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How much contextuality?

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Abstract The amount of contextuality is quantified in terms of the probability of the necessary violations of noncontextual assignments to counterfactual elements of physical reality.

Keywords Quantum measurement theory · Quantum contextuality · Counterfactual observables

Some of the mind boggling features attributed to quantized systems are their alleged ability to counterfactually (Svozil 2009a; Vaidman 2007) respond to complementary queries (Einstein et al. 1935; Clauser and Shimony 1978), as well as their capacity to experimentally render outcomes which have not been encoded into them prior to measurement (Zeilinger 1999). Moreover, under certain "reasonable" assumptions, and by excluding various exotic quasi-classical possibilities (Pitowsky 1982; Meyer 1999), quantum mechanics appears to "outperform" classical correlations by allowing higher-that-classical coincidences of certain events, reflected by violations of Boole–Bell type constraints on classical probabilities (Boole 1862; Froissart 1981; Pitowsky 1989). One of the unresolved issues is the reason (beyond geometric and formal arguments) for the quantitative form of these violations (Cirel'son 1993; Filipp and Svozil 2004); in particular, why Nature should not allow higher-than-quantum or maximal violations (Popescu and Rohrlich 1994; Krenn and Svozil 1998) of Boole's conditions of possible experience (Boole 1862, p. 229).

The Kochen–Specker theorem (Kochen and Specker 1967), stating the impossibility of a consistent truth assignment to potential outcomes of even a finite number of certain interlinked complementary observables, gave further indication for the absence of classical simultaneous omniscience in the quantum domain. From a purely operational point of view, the quantitative predictions that result from Bell- as well as Kochen–Specker-type theorems present a advancement over quantum complementarity. But they do not explicitly indicate the conceivable interpretation of these findings; at least not on the phenomenologic level. Thus the resulting explanations, although sufficient and conceptually desirable and gratifying, lack the necessity.

One possibility to interpret these findings, and the prevalent one among physicists, is in terms of contextuality. Contextuality can be motivated by the benefits of a quasi-classical analysis. In particular, omniscience appears to be corroborated by the feasibility of the potential measurements involved: it is thereby implicitly assumed that all potentially observable elements of physical reality (Einstein et al. 1935) exist prior to any measurement; albeit any such (potential) measurement outcome (the entirety of which could thus consistently pre-exist before the actual measurement) depends on whatever other observables (the context) are co-measured alongside (Bohr 1949; Bell 1966). As, contrary to a very general interpretation of that assumption, the quantum mechanical observables are represented context independently, any such contextual behavior should be restricted to *single* quanta and outcomes within the quantum statistical bounds. This, in essence, is quantum realism in disguise. Nevertheless, it requires very little modifications—indeed, none on the statistical level, and some on the level of

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individual outcomes as described below—both of the quantum as well as of the classical representations.

Einstein–Podolsky–Rosen type experiments (Einstein et al. 1935) for entangled higher than two-dimensional quantized systems seem to indicate that contextuality, if viable, will remain hidden to any direct physical operationalization (and thus might be criticized to be metaphysical) even if counterfactual measurements are allowed (Svozil 2009b). Because "the immense majority of the experimental violations of Bell inequalities does not prove quantum nonlocality, but just quantum contextuality" (Cabello 2008), current claims of proofs of noncontextuality are solely based on violations of classical constraints in Boole–Bell-type, Kochen–Specker-type, or Greenbergerger–Horne–Zeilinger-type configurations.

Nevertheless, insistence on the simultaneous physical contextual coexistence of certain finite sets of counterfactual observables necessarily results in "ambivalent" truth assignments which could be explicitly illustrated by a *forced* tabulation (Peres 1978; Svozil 2010) of contextual truth values for Boole–Bell-type or Kochen–Specker-type configurations. Here contextual means that the truth value of a particular quantum observable depends on whatever other observables are measured alongside this particular observable. Any forced tabulation of truth values would render occurrences of mutually contradicting, potential, counterfactual outcomes of one and the same observable, depending on the measurement context (Svozil 2009c). The amount of this violation of noncontextuality can be quantified by the frequency of occurrence of contextuality. In what follows these frequencies will be calculated for a number of experimental configurations suggested in the literature.

