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LMI-based criterion for robust stability of 2-D discrete systems with interval time-varying delays employing quantisation / overflow nonlinearities

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Abstract This paper is concerned with the problem of global asymptotic stability of a class of nonlinear uncertain two-dimensional (2-D) discrete systems described by the Fornasini-Marchesini second local state-space model with time-varying state delays. The class of systems under investigation involves norm bounded parameter uncertainties, interval-like time-varying delays and various combinations of quantisation and overflow nonlinearities. A linear matrix inequality-based delay-dependent criterion for the global asymptotic stability of such systems is proposed. An example is given to illustrate the effectiveness of the proposed method.

Keywords Delayed system · Linear matrix inequality · Lyapunov stability · Nonlinear system · Two-dimensional system · Uncertain system

1 Introduction

Many physical systems or processes have natural multidimensional characteristics (Paszke et al. 2004). The most investigated systems are two-dimensional (2-D) systems due to their extensive applications in many areas such as geophysics, projective radiography (Mitra and Ekstrom 1978), image data processing and transmission (Bracewell 1995), thermal processes in chemical reactors, 2-D digital control systems (Kaczorek 1985), river pollution modeling (Fornasini 1991), grid based wireless sensor networks (Dewasurendra and Bauer 2008) and process of gas filtration (Bors and Walczak 2012).

Parametric uncertainties, which are intrinsic features of many physical systems, may lead to instability and poor performance of the system. Such uncertainties may arise due

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to the modeling errors, variations in system parameters and some ignored factors. Several publications relating to the stability of uncertain 2-D discrete systems have appeared (Dey and Kar 2011; Du and Xie 1999; Feng et al. 2010; Kar and Singh 2004; Lu 1994a, 1995; Paszke et al. 2004, 2006; Singh 2005; Wang and Liu 2003).

Delay is often present in many physical, industrial and engineering systems as a consequence of the finite capabilities of information processing and data transmission among various parts of the systems (Mahmoud 2000; Malek-Zavarei and Jamshidi 1987). There are many examples of practical 2-D systems containing inherent delays, such as discretisation time in discrete models describing delayed lattice differential equation (Huang et al. 2004) and partial difference equations (Chen and Fong 2006; Zhang and Tian 2004). The presence of delay may cause instability in the designed system. The problem of stability analysis of delayed systems has received much attention in past decades (Chen 2009, 2010a,b; Chen and Fong 2006, 2007, 2010; Feng et al. 2010; He et al. 2008; Huang and Feng 2010; Kandanvli and Kar 2010; Paszke et al. 2004, 2006; Peng and Guan 2009a; Peng and Guan, 2009b; Xu and Yu 2006, 2009a,b). According to dependence of delay, the available stability criteria for time-delay systems can be broadly classified into two categories: delay-independent and delay-dependent. Delay-independent stability criteria can be applied to check the stability of systems without knowing the sizes of the delays. On the other hand, delay-dependent stability criteria utilize the information about the delays in the system. Delay-dependent methods generally yield less conservative stability conditions (Chen 2010a,b; Chen and Fong 2007, 2010; Feng et al. 2010; He et al. 2008; Huang and Feng 2010; Kandanvli and Kar 2010; Paszke et al. 2006; Xu and Yu 2009a,b).

While implementing recursive discrete systems with fixed-point arithmetic, the finite register length of digital hardware or computer generates nonlinearities such as quantisation and overflow. The presence of such nonlinearities may cause instability in the designed system. The common types of overflow nonlinearities are saturation, zeroing, two's complement and triangular. Magnitude truncation, roundoff and value truncation are the frequently occurred quantisation nonlinearities. If the total number of quantisation steps is large or, in other words, the internal wordlength is sufficiently long, then it is usually assumed that quantisation and overflow effects are decoupled or noninteracting. Under this decoupling assumption, several researchers (Aboulnasr and Fahmy 1986; Aravena et al. 1990; Bauer and Jury 1990; Bose and Trautman 1992) have investigated the effects of quantisation in 2-D discrete systems without considering overflow effects, while others (Chen 2009, 2010a,b; Dey and Kar 2011; Du and Xie 1999; El-Agizi and Fahmy 1979; Hinamoto 1997; Kar 2008, 2010, 2012; Kar and Singh 1997, 1999, 2000, 2001b, 2005; Liu 1998; Liu and Michel 1994; Singh 2005; Tzafestas et al. 1992; Wang and Liu 2003; Xiao and Hill 1996) have studied the overflow phenomenon ignoring the effects of quantisation. However, the validity of decoupling assumption has also been queried by several researchers (Johnson and Sandberg 1995; Sim and Pang 1985). Since the practical discrete system operates in the simultaneous presence of both quantisation and overflow nonlinearities, the study of stability of discrete systems involving both types of nonlinearities is considered to be more realistic. A few publications have appeared on the combined effects of quantisation and overflow nonlinearities for 2-D systems (Bose 1995; Kar and Singh 2001a, 2004; Leclerc and Bauer 1994).

