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Modelling a gas pipeline as a repetitive process: controllability, observability and stability

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Abstract In this paper, the gas dynamics within the pipelines is modelled as a repetitive process with smoothing. Controllability and observability criteria when the system is steered through initial and boundary data, which is achieved by an adequate choice of the homogeneity, are obtained. From the point of view of the technical applications, it seems to make more sense to consider boundary data controls as for instance in the management of high pressure gas networks. Stability criteria suitable computer simulations are also included.

Keywords Boundary control · Controllability · Gas networks · Modelling · Observability · Repetitive processes · 2D-systems

1 Introduction

Consider a linear discrete model with inter pass smoothing that has been introduced in Cichy et al. (2007). These repetitive processes (RP) are a distinct class of two dimensional 2-D systems of both systems theoretic and applications interest (Rogers et al. 2007).

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Hence

$$x_{\ell+1}(k+1) = Ax_{\ell+1}(k) + Bu_{\ell+1}(k) + \sum_{p=0}^{\alpha-1} B_{pk} y_\ell(p),$$
$$y_{\ell+1}(k) = Cx_{\ell+1}(k) + Du_{\ell+1}(k) + \sum_{p=0}^{\alpha-1} D_{pk} y_k(p),$$
$$k = 0, 1, \ldots, \quad \ell = 0, 1, \ldots \tag{1}$$

In model (1) the pass number is indexed by ℓ and may be infinite. The time steps per pass-ℓ are indexed by k, $k \in [0, \alpha] \cap \mathbb{Z}$ and α is the pass length. We consider that $\{x_\ell(k) = 0, k = \alpha + 1, \ldots\}$. Therefore, we define the compact support of $x_\ell(k)$, $y_\ell(k)$ and $u_\ell(k)$ as $\mathbb{K} \times \mathbb{L}$.

$x_\ell(k) \in \mathbb{R}^n$ are the state vectors at pass-ℓ. $y_\ell(k) \in \mathbb{R}^m$ are the pass profile vectors at pass-ℓ. $u_\ell(k) \in \mathbb{R}^r$ are the control vectors at pass-ℓ. n, m and r are the dimensions of the state, the pass profile and the control vector, respectively.

The matrices in (1) have the following dimensions $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $B_{pk} \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times r}$, $D_{pk} \in \mathbb{R}^{m\times m}$ and the indices $-_{pk}$ run as stated in (1).

In addition, we define the boundary conditions, from which we understand the pass initial vector sequence, ϕ_ℓ, and the initial pass profile, $v(k)$. Hence

$$x_\ell(0) = \phi(\ell) = \phi_\ell, \quad \ell = 0, 1, \ldots$$
$$y_0(k) = v(k), \qquad k = 0, 1, \ldots, \tag{2}$$

where ϕ_ℓ has the compact support $\mathbb{L}$ and v has the compact support $\mathbb{K}$.

Define $\mathcal{I} = (\mathbb{Z} \cap \mathbb{K}) \times (\mathbb{Z} \cap \mathbb{L})$. The state $x_{(.)}(.) \in \ell^{2,n}(\mathcal{I})$, the pass profile $y_{(.)}(.) \in \ell^{2,m}(\mathcal{I})$ and the controls are admissible if $u_{(.)}(.) \in \ell^{2,r}(\mathcal{I})$, where $\ell^{2,\nu}(\mathcal{I})$ denotes the Hilbert space of ν-dimensional sequences defined on $\mathcal{I}$ with the standard scalar product. The ℓ used to represent $\ell^{2,\nu}(\mathcal{I})$ does not have anything to do with the indexation of every pass, *e.g.* x_ℓ. But we use the same notation, since it looks to us that the different use of this notation becomes clear from the context.

In this work, we use a model of type (1) to represent the gas pipeline. In Dymkou et al. (2007) there was obtained a repetitive process model for a gas distribution network stemming from a discretisation approach in time and space variables and in Azevedo-Perdicoúlis and Jank (2009); Azevedo Perdicoúlis and Jank (2012) a repetitive model with smoothing was used to model a high pressure gas network where the controls used were compressor stations and intakes, hence controlling boundary input levels of the network. In this paper we emphasise boundary control properties of the system.

In Klamka (1997); Rogers et al. (2007) and related articles cited therein, controllability, reachability and observability matters are considered. Only, the criteria obtained are based on an implicit representation of the solutions, since they are defined by some recursions. Also general algebraic methods, i.e., module theoretic or behavioural approaches could be used to obtain controllability results. Controllability properties by boundary data control in continuous time 2-D systems are also obtained in Gyurkovics and Jank (2001) and Jank (2002).

In this article, the dynamics within the pipeline is represented by a repetitive process with smoothing in Sect. 2. This representation allows for studying the sole influence of boundary data on the system dynamics. In Sect. 3 boundary controllability criteria are stated and in Sect. 4 observability is studied. In Sect. 5 we particularise some classical 2D results to analyse asymptotic stability as well as stability along the pass of the pipeline, suitable for simulations.

2 The gas pipeline modelled as a 2D repetitive process

Full models that represent the dynamics of the gas in the pipelines are very complex and, for this reason, it is usual to consider approximations. The derivation of some of the PDE models for gas networks can be seen in Niepłocha (1988), Osiadacz (1987) and Vostry et al. (1988). Usual assumptions in industry are unidimensional flow, which means that the gas pressure, mass flow and velocity are functions of time and distance along the main axis of the pipeline, and also constant temperature and elevation. Taking these assumptions into account, one possible representation for the gas dynamics is the hyperbolic model, that is, following (Niepłocha 1988) consider the set of partial differential equations:

$$\begin{cases} \dfrac{\partial q(t,x)}{\partial t} = -S\dfrac{\partial p(t,x)}{\partial x} - \dfrac{\lambda c^2}{2dS}\dfrac{q^2(t,x)}{p(t,x)}, \\ \dfrac{\partial p(t,x)}{\partial t} = -\dfrac{c^2}{S}\dfrac{\partial q(t,x)}{\partial x}, \end{cases} \tag{3}$$

where x is space, t is time, p is pressure, q is mass flow, S is cross-sectional area, d is the pipe diameter, c is the isothermal speed of sound and λ is the friction factor.

Next, this set of partial differential equations (PDE) is transformed in order to write the network in the repetitive process form.

Model (3) is linearised around the operational levels $(\bar{p}(x), \bar{q})$. Mass flow is assumed constant along a pipeline of length L. $\bar{q}$ is the average mass flow over time in every point of the pipeline. Considering this assumption in the first equation of (3), a representation of the pressure can be obtained:

$$p(t,x) = \sqrt{p^2(t,x_0) - \frac{\lambda c^2}{2dS^2}\bar{q}^2(x-x_0)}, \tag{4}$$

where $x_0 \equiv 0$ and $p(t,x_0)$ is the inlet nodal pressure. Considering $\bar{p}(x)$ as the average pressure over a time interval, i.e., $\bar{p}(x) = \frac{1}{T}\int_0^T p(t,x)dt$, and T is a period of operation, we have:

$$\bar{p}(x) = \sqrt{\bar{p}^2(x_0) - \frac{\lambda c^2}{2dS^2}\bar{q}^2(x-x_0)}, \tag{5}$$

where $\bar{q}$, $\bar{p}(x)$ denote mean values.

