \tag{3.42}$$

where

$$\Delta_1 = \det\begin{pmatrix} 2i\omega_0+1 & 0 & 0 & 0 & -b\text{sech}^2(v^*) & -b\text{sech}^2(w^*) \\ 0 & 2i\omega_0+1 & 0 & -b\text{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & 2i\omega_0+1 & -b\text{sech}^2(u^*) & 0 & 0 \\ -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 & 0 \\ 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 \\ 0 & 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta \end{pmatrix},$$

$$\Delta_{11} = 2\det\begin{pmatrix} H_1 & 0 & 0 & 0 & -b\text{sech}^2(v^*) & -b\text{sech}^2(w^*) \\ H_2 & 2i\omega_0+1 & 0 & -b\text{sech}^2(u^*) & 0 & 0 \\ H_3 & 0 & 2i\omega_0+1 & -b\text{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & 0 & 2i\omega_0+\beta & 0 & 0 \\ 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 \\ 0 & 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta \end{pmatrix},$$

$$\Delta_{12} = 2\det\begin{pmatrix} 2i\omega_0+1 & H_1 & 0 & 0 & -b\text{sech}^2(v^*) & -b\text{sech}^2(w^*) \\ 0 & H_2 & 0 & -b\text{sech}^2(u^*) & 0 & 0 \\ 0 & H_3 & 2i\omega_0+1 & -b\text{sech}^2(u^*) & 0 & 0 \\ -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 2i\omega_0+\beta & 0 \\ 0 & 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta \end{pmatrix},$$

$$\Delta_{13} = 2\det\begin{pmatrix} 2i\omega_0+1 & 0 & H_1 & 0 & -b\text{sech}^2(v^*) & -b\text{sech}^2(w^*) \\ 0 & 2i\omega_0+1 & H_2 & -b\text{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & H_3 & -b\text{sech}^2(u^*) & 0 & 0 \\ -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 & 0 \\ 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 2i\omega_0+\beta \end{pmatrix},$$

$$\Delta_{14} = 2\det\begin{pmatrix} 2i\omega_0+1 & 0 & 0 & H_1 & -b\text{sech}^2(v^*) & -b\text{sech}^2(w^*) \\ 0 & 2i\omega_0+1 & 0 & H_2 & 0 & 0 \\ 0 & 0 & 2i\omega_0+1 & H_3 & 0 & 0 \\ -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 \\ 0 & 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta \end{pmatrix},$$

$$\Delta_{15} = 2\det\begin{pmatrix} 2i\omega_0+1 & 0 & 0 & 0 & H_1 & -b\mathrm{sech}^2(w^*) \\ 0 & 2i\omega_0+1 & 0 & -b\mathrm{sech}^2(u^*) & H_2 & 0 \\ 0 & 0 & 2i\omega_0+1 & -b\mathrm{sech}^2(u^*) & 0 & 0 \\ -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & H_3 & 0 \\ 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta \end{pmatrix},$$

$$\Delta_{16} = 2\det\begin{pmatrix} 2i\omega_0+1 & 0 & 0 & 0 & -b\mathrm{sech}^2(v^*) & H_1 \\ 0 & 2i\omega_0+1 & 0 & -b\mathrm{sech}^2(u^*) & 0 & H_2 \\ 0 & 0 & 2i\omega_0+1 & -b\mathrm{sech}^2(u^*) & 0 & H_3 \\ -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 & 0 \\ 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 2i\omega_0+\beta & 0 \\ 0 & 0 & -\beta e^{i\omega_0\tilde{\tau}_1} & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, substituting (3.32) and (3.37) into (3.35), we have

$$\left(\int_{-\tilde{\tau}_2}^{0} d\eta(\theta)\right) E_2 = 2(-P_1, -P_2, -P_3, 0, 0, 0)^T. \tag{3.43}$$

That is

$$\begin{pmatrix} -1 & 0 & 0 & 0 & b\mathrm{sech}^2(v^*) & b\mathrm{sech}^2(w^*) \\ 0 & -1 & 0 & a\mathrm{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & -1 & a\mathrm{sech}^2(u^*) & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \\ E_2^{(4)} \\ E_2^{(5)} \\ E_2^{(6)} \end{pmatrix} = 2\begin{pmatrix} -P_1 \\ -P_2 \\ -P_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.44}$$

Hence,

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2},\ E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2},\ E_2^{(4)} = \frac{\Delta_{24}}{\Delta_2},\ E_2^{(5)} = \frac{\Delta_{25}}{\Delta_2},\ E_2^{(6)} = \frac{\Delta_{26}}{\Delta_2}, \tag{3.45}$$

where

$$\Delta_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & b\mathrm{sech}^2(v^*) & b\mathrm{sech}^2(w^*) \\ 0 & -1 & 0 & a\mathrm{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & -1 & a\mathrm{sech}^2(u^*) & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \end{pmatrix},$$

$$\Delta_{21} = 2\det\begin{pmatrix} -P_1 & 0 & 0 & 0 & b\mathrm{sech}^2(v^*) & b\mathrm{sech}^2(w^*) \\ -P_2 & -1 & 0 & a\mathrm{sech}^2(u^*) & 0 & 0 \\ -P_3 & 0 & -1 & a\mathrm{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \end{pmatrix} = 0,$$