The design of a 2-D system so as to ensure the stability of the designed system is an interesting and challenging problem. During the past few decades, the stability properties of 2-D discrete systems described by the Fornasini–Marchesini second local state-space (FMSLSS) model (Fornasini and Marchesini 1978) have been studied extensively (Bhaya et al. 2001; Chen 2009, 2010a; Chen and Fong 2006, 2007; Dey and Kar 2011; Du and Xie 1999; Feng et al. 2010; Hinamoto 1993, 1997; Kar and Singh 1999, 2001a,b, 2004;

Liu 1998; Lu 1994a,b, 1995; Ooba 2000; Peng and Guan 2009a; Peng and Guan, 2009b; Singh 2005; Wang and Liu 2003; Xu et al. 2005; Xu and Yu 2009a,b). The problem of global asymptotic stability of 2-D systems described by Roesser (1975) model has also received a considerable attention (Anderson et al. 1986; Chen 2010b; El-Agizi and Fahmy 1979; Kar 2008, 2010, 2012; Kar and Singh 1997, 2000, 2005; Liu and Michel 1994; Tzafestas et al. 1992; Xiao and Hill 1996). The problem of robust stability and stabilisation of 2-D discrete FMSLSS state-delayed systems has been studied in Paszke et al. (2004); Feng et al. (2010). A 2-D filtering approach with H_2/H_∞ performance measure has been developed in Chen and Fong (2006, 2007); Peng and Guan, (2009b); Xu et al. (2005); Xu and Yu (2009a). The guaranteed cost control problem for 2-D discrete state-deayed systems in FMSLSS setting has been considered in Xu and Yu (2009b). In Chen (2010a,b), the stability properties of 2-D discrete systems with time-varying delays using saturation nonlinearities have been investigated.

The stability analysis of 2-D discrete systems in the simultaneous presence of quantisation, overflow, state delay, and parameter uncertainty in their physical models is an important and realistic problem. Since the characterisation of the evolution of nonlinear uncertain dynamical state-delayed systems as a deterministic set of state equations is a formidable task, the stability analysis of such systems is generally difficult. To the best of authors' knowledge, such problem has not been addressed so far in the literature.

This paper, therefore, deals with the problem of global asymptotic stability of a class of uncertain 2-D discrete systems with time-varying state-delays under the influence of various combinations of quantisation and overflow nonlinearities. Parametric uncertainties involved in the system are assumed to be norm bounded. The paper is organized as follows. In Sect. 2, we formulate the problem and recall some useful results. A linear matrix inequality (LMI)-based delay-dependent criterion for the global asymptotic stability of uncertain 2-D discrete systems described by the FMSLSS model with interval-like time-varying state-delays under various combinations of quantisation and overflow nonlinearities is established in Sect. 3. In Sect. 4, an example highlighting the usefulness of the proposed method is given. Finally, conclusions are made in Sect. 5.

2 Problem formulation and preliminaries

The following notations are used throughout the paper:

$\boldsymbol{R}^{n\times n}$	set of $n \times n$ real matrices
$\boldsymbol{R}^{n}$	set of $n\times 1$ real vectors
$\boldsymbol{Z}_{+}$	set of nonnegative integers
$\boldsymbol{I}$	identity matrix of appropriate dimension
$\boldsymbol{0}$	null matrix or null vector of appropriate dimension
$\boldsymbol{B}^{T}$	transpose of the matrix (or vector) $\boldsymbol{B}$
$\boldsymbol{B} > \boldsymbol{0}$	$\boldsymbol{B}$ is positive definite symmetric matrix
$\boldsymbol{B} \geq \boldsymbol{0}$	$\boldsymbol{B}$ is positive semidefinite symmetric matrix
$\boldsymbol{B} < \boldsymbol{0}$	$\boldsymbol{B}$ is negative definite symmetric matrix
$\Vert.\Vert$	any vector or matrix norm
$diag\ \{a_1, a_2, \ldots, a_n\}$	diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$
$\max\ \{v, w\}$	maximum value of scalars v and w
$\boldsymbol{O}(\cdot)$	overflow nonlinearities
$\boldsymbol{Q}(\cdot)$	quantisation nonlinearities
$\boldsymbol{f}(\cdot)$	composite nonlinear functions

We consider a class of uncertain 2-D discrete systems with interval time-varying delays described by the FMSLSS model under various combinations of quantisation and overflow nonlinearities and for the situation where quantisation occurs after summation only. Specifically, the system under consideration is given by

$$\begin{aligned} \boldsymbol{x}(i+1, j+1) &= \boldsymbol{O}\{\boldsymbol{Q}(\boldsymbol{y}(i, j))\} = \boldsymbol{f}(\boldsymbol{y}(i, j)) \\ &= [\, f_1(y_1(i, j))\ f_2(y_2(i, j))\ \ldots\ f_n(y_n(i, j))\,]^T, \end{aligned} \tag{1a}$$

$$\begin{aligned} \boldsymbol{y}(i, j) &= (\boldsymbol{A}_1 + \Delta\boldsymbol{A}_1)\boldsymbol{x}(i, j+1) + (\boldsymbol{A}_2 + \Delta\boldsymbol{A}_2)\boldsymbol{x}(i+1, j) \\ &\quad + (\boldsymbol{A}_{d_1} + \Delta\boldsymbol{A}_{d_1})\boldsymbol{x}(i-\alpha(i), j+1) + (\boldsymbol{A}_{d_2} + \Delta\boldsymbol{A}_{d_2})\boldsymbol{x}(i+1, j-\beta(j)) \\ &= [\, y_1(i, j)\ y_2(i, j)\ \ldots\ y_n(i, j)\,]^T, \end{aligned} \tag{1b}$$