First, consider the generalized Clauser–Horne–Shimony–Holt (CHSH) inequality

$$-\lambda \leq E(a,b) + E(a,b') + E(a',b) - E(a',b') \leq \lambda \qquad (1)$$

which, for $\lambda = 2$ and $\lambda = 2\sqrt{2}$, represents bounds for classical (Clauser and Shimony 1978; Clauser et al. 1969) and quantum (Cirel'son 1980) expectations of dichotomic observables with outcomes "-1" and "$+1$," respectively. The algebraic maximal violation associated with $\lambda = 4$ is attainable only for hypothetical "nonlocal boxes" (Popescu and Rohrlich 1994; Krenn and Svozil 1998; Popescu and Rohrlich 1997; Barrett et al. 2005) or by bit exchange (Svozil 2005a).

Equation 1 can be rewritten in an explicitly contextual form by the substitution

$$E(x,y) \mapsto E(x_y, y_x), \qquad (2)$$

where x_y stands for "observable x measured alongside observable y" (Svozil 2010). Contextuality manifests itself through $x_y \neq x_{y'}$. Because in the particular CHSH configuration there are no other observables measured alongside the ones that appear already in Eq. 1, this form is without ambiguity.

Equation 1 refers to the expectation values for four complementary measurement configurations on the same particles (two particles and two measurement configurations per particle). These expectation values can in principle be computed from the statistical average of the individual two-particle contributions. This requires that all of them exist counterfactually—a requirement that, at least according to the contextuality assumption, is satisfied—because only one of the four configurations can actually be simultaneously measurable; the other three have to be assigned in a consistent manner and contribute to the expectation values $E(a,b) = (1/N)\sum_{i=1}^{N} a_i b_i$. Here, a_i and b_i stand for the outcomes of the dichotomic observables a and b in the ith experiment; N is the number of individual experiments. Suppose we are interested in individual outcomes contributing to a violation of Eq. 1. For the sake of simplicity, suppose further that one would like to force the algebraic maximum of $\lambda = 4$ upon Eq. 1, and suppose that only one observable, say b', is contextual (a highly counterintuitive assumption). Then one obtains, for individual outcomes, say, in the ith experiment,

$$(\pm 1)(\pm 1) + (\pm 1)x + (\pm 1)(\pm 1) - (\pm 1)(-x) = 4, \qquad (3)$$

and thus $x = \pm 1$. That is, the algebraic maximum of $\lambda = 4$ can be reached by a single instance of contextual assignment $b'_a = -b'_{a'}$ per quantum. Table 1 enumerates the two possible truth value assignments associated with this configuration.

It should be stressed that there is no unique correspondence between the proportionality of contextuality and amount of CHSH violation. Indeed, it can be expected that there are several possible sets of truth assignments with relative frequencies with differing amounts of contextuality yielding the same violation. This plasticity is particularly true for more than one instance of contextuality, where two or more violations of noncontextuality may compensate each other. Take, for example, the four-touple ($E(a, b)$, $E(a, b')$, $E(a', b)$, $E(a', b')$) of expectation values contained

Table 1 The first two rows represent contextual assignments associated with an algebraic maximal rendition ($\lambda = 4$) of the CHSH inequality

a_b	$a_{b'}$	a'_b	$a'_{b'}$	b_a	$b_{a'}$	b'_a	$b'_{a'}$
+1	+1	+1	+1	+1	+1	+1	−1
−1	−1	−1	−1	−1	−1	−1	+1
+1	+1	+1	+1	+1	+1	+1	+1
−1	−1	−1	−1	−1	−1	−1	−1

The third and the fourth assignments are noncontextual in Eq. 1, and its transition $(+1,+1,+1,-1) \rightarrow (+1,+1,-1,-1)$, which, for example, can be achieved by changing one instance of contextuality at b' to two instances of contextuality at b' and b, resulting in $E(a,b)+E(a,b')+E(a',b)-E(a',b')=4 \rightarrow 2$.