Thence, we can write $p(t,x) = \bar{p}(x) + \Delta p(t,x)$ and $q(t,x) = \bar{q} + \Delta q(t,x)$, with $\Delta p(t,x)$ and $\Delta q(t,x)$ as the deviations from the operational (reference) values of pressure and mass flow. After neglecting terms of higher order, some simpler calculations yield:

$$\frac{q^2(t,x)}{p(t,x)} = \frac{(\bar{q}+\Delta q(t,x))^2}{\bar{p}(x)+\Delta p(t,x)} \approx \left(\bar{q}^2 + 2\bar{q}\Delta q(t,x) + \Delta q(t,x)^2\right)\frac{1}{\bar{p}(x)}\frac{1}{\left(1+\dfrac{\Delta p(t,x)}{\bar{p}(x)}\right)} \tag{6}$$

where it is assumed that $\dfrac{\Delta p(t,x)}{\bar{p}(x)} \ll 1$ and $\dfrac{\Delta q(t,x)}{\bar{q}} \ll 1$.

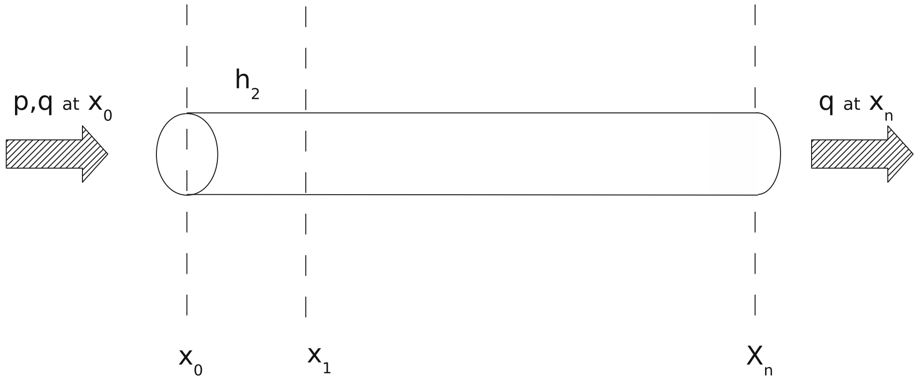

Fig. 1 Space segmentation of the pipeline

Using a first order Taylor approximation and then neglecting terms of higher order, we have:

$$\begin{aligned}\frac{q^2(t,x)}{p(t,x)} &\approx \left(\bar{q}^2 + 2\bar{q}\Delta q(t,x)\right)\frac{1}{\bar{p}(x)}\left(1 - \frac{\Delta p(t,x)}{\bar{p}(x)}\right)\\ &\approx \frac{\bar{q}^2}{\bar{p}(x)} + \frac{2\bar{q}}{\bar{p}(x)}\Delta q(t,x) - \frac{\bar{q}^2}{\bar{p}(x)^2}\Delta p(t,x) - \frac{2\bar{q}}{\bar{p}(x)^2}\Delta p(t,x)\Delta q(t,x).\end{aligned}$$

Neglecting the last term of the approximation, because it is of higher order, we obtain the linear approximation:

$$\frac{q^2(t,x)}{p(t,x)} \approx \frac{\bar{q}^2}{\bar{p}(x)} + \frac{2\bar{q}}{\bar{p}(x)}\Delta q(t,x) - \frac{\bar{q}^2}{\bar{p}(x)^2}\Delta p(t,x) \tag{7}$$

that we substitute in (3):

$$\begin{aligned}\frac{\partial q(t,x)}{\partial t} &= -S\frac{\partial p(t,x)}{\partial x} - \frac{\lambda c^2}{2dS}\left(\frac{\bar{q}^2}{\bar{p}(x)} + \frac{2\bar{q}}{\bar{p}(x)}\Delta q(t,x) - \frac{\bar{q}^2}{\bar{p}(x)^2}\Delta p(t,x)\right),\\ \frac{\partial p(t,x)}{\partial t} &= -\frac{c^2}{S}\frac{\partial q(t,x)}{\partial x}.\end{aligned} \tag{8}$$

The pipeline is divided into several segments $\mathcal{L}_\ell = \left[x_\ell, x_{\ell+1}\right]$, $\ell = 0, 1, \ldots, n-1$, $x_0 = 0$ and $x_n = L$, where L is the length of the pipeline. In every segment, equal mass flow is assumed as well as the above linearisation. Accordingly, the model is discretised according to time and space; h_1 is the time step and h_2 is the space step. Define $\Delta q(t,x) := \Delta q\,(kh_1, \ell h_2)$ where $\ell = 1, \ldots, L$ and $k = 0, 1, 2, \ldots, T$. To simplify notation, consider $f\,(kh_1, \ell h_2) = f(k,\ell) = f_\ell(k)$ (Fig. 1).

In gas networks is usual to consider the main gas period as 1 day, i.e., $T = 24$ hours, and discretisation steps of 1 hour, since the behaviour of the gas can be considered steady during this period. We consider a very long pipeline and, for this reason, the length of the pipeline, L, tends to infinity.

Then, by using a two point forward formula to approximate the time derivatives and a two point backward formula to approximate the space derivatives, we obtain:

$$\begin{aligned}
(\Delta q_\ell(k+1) - \Delta q_\ell(k))\,\frac{1}{h_1} &= -\frac{S}{h_2}\,(\Delta p_\ell(k) - \Delta p_{\ell-1}(k)) \\
&\quad - \frac{S}{h_2}\,(\bar{p}(\ell) - \bar{p}(\ell-1)) \\
&\quad - \frac{\lambda c^2}{2dS}\left(\frac{\bar{q}^2}{\bar{p}(\ell)} + \frac{2\bar{q}}{\bar{p}(\ell)}\Delta q_\ell(k) - \frac{\bar{q}^2}{\bar{p}(\ell)^2}\Delta p_\ell(k)\right), \\
(\Delta p_\ell(k+1) - \Delta p_\ell(k))\,\frac{1}{h_1} &= -\frac{c^2}{Sh_2}\,(\Delta q_\ell(k) - \Delta q_{\ell-1}(k)).
\end{aligned} \tag{9}$$

Associating some terms, we obtain:

$$\begin{aligned}
\Delta q_\ell(k+1) &= \Delta q_\ell(k)\left(1 - \frac{\lambda c^2 h_1}{\not{2}dS}\frac{\not{2}\bar{q}}{\bar{p}(\ell)}\right) + \Delta p_\ell(k)\left(-\frac{Sh_1}{h_2} + \frac{\lambda c^2 h_1}{2dS}\frac{\bar{q}^2}{\bar{p}(\ell)^2}\right) \\
&\quad + \frac{Sh_1}{h_2}\Delta p_{\ell-1}(k) - \frac{Sh_1}{h_2}\,(\bar{p}(\ell) - \bar{p}(\ell-1)) - \frac{\lambda c^2 h_1}{2dS}\frac{\bar{q}^2}{\bar{p}(\ell)}, \\
\Delta p_\ell(k+1) &= \Delta p_\ell(k) - \frac{c^2 h_1}{Sh_2}\Delta q_\ell(k) + \frac{c^2 h_1}{Sh_2}\Delta q_{\ell-1}(k).
\end{aligned} \tag{10}$$

We define:

$$\beta := \frac{Sh_1}{h_2}, \tag{11}$$

$$\xi\,(\ell) := \frac{\lambda c^2 h_1}{2dS}\frac{\bar{q}^2}{\bar{p}(\ell)} = \xi\,(\bar{q}, \bar{p}\,(\ell)), \tag{12}$$

$$\rho := \frac{c^2 h_1}{Sh_2}, \tag{13}$$

$$\gamma\,(\ell) := \left(\frac{\xi(\ell)}{\bar{p}(\ell)} - \beta\right) = \gamma\,(\bar{q}, \bar{p}\,(\ell)), \tag{14}$$

$$\alpha\,(\ell) := \left(1 - \frac{2\xi\,(\ell)}{\bar{q}}\right) = \alpha\,(\bar{q}, \bar{p}\,(\ell)). \tag{15}$$