where $i \in \boldsymbol{Z}_+$ and $j \in \boldsymbol{Z}_+$ are horizontal coordinate and vertical coordinate, respectively; $\boldsymbol{x}(i, j) \in \boldsymbol{R}^n$ is the local state vector; $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_{d_1}, \boldsymbol{A}_{d_2}$ are the known real constant $n \times n$ matrices; $\Delta\boldsymbol{A}_1$, $\Delta\boldsymbol{A}_2$, $\Delta\boldsymbol{A}_{d_1}$, $\Delta\boldsymbol{A}_{d_2}$ are the unknown real $n \times n$ matrices representing parametric uncertainties in the state matrices; $\alpha(i)$ and $\beta(j)$ are time-varying delays along horizontal direction and vertical direction, respectively. It is assumed that (Chen 2010a,b; Feng et al. 2010)

$$\alpha_l \le \alpha(i) \le \alpha_h, \quad \beta_l \le \beta(j) \le \beta_h, \tag{1c}$$

where α_l and β_l are constant nonnegative integers representing the lower delay bounds along horizontal and vertical directions, respectively; α_h and β_h are constant nonnegative integers representing the upper delay bounds along horizontal and vertical directions, respectively.

In the event of $\boldsymbol{Q}(\cdot)$ being either magnitude truncation or roundoff, $\boldsymbol{f}(\cdot)$ turn out to be confined to the sector $[k_o, k_q]$, i.e.,

$$f_k(0) = 0, \quad k_o y_k^2(i, j) \le f_k(y_k(i, j)) y_k(i, j) \le k_q y_k^2(i, j), \quad k = 1, 2, \ldots, n, \tag{2a}$$

where

$$k_q = \begin{cases} 1, & \text{for magnitude truncation} \\ 2, & \text{for roundoff} \end{cases}, \quad k_o = \begin{cases} 0, & \text{for zeroing or saturation} \\ -\frac{1}{3}, & \text{for triangular} \\ -1, & \text{for two's complement} \end{cases} \tag{2b}$$

The uncertainties are assumed to be of the form (Feng et al. 2010; Paszke et al. 2004)

$$\left[\, \Delta\boldsymbol{A}_1\ \Delta\boldsymbol{A}_2\ \Delta\boldsymbol{A}_{d_1}\ \Delta\boldsymbol{A}_{d_2} \,\right] = \left[\, \boldsymbol{HFE}_1\ \boldsymbol{HFE}_2\ \boldsymbol{HFE}_{d_1}\ \boldsymbol{HFE}_{d_2} \,\right], \tag{3a}$$

where $\boldsymbol{H}, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_{d_1}, \boldsymbol{E}_{d_2}$ are known real constant matrices with appropriate dimensions and $\boldsymbol{F}$ is an unknown real matrix satisfying

$$\boldsymbol{F}^T\boldsymbol{F} \le \boldsymbol{I}. \tag{3b}$$

It is assumed (Xu and Yu 2009a,b) that system (1) has a finite set of initial conditions, i.e., there exist two positive integers K and L such that

$$\begin{cases} \boldsymbol{x}(i, j) = \boldsymbol{0}, & \forall i \ge K, \quad j = -\beta_h, -\beta_h + 1, \ldots, 0, \\ \boldsymbol{x}(i, j) = \boldsymbol{0}, & \forall j \ge L, \quad i = -\alpha_h, -\alpha_h + 1, \ldots, 0. \end{cases} \tag{4}$$

Equations (1–4) represent a class of 2-D discrete uncertain state-delayed dynamical systems involving both quantisation and overflow nonlinearities. Examples of such systems are common in engineering and include 2-D discrete systems implemented in a finite register length, digital control systems with finite wordlength nonlinearities, models of various physical phenomena (e.g., compartmental systems, single carriageway traffic flow (Bhaya et al.

2001) etc.), various dynamical processes represented by the Darboux equation (Foda and Agathoklis 1992; Marszalek 1984; Tsai et al. 2002) and so on. A typical example of the system represented by (1–4) can be found in wireless sensor networks (Dewasurendra and Bauer 2008), where the delays induced by information transmission (from one node to its immediate neighbours) are actually time-varying. In such networks, the communication between nodes with zero delay is not possible. Presently, embedded wireless sensor platforms typically use 8-bit or 16-bit fixed-point microprocessors for data processing inside each node. Thus, the nonlinearities due to finite word length are inherently present in such systems.

The main objective of this paper is to develop a delay-dependent and LMI-based global asymptotic stability criterion for system (1–4) using Lyapunov approach. Motivated by the work of 1-D systems presented in He et al. (2008), in the proposed stability analysis, a new 2-D Lyapunov functional is employed. The forward difference of the Lyapunov functional is tightly bounded by making use of slack matrix variables. The proposed conditions are advantageous in terms of less conservativeness, which is achieved by avoiding the utilisation of bounding techniques on some cross product terms.

The following definition and lemma are needed in the proof of our main result.

Definition 1 Paszke et al. (2004) The system (1) is globally asymptotically stable if $\lim_{r\to\infty} X_r = 0$ with initial conditions (4), where $X_r = \sup\{\|\boldsymbol{x}(i,j)\| : i+j=r, \quad i,j \in \boldsymbol{Z}_+\}$.

