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Asynchronous filtering for 2-D switched systems with missing measurements

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Abstract This work is concerned with $\mathscr{H}_\infty$ filter design with missing measurements for a class of two-dimensional (2-D) switched systems represented by Fornasini–Marchesini local state-space model. The switching signal of the switched filters involve time delays, which result in the asynchronism between the filter and the system switching. The issues of asymptotic mean-square stability and ℓ_2-gain analysis for the 2-D switched systems are addressed firstly, based on which mode-dependent filters are designed with mode-dependent average dwell time scheme. Finally, two examples are given to demonstrate the validity of the proposed technique.

Keywords Asynchronous filtering · $\mathscr{H}_\infty$ filtering · Mode-dependent average dwell time · Missing measurements · 2-D switched system

1 Introduction

In the past decade, two-dimensional (2-D) systems have received considerable attention due to both their theoretical significance and wide applications (Ahn and Kar 2015; Kaczorek 1985; Shyu et al. 2014; Xu et al. 2010). In general, 2-D systems can be modeled by the Rosser model, Fornasini–Marchesini (FM) model, and Attasi model (Fornasini and Marchesini 2017; Roesser 2012). Especially, the Fornasini–Marchesini local state-space (FMLSS) model includes the Roesser model and the Attasi model as a special case (Kaczorek 1985; Li et al. 2012).

Meanwhile, quantities of practical systems are subject to abrupt changes, and the switched systems provide a unified framework for characterizing these changes (Fei et al. 2017;

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Gao et al. 2011; Li et al. 2016; Zhao et al. 2012). The switching features may also include in 2-D systems. For example, a 2-D switched representation is needed when modeling the thermal processes in chemical reactors, heat exchangers, and pipe furnaces with multiple modes (Lo et al. 2008; Wu et al. 2015). In general, 2-D switched systems can also be formulated by Roesser model and FM model (Duan and Xiang 2013; Guan et al. 2017). So far, a number of meaningful results for 2-D switched systems have been reported in the literature (Benzaouia et al. 2011; Duan and Xiang 2013; Ghous et al. 2015; Shi et al. 2018; Wu et al. 2015; Xiang and Huang 2013). To mention a few, stability analysis and stabilization problems are discussed on 2-D discrete switched systems represented by FMLSS model in Fei et al. (2017). The stabilization problem formulated by Roesser type is investigated in Xiang and Huang (2013). In Shi et al. (2018), by designing a set of switching signals, 2-D switched systems are stabilized without any controller.

It is well known that filtering problems play significant roles in signal processing. Filtering for 2-D systems have been widely reported (Ahn 2014; Ahn et al. 2017, 2015; Boukili et al. 2016; Du et al. 2000). In Boukili et al. (2016), Du et al. (2000), $\mathscr{H}_\infty$ filter design is addressed for 2-D systems represented by Roesser and FM types, respectively. However, there are few results reported about filtering for 2-D switched systems. When considering filtering problems, one popular assumption is that the measurement always contain consecutive usable signals (Chen et al. 2015; Guan et al. 2016). However, the signals are vulnerable to be corrupted by the noise in practical applications, which results in the inconsecutive observations, namely, the system may have missing measurements (Yang et al. 2014; Zhang et al. 2009). Various factors may lead to inconsecutive measurements such as the high maneuverability of tracked target, intermittent sensor failures, accidental loss of some collected data and so on Sinopoli et al. (2004). One of the common approach to describe the missing measurement phenomenon is using the binary switching sequence. This sequence is specified by a conditional probability distribution which can be described by a Bernoulli distributed white sequence taking on values of 0 and 1. In Zhang et al. (2009), such a model is applied to design robust $\mathscr{H}_\infty$ filters for stochastic time-delay systems.

On the other hand, it takes time to identify the current mode of the system and apply the matched filter for switched systems, which will result in the asynchronous phenomenon between the system mode and the filter (Lian et al. 2013; Mahmoud and Shi 2012; Wang et al. 2013). In Zhang et al. (2011), asynchronous filtering for discrete-time switched systems is investigated under average dwell time (ADT) scheme. However, to the best of the authors' knowledge, the asynchronous filtering is still unsolved for 2-D switched systems.

In this paper, we concentrate on the asynchronous $\mathscr{H}_\infty$ filtering with missing measurements for 2-D switched systems represented by FMLSS model. Mode-dependent average dwell time (MDADT) switching is adopted, which is more general compared with dwell time (DT) and ADT switching (Zhao et al. 2012). The remainder of this paper is organized as follows. In Sect. 2, the model of 2-D switched systems and missing measurements is established, and some definitions are provided. In Sect. 3, sufficient conditions to guarantee the stability and ℓ_2-gain analysis are derived for 2-D switched systems. Then, the filter design is discussed in Sect. 4. In Sect. 5, two examples are presented to illustrate the effectiveness of the developed method. Finally, we conclude the paper in Sect. 6.

Notation The notations used in this paper are fairly standard. The superscript "T" stands for matrix transposition. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{Z}^+$ represents the set of nonnegative integers. Prob$\{\cdot\}$ indicates the occurrence probability of the event "$\cdot$".

$\mathbb{E}\{x\}$ stands for the expectation of x. In addition, in symmetric block matrices or long matrix expressions, we use a "$*$" as an ellipsis for the terms that are introduced by symmetry and $\text{diag}\{\cdots\}$ stands for a block-diagonal matrix. $\|\cdot\|$ refers to the Euclidean vector norm. I and 0 represent identify matrix and zero matrix with appropriate dimensions, respectively. The notation $P > 0$ means that P is real symmetric and positive definite. $\lambda_{\min}\{P\}$ and $\lambda_{\max}\{P\}$ denote the minimum and maximum eigenvalues of matrix P, respectively. The ℓ_2 norm of a 2-D signal $\omega(i, j)$ is defined by $\|\omega(\cdot,\cdot)\|_2 = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega^T(i, j)\omega(i, j)}$. The set of all vector functions such that $\|\omega(\cdot,\cdot)\|_2^2 < \infty$ is denoted by $\ell_2\{[0,\infty), [0,\infty)\}$.

2 Problem formulation and preliminaries

Consider a class of 2-D switched discrete-time systems given by

$$x(i+1, j+1) = A_{1\sigma(i,j+1)}x(i, j+1) + A_{2\sigma(i+1,j)}x(i+1, j) + B_{1\sigma(i,j+1)}\omega(i, j+1) + B_{2\sigma(i+1,j)}\omega(i+1, j), \tag{1}$$

$$z(i, j) = C_{\sigma(i,j)}x(i, j) + D_{\sigma(i,j)}\omega(i, j), \tag{2}$$

where $(i, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $x(i, j) \in \mathbb{R}^n$ is the state vector of the system, $z(i, j) \in \mathbb{R}^p$ is the objective signal to be estimated, and $\omega(i, j) \in \mathbb{R}^r$ is the exogenous disturbance input which belongs to $\ell_2\{[0,\infty), [0,\infty)\}$. $\sigma(i, j) : (\mathbb{Z}^+, \mathbb{Z}^+) \to \mathscr{M}$ is a switching signal, which takes its values in the finite set $\mathscr{M} \triangleq \{1, 2, \ldots, M\}$ with M being the number of subsystems. A_{1p}, A_{2p}, B_{1p}, B_{2p}, C_p and D_p are constant real matrices with appropriate dimensions for $\sigma(i, j) = p \in \mathscr{M}$.

The measurements, which may contain missing data, can be described by

$$y(i, j) = \rho(i, j)C_{y\sigma(i,j)}x(i, j) + D_{y\sigma(i,j)}\omega(i, j), \tag{3}$$

where $y(i, j) \in \mathbb{R}^q$ is the measured output vector, C_{yp} and D_{yp} are constant real matrices with appropriate dimensions for $p \in \mathscr{M}$, the stochastic variable $\rho(i, j) \in \mathbb{R}$ is a Bernoulli distributed white sequence taking the value 0 and 1 with

$$\text{Prob}\{\rho(i, j) = 1\} = \mathbb{E}\{\rho(i, j)\} := \eta,$$
$$\text{Prob}\{\rho(i, j) = 0\} = 1 - \mathbb{E}\{\rho(i, j)\} := 1 - \eta,$$

and $0 < \eta < 1$ is a known scalar.

Assumption 1 For the system (1), the initial condition is satisfied

$$\begin{aligned} x(0, j) &= v_j, \quad \forall 0 \le j \le z_2, \\ x(i, 0) &= w_i, \quad \forall 0 \le i \le z_1, \\ v_0 &= w_0, \quad i = j = 0, \\ x(0, j) &= 0, \quad \forall j > z_2, \\ x(i, 0) &= 0, \quad \forall i > z_1, \end{aligned}$$

where z_1 and z_2 are positive integers, v_j and w_i are given vectors.

Assumption 2 The switching signal is assumed to be only dependent upon $i + j$ (Duan and Xiang 2014; Wu et al. 2015).