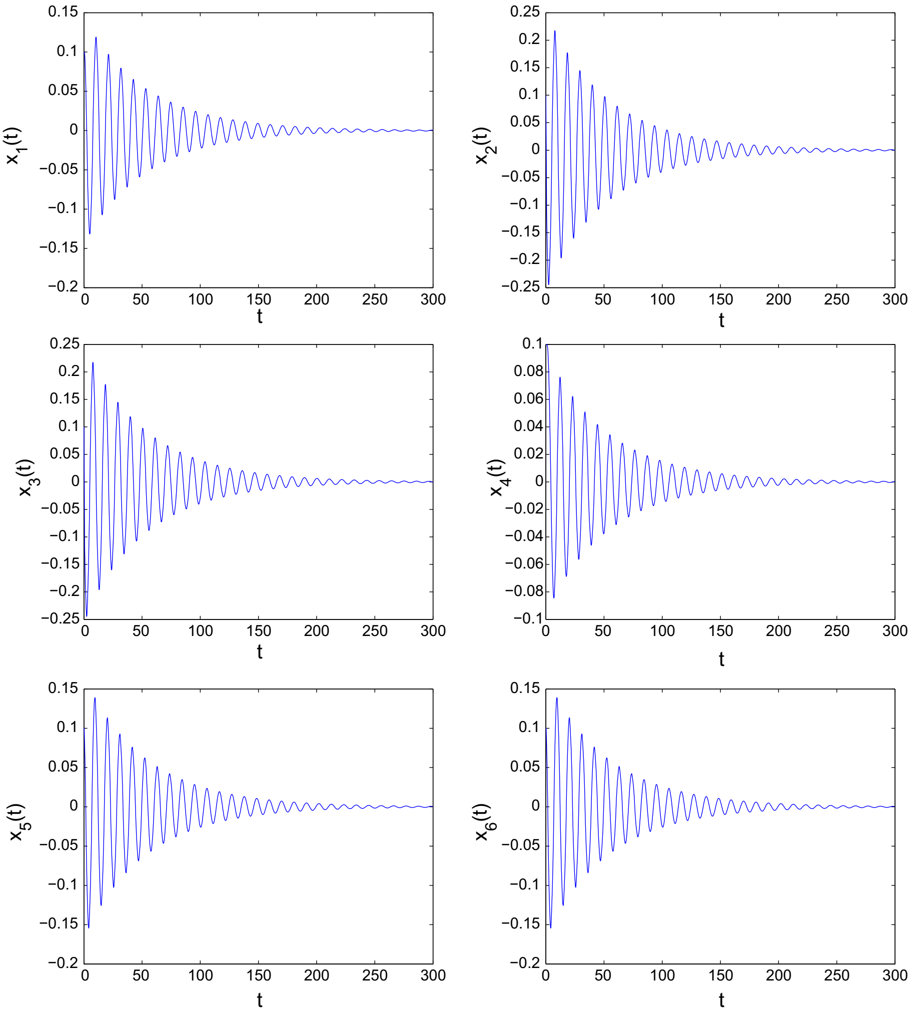

Fig. 1 Dynamic behavior of system (4.1): times series of $x_i (i = 1, 2, 3, 4, 5, 6)$. A Matlab simulation of the asymptotically stable origin to system (4.1) with $\tau \approx 0.6002$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)

$$\Delta_{22} = 2\det\begin{pmatrix} -1 & -P_1 & 0 & 0 & b\text{sech}^2(v^*) & b\text{sech}^2(w^*) \\ 0 & -P_2 & 0 & a\text{sech}^2(u^*) & 0 & 0 \\ 0 & -P_3 & -1 & a\text{sech}^2(u^*) & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \end{pmatrix} = 0,$$

$$\Delta_{23} = 2\det\begin{pmatrix} -1 & 0 & -P_1 & 0 & b\text{sech}^2(v^*) & b\text{sech}^2(w^*) \\ 0 & -1 & -P_2 & a\text{sech}^2(u^*) & 0 & 0 \\ 0 & 0 & -P_3 & a\text{sech}^2(u^*) & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