Lemma 1 Xie et al. (1992) *Let $\boldsymbol{\Sigma}$, $\boldsymbol{\Gamma}$, $\boldsymbol{F}$, and $\boldsymbol{M}$ be real matrices of appropriate dimensions with $\boldsymbol{M}$ satisfying $\boldsymbol{M} = \boldsymbol{M}^T$ then*

$$\boldsymbol{M} + \boldsymbol{\Sigma F \Gamma} + \boldsymbol{\Gamma}^T \boldsymbol{F}^T \boldsymbol{\Sigma}^T < \boldsymbol{0} \tag{5}$$

for all $\boldsymbol{F}^T\boldsymbol{F} \leq \boldsymbol{I}$, if and only if there exists a positive scalar ε such that

$$\boldsymbol{M} + \varepsilon^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T + \varepsilon\boldsymbol{\Gamma}^T\boldsymbol{\Gamma} < \boldsymbol{0}. \tag{6}$$

3 Main result

The main result may be stated as follows.

Theorem 1 *For given nonnegative integers $\alpha_l, \alpha_h, \beta_l, \beta_h$ satisfying $0 < \alpha_l \leq \alpha_h$ and $0 < \beta_l \leq \beta_h$, the system described by (1–4) is globally asymptotically stable if there exist appropriately dimensioned matrices $\boldsymbol{P}_i = \boldsymbol{P}_i^T > \boldsymbol{0}$ $(i = 1, 2)$, $\boldsymbol{Q}_i = \boldsymbol{Q}_i^T > \boldsymbol{0}$ $(i = 1, 2, \ldots, 6)$, $\boldsymbol{Z}_i = \boldsymbol{Z}_i^T > \boldsymbol{0}$ $(i = 1, 2, 3, 4)$, $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} \\ * & \boldsymbol{X}_{22} \end{bmatrix} \geq \boldsymbol{0}$, $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_{11} & \boldsymbol{Y}_{12} \\ * & \boldsymbol{Y}_{22} \end{bmatrix} \geq \boldsymbol{0}$, $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ * & \boldsymbol{C}_{22} \end{bmatrix} \geq \boldsymbol{0}$, $\boldsymbol{D} = \begin{bmatrix} \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ * & \boldsymbol{D}_{22} \end{bmatrix} \geq \boldsymbol{0}$, $\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_1 \\ \boldsymbol{N}_2 \end{bmatrix}$, $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_1 \\ \boldsymbol{M}_2 \end{bmatrix}$, $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix}$, $\boldsymbol{U} = \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{bmatrix}$, $\boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix}$, $\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{bmatrix}$, a diagonal matrix $\boldsymbol{G} = diag\,(g_1, g_2, \ldots, g_n) > \boldsymbol{0}$, and a positive scalar ε such that the following LMIs (7–9) hold*

$$
\begin{bmatrix}
\zeta_{11} & \varepsilon \boldsymbol{E}_1^T \boldsymbol{E}_2 & \zeta_{13} & \varepsilon \boldsymbol{E}_1^T \boldsymbol{E}_{\mathrm{d}_2} & \boldsymbol{S}_1 & -\boldsymbol{M}_1 & \boldsymbol{0} & \boldsymbol{0} & k_q \boldsymbol{A}_1^T \boldsymbol{G} - \bar{\boldsymbol{Z}} & -k_q\sqrt{-2k_o}\boldsymbol{A}_1^T \boldsymbol{G} & \boldsymbol{0} \\
* & \zeta_{22} & \varepsilon \boldsymbol{E}_2^T \boldsymbol{E}_{\mathrm{d}_1} & \zeta_{24} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W}_1 & -\boldsymbol{V}_1 & k_q \boldsymbol{A}_2^T \boldsymbol{G} - \hat{Z} & -k_q\sqrt{-2k_o}\boldsymbol{A}_2{}^T \boldsymbol{G} & \boldsymbol{0} \\
* & * & \zeta_{33} & \varepsilon \boldsymbol{E}_{\mathrm{d}_1}^T \boldsymbol{E}_{\mathrm{d}_2} & \boldsymbol{S}_2 & -\boldsymbol{M}_2 & \boldsymbol{0} & \boldsymbol{0} & k_q \boldsymbol{A}_{\mathrm{d}_1}^T \boldsymbol{G} & -k_q\sqrt{-2k_o}\boldsymbol{A}_{\mathrm{d}_1}^T \boldsymbol{G} & \boldsymbol{0} \\
* & * & * & \zeta_{44} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W}_2 & -\boldsymbol{V}_2 & k_q \boldsymbol{A}_{\mathrm{d}_2}^T \boldsymbol{G} & -k_q\sqrt{-2k_o}\boldsymbol{A}_{\mathrm{d}_2}^T \boldsymbol{G} & \boldsymbol{0} \\
* & * & * & * & -\boldsymbol{Q}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & -\boldsymbol{Q}_2 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & -\boldsymbol{Q}_4 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & * & -\boldsymbol{Q}_5 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & * & * & \begin{bmatrix}\{(k_o/2k_q) \\ -2\}\boldsymbol{G} + \boldsymbol{P}_1 \\ +\boldsymbol{P}_2 + \bar{\boldsymbol{Z}} + \hat{\boldsymbol{Z}}\end{bmatrix} & \sqrt{-k_o/2}\,\boldsymbol{G} & k_q \boldsymbol{G}\boldsymbol{H} \\
* & * & * & * & * & * & * & * & * & -k_q \boldsymbol{G} & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H} \\
* & * & * & * & * & * & * & * & * & * & -\varepsilon \boldsymbol{I}
\end{bmatrix}
< \boldsymbol{0} \tag{7}
$$