According to Assumption 2, the switching signal can be rewritten as $\sigma(\kappa) : \mathbb{Z}^+ \to \mathscr{M}$. The switching sequence can be described as $(\kappa_0, \kappa_1, \ldots, \kappa_l, \kappa_{l+1}, \ldots)$ with $l = 0, 1, 2, \ldots$, κ_l denotes the l-th switching instant. Meanwhile, when $\sigma(\kappa_l) = p \in \mathscr{M}$, the p-th subsystem is active during $[\kappa_l, \kappa_{l+1})$.

We are interested in designing a set of filter for the 2-D switched system (1)–(2) described by

$$\begin{aligned}\hat{x}(i+1, j+1) = {} & G_{1\sigma(\kappa+1-\tau(\kappa+1))}\hat{x}(i, j+1) + G_{2\sigma(\kappa+1-\tau(\kappa+1))}\hat{x}(i+1, j) \\ & + K_{1\sigma(\kappa+1-\tau(\kappa+1))}[y(i, j+1) - \eta C_{y\sigma(\kappa+1-\tau(\kappa+1))}\hat{x}(i, j+1)] \\ & + K_{2\sigma(\kappa+1-\tau(\kappa+1))}[y(i+1, j) - \eta C_{y\sigma(\kappa+1-\tau(\kappa+1))}\hat{x}(i+1, j)], \end{aligned} \tag{4}$$

$$\hat{z}(i, j) = L_{\sigma(\kappa-\tau(\kappa))}\hat{x}(i, j), \tag{5}$$

where $\hat{x}(i, j)$ and $\hat{z}(i, j)$ are the estimate for $x(i, j)$ and $z(i, j)$, respectively, for $\sigma(\kappa - \tau(\kappa)) = p \in \mathscr{M}$, G_{1p}, G_{2p}, K_{1p}, K_{2p} and L_p are filter parameters to be determined. $\tau(\kappa)$ is the uncertain switching delay satisfying $0 < \tau(\kappa) \le \tau_{\max}$. Without loss of generality, we assume that the maximum switching delay $\tau_{\max}$ is a known priori, and $\tau_{\max} < \kappa_{l+1} - \kappa_l$, $l = 1, 2, \ldots$.

Assumption 3 Consider the switching occurs at $\kappa_l, l = 1, 2, \ldots$. Since the delay considered in this paper is only related to the switching, we can assume that $\tau(\kappa) = \tau(\kappa_l)$ for $\forall \kappa \in [\kappa_l, \kappa_{l+1})$.

Here, we consider $\sigma(\kappa_l) = p$, $\sigma(\kappa_l - 1) = q$, $(p, q) \in \mathscr{M} \times \mathscr{M}$, $p \neq q$. When $\kappa \in [\kappa_l, \kappa_l + \tau(\kappa_l))$, the q-th subsystem has switched to the p-th subsystem, but the q-th filter is still active because it takes time to identify the system modes and apply the matched filter.

Combing (1)–(3) and (4)–(5), $\forall (p, q) \in \mathscr{M} \times \mathscr{M}$, $p \neq q$, we obtain the augmented filtering error system

$$\begin{aligned}\xi(i+1, j+1) = {} & \mathscr{A}_{1p}\xi(i, j+1) + \mathscr{A}_{2p}\xi(i+1, j) \\ & + (\rho(i, j+1) - \eta)\mathscr{A}_{\eta 1p}\xi(i, j+1) + (\rho(i+1, j) - \eta)\mathscr{A}_{\eta 2p}\xi(i+1, j) \\ & + \mathscr{B}_{1p}\omega(i, j+1) + \mathscr{B}_{2p}\omega(i+1, j), \end{aligned} \tag{6}$$

$$e(i, j) = \mathscr{C}_p\xi(i, j) + \mathscr{D}_p\omega(i, j), \tag{7}$$

where $\xi(i, j) = \left[x^T(i, j)\ \hat{x}^T(i, j)\right]^T$, $e(i, j) = z(i, j) - \hat{z}(i, j)$, and when $\kappa \in [\kappa_l, \kappa_l + \tau(\kappa_l))$,

$$\begin{aligned} &\mathscr{A}_{1p} = \tilde{A}_{1p} = \begin{bmatrix} A_{1p} & 0 \\ \eta K_{1q}C_{yp} & G_{1q} - \eta K_{1q}C_{yp} \end{bmatrix}, \mathscr{A}_{\eta 1p} = \tilde{A}_{\eta 1p} = \begin{bmatrix} 0 & 0 \\ K_{1q}C_{yp} & 0 \end{bmatrix}, \\ &\mathscr{A}_{2p} = \tilde{A}_{2p} = \begin{bmatrix} A_{2p} & 0 \\ \eta K_{2q}C_{yp} & G_{2q} - \eta K_{2q}C_{yp} \end{bmatrix}, \mathscr{A}_{\eta 2p} = \tilde{A}_{\eta 2p} = \begin{bmatrix} 0 & 0 \\ K_{2q}C_{yp} & 0 \end{bmatrix}, \\ &\mathscr{B}_{1p} = \tilde{B}_{1p} = \begin{bmatrix} B_{1p} \\ K_{1q}D_{yp} \end{bmatrix}, \mathscr{B}_{2p} = \tilde{B}_{2p} = \begin{bmatrix} B_{2p} \\ K_{2q}D_{yp} \end{bmatrix}, \\ &\mathscr{C}_p = \tilde{C}_p = \begin{bmatrix} C_p & -L_q \end{bmatrix}, \mathscr{D}_p = \tilde{D}_p = D_p, \end{aligned}$$

when $\kappa \in [\kappa_l + \tau(\kappa_l), \kappa_{l+1})$,

$$\mathscr{A}_{1p} = \bar{A}_{1p} = \begin{bmatrix} A_{1p} & 0 \\ \eta K_{1p} C_{yp} & G_{1p} - \eta K_{1p} C_{yp} \end{bmatrix}, \mathscr{A}_{\eta 1p} = \bar{A}_{\eta 1p} = \begin{bmatrix} 0 & 0 \\ K_{1p} C_{yp} & 0 \end{bmatrix},$$
$$\mathscr{A}_{2p} = \bar{A}_{2p} = \begin{bmatrix} A_{2p} & 0 \\ \eta K_{2p} C_{yp} & G_{2p} - \eta K_{2p} C_{yp} \end{bmatrix}, \mathscr{A}_{\eta 2p} = \bar{A}_{\eta 2p} = \begin{bmatrix} 0 & 0 \\ K_{2p} C_{yp} & 0 \end{bmatrix},$$
$$\mathscr{B}_{1p} = \bar{B}_{1p} = \begin{bmatrix} B_{1p} \\ K_{1p} D_{yp} \end{bmatrix}, \mathscr{B}_{2p} = \bar{B}_{2p} = \begin{bmatrix} B_{2p} \\ K_{2p} D_{yp} \end{bmatrix},$$
$$\mathscr{C}_p = \bar{C}_p = \begin{bmatrix} C_p & -L_p \end{bmatrix}, \mathscr{D}_p = \bar{D}_p = D_p.$$

Here, we present the following definitions for further development.

Definition 1 Fei et al. (2017) For any $D \geq r$ and switching signal σ, let $N_{\sigma p}(r, D)$ denote the switching numbers of the p-th subsystem activated during the interval $[r, D]$ and $H_p(r, D)$ denote the total running time of the p-th subsystem in $[r, D]$, $p \in \mathscr{M}$. We say that σ has a mode-dependent average dwell time τ_{ap} if there exist positive numbers N_{0p} (we call N_{0p} the mode-dependent chatter bounds here) and τ_{ap} such that

$$N_{\sigma p}(r, D) \leq N_{0p} + \frac{H_p(r, D)}{\tau_{ap}}.$$

Definition 2 Duan and Xiang (2014) The 2-D switched system (1) with $\omega(i, j) \equiv 0$ is said to be asymptotically mean-square stable under the switching signal $\sigma(i, j)$, if for a given $r \geq 0$, there exist $\varsigma > 0$ and $0 < \varepsilon < 1$, such that the solution $x(i, j)$ satisfies

$$\mathbb{E}\left\{ \sum_{i+j=D} \|x(i, j)\|^2 \right\} \leq \varsigma \varepsilon^{D-r} \mathbb{E}\left\{ \sum_{i+j=r} \|x(i, j)\|_r^2 \right\},$$

for all $D \geq r$, where $\|x(i, j)\|_r = \sup\{\|x(i, j)\| : i + j = r, i \leq z_1, j \leq z_2\}$.

Assumption 4 The 2-D switched system (1) is asymptotically mean-square stable.

Remark 1 To guarantee the asymptotic mean-square stability for the filtering error system (6)–(7), the prerequisite is that the original system (1) to be estimated, which exists no control, has to satisfy Assumption 4.