That is:

$$\begin{aligned}
\Delta q_\ell(k+1) &= \Delta q_\ell(k)\alpha\,(\ell) + \Delta p_\ell(k)\gamma\,(\ell) + \beta\Delta p_{\ell-1}(k) \\
&\quad - \beta\,(\bar{p}(\ell) - \bar{p}(\ell-1)) - \xi(\ell), \\
\Delta p_\ell(k+1) &= \Delta p_\ell(k) - \rho\Delta q_\ell(k) + \rho\Delta q_{\ell-1}(k).
\end{aligned} \tag{16}$$

If we define:

$$x_1(k,\ell) = \Delta q_\ell(k), \tag{17}$$

$$x_2(k,\ell) = \Delta p_\ell(k), \tag{18}$$

hence:

$$\begin{aligned}
x_1(k+1,\ell) &= x_1(k,\ell)\alpha\,(\ell) + x_2(k,\ell)\gamma\,(\ell) + \beta x_2(k,\ell-1) \\
&\quad - \beta\,(\bar{p}(\ell) - \bar{p}(\ell-1)) - \xi(\ell), \\
x_2(k+1,\ell) &= x_2(k,\ell) - \rho x_1(k,\ell) + \rho x_1(k,\ell-1).
\end{aligned} \tag{19}$$

Further, we define:

$$F_1(\bar{q}, \bar{p}(\ell-1)) := \beta \bar{p}(\ell-1), \tag{20}$$

$$F_2(\bar{q}, \bar{p}(\ell)) := -\beta \bar{p}(\ell) - \xi(\ell) \tag{21}$$

and consequently:

$$\begin{aligned} x_1(k+1,\ell) &= x_1(k,\ell)\alpha(\ell) + x_2(k,\ell)\gamma(\ell) + \beta x_2(k,\ell-1) \\ &\quad + F_1(\bar{q}, \bar{p}(\ell-1)) + F_2(\bar{q}, \bar{p}(\ell)), \\ x_2(k+1,\ell) &= x_2(k,\ell) - \rho x_1(k,\ell) + \rho x_1(k,\ell-1). \end{aligned} \tag{22}$$

Now, we write model (22) in matricial form:

$$\begin{pmatrix} x_1(k+1,\ell) \\ x_2(k+1,\ell) \end{pmatrix} = \begin{pmatrix} \alpha(\ell) & \gamma(\ell) \\ -\rho & 1 \end{pmatrix} \begin{pmatrix} x_1(k,\ell) \\ x_2(k,\ell) \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ \rho & 0 \end{pmatrix} \begin{pmatrix} x_1(k,\ell-1) \\ x_2(k,\ell-1) \end{pmatrix} + \begin{pmatrix} F_1(\bar{q}, \bar{p}(\ell-1)) + F_2(\bar{q}, \bar{p}(\ell)) \\ 0 \end{pmatrix}.$$

We further define:

$$x_\ell(k) := \begin{pmatrix} x_1(k,\ell) \\ x_2(k,\ell) \end{pmatrix}, \tag{23}$$

$$A(\ell) := \begin{pmatrix} \alpha(\ell) & \gamma(\ell) \\ -\rho & 1 \end{pmatrix}, \tag{24}$$

$$B_0 := \begin{pmatrix} 0 & \beta \\ \rho & 0 \end{pmatrix}, \tag{25}$$

$$\mathbb{F}_{\ell-1,\ell} := \begin{pmatrix} F_1(\bar{q}, \bar{p}(\ell-1)) + F_2(\bar{q}, \bar{p}(\ell)) \\ 0 \end{pmatrix}. \tag{26}$$

and finally, we can write:

$$x_\ell(k+1) = A(\ell)x_\ell(k) + B_0 x_{\ell-1}(k) + \mathbb{F}_{\ell-1,\ell}. \tag{27}$$

To this model we add initial and boundary conditions:

$$x_\ell(0) = \phi(\ell) = \phi_\ell, \qquad \ell = 0, 1, \ldots, L, \tag{28}$$

$$x_0(k) = x(k,0) = v(k), \quad k = 0, 1, \ldots, T. \tag{29}$$

ϕ_ℓ can be interpreted as the most convenient regime of operation along the pipeline, i.e., gas pressure and mass flow need to be kept at some desirable levels at every point of the pipeline. These initialisation values are the operational levels $(\bar{q}, \bar{p}(x))$ considered in the linearisation of the PDE system (3). The initial conditions, $v(k)$, can be seen as the set inlet pumping regime along the operation period.

Since we have $\mathbb{F}_{\ell-1,\ell} = f(\phi_{\ell-1}, \phi_\ell)$, we assume a linearisation for $F_1(\ell-1)$ and $F_2(\ell)$, that is $\mathbb{F}_{\ell-1,\ell} = B_{\ell-1}\phi_{\ell-1} + B_\ell \phi_\ell$.

Considering that C^{-1} exists, (27) can be written as:

$$x_\ell(k+1) = A(\ell)x_\ell(k) + B_0 C^{-1} y_{\ell-1}(k) + B_{\ell-1}\phi_{\ell-1} + B_\ell \phi_\ell \tag{30}$$

Defining $\tilde{B}_0 := B_0 C^{-1}$, we end up with the pipeline represented by a model of type (1), where the boundary conditions drive the system and the state of the system at every pass depends on the profile at the previous pass, specially at closed instants:

$$x_\ell(k+1) = A(\ell)x_\ell(k) + \tilde{B}_0 y_{\ell-1}(k) + B_{\ell-1}\phi_{\ell-1} + B_\ell \phi_\ell, \tag{31}$$

$$y_\ell(k) = Cx_\ell(k), \tag{32}$$

$$x_\ell(0) = \phi_\ell, \tag{33}$$

$$x_0(k) = v(k). \tag{34}$$

The compact support of ϕ_ℓ is $\mathbb{L} = \{0, 1, \ldots, L\}$, L is the length of the pipeline, and the compact support of $v(k)$ is $\mathbb{K} = \{0, 1, \ldots, T\}$, T is the gas period of operation. The compact support of $x_\ell(k)$, $y_\ell(k)$ and $u_\ell(k)$ is also $\mathbb{K} \times \mathbb{L}$.

Every space discretisation point of the pipeline is the pass-ℓ, $\ell = 0, 1, \ldots$, and $\alpha = T$ is the pass length. $x_\ell(k), y_\ell(k) \in \mathbb{R}^2$ and matrices $A, B_0, C, B \in \mathbb{R}^{2\times 2}$. The time steps (every hour) are indexed by $k \in [0, \alpha] \cap \mathbb{Z}$ and the pass (every sensor point) are indexed by $\ell \in [0, L] \cap \mathbb{Z}$. Then $\mathcal{I} = [0, T] \times [0, L] \cap \mathbb{Z} \times \mathbb{Z}$ and $x_{(\cdot)}(\cdot), y_{(\cdot)}(\cdot) \in \ell^{2,2}(\mathcal{I})$.

In fact, because we use a very simple approximation of the time derivative, we obtain a model much simpler than model (1) [see model (1) in Cichy et al. 2013] where coefficients $B_{pk} = \begin{cases} 0, & p \neq k \\ B_0, & p = k \end{cases}$, $D_{pk} = 0$, $p = 0, \ldots, \alpha - 1$, and $D = 0$.

3 Boundary control of the system

In this section we start by defining what we understand by boundary control of the network. In what follows in this section, we consider the output matrix as the identity matrix in equation (32).