$$
\boldsymbol{\psi}_1 = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{N} \\ * & \boldsymbol{Z}_1 \end{bmatrix} \geq \boldsymbol{0}, \quad \boldsymbol{\psi}_2 = \begin{bmatrix} \boldsymbol{Y} & \boldsymbol{S} \\ * & \boldsymbol{Z}_2 \end{bmatrix} \geq \boldsymbol{0}, \quad \boldsymbol{\psi}_3 = \begin{bmatrix} \boldsymbol{X}+\boldsymbol{Y} & \boldsymbol{M} \\ * & \boldsymbol{Z}_1 + \boldsymbol{Z}_2 \end{bmatrix} \geq \boldsymbol{0} \tag{8}
$$

$$
\boldsymbol{\psi}_4 = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{U} \\ * & \boldsymbol{Z}_3 \end{bmatrix} \geq \boldsymbol{0}, \quad \boldsymbol{\psi}_5 = \begin{bmatrix} \boldsymbol{D} & \boldsymbol{W} \\ * & \boldsymbol{Z}_4 \end{bmatrix} \geq \boldsymbol{0}, \quad \boldsymbol{\psi}_6 = \begin{bmatrix} \boldsymbol{C}+\boldsymbol{D} & \boldsymbol{V} \\ * & \boldsymbol{Z}_3 + \boldsymbol{Z}_4 \end{bmatrix} \geq \boldsymbol{0} \tag{9}
$$

where

$$\bar{\boldsymbol{Z}} = \alpha_h \boldsymbol{Z}_1 + (\alpha_h - \alpha_l)\boldsymbol{Z}_2, \tag{10}$$

$$\hat{\boldsymbol{Z}} = \beta_h \boldsymbol{Z}_3 + (\beta_h - \beta_l)\boldsymbol{Z}_4, \tag{11}$$

$$\zeta_{11} = -\boldsymbol{P}_1 + \boldsymbol{Q}_1 + \boldsymbol{Q}_2 + (\alpha_h - \alpha_l + 1)\boldsymbol{Q}_3 + \bar{\boldsymbol{Z}} + \boldsymbol{N}_1 + \boldsymbol{N}_1^T + \alpha_h \boldsymbol{X}_{11} + (\alpha_h - \alpha_l)\boldsymbol{Y}_{11} + \varepsilon \boldsymbol{E}_1^T \boldsymbol{E}_1, \tag{12}$$

$$\zeta_{13} = -\boldsymbol{N}_1 + \boldsymbol{N}_2^T + \boldsymbol{M}_1 - \boldsymbol{S}_1 + \alpha_h \boldsymbol{X}_{12} + (\alpha_h - \alpha_l)\boldsymbol{Y}_{12} + \varepsilon \boldsymbol{E}_1^T \boldsymbol{E}_{\mathrm{d}_1}, \tag{13}$$

$$\zeta_{22} = -\boldsymbol{P}_2 + \boldsymbol{Q}_4 + \boldsymbol{Q}_5 + (\beta_h - \beta_l + 1)\boldsymbol{Q}_6 + \hat{\boldsymbol{Z}} + \boldsymbol{U}_1 + \boldsymbol{U}_1^T + \beta_h \boldsymbol{C}_{11} + (\beta_h - \beta_l)\boldsymbol{D}_{11} + \varepsilon \boldsymbol{E}_2^T \boldsymbol{E}_2, \tag{14}$$

$$\zeta_{24} = -\boldsymbol{U}_1 + \boldsymbol{U}_2^T + \boldsymbol{V}_1 - \boldsymbol{W}_1 + \beta_h \boldsymbol{C}_{12} + (\beta_h - \beta_l)\boldsymbol{D}_{12} + \varepsilon \boldsymbol{E}_2^T \boldsymbol{E}_{\mathrm{d}_2}, \tag{15}$$

$$\zeta_{33} = -\boldsymbol{Q}_3 - \boldsymbol{N}_2 - \boldsymbol{N}_2^T + \boldsymbol{M}_2 + \boldsymbol{M}_2^T - \boldsymbol{S}_2 - \boldsymbol{S}_2^T + \alpha_h \boldsymbol{X}_{22} + (\alpha_h - \alpha_l)\boldsymbol{Y}_{22} + \varepsilon \boldsymbol{E}_{\mathrm{d}_1}^T \boldsymbol{E}_{\mathrm{d}_1}, \tag{16}$$

$$\zeta_{44} = -\boldsymbol{Q}_6 - \boldsymbol{U}_2 - \boldsymbol{U}_2^T + \boldsymbol{V}_2 + \boldsymbol{V}_2^T - \boldsymbol{W}_2 - \boldsymbol{W}_2^T + \beta_h \boldsymbol{C}_{22} + (\beta_h - \beta_l)\boldsymbol{D}_{22} + \varepsilon \boldsymbol{E}_{\mathrm{d}_2}^T \boldsymbol{E}_{\mathrm{d}_2}. \tag{17}$$

Proof The proof of Theorem 1 is based on standard Lyapunov theory. It consists of several steps. First, we construct a 2-D Lyapunov functional $V(\boldsymbol{x}(i, j))$. Second, we estimate the forward difference of the Lyapunov functional along the trajectories of the system (1a), i.e., $\Delta V(\boldsymbol{x}(i, j))$ by introducing slack matrix variables and using sector based characterisation of the nonlinearities. Third, the condition under which $\Delta V(\boldsymbol{x}(i, j)) < 0$ is determined. Finally, Lemma 1 is used to remove the dependency of uncertain parameters in the stability condition. □