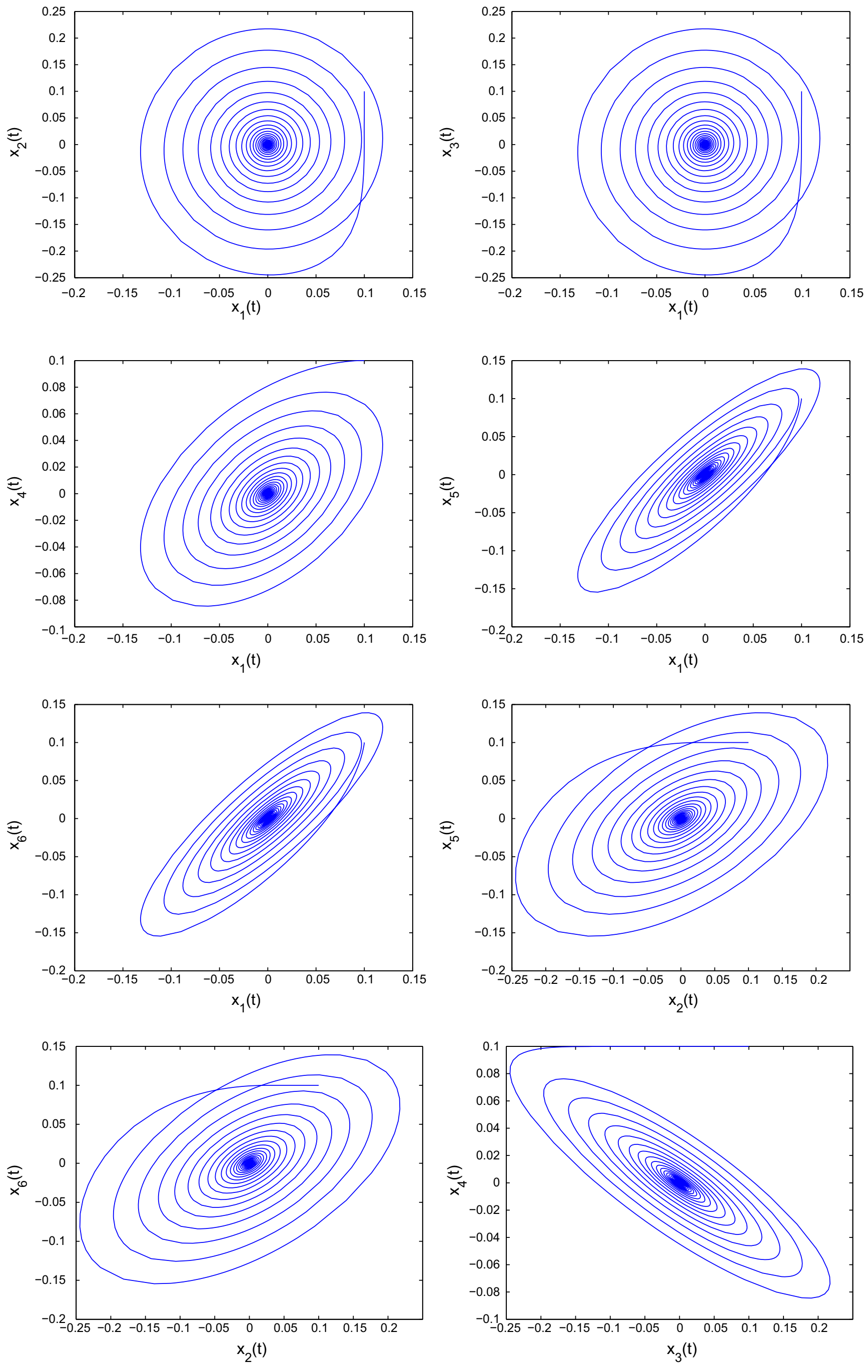

Fig. 2 Dynamic behavior of system (4.1): projection on $x_1 - x_2, x_1 - x_3, x_1 - x_4, x_1 - x_5, x_1 - x_6, x_2 - x_5, x_2 - x_6, x_3 - x_4, x_3 - x_5, x_3 - x_6, x_4 - x_5, x_4 - x_6$ plane, respectively. A Matlab simulation of the asymptotically stable origin to system (4.1) with $\tau_1 = 0.3$, $\tau_2 = 0.2$ and $\tau_1 + \tau_2 = \tau = 0.5 < \tau_0 \approx 0.6002$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)

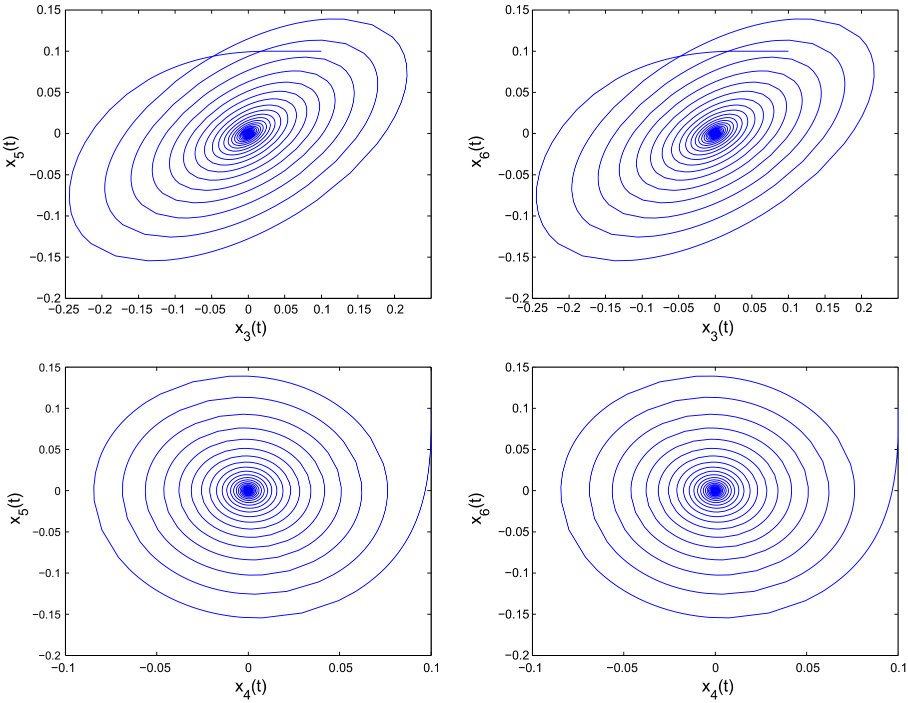

Fig. 2 continued

$$\Delta_{24} = 2\det\begin{pmatrix} -1 & 0 & 0 & -P_1 & b\mathrm{sech}^2(v^*) & b\mathrm{sech}^2(w^*) \\ 0 & -1 & 0 & -P_2 & 0 & 0 \\ 0 & 0 & -1 & -P_3 & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \end{pmatrix},$$

$$\Delta_{25} = 2\det\begin{pmatrix} -1 & 0 & 0 & 0 & -P_1 & b\mathrm{sech}^2(w^*) \\ 0 & -1 & 0 & a\mathrm{sech}^2(u^*) & -P_2 & 0 \\ 0 & 0 & -1 & a\mathrm{sech}^2(u^*) & -P_3 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \end{pmatrix},$$

$$\Delta_{26} = 2\det\begin{pmatrix} -1 & 0 & 0 & 0 & b\mathrm{sech}^2(v^*) & -P_1 \\ 0 & -1 & 0 & a\mathrm{sech}^2(u^*) & 0 & -P_2 \\ 0 & 0 & -1 & a\mathrm{sech}^2(u^*) & 0 & -P_3 \\ \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \end{pmatrix}.$$