Matrix C is defined according to the instrumentation of the network. That is, we consider a state to be observable whenever it can be measured. Common practice in gas networks is to install mass flow and pressure sensors at both pipeline ends, but only pressure sensors are installed along the pipeline. If we make the discretisation points to coincide with the location of the metering devices and consider that mass flows can always be calculated from pressures using a simulator, we can assume all the states to be observable and therefore $C = I_2$, i.e., the identity matrix.

Definition 1 (*Boundary control by pass initial data*) System (31)–(34) is completely pass boundary controllable on pass-$\ell_0, \ell_0 + 1, \ldots, \ell_0 + L = \ell_1$ if for any initial conditions, $\phi_0 = v(0), v(1), \ldots, v(\alpha)$ (the inlet pumping regime), and any given terminal pass, $x_f(k)$, with compact support $\mathbb{K} = \{0, 1, \ldots, \alpha\}$, there exists a sequence of boundary data, ϕ_ℓ, with compact support $\mathbb{L} = \{\ell_0, \ell_0 + 1, \ldots, \ell_1\}$, such that $x_{\ell_1}(k) = x_f(k)$ in the whole $\mathbb{K}$.

Theorem 1 *System* (31)–(34) *is completely pass boundary controllable on* $\ell \in \mathbb{L}$ *if and only if the grammian matrix* Γ_L *is positive definite, with*

$$\Gamma_L := \sum_{s=1}^{L-1} (\mathcal{A} \otimes B_0)^{s-1} \left((\mathcal{A} \otimes B_0)\mathbb{B}_2 + \mathbb{B}_1\right)\left((\mathcal{A} \otimes B_0)\mathbb{B}_2 + \mathbb{B}_1\right)^T \left(B_0^T \otimes \mathcal{A}^T\right)^{s-1} \tag{35}$$

with $\mathcal{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ I_2 & 0 & \cdots & 0 & 0 & 0 \\ A(\ell) & I_2 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & \\ A^{\alpha-3}(\ell) & A^{\alpha-4}(\ell) & \cdots & I_2 & 0 & 0 \\ A^{\alpha-2}(\ell) & A^{\alpha-3}(\ell) & \cdots & A(\ell) & I_2 & 0 \end{pmatrix} \in \mathbb{R}^{2\alpha\times 2\alpha}$ *and* $A(\ell)$ *and* B_0 *are defined in* (24)–(25).

The i*th row of matrix* $\mathbb{B}_1$ *is defined as* $\mathbb{B}_{i1} := A^{i-1}(\ell)B_0 + \sum_{s=1}^{i-1} A^{s-1}(\ell)B_{\ell-1}$ *and the* i*th row of matrix* $\mathbb{B}_2$ *is defined as* $\mathbb{B}_{i2} := A^{i}(\ell) + \sum_{s=1}^{i} A^{s-1}(\ell)B_{\ell}$*,* $i = 1, 2, \ldots, \alpha$*, and* $\mathbb{B}_1, \mathbb{B}_2 \in \mathbb{R}^{2\alpha\times 2}$.

Proof Consider model (31)–(34). For notation simplicity sake, in what follows we always write A instead of $A(\ell)$.

Consider the state equation of the repetitive process (31) at every instant of the pass, after assuming $C = I_2$:

$$x_\ell(k+1) = Ax_\ell(k) + B_0 x_{\ell-1}(k) + B_{\ell-1}\phi_{\ell-1} + B_\ell\phi_\ell, \quad k = 0, 1, \ldots, \alpha.$$

In matrix form, this model may be written as:

$$\begin{pmatrix} x_\ell(1) \\ x_\ell(2) \\ x_\ell(3) \\ \vdots \\ x_\ell(k) \\ \vdots \\ x_\ell(\alpha) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ B_0 & 0 & \cdots & 0 & 0 & 0 \\ AB_0 & B_0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & \\ A^{k-2}B_0 & A^{k-3}B_0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & \\ A^{\alpha-2}B_0 & A^{\alpha-3}B_0 & \cdots & AB_0 & B_0 & 0 \end{pmatrix} \begin{pmatrix} x_{\ell-1}(1) \\ x_{\ell-1}(2) \\ x_{\ell-1}(3) \\ \vdots \\ x_{\ell-1}(k) \\ \vdots \\ x_{\ell-1}(\alpha) \end{pmatrix}$$
$$+ \begin{pmatrix} B_0 + B_{\ell-1} & A + B_\ell \\ AB_0 + (A+I)B_{\ell-1} & A^2 + (A+I)\,B_\ell \\ A^2B_0 + (A^2+A+I)B_{\ell-1} & A^3 + \left(A^2+A+I\right)B_\ell \\ \vdots & \vdots \\ A^{k-1}B_0 + (A^{k-1}+\cdots+A+I)B_{\ell-1} & A^k + \left(A^{k-1}+\cdots+A+I\right)B_\ell \\ \vdots & \vdots \\ A^{\alpha-1}B_0 + (A^{\alpha-1}+\cdots+A+I)B_{\ell-1} & A^\alpha + \left(A^{\alpha-1}+\cdots+A+I\right)B_\ell \end{pmatrix} \begin{pmatrix} \phi_{\ell-1} \\ \phi_\ell \end{pmatrix}. \tag{36}$$

Next, we define some compact notation: $X_\ell := \left(x_\ell(1), x_\ell(2), \ldots, x_\ell(\alpha)\right)^T \in \mathbb{R}^{2\times\alpha}$ and in particular $X_0 = v$ and $v = \left(v(1), v(2), \ldots, v(\alpha)\right)^T$. Hence

$$\begin{aligned} X_\ell &= (\mathcal{A} \otimes B_0)\,X_{\ell-1} + \mathbb{B}_1\phi_{\ell-1} + \mathbb{B}_2\phi_\ell, \\ X_0 &= v. \end{aligned} \tag{37}$$

$\mathbb{L} = \{0, 1, 2, \ldots, L\}$ is the compact support of X_ℓ and ϕ_ℓ, where $\mathbb{B}_1, \mathbb{B}_2$ are the first and second column of the input matrix of (36), respectively.

In alternative, we can write a non-recursive representation for the solution that depends solely on the initial and boundary conditions. Thus:

$$\begin{aligned} X_\ell &= (\mathcal{A}\otimes B_0)^\ell\, v + (\mathcal{A}\otimes B_0)^{\ell-1}\,\mathbb{B}_1\phi_0 \\ &\quad + \sum_{s=1}^{\ell-1} (\mathcal{A}\otimes B_0)^{s-1}\left((\mathcal{A}\otimes B_0)\,\mathbb{B}_2 + \mathbb{B}_1\right)\phi_{\ell-s} + \mathbb{B}_2\phi_\ell \end{aligned} \tag{38}$$

with the same compact support $\mathbb{L}$ of X_ℓ.