Define

$$\boldsymbol{\eta}_1(i, j+1) = \boldsymbol{x}(i+1, j+1) - \boldsymbol{x}(i, j+1) = \boldsymbol{f}(\boldsymbol{y}(i, j)) - \boldsymbol{x}(i, j+1), \tag{18}$$

$$\boldsymbol{\eta}_2(i+1, j) = \boldsymbol{x}(i+1, j+1) - \boldsymbol{x}(i+1, j) = \boldsymbol{f}(\boldsymbol{y}(i, j)) - \boldsymbol{x}(i+1, j) \tag{19}$$

and consider a quadratic 2-D Lyapunov function

$$V(\boldsymbol{x}(i, j)) = \bar{V}(\boldsymbol{x}(i, j)) + \tilde{V}(\boldsymbol{x}(i, j)), \tag{20a}$$

$$\bar{V}(\boldsymbol{x}(i, j)) = \sum_{k=1}^{4} \bar{V}_k(\boldsymbol{x}(i, j)), \quad \tilde{V}(\boldsymbol{x}(i, j)) = \sum_{k=1}^{4} \tilde{V}_k(\boldsymbol{x}(i, j)), \tag{20b}$$

where

$$\bar{V}_1(\boldsymbol{x}(i,j)) = \boldsymbol{x}^T(i,j)\boldsymbol{P}_1\boldsymbol{x}(i,j), \tag{21}$$

$$\begin{aligned}\bar{V}_2(\boldsymbol{x}(i,j)) &= \sum_{\theta=-\alpha_h+1}^{0}\sum_{l=\theta-1}^{-1}\boldsymbol{\eta}_1^T(i+l,j)Z_1\boldsymbol{\eta}_1(i+l,j)\\ &\quad+\sum_{\theta=-\alpha_h+1}^{-\alpha_l}\sum_{l=\theta-1}^{-1}\boldsymbol{\eta}_1^T(i+l,j)\boldsymbol{Z}_2\boldsymbol{\eta}_1(i+l,j),\end{aligned} \tag{22}$$

$$\begin{aligned}\bar{V}_3(\boldsymbol{x}(i,j)) &= \sum_{l=-\alpha_l}^{-1}\boldsymbol{x}^T(i+l,j)\boldsymbol{Q}_1\boldsymbol{x}(i+l,j)\\ &\quad+\sum_{l=-\alpha_h}^{-1}\boldsymbol{x}^T(i+l,j)\boldsymbol{Q}_2\boldsymbol{x}(i+l,j),\end{aligned} \tag{23}$$

$$\bar{V}_4(\boldsymbol{x}(i,j)) = \sum_{\theta=-\alpha_h+1}^{-\alpha_l+1}\sum_{l=\theta-1}^{-1}\boldsymbol{x}^T(i+l,j)\boldsymbol{Q}_3\boldsymbol{x}(i+l,j), \tag{24}$$

$$\tilde{V}_1(\boldsymbol{x}(i,j)) = \boldsymbol{x}^T(i,j)\boldsymbol{P}_2\boldsymbol{x}(i,j), \tag{25}$$

$$\begin{aligned}\tilde{V}_2(\boldsymbol{x}(i,j)) &= \sum_{\theta=-\beta_h+1}^{0}\sum_{l=\theta-1}^{-1}\boldsymbol{\eta}_2^T(i,j+l)\boldsymbol{Z}_3\boldsymbol{\eta}_2(i,j+l)\\ &\quad+\sum_{\theta=-\beta_h+1}^{-\beta_l}\sum_{l=\theta-1}^{-1}\boldsymbol{\eta}_2^T(i,j+l)\boldsymbol{Z}_4\boldsymbol{\eta}_2(i,j+l),\end{aligned} \tag{26}$$

$$\tilde{V}_3(\boldsymbol{x}(i,j)) = \sum_{l=-\beta_l}^{-1}\boldsymbol{x}^T(i,j+l)\boldsymbol{Q}_4\boldsymbol{x}(i,j+l)+\sum_{l=-\beta_h}^{-1}\boldsymbol{x}^T(i,j+l)\boldsymbol{Q}_5\boldsymbol{x}(i,j+l), \tag{27}$$

$$\tilde{V}_4(\boldsymbol{x}(i,j)) = \sum_{\theta=-\beta_h+1}^{-\beta_l+1}\sum_{l=\theta-1}^{-1}\boldsymbol{x}^T(i,j+l)\boldsymbol{Q}_6\boldsymbol{x}(i,j+l). \tag{28}$$

The 2-D Lyapunov function (20) may be treated as an extension of the 1-D Lyapunov function used in He et al. (2008) for output feedback control of linear 1-D discrete time systems with a time-varying delay.