That contextuality could accommodate any bound $0<\lambda<4$ can be demonstrated by interpreting all possible noncontextual and contextual assignments, as well as the resulting corresponding joint expectations enumerated in Table 2 as vertices of a convex correlation polytope. According to the Minkoswki-Weyl representation theorem (Ziegler 1994, p. 29), an equivalent (hull) representation of the associated convex polyhedron is in terms of the half-spaces defined by Boole–Bell type inequalities of the form

$$\begin{aligned} -1 &\leq E(a_b)+E(b_a)+E(a_b b_a),\\ -1 &\leq E(a_b)-E(b_a)-E(a_b b_a),\\ -1 &\leq -E(a_b)+E(b_a)-E(a_b b_a),\\ -1 &\leq -E(a_b)-E(b_a)+E(a_b b_a), \end{aligned} \tag{4}$$

(and the inequalities resulting from permuting $a \leftrightarrow a'$, $b \leftrightarrow b'$) which, for $E(a_b)=E(b_a)=0$, reduce to $-1 \leq E(a_b\, b_a) \leq 1$. Note that, by taking only the 16 context-independent $(x_y = x_{y'})$ from all the 256 assignments, the CHSH inequality (1) with $\lambda = 2$ is recovered.

Next, for the sake of demonstration, an example configuration will be given that conforms to Tsirel'son's maximal quantum bound of $\lambda = 2\sqrt{2}$ (Cirel'son 1993). Substituting this for $2\sqrt{2}$ in Eq. (3) yields $x = \pm(\sqrt{2}-1)$;

Table 2 (Color online) Contextual (bold) and noncontextual value assignments, and the associated joint values

a_b	$a_{b'}$	a'_b	$a'_{b'}$	b_a	$b_{a'}$	b'_a	$b'_{a'}$	$a_b b_a$	$a_{b'} b'_a$	$a'_b b_{a'}$	$a'_{b'} b'_{a'}$
−1	−1	−1	−1	−1	−1	−1	−1	+1	+1	+1	+1
−1	−1	−1	−1	−1	−1	**−1**	**+1**	+1	+1	+1	−1
−1	−1	−1	−1	−1	−1	**+1**	**−1**	+1	−1	+1	+1
−1	−1	−1	−1	−1	−1	+1	+1	+1	−1	+1	−1
−1	−1	−1	−1	**−1**	**+1**	−1	−1	+1	+1	−1	+1
−1	−1	−1	−1	**−1**	**+1**	**−1**	**+1**	+1	+1	−1	−1
−1	−1	−1	−1	**−1**	**+1**	**+1**	**−1**	+1	−1	−1	+1
−1	−1	−1	−1	**−1**	**+1**	+1	+1	+1	−1	−1	−1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
−1	**+1**	**−1**	**+1**	**−1**	**+1**	**−1**	**+1**	+1	−1	−1	+1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
+1	**−1**	**+1**	**−1**	**+1**	**−1**	**+1**	**−1**	+1	−1	−1	+1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
+1	+1	+1	+1	+1	+1	**−1**	**+1**	+1	−1	+1	+1
+1	+1	+1	+1	+1	+1	**+1**	**−1**	+1	+1	+1	−1
+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1

that is, the (limit) frequency for the occurrence of contextual assignments $b'_a = -b'_{a'}$ as enumerated in Table 1 with respect to the associated noncontextual assignments $b'_a = b'_{a'}$ (rendering 2 to the sum of terms in the CHSH expression) should be $(\sqrt{2}-1):(2-\sqrt{2})$. More explicitly, if there are four different assignments, enumerated in Table 1, which may contribute quantum mechanically by the correct (limiting) frequency, then Table 3 is a simulation of 20 assignments rendering the maximal quantum bound for the CHSH inequalities.

With regards to Kochen–Specker type configurations (Kochen and Specker 1967; Cabello et al. 1996) with no two-valued state, any co-existing set of observables (associated with the configuration) must breach noncontextuality at least once. Other Kochen–Specker type configurations (Kochen and Specker 1967; Svozil 1998; Calude et al. 1999) still allowing two-valued states, albeit an insufficient number for a homeomorphic embedding into Boolean algebras, might require contextual value assignments for quantum statistical reasons; but this question remains unsolved at present.

In summary, several concrete, quantitative examples of contextual assignments for co-existing complementary—and thus strictly counterfactual—observables have been given. The amount of noncontextuality can be characterized quantitatively by the required relative amount of contextual assignments versus noncontextual ones reproducing quantum mechanical predictions; or, alternatively, by the required relative amount of contextual assignment versus all assignments. One may thus consider the average number of contextual assignments per quantum as a criterion.