Definition 3 Duan et al. (2013) For a given scalar $0 < \alpha < 1$, the 2-D switched system (1)–(2) is said to be with a weighted $\mathscr{H}_\infty$ disturbance attenuation γ under switching signal σ if it satisfies the following conditions:

1. System (1) with $\omega(i, j) \equiv 0$ is asymptotically mean-square stable;
2. Under zero boundary condition, it holds that

$$\mathbb{E}\left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (\alpha^{i+j} \|\bar{z}\|_2^2) \right\} < \gamma^2 \mathbb{E}\left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|\bar{\omega}\|_2^2 \right\},$$

for all $0 \neq \omega \in \ell_2 \{[0, \infty), [0, \infty)\}$, where the ℓ_2-norm of 2-D discrete signal $z(i, j)$ and $\omega(i, j)$ are defined as

$$\|\bar{z}\|_2^2 = \|z(i + 1, j)\|_2^2 + \|z(i, j + 1)\|_2^2,$$
$$\|\bar{\omega}\|_2^2 = \|\omega(i + 1, j)\|_2^2 + \|\omega(i, j + 1)\|_2^2.$$

3 Stability and ℓ_2-gain analysis

In this section, an improved approach will be developed to solve the stability and ℓ_2-gain analysis for (6)–(7). The switching signal is with the form of MDADT, and the asynchronous switching is taken into consideration.

Lemma 1 *Consider the 2-D switched system* (6)–(7). *For any* $(p, q) \in \mathscr{M} \times \mathscr{M}$, $p \neq q$, *let* $0 < \alpha_p < 1$, $\beta_p > 0$ *and* $\mu_p > 1$ *be given constants, if there exist matrices* $P_p > 0$, $P_q > 0$, $Q_p > 0$, $Q_q > 0$, *and a scalar* $\gamma > 0$, *such that*

$$\bar{\Phi} = \begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} & \bar{\Phi}_{13} & \bar{\Phi}_{14} \\ * & \bar{\Phi}_{22} & \bar{\Phi}_{23} & \bar{\Phi}_{24} \\ * & * & \bar{\Phi}_{33} & \bar{\Phi}_{34} \\ * & * & * & \bar{\Phi}_{44} \end{bmatrix} < 0, \tag{8}$$

$$\tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} & \tilde{\Phi}_{13} & \tilde{\Phi}_{14} \\ * & \tilde{\Phi}_{22} & \tilde{\Phi}_{23} & \tilde{\Phi}_{24} \\ * & * & \tilde{\Phi}_{33} & \tilde{\Phi}_{34} \\ * & * & * & \tilde{\Phi}_{44} \end{bmatrix} < 0, \tag{9}$$

$$P_q \leq \mu_p P_p, \tag{10}$$

$$Q_q \leq \mu_p Q_p, \tag{11}$$

where $\bar{\alpha}_p = 1 - \alpha_p$, $\bar{\beta}_p = 1 + \beta_p$, $\theta_p = \bar{\beta}_p / \bar{\alpha}_p$, *and*

$\bar{\Phi}_{11} = \bar{A}_{1p}^T (P_p + Q_p) \bar{A}_{1p} + \eta(1-\eta) \bar{A}_{\eta 1p}^T (P_p + Q_p) \bar{A}_{\eta 1p} - \bar{\alpha}_p P_p + \bar{C}_p^T \bar{C}_p$,
$\bar{\Phi}_{12} = \bar{A}_{1p}^T (P_p + Q_p) \bar{A}_{2p}$, $\bar{\Phi}_{13} = \bar{A}_{1p}^T (P_p + Q_p) \bar{B}_{1p} + \bar{C}_p^T \bar{D}_p$, $\bar{\Phi}_{14} = \bar{A}_{1p}^T (P_p + Q_p) \bar{B}_{2p}$,
$\bar{\Phi}_{22} = \bar{A}_{2p}^T (P_p + Q_p) \bar{A}_{2p} + \eta(1-\eta) \bar{A}_{\eta 2p}^T (P_p + Q_p) \bar{A}_{\eta 2p} - \bar{\alpha}_p Q_p + \bar{C}_p^T \bar{C}_p$,
$\bar{\Phi}_{23} = \bar{A}_{2p}^T (P_p + Q_p) \bar{B}_{1p}$, $\bar{\Phi}_{24} = \bar{A}_{2p}^T (P_p + Q_p) \bar{B}_{2p} + \bar{C}_p^T \bar{D}_p$,
$\bar{\Phi}_{33} = \bar{B}_{1p}^T (P_p + Q_p) \bar{B}_{1p} + \bar{D}_p^T \bar{D}_p - \gamma^2 I$, $\bar{\Phi}_{34} = \bar{B}_{1p}^T (P_p + Q_p) \bar{B}_{2p}$,
$\bar{\Phi}_{44} = \bar{B}_{2p}^T (P_p + Q_p) \bar{B}_{2p} + \bar{D}_p^T \bar{D}_p - \gamma^2 I$,
$\tilde{\Phi}_{11} = \tilde{A}_{1p}^T (P_p + Q_p) \tilde{A}_{1p} + \eta(1-\eta) \tilde{A}_{\eta 1p}^T (P_p + Q_p) \tilde{A}_{\eta 1p} - \bar{\beta}_p P_p + \tilde{C}_p^T \tilde{C}_p$,
$\tilde{\Phi}_{12} = \tilde{A}_{1p}^T (P_p + Q_p) \tilde{A}_{2p}$, $\tilde{\Phi}_{13} = \tilde{A}_{1p}^T (P_p + Q_p) \tilde{B}_{1p} + \tilde{C}_p^T \tilde{D}_p$, $\tilde{\Phi}_{14} = \tilde{A}_{1p}^T (P_p + Q_p) \tilde{B}_{2p}$,
$\tilde{\Phi}_{22} = \tilde{A}_{2p}^T (P_p + Q_p) \tilde{A}_{2p} + \eta(1-\eta) \tilde{A}_{\eta 2p}^T (P_p + Q_p) \tilde{A}_{\eta 2p} - \bar{\beta}_p Q_p + \tilde{C}_p^T \tilde{C}_p$,
$\tilde{\Phi}_{23} = \tilde{A}_{2p}^T (P_p + Q_p) \tilde{B}_{1p}$, $\tilde{\Phi}_{24} = \tilde{A}_{2p}^T (P_p + Q_p) \tilde{B}_{2p} + \tilde{C}_p^T \tilde{D}_p$,
$\tilde{\Phi}_{33} = \tilde{B}_{1p}^T (P_p + Q_p) \tilde{B}_{1p} + \tilde{D}_p^T \tilde{D}_p - \gamma^2 I$, $\tilde{\Phi}_{34} = \tilde{B}_{1p}^T (P_p + Q_p) \tilde{B}_{2p}$,
$\tilde{\Phi}_{44} = \tilde{B}_{2p}^T (P_p + Q_p) \tilde{B}_{2p} + \tilde{D}_p^T \tilde{D}_p - \gamma^2 I$.

Then, for any switching signal with MDADT satisfying

$$\tau_{ap} > \tau_{ap}^* = -\frac{\ln \mu_p + \tau_{\max} \ln \theta_p}{\ln \bar{\alpha}_p}, \tag{12}$$

the 2-D switched system (6)–(7) *is asymptotically mean-square stable with a prescribed weighted* $\mathscr{H}_\infty$ *disturbance attenuation level* $\gamma_s = \sqrt{\prod_{p=1}^{M} (\theta_p^{\tau_p} \mu_p)^{N_{0p}} (\alpha_{\max}/\alpha_{\min}) \theta_{\max}^{\tau_{\max}-1}}\gamma$ *and* $\alpha_{\max} = \max_{p \in \mathscr{M}} \{\alpha_p\}$, $\alpha_{\min} = \min_{p \in \mathscr{M}} \{\alpha_p\}$, $\theta_{\max} = \max_{p \in \mathscr{M}} \{\theta_p\}$.

Proof To establish the stability and $\mathscr{H}_\infty$ performance of the 2-D switched system (6)–(7), we construct the following Lyapunov function:

$$\begin{aligned} &V_{\sigma(\kappa)}(x(i,j)) = V^1_{\sigma(\kappa)}(x(i,j)) + V^2_{\sigma(\kappa)}(x(i,j)),\\ &V^1_{\sigma(\kappa)}(x(i,j)) = x^T(i,j)P_{\sigma(\kappa)}x(i,j),\\ &V^2_{\sigma(\kappa)}(x(i,j)) = x^T(i,j)Q_{\sigma(\kappa)}x(i,j), \end{aligned} \tag{13}$$

where $P_{\sigma(\kappa)}$ and $Q_{\sigma(\kappa)}$ are positive definite matrices for $\forall\sigma(\kappa)\in\mathscr{M}$.