Taking the forward difference of Lyapunov functional along the trajectories of system (1a), we obtain

$$\begin{aligned}\Delta V(\boldsymbol{x}(i,j)) &= \bar{V}(\boldsymbol{x}(i+1,j+1))-\bar{V}(\boldsymbol{x}(i,j+1))+\tilde{V}(\boldsymbol{x}(i+1,j+1))-\tilde{V}(\boldsymbol{x}(i+1,j))\\ &=\sum_{k=1}^{4}\Delta\bar{V}_k(\boldsymbol{x}(i,j))+\sum_{k=1}^{4}\Delta\tilde{V}_k(\boldsymbol{x}(i,j)),\end{aligned} \tag{29}$$

where

$$\begin{aligned}\Delta\bar{V}_1(\boldsymbol{x}(i,j)) &= \boldsymbol{x}^T(i+1,j+1)\boldsymbol{P}_1\boldsymbol{x}(i+1,j+1)-\boldsymbol{x}^T(i,j+1)\boldsymbol{P}_1\boldsymbol{x}(i,j+1)\\ &=\boldsymbol{f}^T(\boldsymbol{y}(i,j))\boldsymbol{P}_1\boldsymbol{f}(\boldsymbol{y}(i,j))-\boldsymbol{x}^T(i,j+1)\boldsymbol{P}_1\boldsymbol{x}(i,j+1),\end{aligned} \tag{30}$$

$$\begin{aligned}\Delta\bar{V}_2(\boldsymbol{x}(i,j)) &= \alpha_h\boldsymbol{\eta}_1^T(i,j+1)\boldsymbol{Z}_1\boldsymbol{\eta}_1(i,j+1)-\sum_{\theta=-\alpha_h}^{-1}\boldsymbol{\eta}_1^T(i+\theta,j+1)\boldsymbol{Z}_1\boldsymbol{\eta}_1(i+\theta,j+1)\\ &\quad+(\alpha_h-\alpha_l)\boldsymbol{\eta}_1^T(i,j+1)\boldsymbol{Z}_2\boldsymbol{\eta}_1(i,j+1)-\sum_{\theta=-\alpha_h}^{-\alpha_l-1}\boldsymbol{\eta}_1^T(i+\theta,j+1)\boldsymbol{Z}_2\boldsymbol{\eta}_1(i+\theta,j+1)\end{aligned}$$

$$
\begin{aligned}
&= \eta_1^T(i,j+1)\bar{Z}\eta_1(i,j+1) - \sum_{\theta=-\alpha_h}^{-\alpha(i)-1}\eta_1^T(i+\theta,j+1)(Z_1+Z_2)\eta_1(i+\theta,j+1)\\
&\quad - \sum_{\theta=-\alpha(i)}^{-1}\eta_1^T(i+\theta,j+1)Z_1\eta_1(i+\theta,j+1) - \sum_{\theta=-\alpha(i)}^{-\alpha_l-1}\eta_1^T(i+\theta,j+1)Z_2\eta_1(i+\theta,j+1), \quad (31)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\bar{V}_3(x(i,j)) &= x^T(i,j+1)(Q_1+Q_2)x(i,j+1) - x^T(i-\alpha_l,j+1)Q_1x(i-\alpha_l,j+1)\\
&\quad - x^T(i-\alpha_h,j+1)Q_2x(i-\alpha_h,j+1), \quad (32)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\bar{V}_4(x(i,j)) &= (\alpha_h-\alpha_l+1)x^T(i,j+1)Q_3x(i,j+1) - \sum_{\theta=-\alpha_h}^{-\alpha_l}x^T(i+\theta,j+1)Q_3x(i+\theta,j+1)\\
&\le (\alpha_h-\alpha_l+1)x^T(i,j+1)Q_3x(i,j+1) - x^T(i-\alpha(i),j+1)Q_3x(i-\alpha(i),j+1), \quad (33)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{V}_1(x(i,j)) &= x^T(i+1,j+1)P_2x(i+1,j+1) - x^T(i+1,j)P_2x(i+1,j)\\
&= f^T(y(i,j))P_2f(y(i,j)) - x^T(i+1,j)P_2x(i+1,j), \quad (34)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{V}_2(x(i,j)) &= \beta_h\eta_2^T(i+1,j)Z_3\eta_2(i+1,j) - \sum_{\theta=-\beta_h}^{-\beta_l-1}\eta_2^T(i+1,j+\theta)Z_4\eta_2(i+1,j+\theta)\\
&\quad - \sum_{\theta=-\beta_h}^{-1}\eta_2^T(i+1,j+\theta)Z_3\eta_2(i+1,j+\theta) + (\beta_h-\beta_l)\eta_2^T(i+1,j)Z_4\eta_2(i+1,j)\\
&= \eta_2^T(i+1,j)\hat{Z}\eta_2(i+1,j) - \sum_{\theta=-\beta_h}^{-\beta(j)-1}\eta_2^T(i+1,j+\theta)(Z_3+Z_4)\eta_2(i+1,j+\theta)\\
&\quad - \sum_{\theta=-\beta(j)}^{-\beta_l-1}\eta_2^T(i+1,j+\theta)Z_4\eta_2(i+1,j+\theta) - \sum_{\theta=-\beta(j)}^{-1}\eta_2^T(i+1,j+\theta)Z_3\eta_2(i+1,j+\theta), \quad (35)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{V}_3(x(i,j)) &= x^T(i+1,j)(Q_4+Q_5)x(i+1,j) - x^T(i+1,j-\beta_l)Q_4x(i+1,j-\beta_l)\\
&\quad - x^T(i+1,j-\beta_h)Q_5x(i+1,j-\beta_h), \quad (36)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{V}_4(x(i,j)) &= (\beta_h-\beta_l+1)x^T(i+1,j)Q_6x(i+1,j) - \sum_{\theta=-\beta_h}^{-\beta_l}x^T(i+1,j+\theta)Q_6x(i+1,j+\theta)\\
&\le (\beta_h-\beta_l+1)x^T(i+1,j)Q_6x(i+1,j) - x^T(i+1,j-\beta(j))Q_6x(i+1,j-\beta(j)). \quad (37)
\end{aligned}
$$