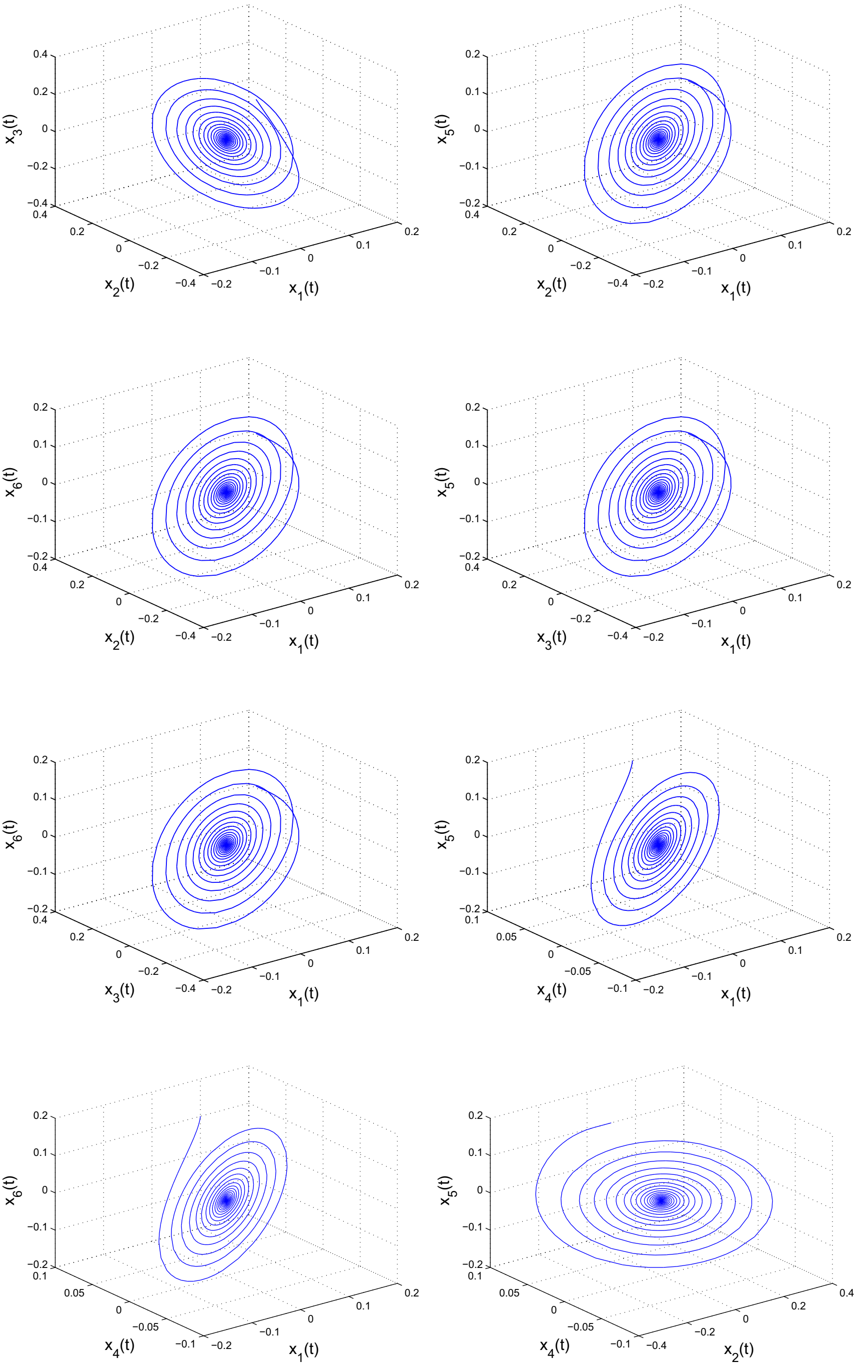

Fig. 3 Dynamic behavior of system (4.1): projection on $x_1-x_2-x_3$, $x_1-x_2-x_5$, $x_1-x_2-x_6$, $x_1-x_3-x_5$, $x_1-x_3-x_6$, $x_1-x_4-x_5$, $x_1-x_4-x_6$, $x_2-x_4-x_5$, $x_2-x_4-x_6$, $x_2-x_5-x_6$, $x_3-x_5-x_6$, $x_4-x_5-x_6$ space, respectively. A Matlab simulation of the asymptotically stable origin to system (4.1) with $\tau_1 = 0.3$, $\tau_2 = 0.2$ and $\tau_1 + \tau_2 = \tau = 0.5 < \tau_0 \approx 0.6002$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)