Here, we define

$$\Delta V_{\sigma(\kappa)}(x(i,j)) = V_{\sigma(\kappa)}(x(i+1,j+1)) - V^1_{\sigma(\kappa)}(x(i,j+1)) - V^2_{\sigma(\kappa)}(x(i+1,j)). \tag{14}$$

Noting that $\mathbb{E}\left\{[\rho(i,j)-\eta]^2\right\} = (1-\eta)\eta$. When $\kappa\in\left[\kappa_l+\tau(\kappa_l),\kappa_{l+1}\right)$, the filter is matched with the mode. From (8), we obtain that

$$\begin{aligned} &\mathbb{E}\{\Delta V_{\sigma(\kappa)}(x(i,j)) + \alpha_{\sigma(\kappa)}\left[V^1_{\sigma(\kappa)}(x(i,j+1)) + V^2_{\sigma(\kappa)}(x(i+1,j))\right] + \Gamma(i,j)\}\\ &= \mathbb{E}\left\{\zeta^T(i,j)\bar{\Phi}\zeta(i,j)\right\} < 0, \end{aligned}$$

where $\zeta(i,j) = \left[\xi(i,j+1)\ \xi(i+1,j)\ \omega^T(i,j+1)\ \omega^T(i,j+1)\right]^T$, $\Gamma(i,j) = \bar{e}^T\bar{e} - \gamma^2\bar{\omega}^T\bar{\omega}$ with $\bar{e} = \left[e^T(i,j+1)\ e^T(i+1,j)\right]^T$, $\bar{\omega} = \left[\omega^T(i,j+1)\ \omega^T(i+1,j)\right]^T$.

Thus, we can get

$$\begin{aligned} &\mathbb{E}\{V_{\sigma(\kappa)}(x(i+1,j+1))\}\\ &\quad < \mathbb{E}\{\bar{\alpha}_{\sigma(\kappa)}[V^1_{\sigma(\kappa)}(x(i,j+1)) + V^2_{\sigma(\kappa)}(x(i+1,j))] - \Gamma(i,j)\}, \end{aligned} \tag{15}$$

Consider $\kappa_l+\tau(\kappa_l) < D < \kappa_{l+1}$. From (15), we obtain for $\kappa\in[\kappa_l+\tau(\kappa_l),D)$,

$$\begin{cases} \mathbb{E}\{V_{\sigma(\kappa)}(x(1,D-1))\} < \mathbb{E}\{\bar{\alpha}_{\sigma(\kappa)}[V^1_{\sigma(\kappa)}(x(0,D-1)) + V^2_{\sigma(\kappa)}(x(1,D-2))]\\ \qquad\qquad -\Gamma(0,D-2)\},\\ \mathbb{E}\{V_{\sigma(\kappa)}(x(2,D-2))\} < \mathbb{E}\{\bar{\alpha}_{\sigma(\kappa)}[V^1_{\sigma(\kappa)}(x(1,D-2)) + V^2_{\sigma(\kappa)}(x(2,D-3))]\\ \qquad\qquad -\Gamma(1,D-3)\},\\ \qquad\qquad \vdots\\ \mathbb{E}\{V_{\sigma(\kappa)}(x(D-1,1))\} < \mathbb{E}\{\bar{\alpha}_{\sigma(\kappa)}[V^1_{\sigma(\kappa)}(x(D-2,1)) + V^2_{\sigma(\kappa)}(x(D-1,0))]\\ \qquad\qquad -\Gamma(D-2,0)\}. \end{cases}$$

According to Assumption 1, we can get for $\kappa\in[\kappa_l+\tau(\kappa_l),D)$,

$$\begin{aligned} &\mathbb{E}\left\{\sum_{i+j=D} V_{\sigma(\kappa)}(x(i,j))\right\}\\ &< \mathbb{E}\left\{\bar{\alpha}_{\sigma(\kappa)}\sum_{i+j=D-1} V_{\sigma(\kappa)}(x(i,j)) - \sum_{i+j=D-2}\Gamma(i,j)\right\}\\ &< \ldots\\ &< \mathbb{E}\left\{\bar{\alpha}_{\sigma(\kappa_l)}^{D-(\kappa_l+\tau(\kappa_l))}\sum_{i+j=\kappa_l+\tau(\kappa_l)} V_{\sigma(\kappa_l)}(x(i,j)) - \sum_{s=\kappa_l+\tau(\kappa_l)-1}^{D-2}\sum_{i+j=s}\bar{\alpha}_{\sigma(\kappa_l)}^{D-2-s}\Gamma(i,j)\right\}. \end{aligned} \tag{16}$$

Similarly, consider $\kappa \in [\kappa_l, \kappa_l + \tau(\kappa_l))$, in this situation the filter is mismatched with the subsystem. From (9), it holds that

$$\mathbb{E}\left\{\sum_{i+j=\kappa_l+\tau(\kappa_l)} V_{\sigma(\kappa)}(x(i,j))\right\} < \mathbb{E}\left\{\bar{\beta}_{\sigma(\kappa_l)}^{\tau(\kappa_l)} \sum_{i+j=\kappa_l} V_{\sigma(\kappa_l)}(x(i,j)) - \sum_{s=\kappa_l-1}^{\kappa_l+\tau(\kappa_l)-2} \sum_{i+j=s} \bar{\beta}_{\sigma(\kappa_l)}^{\kappa_l+\tau(\kappa_l)-2-s} \Gamma(i,j)\right\}. \quad (17)$$

Combining (16) with (17), we obtain that for $\kappa \in \left[\kappa_l, \kappa_{l+1}\right)$,

$$\begin{aligned}\mathbb{E}\left\{\sum_{i+j=D} V_{\sigma(\kappa)}(x(i,j))\right\} &\leq \mathbb{E}\left\{\bar{\alpha}_{\sigma(\kappa_l)}^{D-\kappa_l} \theta_{\sigma(\kappa_l)}^{\tau(\kappa_l)} \sum_{i+j=\kappa_l} V_{\sigma(\kappa_l)}(x(i,j)) - \sum_{s=\kappa_l+\tau(\kappa_l)-1}^{D-2} \sum_{i+j=s} \bar{\alpha}_{\sigma(\kappa_l)}^{D-2-s} \Gamma(i,j)\right. \\ &\left. - \sum_{s=\kappa_l-1}^{\kappa_l+\tau(\kappa_l)-2} \sum_{i+j=s} \bar{\alpha}_{\sigma(\kappa_l)}^{D-2-s} \theta_{\sigma(\kappa_l)}^{\kappa_l+\tau(\kappa_l)-2-s} \Gamma(i,j)\right\}. \quad (18)\end{aligned}$$

On the other hand, according to (10)–(11), we conclude for all $(\sigma(\kappa_l) = p, \sigma(k_{l-1}) = q) \in \mathscr{M} \times \mathscr{M}$, $p \neq q$,

$$\mathbb{E}\left\{\sum_{i+j=\kappa_l} V_{\sigma(\kappa_l)}(x(i,j))\right\} \leq \mathbb{E}\left\{\mu_{\sigma(\kappa_l)} \sum_{i+j=\kappa_l} V_{\sigma(\kappa_{l-1})}(x(i,j))\right\}. \quad (19)$$

We prove the stability firstly. Consider $\omega(i,j) \equiv 0$, from (18), we have,

$$\mathbb{E}\left\{\sum_{i+j=D} V_{\sigma(\kappa)}(x(i,j))\right\} \leq \mathbb{E}\left\{\bar{\alpha}_{\sigma(\kappa_1)}^{D-\kappa_l} \theta_{\sigma(\kappa_l)}^{\tau(\kappa_l)} \sum_{i+j=\kappa_l} V_{\sigma(\kappa_l)}(x(i,j))\right\}. \quad (20)$$

Denote $\tau(\kappa_l)$ as τ_l, and $\tau_{\max} = \max_{p \in \mathscr{M}} \{\tau_p\}$. From (19)–(20), we obtain

$$\begin{aligned}\mathbb{E}\left\{\sum_{i+j=D} V_{\sigma(\kappa)}(x(i,j))\right\} &< \mathbb{E}\left\{\mu_{\sigma(\kappa_l)} \bar{\alpha}_{\sigma(\kappa_l)}^{D-\kappa_l} \theta_{\sigma(\kappa_l)}^{\tau(\kappa_l)} \sum_{i+j=\kappa_l} V_{\sigma(\kappa_{l-1})}(x(i,j))\right\} \\ &< \ldots \\ &\leq \mathbb{E}\left\{\prod_{p=1}^{M} (\mu_p \theta_p^{\tau_p})^{N_{0p} + H_p(r,D)/\tau_{ap}} \prod_{p=1}^{M} \bar{\alpha}_p^{H_p(r,D)} \sum_{i+j=r} V_{\sigma(r)}(x(i,j))\right\} \\ &\leq \mathbb{E}\left\{\prod_{p=1}^{M} (\mu_p \theta_p^{\tau_p})^{N_{0p}} \left\{\max_{p \in \mathscr{M}} \left[(\mu_p \theta_p^{\tau_{\max}})^{1/\tau_{ap}} \bar{\alpha}_p\right]\right\}^{D-r} \sum_{i+j=r} V_{\sigma(r)}(x(i,j))\right\}.\end{aligned}$$

From (13), we know that there exist two positive scalars λ_1 and λ_2 such that $\forall\sigma(\kappa) = p \in \mathscr{M}$,

$$\lambda_1\mathbb{E}\left\{\|x(i,j)\|^2\right\} \le \mathbb{E}\left\{V_p(x(i,j))\right\} \le \lambda_2\mathbb{E}\left\{\|x(i,j)\|^2\right\},$$

where

$$\lambda_1 = \min\left\{\lambda_{\min}(P_p) + \lambda_{\min}(Q_p)\right\},$$
$$\lambda_2 = \max\left\{\lambda_{\max}(P_p) + \lambda_{\max}(Q_p)\right\}.$$

Thus, we have

$$\mathbb{E}\left\{\sum_{i+j=D}\|x(i,j)\|^2\right\} \le \varsigma\varepsilon^{D-r}\mathbb{E}\left\{\sum_{i+j=r}\|x(i,j)\|_r^2\right\},$$

where

$$\varsigma = \frac{\lambda_2}{\lambda_1}\prod_{p=1}^{M}(\mu_p\theta_p^{\tau_p})^{N_{0p}} > 0,\ \varepsilon = \max_{p\in\mathscr{M}}\left[(\mu_p\theta_p^{\tau_p})^{1/\tau_{ap}}\bar{\alpha}_p\right],$$

then if there exist constants τ_{ap}, $p \in \mathscr{M}$ satisfying (12), we can get $0 < \varepsilon < 1$. From Definition 2, it can be concluded that the system (6) under asynchronous switching is asymptotically mean-square stable.