With regard to the above criteria, as could be expected, Kochen–Specker type configurations require assignments which violate noncontextuality for every single quantum, whereas Boole–Bell-type configurations, such as CHSH, would still allow occasional noncontextual assignments. In this sense, Kochen–Specker-type arguments violate noncontextuality stronger than Boole–Bell-type ones.

These considerations are relevant under the assumption that contextuality is a viable concept for explaining the experiments (Cabello 2008; Hasegawa et al. 2006; Bartosik et al. 2009; Amselem et al. 2009; Kirchmair et al. 2009). As I have argued elsewhere (Svozil 2009a, b, 2010, 2005b), this might not be the case; at least contextuality might not be a necessary quantum feature. In particular the abandonment of quantum omniscience, in the sense that a quantum system can carry information about its state with regard to only a *single* context (Zeilinger 1999), in conjunction with a *context translation principle* (Svozil 2004, 2009c) might yield an alternative approach to the quantum formalism. Thereby the many degrees of freedom of the "quasi-classical" measurement apparatus effectively introduce stochasticity in the

Table 3 (Color online) 20 Counterfactual assignments of contextual (bold) and noncontextual values, and the associated joint values, rendering an approximation 2.95 for Tsirel'son's maximal quantum bound $2\sqrt{2}$ for the CHSH sum

a_b	$a_{b'}$	a'_b	$a'_{b'}$	b_a	$b_{a'}$	b'_a	$b'_{a'}$
+1	+1	+1	+1	+1	+1	+1	+1
−1	−1	−1	−1	−1	−1	−1	−1
+1	+1	+1	+1	+1	+1	**+1**	**−1**
−1	−1	−1	−1	−1	−1	**−1**	**+1**
−1	−1	−1	−1	−1	−1	−1	−1
+1	+1	+1	+1	+1	+1	**+1**	**−1**
+1	+1	+1	+1	+1	+1	**+1**	**−1**
+1	+1	+1	+1	+1	+1	+1	+1
−1	−1	−1	−1	−1	−1	**−1**	**+1**
+1	+1	+1	+1	+1	+1	+1	+1
−1	−1	−1	−1	−1	−1	−1	−1
+1	+1	+1	+1	+1	+1	+1	+1
+1	+1	+1	+1	+1	+1	+1	+1
+1	+1	+1	+1	+1	+1	**+1**	**−1**
+1	+1	+1	+1	+1	+1	**+1**	**−1**
+1	+1	+1	+1	+1	+1	+1	+1
−1	−1	−1	−1	−1	−1	**−1**	**+1**
−1	−1	−1	−1	−1	−1	−1	−1
+1	+1	+1	+1	+1	+1	**+1**	**−1**
+1	+1	+1	+1	+1	+1	+1	+1
+1	+1	+1	+1	+1	+1	**+1**	**−1**
−1	−1	−1	−1	−1	−1	−1	−1
−1	−1	−1	−1	−1	−1	**−1**	**+1**
+1	+1	+1	+1	+1	+1	+1	+1
+1	+1	+1	+1	+1	+1	**+1**	**−1**
−1	−1	−1	−1	−1	−1	−1	−1
+1	+1	+1	+1	+1	+1	+1	+1
−1	−1	−1	−1	−1	−1	**−1**	**+1**
−1	−1	−1	−1	−1	−1	−1	−1
−1	−1	−1	−1	−1	−1	**−1**	**+1**
−1	−1	−1	−1	−1	−1	−1	−1
−1	−1	−1	−1	−1	−1	−1	−1
−1	−1	−1	−1	−1	−1	−1	−1
+1	+1	+1	+1	+1	+1	**+1**	**−1**
−1	−1	−1	−1	−1	−1	**−1**	**+1**
−1	−1	−1	−1	−1	−1	−1	−1
−1	−1	−1	−1	−1	−1	**−1**	**+1**
−1	−1	−1	−1	−1	−1	−1	−1
+1	+1	+1	+1	+1	+1	**+1**	**−1**
−1	−1	−1	−1	−1	−1	**−1**	**+1**

case of a mismatch between preparation and measurement context.

Clearly, these considerations have large consequences for the type of randomness that could be rendered by quantum random number generators based on beam splitters, and on quantum oracles in general (Calude and Svozil 2008; Calude et al. 2010), as context translation schemes may still be deterministic and even computable, whereas irreducible indeterminism can be postulated only from a complete lawlessness (Zeilinger 2005) of the underlying processes.

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