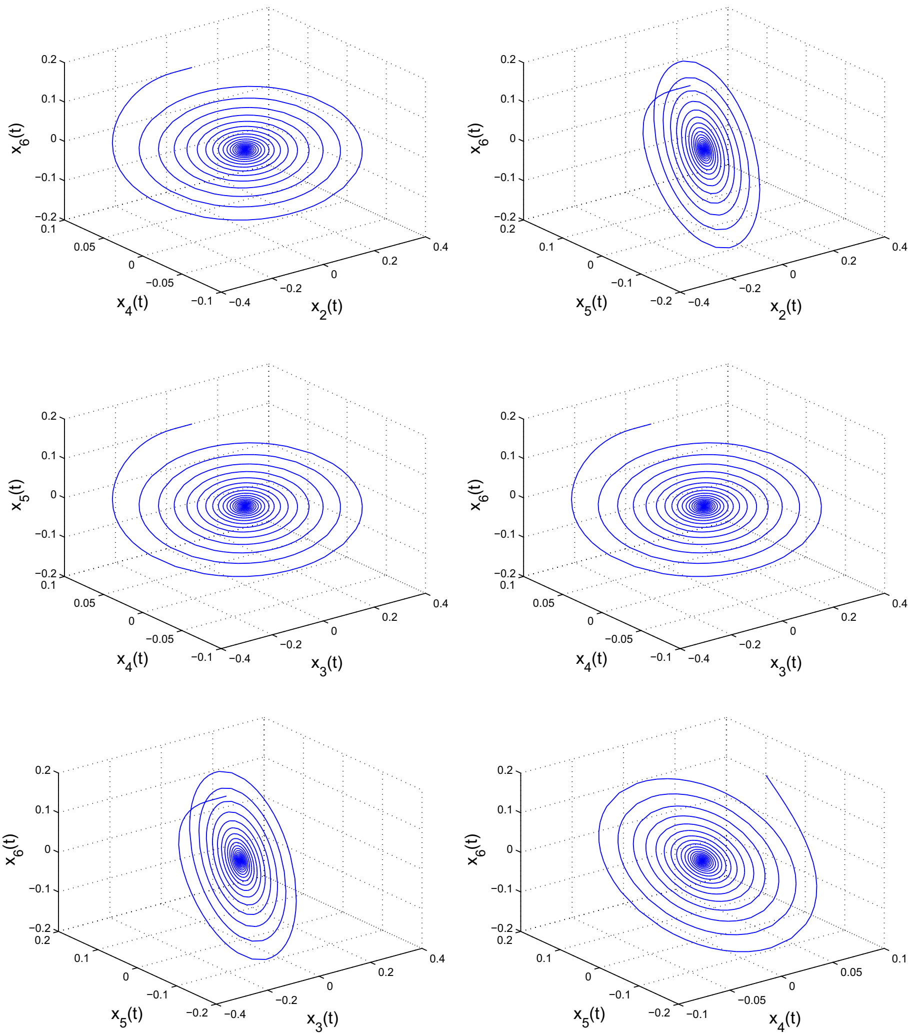

Fig. 3 continued

From (3.31), (3.33), (3.42), (3.45), we can calculate g_{21} and derive the following values:

$$
\begin{aligned}
c_1(0) &= \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda^{'}(\tau_0)\}}, \\
\beta_2 &= 2\mathrm{Re}(c_1(0)), \\
T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2 Im\{\lambda^{'}(\tau_0)\}}{\omega_0}.
\end{aligned}
$$

These formulaes give a description of the Hopf bifurcation periodic solutions of (2.3) at $\tau = \tau_0$ on the center manifold. From the discussion above, we have the following result: □

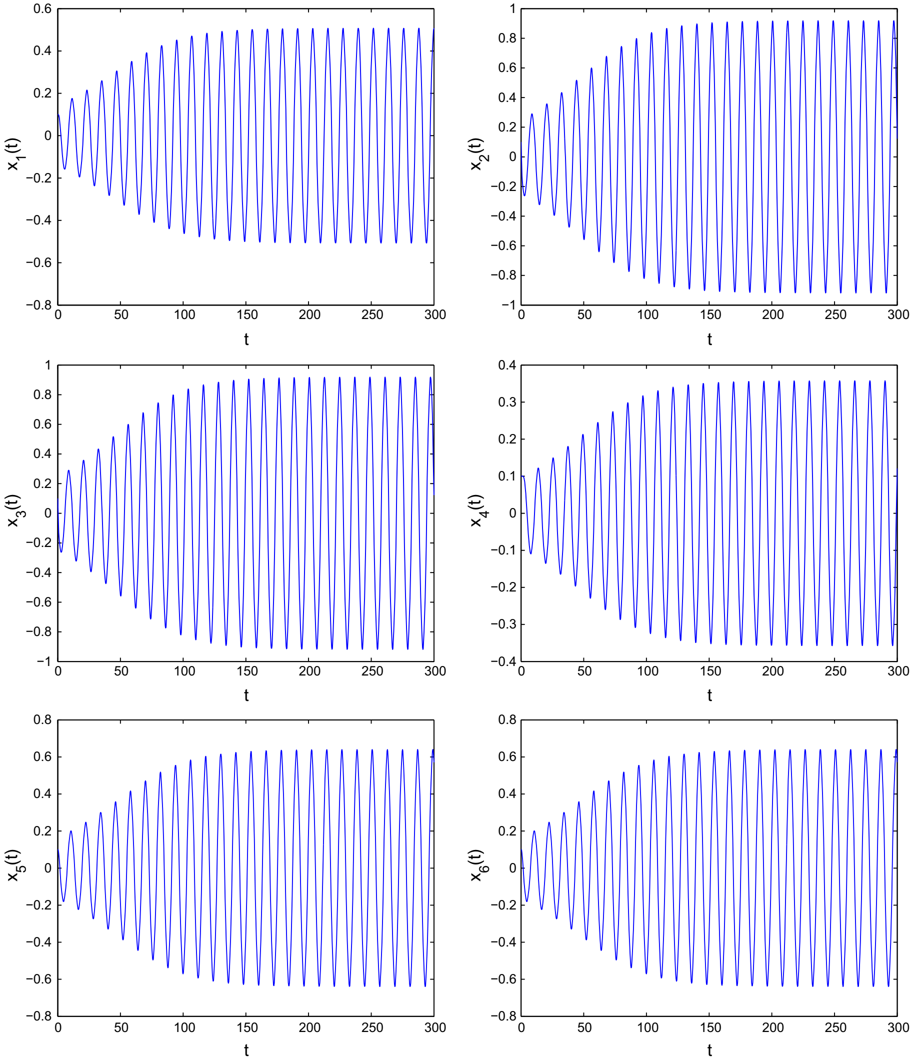

Fig. 4 Dynamic behavior of system (4.1): times series of x_i $(i = 1, 2, 3, 4, 5, 6)$. A Matlab simulation of the Hopf bifurcation of system (4.1) with $\tau_1 = 0.8$, $\tau_2 = 0.3$ and $\tau_1 + \tau_2 = \tau = 1.1 > \tau_0 \approx 0.6002$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)

Theorem 3.3 *For system (2.3), if (H1)–(H4) hold, the periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$); The bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); The periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*

Remark 3.1 In [44], authors considered the stability switches and bifurcation for a neural networks with continuous delay and strong kernel by regarding the mean time delay as bifurcation parameter. In [45], authors discussed the delay-dependent asymptotic stability for neural networks with distributed delays by employing suitable Lyapunov functionals and delay-dependent criteria. In [46], authors analyzed bifurcation behavior for a two-neuron