Then, we address the ℓ_2-gain analysis. From (18)–(19), we obtain

$$\begin{aligned}
&\mathbb{E}\left\{\sum_{i+j=D}V_{\sigma(\kappa_l)}(x(i,j))\right\}\\
&< \mathbb{E}\Bigg\{\bar{\alpha}_{\sigma(\kappa_l)}^{D-\kappa_l}\theta_{\sigma(\kappa_l)}^{\tau(\kappa_l)}\mu_{\sigma(\kappa_l)}\sum_{i+j=\kappa_l}V_{\sigma(\kappa_{l-1})}(x(i,j)) - \sum_{s=\kappa_l+\tau(\kappa_l)-1}^{D-2}\sum_{i+j=s}\bar{\alpha}_{\sigma(\kappa_l)}^{D-2-s}\Gamma(i,j)\\
&\quad - \sum_{s=\kappa_l-1}^{\kappa_l+\tau(\kappa_l)-2}\sum_{i+j=s}\bar{\alpha}_{\sigma(\kappa_l)}^{D-2-s}\theta_{\sigma(\kappa_l)}^{\kappa_l+\tau(\kappa_l)-2-s}\Gamma(i,j)\Bigg\}\\
&< \ldots\\
&< \mathbb{E}\Bigg\{\bar{\alpha}_{\sigma(\kappa_l)}^{D-\kappa_l}\bar{\alpha}_{\sigma(\kappa_{l-1})}^{\kappa_l-\kappa_{l-1}}\cdots\bar{\alpha}_{\sigma(r)}^{\kappa_1-r}\theta_{\sigma(\kappa_l)}^{\tau(\kappa_l)}\ldots\theta_{\sigma(r)}^{\tau(r)}\mu_{\sigma(\kappa_l)}\ldots\mu_{\sigma(r)}\sum_{i+j=r}V_{\sigma(r)}(x(i,j))\\
&\quad - \bar{\alpha}_{\sigma(\kappa_l)}^{D-\kappa_l}\ldots\bar{\alpha}_{\sigma(r)}^{\kappa_1-r}\theta_{\sigma(\kappa_l)}^{\tau(\kappa_l)}\ldots\theta_{\sigma(r)}^{\tau(r)}\mu_{\sigma(\kappa_l)}\ldots\mu_{\sigma(r)}\sum_{s=r+\tau(r)-1}^{\kappa_1-2}\sum_{i+j=s}\bar{\alpha}_{\sigma(r)}^{\kappa_1-2-s}\Gamma(i,j)\\
&\quad - \bar{\alpha}_{\sigma(\kappa_l)}^{D-\kappa_l}\ldots\bar{\alpha}_{\sigma(r)}^{\kappa_1-r}\theta_{\sigma(\kappa_l)}^{\tau(\kappa_l)}\ldots\theta_{\sigma(r)}^{\tau(r)}\mu_{\sigma(\kappa_l)}\ldots\mu_{\sigma(r)}\\
&\quad \times\sum_{s=r-1}^{r+\tau(r)-2}\sum_{i+j=s}\bar{\alpha}_{\sigma(r)}^{\kappa_1-2-s}\theta_{\sigma(r)}^{r+\tau(r)-2-s}\Gamma(i,j) - \ldots\\
&\quad - \sum_{s=\kappa_l+\tau(\kappa_l)-1}^{D-2}\sum_{i+j=s}\bar{\alpha}_{\sigma(\kappa_l)}^{D-2-s}\Gamma(i,j) - \sum_{s=\kappa_l-1}^{\kappa_l+\tau(\kappa_l)-2}\sum_{i+j=s}\bar{\alpha}_{\sigma(\kappa_l)}^{D-2-s}\theta_{\sigma(\kappa_l)}^{\kappa_l+\tau(\kappa_l)-2-s}\Gamma(i,j)\Bigg\}.
\end{aligned} \tag{21}$$

Denote $\alpha_{\max} = \max\{\alpha_p\}$, $\alpha_{\min} = \min\{\alpha_p\}$ and $\bar{\alpha}_{\max} = 1 - \alpha_{\max}$, $\bar{\alpha}_{\min} = 1 - \alpha_{\min}$. Consider zero initial condition, i.e., $\sum_{i+j=r} V_{\sigma(r)}(x(i,j)) = 0$, and $\sum_{i+j=\kappa_l} V_{\sigma(\kappa_l)}(x(i,j)) \geq 0$. Owing to $1 < \theta_p^{\kappa_p+\tau_p-2-s} < \theta_p^{\tau_p-1}$, $s \in [\kappa_p - 1, \kappa_p + \tau_p - 2)$, then we have

$$\mathbb{E}\left\{\sum_{s=r-1}^{D-2}\sum_{i+j=s}\left[\bar{\alpha}_{\max}^{D-2-s}\prod_{p=1}^{M}(\theta_p^{\tau_p}\mu_p)^{N_{\sigma p}(s,D)}\bar{e}^T\bar{e}\right]\right\}$$
$$\leq \mathbb{E}\left\{\sum_{s=r-1}^{D-2}\sum_{i+j=s}\left[\bar{\alpha}_{\min}^{D-2-s}\prod_{p=1}^{M}(\theta_p^{\tau_p}\mu_p)^{N_{\sigma p}(s,D)}\theta_{\sigma(s)}^{\tau_{\sigma(s)}-1}\gamma^2\bar{\omega}^T\bar{\omega}\right]\right\}.$$

Multiplying both sides of the above inequality by $(\theta_p^{\tau_p}\mu_p)^{-N_{\sigma p}(r,D)}$, it follows that

$$\mathbb{E}\left\{\sum_{s=r-1}^{D-2}\sum_{i+j=s}\left[\bar{\alpha}_{\max}^{D-2-s}\prod_{p=1}^{M}(\theta_p^{\tau_p}\mu_p)^{-N_{\sigma p}(r,s)}\bar{e}^T\bar{e}\right]\right\}$$
$$\leq \mathbb{E}\left\{\sum_{s=r-1}^{D-2}\sum_{i+j=s}\left[\bar{\alpha}_{\min}^{D-2-s}\prod_{p=1}^{M}(\theta_p^{\tau_p}\mu_p)^{-N_{\sigma p}(r,s)}\theta_{\sigma(s)}^{\tau_{\sigma(s)}-1}\gamma^2\bar{\omega}^T\bar{\omega}\right]\right\}.$$

From Definition 1, we know $-N_{0p} - H_p(r,s)/\tau_{ap} \leq -N_{\sigma p}(r,s) \leq 0$. Meanwhile, by noticing (12) we have

$$\mathbb{E}\left\{\sum_{s=r-1}^{D-2}\sum_{i+j=s}\left[\bar{\alpha}_{\max}^{D-2-s}\prod_{p=1}^{M}(\theta_p^{\tau_p}\mu_p)^{-N_{0p}+H_p(r,s)\ln\bar{\alpha}_p/\ln(\theta_p^{\tau_p}\mu_p)}\bar{e}^T\bar{e}\right]\right\}$$
$$\leq \mathbb{E}\left\{\sum_{s=r-1}^{D-2}\sum_{i+j=s}\left(\bar{\alpha}_{\min}^{D-2-s}\theta_{\max}^{\tau_{\max}-1}\gamma^2\bar{\omega}^T\bar{\omega}\right)\right\}.$$