Using this representation, we prove the controllability of the system by boundary data. Consider

$$\begin{aligned} \tilde{X}_\ell &= X_\ell - (\mathcal{A}\otimes B_0)^\ell\, v - (\mathcal{A}\otimes B_0)^{\ell-1}\,\mathbb{B}_1\phi_0 - \mathbb{B}_2\phi_\ell \\ &= \sum_{s=1}^{\ell-1} (\mathcal{A}\otimes B_0)^{s-1}\left((\mathcal{A}\otimes B_0)\,\mathbb{B}_2 + \mathbb{B}_1\right)\phi_{\ell-s}. \end{aligned} \tag{39}$$

Define $\Phi_{s-1} := (\mathcal{A}\otimes B_0)^{s-1}\left((\mathcal{A}\otimes B_0)\,\mathbb{B}_2 + \mathbb{B}_1\right)$ and

$$\tilde{X}_\ell = \sum_{s=1}^{\ell-1} \Phi_{s-1}\phi_{\ell-s}. \tag{40}$$

Define also $\phi_{\ell-s} := \Phi_{s-1}^T\mathcal{L}_\ell$ and then (39) can be written as:

$$\tilde{X}_\ell = \sum_{s=1}^{\ell-1} \Phi_{s-1}\Phi_{s-1}^T\mathcal{L}_\ell. \tag{41}$$

In (41) consider $\ell = L$ and hence:

$$\tilde{X}_L = \sum_{s=1}^{L-1} \Phi_{s-1}\Phi_{s-1}^T\mathcal{L}_L. \tag{42}$$

Define the grammian $\Gamma_L := \sum_{s=1}^{L-1}\Phi_{s-1}\Phi_{s-1}^T$, from what we have $\tilde{X}_L = \Gamma_L\mathcal{L}_L$. This means that if the grammian Γ_L is positive definite, then its inverse exists and we can write

$$\mathcal{L}_L = \Gamma_L^{-1}\tilde{X}_L,$$

which implies that we have a representation for the boundary conditions to steer the system to $\tilde{X}_L$ (and also to X_L):

$$\phi_{L-s} = \Phi_{s-1}^T\Gamma_L^{-1}\tilde{X}_L,\, s = 1,\ldots,L.$$

That is, the system is controllable.

On the other hand, if system (31)–(34) is pass controllable then there exists, for any given X_L (or equivalently $\tilde{X}_L$), a sequence of boundary data $\phi_{L-s}, s = 1, 2, \ldots, L$, such that $\tilde{X}_L = \sum_{s=1}^{L-1}\Phi_{s-1}\phi_{L-s}$.

Let us assume that $\tilde{X}_L \neq 0$ and is in the kernel of $\Gamma_L \geq 0$, that is $\Gamma_L\tilde{X}_L = 0$. Hence:

$$\begin{aligned} \tilde{X}_L^T\Gamma_L\tilde{X}_L = \tilde{X}_L^T\sum_{s=1}^{L-1}\Phi_{s-1}\Phi_{s-1}^T\tilde{X}_L &= \sum_{s=1}^{L-1}\left|\tilde{X}_L^T\Phi_{s-1}\right|^2 = 0 \\ \Longrightarrow \quad \tilde{X}_L^T\Phi_{s-1} = 0,\, s = 1,\ldots,L. \end{aligned}$$

Using (40) we write: $\left|\tilde{X}_L^T\right|^2 = \tilde{X}_L^T\left(\sum_{s=1}^{L-1}\Phi_{s-1}\phi_{L-s}\right)^T = 0$, that is $\tilde{X}_L^T = 0$.

This is a contradiction! This contradiction proves that $\Gamma_L > 0$. □

Remark 1 Notice that if $\{\phi_\ell\}_{\ell\in\mathbb{N}} \in \ell^{2,n}$ and $\|\mathcal{A}\otimes B_0\|_2 < 1$ the solution also exists on an infinitely long pipe.

Proof Considering the norm of representation (38), where $\ell \to \infty$. Then, from our assumption, we can guarantee convergence of the series. If there exists $M \geq 0$ such that

$$\|((\mathcal{A}\otimes B_0)\,\mathbb{B}_2 + \mathbb{B}_1)\,\phi_{\ell-s}\|_2 \leq M \quad \text{for all } \ell, s \in \mathbb{N}.$$

then it follows that $|X_\ell|_2 \leq \|\mathcal{A}\otimes B_0\|_2^\ell\,|v|_2 + \dfrac{M}{1-\|\mathcal{A}\otimes B_0\|_2}$.

This proves that $\{X_\ell\}_{\ell\in\mathbb{N}} \in \ell^{2,n(\alpha)}$. □

Another possibility for boundary control of system (31)–(34) is to steer it with data at the initial pass, that is, by choosing appropriately $x_0(0) = v(0), x_0(1) = v(1), \ldots, x_0(\alpha) = v(\alpha)$, while keeping the boundary data fixed. From the operational point of view this makes a lot of sense and corresponds to choosing the most adequate inlet pumping regime to steer the system to a required state.

Definition 2 *(Boundary control through initial data)* System (31)–(34) is completely pass controllable by initial data if for any boundary conditions ϕ_ℓ, whose compact support is $\mathbb{L}$, and any given terminal pass $x_f(k)$, whose compact support is $\mathbb{K}$, there exists a sequence $v(0), v(1), \ldots, v(\alpha)$ such that $x_L = x_f$, for every k.

A criterion for this type of controllability is stated in the following theorem:

Theorem 2 *System* (31)–(34) *is completely pass controllable by initial data if $\mathcal{A}\otimes B_0 \in \mathbb{R}^{2\alpha\times 2\alpha}$ has full rank, where $\mathcal{A}$ and B_0 are the matrices defined in Theorem 1.*

Proof We consider the state equation (31) for every point of the pass and write it in the matrix form, in a similar manner as we have done in the proof of Theorem 1. That is, we consider the representation of the solution (38):

$$\begin{aligned} X_\ell &= (\mathcal{A}\otimes B_0)^\ell\, v + (\mathcal{A}\otimes B_0)^{\ell-1}\,\mathbb{B}_1\phi_0 \\ &\quad + \sum_{s=1}^{\ell-1} (\mathcal{A}\otimes B_0)^{s-1}\,((\mathcal{A}\otimes B_0)\,\mathbb{B}_2 + \mathbb{B}_1)\,\phi_{\ell-s} + \mathbb{B}_2\phi_\ell. \end{aligned}$$

Hence

$$\begin{aligned} \tilde{X}_\ell &= X_\ell - (\mathcal{A}\otimes B_0)^{\ell-1}\,\mathbb{B}_1\phi_0 \\ &\quad - \sum_{s=1}^{\ell-1} (\mathcal{A}\otimes B_0)^{s-1}\,((\mathcal{A}\otimes B_0)\,\mathbb{B}_2 + \mathbb{B}_1)\,\phi_{\ell-s} - \mathbb{B}_2\phi_\ell \\ &= (\mathcal{A}\otimes B_0)^\ell\, v \end{aligned}$$

with $\ell = L$, we have $\tilde{X}_L = (\mathcal{A}\otimes B_0)^L\, v$, that is, $v = (\mathcal{A}\otimes B_0)^{-L}\,\tilde{X}_L$ provided that the inverse of the matrix $(\mathcal{A}\otimes B_0)$ exists. Whenever this happens (i.e., the matrix is of full rank), v represented in this manner steers the system to the required state. □

Remark 2 Reachability is not treated in this work since, from the operational point of view, it does not make sense to go back to the inlet of the pipeline nor to go back in time.

4 Observability of the system

Consider the full model (31)–(34), that is a general output equation where $C \in \mathbb{R}^{2\times 2}$.

Definition 3 *(pass boundary observable)* System (31)–(34) is pass boundary observable in $\mathbb{L} = \{0, 1, \ldots, L\}$ if for all $\ell_1 \in \mathbb{L}$, and initial data $\phi_0 = v(0), v(1), \ldots, v(\alpha)$, whenever having two trajectories $x_\ell(k), \tilde{x}_\ell(k)$, whose compact support is $\mathbb{K} \times \mathbb{L}$, originated from the same boundary data, ϕ_ℓ, whose compact support is $\mathbb{L}$, then $Cx_\ell(k) = C\tilde{x}_\ell(k), 0 < \ell \leq \ell_1$, it follows necessarily that $x_\ell(k) = \tilde{x}_\ell(k)$.