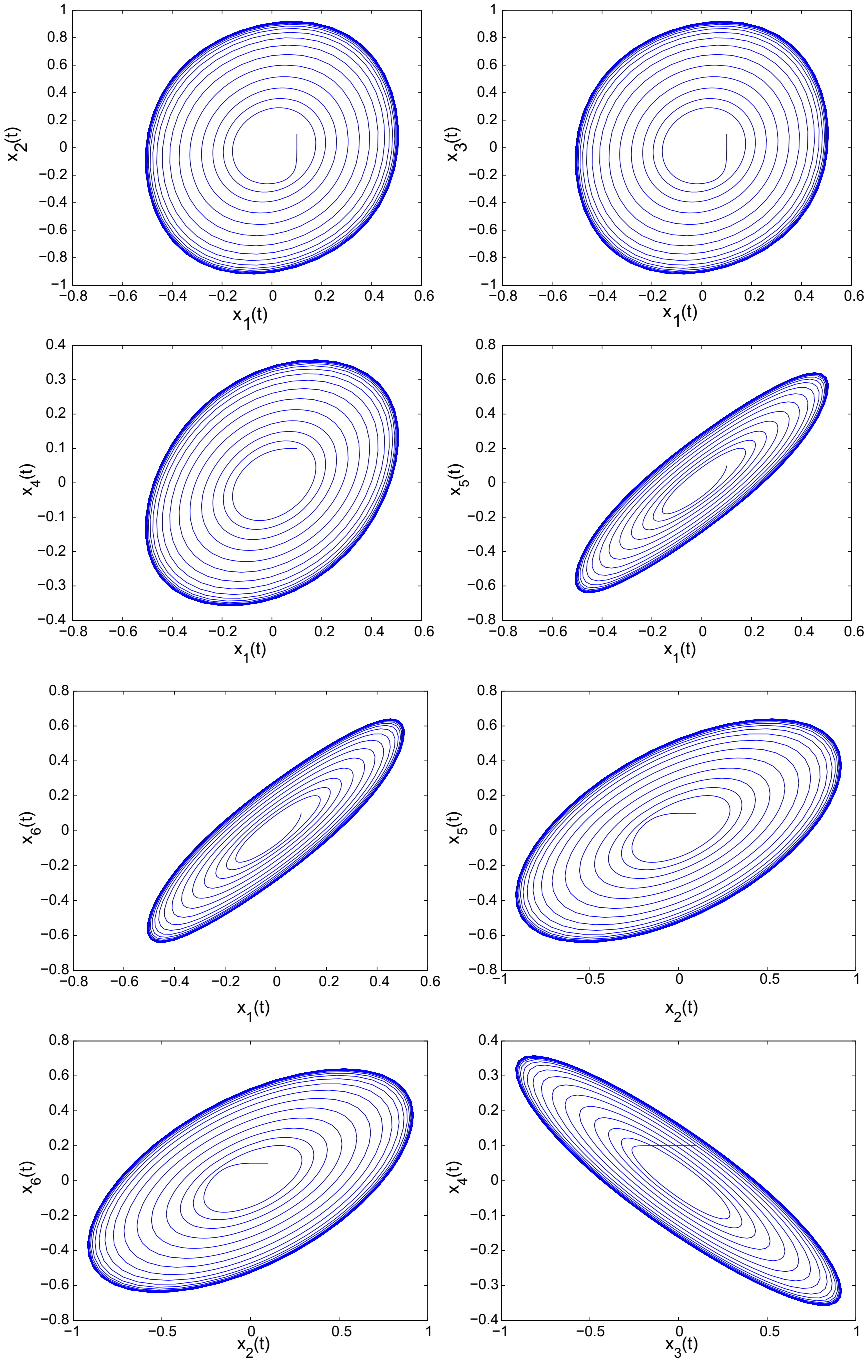

Fig. 5 Dynamic behavior of system (4.1): projection on $x_1 - x_2 - x_3$, $x_1 - x_2 - x_5$, $x_1 - x_2 - x_6$, $x_1 - x_3 - x_5$, $x_1 - x_3 - x_6$, $x_1 - x_4 - x_5$, $x_1 - x_4 - x_6$, $x_2 - x_4 - x_5$, $x_2 - x_4 - x_6$, $x_2 - x_5 - x_6$, $x_3 - x_5 - x_6$, $x_4 - x_5 - x_6$ space, respectively. A Matlab simulation of the Hopf bifurcation of system (4.1) with $\tau_1 = 0.8$, $\tau_2 = 0.3$ and $\tau_1 + \tau_2 = \tau = 1.1 > \tau_0 \approx 0.6002$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)