Furthermore,

$$\mathbb{E}\left\{\sum_{D=2}^{\infty}\sum_{s=r-1}^{D-2}\sum_{i+j=s}\left[\bar{\alpha}_{\max}^{D-2-s}\prod_{p=1}^{M}\bar{\alpha}_p^{H_p(r,s)}\bar{e}^T\bar{e}\right]\right\}$$
$$\leq \Theta_p\gamma^2\mathbb{E}\left\{\sum_{D=2}^{\infty}\sum_{s=r-1}^{D-2}\sum_{i+j=s}\bar{\alpha}_{\min}^{D-2-s}\theta_{\max}^{\tau_{\max}-2}\gamma^2\bar{\omega}^T\bar{\omega}\right\},$$

and

$$\mathbb{E}\left\{\sum_{s=r-1}^{\infty}\sum_{D=s+2}^{\infty}\sum_{i+j=s}\bar{\alpha}_{\max}^{D-2-s}\bar{\alpha}_{\max}^{s}\bar{e}^T\bar{e}\right\}$$
$$\leq \Theta_p\gamma^2\mathbb{E}\left\{\sum_{s=r-1}^{\infty}\sum_{D=s+2}^{\infty}\sum_{i+j=s}\bar{\alpha}_{\min}^{D-2-s}\theta_{\max}^{\tau_{\max}-2}\gamma^2\bar{\omega}^T\bar{\omega}\right\},$$

where $\Theta_p = \prod_{p=1}^{M}(\theta_p^{\tau_p}\mu_p)^{N_{0p}}$. Since

$$\sum_{D=s+2}^{\infty}\sum_{i+j=s}\bar{\alpha}_{\max}^{D-2-s} = \sum_{i+j=s} 1/\alpha_{\max}, \sum_{D=s+2}^{\infty}\sum_{i+j=s}\bar{\alpha}_{\min}^{D-2-s} = \sum_{i+j=s} 1/\alpha_{\min},$$

thus, we conclude

$$\mathbb{E}\left\{\sum_{s=r-1}^{\infty}\sum_{i+j=s}\alpha^s\bar{e}^T\bar{e}\right\} \le \prod_{p=1}^{M}(\theta_p^{\tau_p}\mu_p)^{N_{0p}}\frac{\alpha_{\max}}{\alpha_{\min}}\theta_{\max}^{\tau_{\max}-1}\gamma^2\mathbb{E}\left\{\sum_{s=r-1}^{\infty}\sum_{i+j=s}\bar{\omega}^T\bar{\omega}\right\},$$

$$\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha^{i+j}\|\bar{e}\|_2^2\right\} \le \gamma_s^2\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|\bar{\omega}\|_2^2\right\},$$

where $\alpha = 1-\alpha_{\max}$ and $\gamma_s = \sqrt{\prod_{p=1}^{M}(\theta_p^{\tau_p}\mu_p)^{N_{0p}}(\alpha_{\max}/\alpha_{\min})\theta_{\max}^{\tau_{\max}-1}}\gamma$.

According to Definition 3, we conclude that the 2-D switched system (6)–(7) is asymptotically stable with a prescribed weighted $\mathscr{H}_\infty$ disturbance attenuation level γ_s, which ends the proof. □

4 $\mathscr{H}_\infty$ Filter Design

According to ℓ_2-gain analysis in the preceding section, we address asynchronous $\mathscr{H}_\infty$ filter design with missing measurements for 2-D switched systems.

Theorem 1 *Consider the 2-D switched system (1)–(3). For any $(p,q)\in\mathscr{M}\times\mathscr{M}$, $p\neq q$, let $0<\alpha_p<1$, $\beta_p>0$ and $\mu_p>1$ be given constants, if there exist matrices $P_p>0$, $Q_p>0$, X_p, Y_p, Z_p, G_{F1p}, G_{F2p}, K_{F1p}, K_{F2p}, L_{Fp}, $p\in\mathscr{M}$, and a scalar $\gamma>0$, such that (10)–(11) hold, and*

$$\begin{bmatrix} \bar{\Xi}_{11} & 0 & \bar{\Xi}_{13} & \bar{\Xi}_{14} & \bar{\Xi}_{15} \\ * & -\gamma^2 I & \bar{\Xi}_{23} & 0 & \bar{\Xi}_{25} \\ * & * & \bar{\Xi}_{33} & 0 & 0 \\ * & * & * & \bar{\Xi}_{44} & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} \tilde{\Xi}_{11} & 0 & \tilde{\Xi}_{13} & \tilde{\Xi}_{14} & \tilde{\Xi}_{15} \\ * & -\gamma^2 I & \tilde{\Xi}_{23} & 0 & \tilde{\Xi}_{25} \\ * & * & \tilde{\Xi}_{33} & 0 & 0 \\ * & * & * & \tilde{\Xi}_{44} & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \tag{23}$$

where

$$\bar{\Xi}_{11} = \mathrm{diag}\{-\bar{\alpha}_p P_p, -\bar{\alpha}_p Q_p\}, \bar{\Xi}_{33} = \mathrm{diag}\{P_p - R_p - R_p^T, Q_p - R_p - R_p^T\},$$
$$\bar{\Xi}_{44} = \bar{\eta}\mathrm{diag}\{\bar{\Xi}_{33}, \bar{\Xi}_{33}\}, \bar{\Xi}_{14} = \mathrm{diag}\{\bar{\Xi}_{14}^1, \bar{\Xi}_{14}^2\},$$
$$\bar{\Xi}_{13} = \begin{bmatrix} \bar{\Xi}_{13}^1 & \bar{\Xi}_{13}^1 \\ \bar{\Xi}_{13}^2 & \bar{\Xi}_{13}^2 \end{bmatrix}, \bar{\Xi}_{23} = \begin{bmatrix} \bar{\Xi}_{23}^1 & \bar{\Xi}_{23}^1 \\ \bar{\Xi}_{23}^2 & \bar{\Xi}_{23}^2 \end{bmatrix}, \bar{\Xi}_{15} = \begin{bmatrix} C_p^T & 0 \\ -L_{Fp}^T & 0 \end{bmatrix}, \bar{\Xi}_{25} = \begin{bmatrix} 0 & C_p^T \\ 0 & -L_{Fp}^T \end{bmatrix},$$
$$\tilde{\Xi}_{11} = \mathrm{diag}\{-\bar{\beta}_p P_p, -\bar{\beta}_p Q_p\}, \tilde{\Xi}_{33} = \mathrm{diag}\{P_p - R_q - R_q^T, Q_p - R_q - R_q^T\},$$

$$\tilde{\Xi}_{44} = \bar{\eta}\text{diag}\{\tilde{\Xi}_{33}, \tilde{\Xi}_{33}\}, \tilde{\Xi}_{14} = \text{diag}\{\tilde{\Xi}^1_{14}, \tilde{\Xi}^2_{14}\},$$

$$\tilde{\Xi}_{13} = \begin{bmatrix} \tilde{\Xi}^1_{13} & \tilde{\Xi}^1_{13} \\ \tilde{\Xi}^2_{13} & \tilde{\Xi}^2_{13} \end{bmatrix}, \tilde{\Xi}_{23} = \begin{bmatrix} \tilde{\Xi}^1_{23} & \tilde{\Xi}^1_{23} \\ \tilde{\Xi}^2_{23} & \tilde{\Xi}^2_{23} \end{bmatrix}, \tilde{\Xi}_{15} = \begin{bmatrix} C_p^T & 0 \\ -L_{Fq}^T & 0 \end{bmatrix}, \tilde{\Xi}_{25} = \begin{bmatrix} 0 & C_p^T \\ 0 & -L_{Fq}^T \end{bmatrix},$$

with $\bar{\alpha}_p = 1 - \alpha_p$, $\bar{\beta}_p = 1 + \beta_p$, $\theta_p = \bar{\beta}_p / \bar{\alpha}_p$, $\bar{\eta} = (1 - \eta)\eta$, *and for* $r = 1, 2$,

$$R_p = \begin{bmatrix} X_p & Y_p \\ Z_p & Y_p \end{bmatrix},$$

$$\bar{\Xi}^r_{13} = \begin{bmatrix} A^T_{rp} X^T_p + \eta C^T_{yp} K_{Frp} & A^T_{rp} Z^T_p + \eta C^T_{yp} K_{Frp} \\ G^T_{Frp} - \eta C^T_{yp} K_{Frp} & G^T_{Frp} - \eta C^T_{yp} K_{Frp} \end{bmatrix},$$

$$\bar{\Xi}^r_{14} = \begin{bmatrix} C^T_{yp} K_{Frp} & C^T_{yp} K_{Frp} & C^T_{yp} K_{Frp} & C^T_{yp} K_{Frp} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{\Xi}^r_{23} = \begin{bmatrix} B^T_{rp} X^T_p + D^T_{yp} K_{Frp} & B^T_{rp} Z^T_p + D^T_{yp} K_{Frp} \end{bmatrix},$$

$$\tilde{\Xi}^r_{13} = \begin{bmatrix} A^T_{rp} X^T_q + \eta C^T_{yp} K_{Frq} & A^T_{rp} Z^T_q + \eta C^T_{yp} K_{Frq} \\ G^T_{Frq} - \eta C^T_{yp} K_{Frq} & G^T_{Frq} - \eta C^T_{yp} K_{Frq} \end{bmatrix},$$

$$\tilde{\Xi}^r_{14} = \begin{bmatrix} C^T_{yp} K_{Frq} & C^T_{yp} K_{Frq} & C^T_{yp} K_{Frq} & C^T_{yp} K_{Frq} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\Xi}^r_{23} = \begin{bmatrix} B^T_{rp} X^T_q + D^T_{yp} K_{Frq} & B^T_{rp} Z^T_q + D^T_{yp} K_{Frq} \end{bmatrix},$$

Then, for any switching signal with MDADT satisfying (12)*, the 2-D switched system* (6)–(7) *is asymptotically mean-square stable with a prescribed weighted* $\mathscr{H}_\infty$ *disturbance attenuation level* $\gamma_s = \sqrt{\prod_{p=1}^{M} (\theta_p^{\tau_p} \mu_p)^{N_{0p}} (\alpha_{\max}/\alpha_{\min}) \theta_{\max}^{\tau_{\max}-1}}\gamma$ *and* $\theta_{\max} = \max_{p \in \mathscr{M}} \{\theta_p\}$*. Moreover, the filter parameters are given by* $G_{1p} = Y_p^{-1} G_{F1p}$, $G_{2p} = Y_p^{-1} G_{F2p}$, $K_{1p} = Y_p^{-1} K_{F1p}$, $K_{2p} = Y_p^{-1} K_{F2p}$*, and* $L_p = L_{Fp}$.