Setting $\hat{x}_\ell(k) = x_\ell(k) - \tilde{x}_\ell(k)$, whose compact support is $\mathbb{K}\times\mathbb{L}$, then pass boundary observability is equivalent to the condition that $C\hat{x}_\ell(k) = 0, \implies \hat{x}_\ell(k) = 0, 0 < \ell \leq \ell_1$, where $\hat{x}_\ell(k)$ is any solution of the homogeneous equation, i.e., with $\phi_\ell = 0, 0 < \ell \leq \ell_1$, and $v(k) = 0$ for every k.

We obtain the following sufficient pass boundary observability criterion.

Theorem 3 *System (31)–(34) is pass boundary observable in $\mathbb{L}$ if matrix $\mathbb{C}\mathbb{B}_2$ is of full rank, where $\mathbb{C} = \mathrm{diag}(C, C, \ldots, C)$.*

Proof Consider the output equation (32) and use recursively (31):

$$
\begin{aligned}
y_\ell(k) = Cx_\ell(k) = & \, CA^{k-2}\tilde{B}_0 y_{\ell-1}(1) + \cdots + \\
& + CA\tilde{B}_0 y_{\ell-1}(k-2) + C\tilde{B}_0 y_{\ell-1}(k-1) \\
& + C\left(A^{k-1}\tilde{B}_0 C + \left(A^{k-1} + \cdots + I\right) B_{\ell-1}\right)\phi_{\ell-1} \\
& + C\left(A^{k} + \left(A^{k-2} + \cdots + I\right) B_{\ell}\right)\phi_{\ell},
\end{aligned}
$$

where $A := A(\ell)$ and compact support $\mathbb{K}\times\mathbb{L}$. In order to write this in matrix form, we define: $Y_\ell := \left(y_\ell(1) \ y_\ell(2) \ \ldots \ y_\ell(\alpha)\right)^T$ and hence:

$$
Y_\ell = \mathbb{C}\left(\mathcal{A} \otimes \tilde{B}_0\right) Y_{\ell-1} + \mathbb{C}\mathbb{B}_1\phi_{\ell-1} + \mathbb{C}\mathbb{B}_2\phi_\ell \tag{43}
$$

whose compact support is $\mathbb{L}$.

Now, we obtain a representation for the state vector such that (37) is a particular case when we consider $C = I_2$:

$$
X_\ell = \left(\mathcal{A} \otimes \tilde{B}_0\right)\mathbb{C}X_{\ell-1} + \bar{\mathbb{B}}_1\phi_{\ell-1} + \mathbb{B}_2\phi_\ell, \tag{44}
$$

where $\bar{\mathbb{B}}_{i1} := A^{i-1}(\ell)\tilde{B}_0 C + \sum_{s=1}^{i} A^{s-1}(\ell)B_{\ell-1}$ and the compact support of X_ℓ is $\mathbb{L}$.

We also obtain a representation that depends only on the initial and boundary conditions:

$$
\begin{aligned}
X_\ell = & \left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell} v + \left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell-1}\bar{\mathbb{B}}_1\phi_0 \\
& + \sum_{s=1}^{\ell-1}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell-(s+1)}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\mathbb{B}_2 + \bar{\mathbb{B}}_1\right]\phi_s + \mathbb{B}_2\phi_\ell.
\end{aligned} \tag{45}
$$

Substituting (45) in (43) in order to obtain a recursive representation for the output pass vector that depends only on initial and boundary conditions, hence:

$$
\begin{aligned}
Y_\ell = {} & \mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell} v + \mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell-1}\bar{\mathbb{B}}_1\phi_0 \\
& + \mathbb{C}\sum_{s=1}^{\ell-2}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell-(s+1)}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\mathbb{B}_2+\bar{\mathbb{B}}_1\right]\phi_s + \mathbb{B}_2\phi_\ell \\
& + \mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\mathbb{B}_2+\mathbb{B}_1\right]\phi_{\ell-1} + \mathbb{C}\mathbb{B}_2\phi_\ell
\end{aligned}
$$

Without loss of generality, we assume that $\mathbb{B}_1 \approx \bar{\mathbb{B}}_1$, since this does not matter for the proof. Then, we can write:

$$
\begin{aligned}
Y_\ell = {} & \mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell} v + \mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell-1}\bar{\mathbb{B}}_1\phi_0 \\
& + \mathbb{C}\sum_{s=1}^{\ell-1}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell-(s+1)}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\bar{\mathbb{B}}_2+\bar{\mathbb{B}}_1\right]\phi_s + \mathbb{B}_2\phi_\ell + \mathbb{C}\mathbb{B}_2\phi_\ell. \qquad (46)
\end{aligned}
$$

Consider two different trajectories where we fix the initial data. Hence:

$$
\begin{aligned}
\hat{Y}_\ell = {} & \mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell-1}\bar{\mathbb{B}}_1\left(\phi_0-\tilde{\phi}_0\right) \\
& + \mathbb{C}\sum_{s=1}^{\ell-1}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell-(s+1)}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\bar{\mathbb{B}}_2+\bar{\mathbb{B}}_1\right]\left(\phi_s-\tilde{\phi}_s\right) \\
& + \mathbb{C}\mathbb{B}_2\left(\phi_\ell-\tilde{\phi}_\ell\right),
\end{aligned}
$$

where ϕ_i, $\tilde{\phi}_i$ denote the boundary data for trajectory X_ℓ, $\tilde{X}_\ell$, respectively.

Setting $\hat{Y}_\ell = 0$, $\ell \in \mathbb{L}$, we obtain:

$$
\begin{aligned}
& \ell = 0 : 0 = \mathbb{C}\mathbb{B}_2\left(\phi_0-\tilde{\phi}_0\right) \implies \left(\phi_0-\tilde{\phi}_0\right) = 0, \text{ because } C\mathbb{B}_2 \text{ is of full rank} \\
& \ell = 1 : 0 = \mathbb{C}\bar{\mathbb{B}}_1\left(\phi_0-\tilde{\phi}_0\right) + \mathbb{C}\mathbb{B}_2\left(\phi_1-\tilde{\phi}_1\right) = \mathbb{C}\mathbb{B}_2\left(\phi_1-\tilde{\phi}_1\right) \\
& \qquad \implies \left(\phi_1-\tilde{\phi}_1\right) = 0 \\
& \ell = 2 : 0 = \mathbb{C}\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\bar{\mathbb{B}}_1\left(\phi_0-\tilde{\phi}_0\right) + \left[\mathbb{C}\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\mathbb{B}_2+\bar{\mathbb{B}}_1\right]\left(\phi_1-\tilde{\phi}_1\right) \\
& \qquad + \mathbb{C}\mathbb{B}_2\left(\phi_2-\tilde{\phi}_2\right) = \mathbb{C}\mathbb{B}_2\left(\phi_2-\tilde{\phi}_2\right) \implies \left(\phi_2-\tilde{\phi}_2\right) = 0 \\
& \qquad \vdots
\end{aligned}
$$

Continuing in this way, we obtain $\phi_\ell - \tilde{\phi}_\ell = 0$, for every ℓ in the compact support $\mathbb{L}$, and this yields $\hat{X}_\ell = 0$; from what we conclude that, at every pass-ℓ, the boundary data is uniquely defined by the measured output Y_ℓ. □

Definition 4 *(Initial pass observable)* System (31)–(34) is initial pass observable in $\mathbb{L}$ if for all boundary data ϕ_ℓ, with ℓ in compact support $\mathbb{L}$, whenever having two trajectories $x_\ell(k)$, $\tilde{x}_\ell(k)$, whose compact support is $\mathbb{K}\times\mathbb{L}$ and $\ell_1 \in \mathbb{L}$, originated from the same initial data $v(0), v(1), \ldots, v(\alpha)$, then $Cx_\ell(k) = C\tilde{x}_\ell(k)$ always implies $x_\ell(k) = \tilde{x}_\ell(k)$, $0 < \ell \leq \ell_1$.