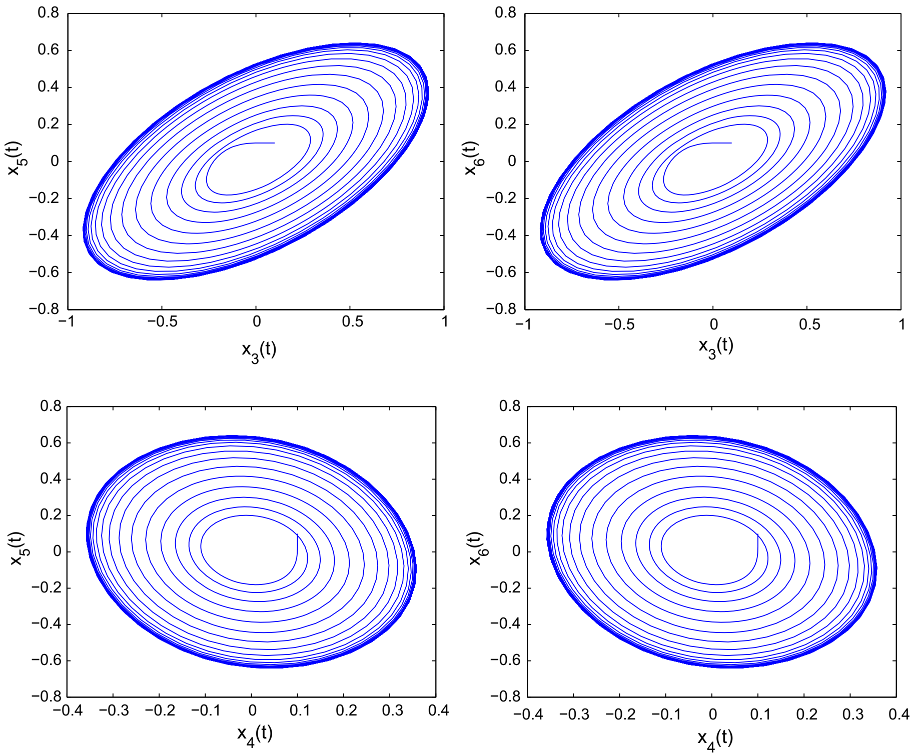

Fig. 5 continued

system with distributed delays by applying frequency domain method. All the analysis methods above are different from those in this paper. Thus our work complements the previous studies.

4 Numerical Examples

To illustrate the analytical results, we consider the following special case of system (2.3).

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1(t) + 0.5\tanh[x_5(t)] + 0.5\tanh[x_6(t)], \\ \dfrac{dx_2}{dt} = -x_2(t) + 3\tanh[x_4(t)], \\ \dfrac{dx_3}{dt} = -x_3(t) + 3\tanh[x_4(t)], \\ \dfrac{dx_4}{dt} = 0.5[x_1(t-\tau_1) - x_4(t)], \\ \dfrac{dx_5}{dt} = 0.5[x_2(t-\tau_2) - x_5(t)], \\ \dfrac{dx_6}{dt} = 0.5[x_3(t-\tau_2) - x_6(t)], \end{cases} \tag{4.1}$$

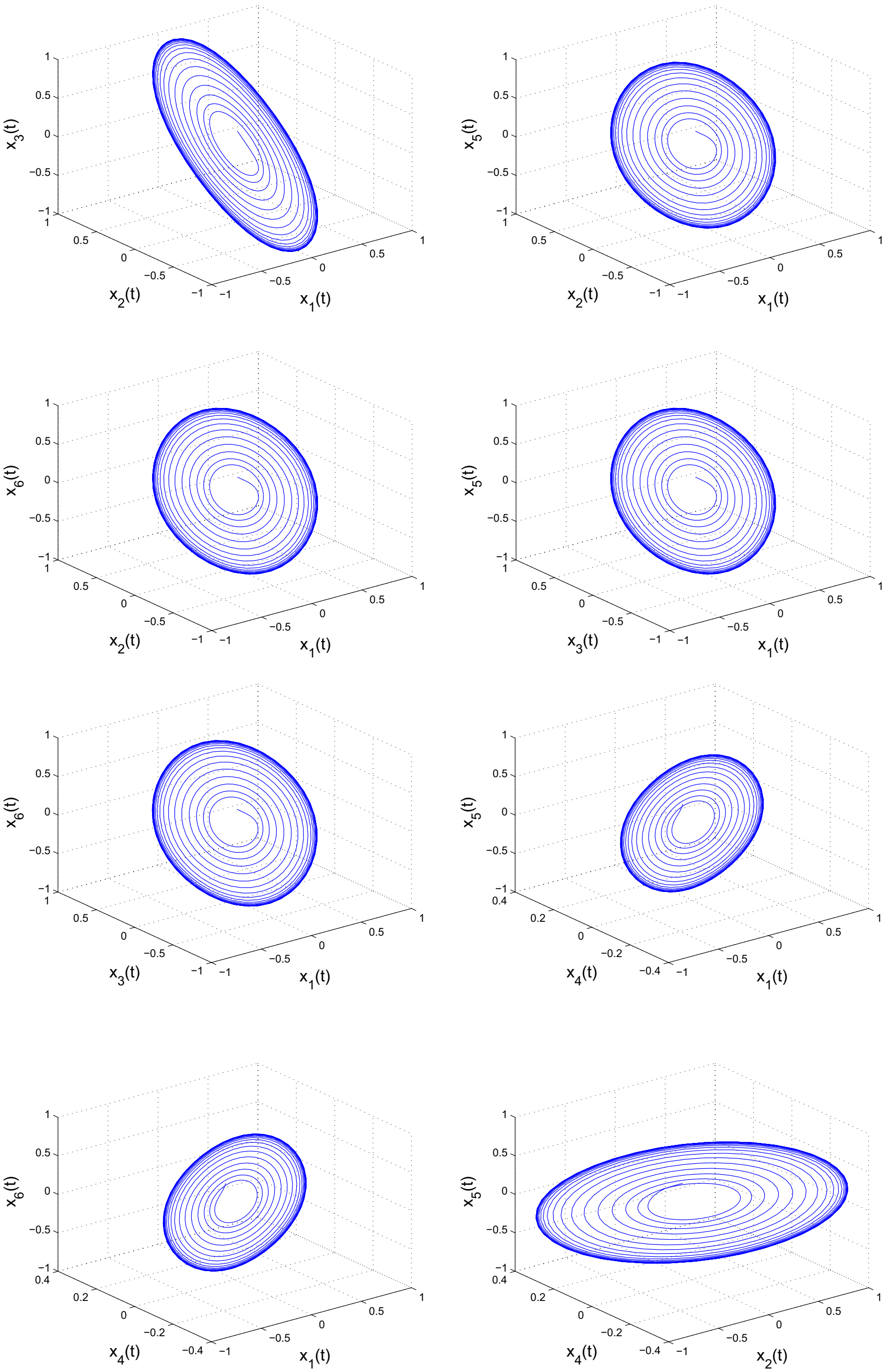

Fig. 6 Dynamic behavior of system (4.1): projection on $x_1-x_2-x_3$, $x_1-x_2-x_5$, $x_1-x_2-x_6$, $x_1-x_3-x_5$, $x_1-x_3-x_6$, $x_1-x_4-x_5$, $x_1-x_4-x_6$, $x_2-x_4-x_5$, $x_2-x_4-x_6$, $x_2-x_5-x_6$, $x_3-x_5-x_6$, $x_4-x_5-x_6$ space, respectively. A Matlab simulation of the Hopf bifurcation of system (4.1) with $\tau_1 = 0.8$, $\tau_2 = 0.3$ and $\tau_1 + \tau_2 = \tau = 1.1 > \tau_0 \approx 0.6002$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)