Proof Choose (13) as the Lyapunov function.

Noticing that $-R_p P_p^{-1} R_p^T < P_p - R_p - R_p^T$ and $-R_p^{-1} Q_p R_p^T < Q_p - R_p - R_p^T$, denote $G_{F1p} = Y_p G_{1p}$, $G_{F2p} = Y_p G_{2p}$, $K_{F1p} = Y_p K_{1p}$, $K_{F2p} = Y_p K_{2p}$, and $L_{Fp} = L_p$. Then taking congruent transformation and using Schur complement, we know that (22) ensures (8). By a similar procedure, we find that (9) can be guaranteed by (23). According to Lemma 1, we conclude that system (6)–(7) is asymptotically mean-square stable with a prescribed weighted $\mathscr{H}_\infty$ disturbance attenuation level γ_s, which ends the proof. □

When taking no account of the asynchronous switching, i.e., $\tau(\kappa_l) = 0$, we have the following corollary.

Corollary 1 *Consider the 2-D switched system* (1)–(3)*. For any* $(p, q) \in \mathscr{M} \times \mathscr{M}$, $p \neq q$*, let* $0 < \alpha_p < 1$ *and* $\mu_p > 1$ *be given constants, if there exist a set of matrices* $P_p > 0$, $Q_p > 0$, X_p, Y_p, Z_p, G_{F1p}, G_{F2p}, K_{F1p}, K_{F2p}, L_{Fp}*, and a scalar* $\gamma > 0$*, such that* (10)–(11) *and* (22) *hold, Then, for any switching signal with MDADT satisfying*

$$\tau_{ap} > \tau^*_{ap} = -\frac{\ln \mu_p}{\ln \bar{\alpha}_p},$$

where $\bar{\alpha}_p = 1 - \alpha_p$*, the 2-D switched system* (6)–(7) *is asymptotically mean-square stable with a prescribed weighted* $\mathscr{H}_\infty$ *disturbance attenuation level* γ_s

$= \sqrt{\prod_{p=1}^{M} \mu_p^{N_{0p}} (\alpha_{\max}/\alpha_{\min})}\gamma$, $\alpha_{\max} = \max_{p\in\mathscr{M}}\{\alpha_p\}$, $\alpha_{\min} = \min_{p\in\mathscr{M}}\{\alpha_p\}$. *Moreover, the filter parameters are given by* $G_{1p} = Y_p^{-1}G_{F1p}$, $G_{2p} = Y_p^{-1}G_{F2p}$, $K_{1p} = Y_p^{-1}K_{F1p}$, $K_{2p} = Y_p^{-1}K_{F2p}$, *and* $L_p = L_{Fp}$.

Remark 2 Only switching delay is considered in this paper. As a matter of fact, state delay may also be involved in practical systems. There exist some results for 2-D switched delay systems (Duan et al. 2013; Ghous et al. 2015), one can develop the filter design for 2-D switched delay systems by using the similar methods.

Remark 3 All above results are based on MDADT switching, which is more general than ADT switching (Fei et al. 2017). Recently, a more flexible switching logic, persistent dwell time (PDT) switching is applied in 1-D switching systems (Zhang et al. 2015). In future works, it is meaningful to extended PDT switching to 2-D switched systems.

5 Illustrative examples

In this section, we use two examples to illustrate the effectiveness of the results developed in the above section.

Example 1 Consider the 2-D switched systems (1)–(3) consisting of two subsystems as follow:
Subsystem 1:

$$A_{11} = \begin{bmatrix} 0.2352 & 0.0019 \\ -0.9648 & 0.2019 \end{bmatrix}, B_{11} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, A_{21} = \begin{bmatrix} -0.0965 & 0.1002 \\ -0.0965 & 0.0002 \end{bmatrix}, B_{21} = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}, D_1 = -0.1, C_{y1} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, D_{y1} = 0.1;$$

Subsystem 2:

$$A_{12} = \begin{bmatrix} 0.0313 & 0.1757 \\ -0.5773 & 0.0515 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 \\ 0.1254 & -0.0970 \end{bmatrix}, B_{22} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}, D_2 = -0.1, C_{y2} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, D_{y2} = 0.1.$$

Our purpose here is to design a set of mode-dependent filters with missing measurements such that the filtering error system is asymptotically mean-square stable and has an prescribed $\mathscr{H}_\infty$ performance.

Set $\mu_1 = \mu_2 = 1.1$ and $\alpha_1 = 0.4$, $\alpha_2 = 0.3$, $\beta_1 = 1.2$, $\beta_2 = 1.1$, and assume $\tau_{\max} = 2$, $\eta = 0.9$, $\gamma = 1.2247$. Then we find an admissible solution by using standard softwares. The corresponding $\tau_{a1}^* = 5.2736$, $\tau_{a2}^* = 6.4275$, and a set of filters are designed:

$$G_{11} = \begin{bmatrix} 0.0379 & -0.0593 \\ -0.1537 & 0.2619 \end{bmatrix}, G_{21} = \begin{bmatrix} -0.0140 & 0.0373 \\ 0.0256 & -0.0586 \end{bmatrix},$$
$$K_{11} = \begin{bmatrix} 0.0005 \\ -0.0411 \end{bmatrix}, K_{21} = \begin{bmatrix} -0.0388 \\ 0.1252 \end{bmatrix}, L_1 = \begin{bmatrix} 0.0050 & -0.0986 \end{bmatrix},$$
$$G_{12} = \begin{bmatrix} -0.0068 & 0.0447 \\ -0.0078 & 0.0018 \end{bmatrix}, G_{22} = \begin{bmatrix} -0.0030 & 0.0103 \\ 0.0156 & -0.0542 \end{bmatrix},$$
$$K_{12} = \begin{bmatrix} -0.0139 \\ -0.0035 \end{bmatrix}, K_{22} = \begin{bmatrix} -0.0007 \\ 0.0014 \end{bmatrix}, L_2 = \begin{bmatrix} 0.0160 & -0.1535 \end{bmatrix}.$$

Here, we choose the switching signal as Fig. 1. The missing measurements satisfying $\eta = 0.9$ is shown in Fig. 2. Under the zero initial condition, the disturbance input is assumed to be $\omega(i, j) = \sin(0.1\pi(i + j))e^{-0.2(i+j)}$. Using the filters obtained by Theorem 1, we can get the filter state responses shown in Figs. 3, 4, and filter error responses shown in Fig. 5. It can be observed the augmented system is asymptotically stable.