Theorem 4 *System* (31)–(34) *is initial pass observable in* $\mathbb{L}$ *if matrix*

$$\mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell}$$

is of full rank.

Proof Recall the compact output representation (46) and consider two of these trajectories where the boundary conditions are the same for both trajectories. Hence:

$$\hat{Y}_\ell = \mathbb{C}\left(X_\ell - \tilde{X}_\ell\right) = \mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell}(v-\tilde{v}),$$

where v, $\tilde{v}$ denote the initial data for trajectory X_ℓ, $\tilde{X}_\ell$, respectively. Setting $\hat{Y}_\ell = 0$, we obtain $\mathbb{C}\left(X_\ell - \tilde{X}_\ell\right) = \mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell}(v-\tilde{v}) \implies v-\tilde{v}=0$ provided $\mathbb{C}\left[\left(\mathcal{A}\otimes\tilde{B}_0\right)\mathbb{C}\right]^{\ell}$ is invertible. □

5 Stability of the system

To state the gas pipeline as a repetitive process in operator terms, recall model (43):

$$Y_{\ell+1} = \mathbb{C}\left(\mathcal{A}\otimes\tilde{B}_0\right)Y_\ell + \mathbb{C}\mathbb{B}_1\phi_\ell + \mathbb{C}\mathbb{B}_2\phi_{\ell+1}.$$

Definition 5 Define $L_\alpha := \mathbb{C}\left(\mathcal{A}\otimes\tilde{B}_0\right)$ such that $L_\alpha \in \mathcal{B}(E_\alpha, E_\alpha)$, that is, L_α is a bounded linear operator mapping E_α into itself, where E_α is a Banach space, that is, the Hilbert space $\ell^{2,0}$ equipped with the usual vector norm in $\mathbb{R}^2$. Consider W_α a linear subspace of E_α. Thence, the system dynamics (43) is described by a linear recursion relation of the form:

$$Y_{\ell+1} = L_\alpha Y_\alpha + b_{\ell+1}, \tag{47}$$

where $Y_\ell \in E_\alpha$ is the pass profile at pass-ℓ and is defined in a compact support $\ell \in \mathbb{L}$. α is the pass length. $L_\alpha Y_\alpha$ is the contribution from pass-ℓ to pass-$(\ell+1)$. Moreover, we define the disturbance sequence $\{b_{\ell+1}\}_{\ell\geq 0}$, where $b_{\ell+1} := \mathbb{C}\mathbb{B}_1\phi_\ell + \mathbb{C}\mathbb{B}_2\phi_{\ell+1} \in W_\alpha$ and compact support $\mathbb{L}$; it represents the initial/boundary conditions and the system disturbances.

Definition 6 *(Asymptotic stability)* A linear repetitive process of type (47) is said to be asymptotically stable if $\exists\delta > 0$ such that, given any initial profile Y_0 and any strongly convergent disturbance sequence $\{b_\ell\}_{\ell\geq 1} \in W_\alpha$, the sequence $\{Y_\ell\}_{\ell\geq 1}$ generated by the disturbed process

$$Y_{\ell+1} = (L_\alpha + \gamma)\,Y_\alpha + b_{\ell+1}, \tag{48}$$

and whose compact support is $\mathbb{L}$, converges strongly to a limited profile $Y_\infty \in E_\alpha$ when $\|\gamma\| < \delta$.

In a gas pipeline, γ can be interpreted as the disturbances on the measurements at every pass-ℓ, enviable to be as small as possible. From which we may have $\|\gamma\| < \delta$, with δ a very small positive value.

Next, we look for stability criteria suitable for implementation for the gas RP model. To do this, we apply the general theory of stability in Rogers et al. (2007).

Theorem 5 *The linear repetitive process* (47) *is asymptotically stable if and only if* $\rho(L_\alpha) < 1$.

Proof See page 44 of Rogers et al. (2007, Chapter 2). □

Also, by manipulating the boundary conditions, ϕ_ℓ, the disturbance sequence $\{b_\ell\}_{\ell\geq 1}$ can always be driven to a stable regime of operation.

Theorem 6 *Suppose that the linear repetitive process (47) is asymptotically stable and $\{b_\ell\}_{\ell\geq 1}$ converges strongly to b_∞. Then the strong limit $Y_\infty := \lim_{\ell\to\infty} Y_\ell$ is designated the limit profile corresponding to $\{b_\ell\}_{\ell\geq 1}$. Furthermore, the limit profile corresponding to this disturbance sequence is the unique solution of the compact equation $Y_\infty = L_\alpha Y_\infty + b_\infty$. That is $Y_\infty = (I - L_\alpha)^{-1}\, b_\infty$.*

Moreover, the limit profile, Y_∞, is independent of the initial pass profile, Y_0, and the direction of approach to b_∞.

Proof See page 45 of Rogers et al. (2007, Chapter 2). □

It is also possible to interpret asymptotic stability in BIBO terms.

Definition 7 *(BIBO stability)* The linear repetitive process (47) is said to be BIBO stable, over the finite compact support $\mathbb{K}$ of $y_\ell(k)$, if $\exists\delta > 0$ such that, given any initial pass profile Y_0 and any disturbance sequence $\{b_\ell\}_{\ell\geq 1}$ bounded in norm $\big($i.e., $\{b_\ell\}_{\ell\geq 1} \leq c_1$ for some $c_1 \geq 0$ and $\forall \ell \geq 1\big)$, the output sequence generated by the disturbance process (48) is bounded in norm provided $\|\gamma\| \leq \delta$.

Remark 3 The equivalence between asymptotic stability and BIBO stability for this type of processes has been established in Theorem 2.1.9 of Rogers et al. (2007, Chapter 2).

We recall the output equation from model (1):

$$y_\ell(k) = Cx_\ell(k) + \sum_{p=0}^{\alpha} D_{pk}\, y_\ell(p)$$

and define $D_0 := \begin{pmatrix} D_{11} & D_{21} & \cdots & D_{(\alpha-1)1} \\ D_{12} & D_{22} & \cdots & D_{(\alpha-1)2} \\ \vdots & & & \\ D_{1\alpha} & D_{2\alpha} & \cdots & D_{(\alpha-1)\alpha} \end{pmatrix}$.

In order to write the output equation as an equivalent 1D system:

$$Y_{\ell+1} = CX_\ell + \begin{pmatrix} D_{01} \\ D_{02} \\ \vdots \\ D_{0\alpha} \end{pmatrix} \otimes (\mathbb{C} \otimes \phi_{\ell-1}) + D_0 Y_{\ell-1}. \tag{49}$$

Remark 4 A necessary and sufficient condition for stability equivalent to the spectral radius is $\rho\,(D_0) < 1$.

From model (32), we have that for gas networks D_0 is always zero, then we conclude that the gas pipeline system is asymptotically stable.