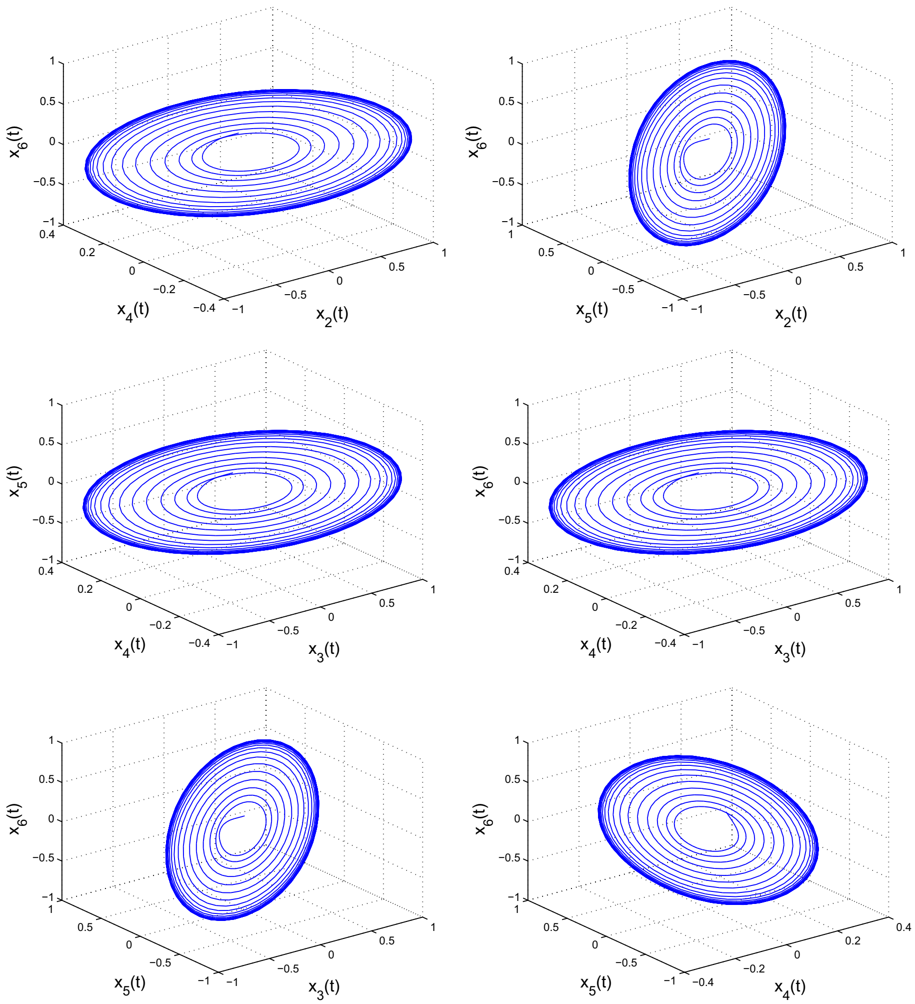

Fig. 6 continued

which has a unique steady state $E(0, 0, 0, 0, 0, 0)$. By means of Matlab 7.0, we obtain $\omega_0 \approx 0.9059$, $\tau_0 \approx 0.6002$, $\lambda^{'}(0) \approx 3.4023 - 2.6576i$. Thus we can compute these values as follows: $c_1(0) \approx -4.1128 - 165.5123i$, $\mu_2 \approx 2.0213 > 0$, $\beta_2 \approx -8.2256 < 0$, $T_2 \approx 16.4719$. Then all the conditions indicated in Theorem 2.2 hold true. From Theorem 2.1, we know that the zero steady state of system (4.1) is asymptotically stable when $\tau_1 + \tau_2 \in [0, 0.6)$ which is illustrated by the numerical simulations shown in Figs. 1, 2 and 3 in which $\tau_1 = 0.3$ and $\tau_2 = 0.2$. When $\tau_1 + \tau_2$ is increased to the critical value 0.6, the equilibrium $E(0, 0, 0, 0, 0, 0)$ loses its stability and a Hopf bifurcation occurs. Since $\mu_2 > 0$ and $\beta_2 < 0$ it follows from Theorem 3.3 that the Hopf bifurcation is supercritical and bifurcating periodic solution is asymptotically stable which is depicted in Figs. 4, 5 and 6.

5 Conclusions

In this paper, we have investigated three-neuron artificial neural network model with distributed delays. Using Hopf bifurcation theory and numerical method of functional differential equation, we have analyzed the local stability of the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$ and oscillatory behavior of the system. We have showed that if some suitable conditions hold and $\tau \in [0, \tau_0)$, then the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$ of system (2.3) is asymptotically stable and unstable when $\tau > \tau_0$. It is also showed that if some other suitable conditions are fulfilled and when the delay τ increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occur around $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$, i.e., a family of periodic orbits bifurcate from the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)$. In addition, explicit algorithm for determining the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are derived by applying the normal form theory and the center manifold theorem. Numerical simulation results are agreeable with the theoretical findings. In addition, we would like to point out that system (2.3) is obtained using the weak kernel. If we use the general kernel, then it is difficult for us to simplify system (1.4) by the similar variable changes (see (2.1)), then our results obtained in this paper would not hold true. We leave this topic for future work.

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