Example 2 Consider the thermal processes in heat exchangers, which can be described by a partial differential equation (Ghous et al. 2015):

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - a_{\sigma(x,t)}T(x,t) + b_{\sigma(x,t)}f(x,t), \tag{24}$$

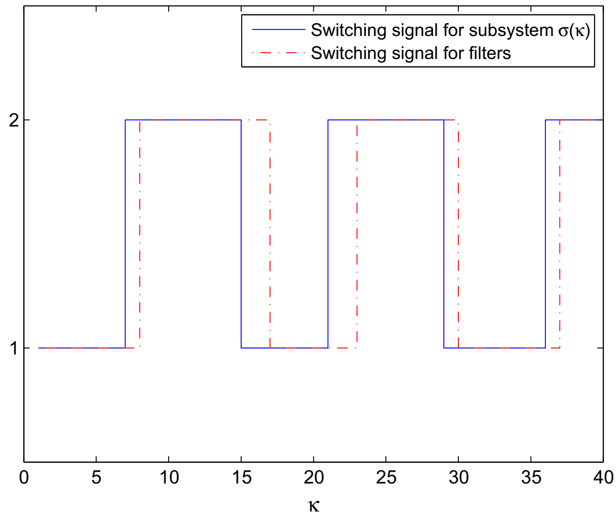

Fig. 1 The switching signal $\sigma(\kappa)$

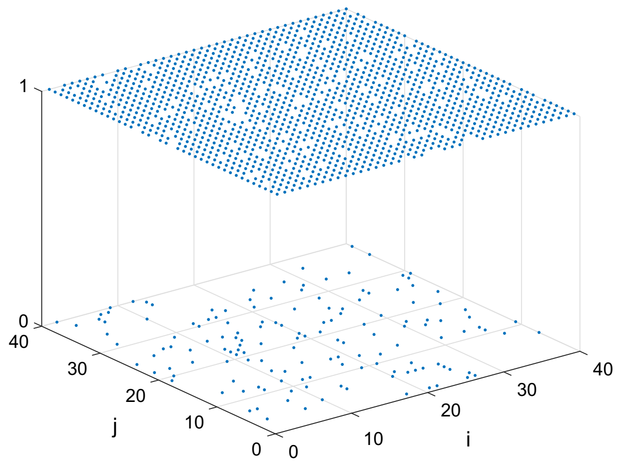

Fig. 2 The missing measurement distribution $\rho(i, j)$

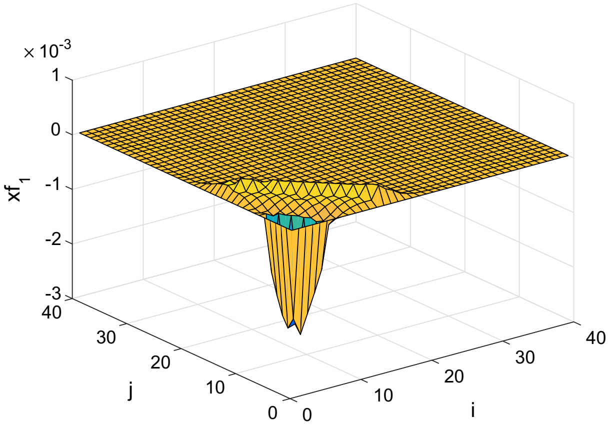

Fig. 3 The filter state response $\hat{x}_1(i, j)$

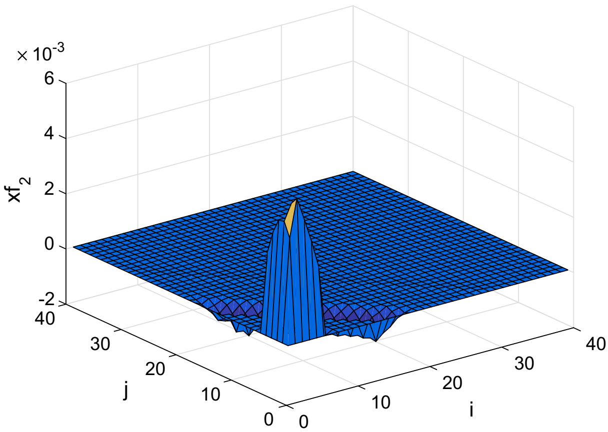

Fig. 4 The filter state response $\hat{x}_2(i, j)$

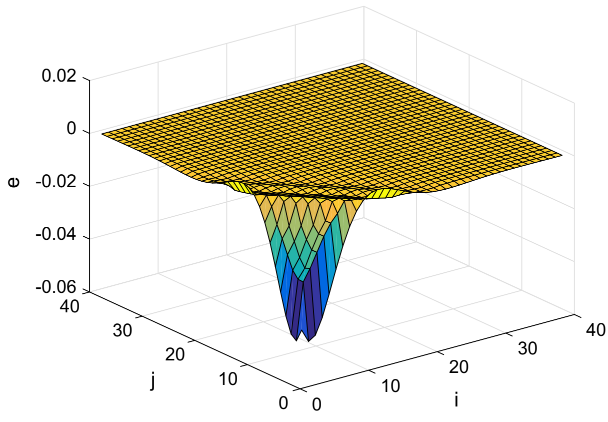

Fig. 5 The filter error response $e(i, j)$

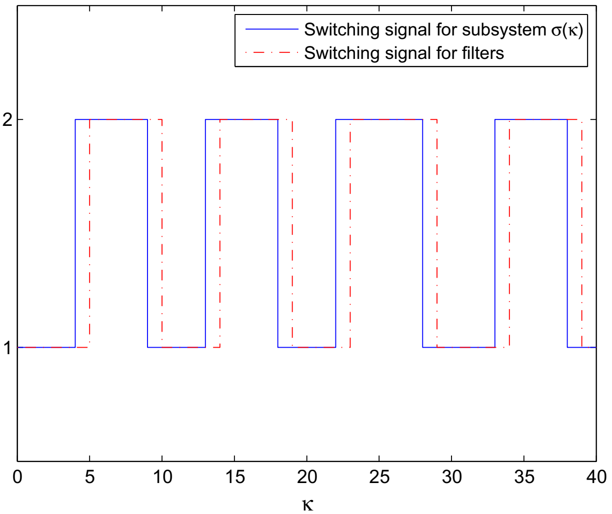

Fig. 6 The switching signal $\sigma(\kappa)$

where $T(x,t)$ is the temperature at x(space) $\in [0, x_f]$ and t(time) $\in [0,\infty)$, $f(x,t)$ is the input function, and $a_{\sigma(x,t)}$, $b_{\sigma(x,t)}$ are real coefficients, which are functions of $\sigma(x,t)$. Similar to the technique used in Kaczorek (1985), here, define

$$\begin{aligned}\frac{\partial T(x,t)}{\partial t} &\approx \frac{T(i,j+1)-T(i,j)}{\Delta t},\\ \frac{\partial T(x,t)}{\partial x} &\approx \frac{T(i,j)-T(i-1,j)}{\Delta x},\\ u(x,t) &\approx u(i,j),\end{aligned}$$

where $T(i,j) = T(i\Delta x, j\Delta t)$, $u(i,j) = u(i\Delta x, j\Delta t)$, Δx and Δt are space and time discretization periods, respectively. Then, when viewing the disturbance input as the input function, we obtain that (24) can be rewritten in the form of (1) with

$$A_{1\sigma(i,j)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_{1\sigma(i,j)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_{2\sigma(i,j)} = \begin{bmatrix} 0 & 0 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - a_{\sigma(i,j)}\Delta t \end{bmatrix}, B_{2\sigma(i,j)} = \begin{bmatrix} 0 \\ b_{\sigma(i,j)}\Delta t \end{bmatrix}.$$

Consider this 2-D switched system with two modes and $\Delta t = 0.2$, $\Delta x = 0.5$, $a_1 = 2$, $a_2 = 2.5$, $b_1 = 0.25$, $b_2 = 0.5$. Here, assume that

$$C_1 = C_2 = C_{y1} = C_{y2} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, D_1 = D_2 = -0.1, D_{y1} = D_{y2} = 0.1.$$

Set $\mu_1 = \mu_2 = 1.1$, $\alpha_1 = \alpha_2 = 0.35$, $\beta_1 = 1.2$, $\beta_2 = 1.1$, and $\tau_{max} = 1$, $\eta = 0.9$, $\gamma = 3.1623$. According to Theorem 1, we have $\tau_{a1}^* = 2.9435$, $\tau_{a2}^* = 3.0515$. Choose the switching signal as Fig. 6. Under the zero initial condition, the disturbance input is assumed to be $\omega(i,j) = \frac{\sin(0.1\pi(i+j))}{0.1(i+j)^2}$. Using the filters obtained by Theorem 1, we can get the filter error responses as shown in Fig. 7, which satisfies above constraints. This illustrate the effectiveness of the proposed method.

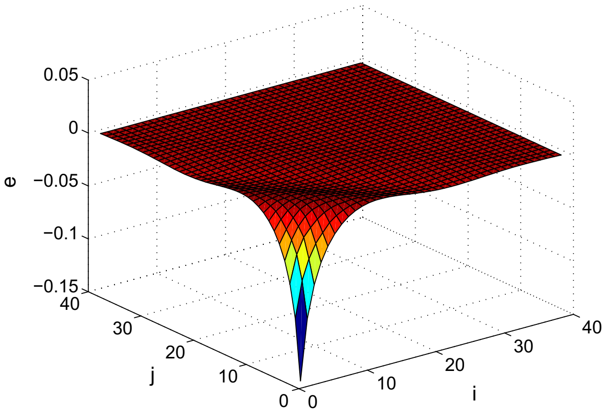

Fig. 7 The filter error response $e(i, j)$

6 Conclusions

In this paper, the problem of asynchronous $\mathscr{H}_\infty$ filtering has been solved for 2-D switched systems with missing measurements. By constructing a class of mode-dependent Lyapunov function, sufficient conditions are proposed to guarantee the asymptotic mean-square stability and ℓ_2-gain of the 2-D switched systems. Based on the results, a set of filters are designed. For future study, since the state delay is prevalent in practical systems, it would be interesting to extend the above results to 2-D switched delay systems. Meanwhile, more general switching logics could be applied to study 2-D switched delay systems.

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