5.1 Stability along the pass

Asymptotic stability (or equivalently BIBO stability over the whole pass length) guarantees the existence of a limit profile. However, this pass profile may become unacceptable along the pass dynamics. In order to express stability along the pass of the repetitive process (31)–(34) in terms of the rate of approach to the limit profile, as the time horizon goes to infinity, $\alpha \to \infty$, we recall Lemma 2.2.1 from Rogers et al. (2007). This result states that, under asymptotic stability, the output sequence $\{y_\ell(k)\}$, whose compact support is $\mathbb{K} \times \mathbb{L}$, approaches the limit profile at a geometric rate governed by a scalar $\lambda := \left(1 + \frac{\delta}{\|L_\alpha\|}\right)^{-1} \in (0, 1)$. Being α the pass length, then the definition along the pass is expressed in terms of finite bounds on the scalars M_α and Λ_α, $\alpha \to \infty$. That is, the rate of approach of the output sequence to the limit profile has a guaranteed geometric upper bound independent of the pass length.

Definition 8 *(Stable along the pass length)* The linear repetitive process (31)–(34) is said to be stable along the pass if finite scalars exist, M_∞ and $\lambda_\infty \in (0, 1)$, independent of the pass length, which for each constant disturbance $b_{\ell+1} = b_\infty$ with compact support $\mathbb{L}$, ensure that the output sequence satisfies:

$$\|y_\ell(k) - y_\infty(k)\| \leq M_\infty \lambda_\infty^\ell \left\{ \left\| y_0(k) + \frac{\|b_\infty(k)\|}{1 - \lambda_\infty} \right\| \right\},$$

for the compact support of $y_\ell(k)$ to be $\mathbb{K} \times \mathbb{L}$.

Next, in order to state a more useful condition in the sense of being more appropriate to the development of conditions under which the stability holds, we recall a result that follows immediately from Theorem 2.2.4 of Rogers et al. (2007, Chapter 2).

Theorem 7 *Suppose the pair* $\left(A(\ell), \tilde{B}_0\right)$ *to be controllable and the pair* $(A(\ell), C)$ *to be observable. Then the linear repetitive process* (31)–(34) *is stable along the pass if and only if there exist real numbers* $\varepsilon > 0$ *and* $\lambda \in (0, 1)$ *such that for every choice of* $|z| \geq \lambda \rho \left(A(\ell) + \tilde{B}_0 z I_2 C\right) \leq 1 - \varepsilon$.

Proof The proof follows immediately from substituting the matrices of the gas model into the classical result stated on page 62 of Rogers et al. (2007, Chapter 2). □

To look for equivalent conditions to develop computer implementable stability tests, we define the forward shift operator along the pass as $x_\ell(k + 1) = z_1 x_\ell(k)$ as well as z_2 as the forward shift operator pass-to-pass applied to $y_\ell(k)$, i.e., $y_{\ell+1}(k) = z_2 y_\ell(k)$. Then, the process dynamics along the pass can be expressed as:

$$\begin{aligned} y_{\ell+1}(k) &= C\,(z_1 I - A(\ell+1))^{-1}\,\tilde{B}_0 y_\ell(k) \\ &\quad + \left(C\,(z_1 I - A(\ell+1))^{-1}\,B_\ell \;\; C\,(z_1 I - A(\ell+1))^{-1}\,B_{\ell+1}\right)\begin{pmatrix}\phi_\ell \\ \phi_{\ell+1}\end{pmatrix}, \end{aligned}$$

and defining $G\,(z_1) := C\,(z_1 I - A(\ell+1))^{-1}$ it becomes:

$$y_{\ell+1}(k) = G\,(z_1)\,\tilde{B}_0 y_\ell(k) + \left(G\,(z_1)\,B_\ell \;\; G\,(z_1)\,B_{\ell+1}\right)\begin{pmatrix}\phi_\ell \\ \phi_{\ell+1}\end{pmatrix}.$$

From the structure of $G\,(z_1)$, we obtain a definition of the inter pass characteristic polynomial:

$$\mathcal{C}\,(z_1, z_2) = \det\,(z_1 I - A(\ell+1))\,. \tag{50}$$

Remark 5 The finite length of the pass does not cause a problem in the application of the z_1 transform provided this is extended from α to ∞. We assume this to be the case.

Then a theorem equivalent to Theorem 7 can be formulated:

Theorem 8 *Suppose that*

(i) $\rho\,(A(\ell)) < 1$
(ii) all poles of the transfer function $G\,(z_1)$ *are inside of the unit circle.*

Then the linear repetitive process (31)–(34) *is stable along the pass.*

Proof See page of Rogers and Owens (1992). □

It is also possible to express stability along the pass in terms of the characteristic polynomial.

Theorem 9 *Suppose the pair* $\left(A(\ell), \tilde{B}_0\right)$ *to be controllable and the pair* $(A(\ell), C)$ *to be observable. Then the linear repetitive process* (31)–(34) *is stable along the pass if and only if its characteristic polynomial* (50) *satisfies* $\mathcal{C}\,(z_1, z_2) \neq 0 \in \mathbb{U}_C^2$, *with* $\mathbb{U}_C^2 = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}$ *as the closed bidisc.*

Proof See page of Rogers and Owens (1994). □

The conditions of Theorem 9 can easily be tested since they are straightforward, specially when aided by computer analysis. Also the last conditions can be tested by generating a Nyquist plot of $G\,(z_1)$.

6 Illustrating example

A gas pipeline section located near Sines, in the south of Portugal, cylindrical with a diameter of $d = 793$ mm, a length of $L = 35.58$ Km, and a roughness factor of $\lambda = 0.005$ mm is used to illustrated some of the results of this article. Also $c = 340$ m/s. Operational field measurements are the supply and offtake mass-flows as well as the pressures along the entire network. Then we calculated $\bar{q} = 60$ kg/s and $\bar{p} = 8 \times 10^6$ Pa from this data. This data has been supplied by REN-Gasodutos, Portugal.

Then, we substitute this data into formulas (11)–(15), obtaining $\beta = 0.4939$, $\xi(\ell) = 2.6564 \times 10^6 \dfrac{1}{\bar{p}(\ell)}$, $\rho = 2.3406 \times 10^5$, $\gamma(\ell) = 2.6564 \times 10^6 \dfrac{1}{\bar{p}(\ell)^2} - 0.4939$, $\alpha(\ell) = 1 - 8.8546 \times 10^4 \dfrac{1}{\bar{p}(\ell)}$case study. The linearisation mentioned before formula (30) is done graphically.

Then:

$$x_\ell(k+1) = \begin{pmatrix} 1 - 8.8546 \times 10^4 \dfrac{1}{\bar{p}(\ell)} & 2.6564 \times 10^6 \dfrac{1}{\bar{p}(\ell)^2} - 0.4939 \\ -2.3406 \times 10^5 & 1 \end{pmatrix} x_\ell(k) + \begin{pmatrix} 0 & 0.4939 \\ 2.3406 \times 10^5 & 0 \end{pmatrix} x_{\ell-1}(k) + \mathbb{F}_{\ell-1,\ell}.$$

7 Conclusions and future work

In this work we model the gas pipeline as a repetitive process with smoothing. The system is steered to any demanded operational level through an adequate initial pumping regime and/or operational boundary conditions. For this reason, we study boundary control of the system. In addition, the observability of the system is also studied and it is found this to be highly dependent on the network instrumentation. The system can always be made virtually observable by using advanced networks simulators. Based on classical 2D results, we also study asymptotic stability along the pass stability for the network, withdrawing inclusive some applicable criteria.

In the immediate future, we would like to reformulate the gas game described in (Azevedo Perdicoúlis and Jank 2012) using boundary control.

Furthermore, a study of the same problem using an operator approach needs to be done, as well as a study of the stability for boundary control. These ideas should be then extended to Nash games.

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