Next, we need to prove that $\Delta V(x(i,j)) < 0$. Before showing $\Delta V(x(i,j)) < 0$, we consider the following null products:

$$0 = 2\xi_1^T(i,j+1)N\left[x(i,j+1) - x(i-\alpha(i),j+1) - \sum_{l=-\alpha(i)}^{-1}\eta_1(i+l,j+1)\right], \quad (38)$$

$$0 = 2\xi_1^T(i,j+1)M\left[x(i-\alpha(i),j+1) - x(i-\alpha_h,j+1) - \sum_{l=-\alpha_h}^{-\alpha(i)-1}\eta_1(i+l,j+1)\right], \quad (39)$$

$$0 = 2\xi_1^T(i,j+1)S\left[x(i-\alpha_l,j+1) - x(i-\alpha(i),j+1) - \sum_{l=-\alpha(i)}^{-\alpha_l-1}\eta_1(i+l,j+1)\right], \quad (40)$$

$$0 = 2\xi_2^T(i+1,j)U\left[x(i+1,j) - x(i+1,j-\beta(j)) - \sum_{l=-\beta(j)}^{-1}\eta_2(i+1,j+l)\right], \quad (41)$$

$$0 = 2\xi_2^T(i+1,j)V\left[x(i+1,j-\beta(j)) - x(i+1,j-\beta_h) - \sum_{l=-\beta_h}^{-\beta(j)-1}\eta_2(i+1,j+l)\right], \quad (42)$$

$$0 = 2\xi_2^T(i+1,j)W\left[x(i+1,j-\beta_l) - x(i+1,j-\beta(j)) - \sum_{l=-\beta(j)}^{-\beta_l-1}\eta_2(i+1,j+l)\right], \quad (43)$$

where

$$\xi_1(i,j+1) = \left[x^T(i,j+1) \;\; x^T(i-\alpha(i),j+1)\right]^T, \quad (44)$$

$$\xi_2(i+1,j) = \left[x^T(i+1,j) \;\; x^T(i+1,j-\beta(j))\right]^T. \quad (45)$$

Relations (38–43) can be obtained directly by using (18) and (19). The following equations hold for any appropriately dimensioned positive semi-definite symmetric matrices $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{C}$ and $\boldsymbol{D}$:

$$0 = \alpha_h \boldsymbol{\xi}_1^T(i, j+1)\boldsymbol{X}\boldsymbol{\xi}_1(i, j+1) - \sum_{l=i-\alpha(i)}^{i-1} \boldsymbol{\xi}_1^T(i, j+1)\boldsymbol{X}\boldsymbol{\xi}_1(i, j+1) - \sum_{l=i-\alpha_h}^{i-\alpha(i)-1} \boldsymbol{\xi}_1^T(i, j+1)\boldsymbol{X}\boldsymbol{\xi}_1(i, j+1), \tag{46}$$

$$0 = (\alpha_h - \alpha_l)\boldsymbol{\xi}_1^T(i, j+1)\boldsymbol{Y}\boldsymbol{\xi}_1(i, j+1) - \sum_{l=i-\alpha(i)}^{i-\alpha_l-1} \boldsymbol{\xi}_1^T(i, j+1)\boldsymbol{Y}\boldsymbol{\xi}_1(i, j+1) - \sum_{l=i-\alpha_h}^{i-\alpha(i)-1} \boldsymbol{\xi}_1^T(i, j+1)\boldsymbol{Y}\boldsymbol{\xi}_1(i, j+1), \tag{47}$$

$$0 = \beta_h \boldsymbol{\xi}_2^T(i+1, j)\boldsymbol{C}\boldsymbol{\xi}_2(i+1, j) - \sum_{l=j-\beta(j)}^{j-1} \boldsymbol{\xi}_2^T(i+1, j)\boldsymbol{C}\boldsymbol{\xi}_2(i+1, j) - \sum_{l=j-\beta_h}^{j-\beta(j)-1} \boldsymbol{\xi}_2^T(i+1, j)\boldsymbol{C}\boldsymbol{\xi}_2(i+1, j), \tag{48}$$

$$0 = (\beta_h - \beta_l)\boldsymbol{\xi}_2^T(i+1, j)\boldsymbol{D}\boldsymbol{\xi}_2(i+1, j) - \sum_{l=j-\beta(j)}^{j-\beta_l-1} \boldsymbol{\xi}_2^T(i+1, j)\boldsymbol{D}\boldsymbol{\xi}_2(i+1, j) - \sum_{l=j-\beta_h}^{j-\beta(j)-1} \boldsymbol{\xi}_2^T(i+1, j)\boldsymbol{D}\boldsymbol{\xi}_2(i+1, j), \tag{49}$$

Adding the terms on the right sides of (38–43) and (46–49) to $\Delta V(\boldsymbol{x}(i, j))$ yields

$$\begin{aligned} \Delta V(\boldsymbol{x}(i, j)) \leq\ & \boldsymbol{\xi}_3^T(i, j)\boldsymbol{\mu}\boldsymbol{\xi}_3(i, j) - \sum_{l=i-\alpha(i)}^{i-1} \boldsymbol{\xi}_4^T(i, j, l)\boldsymbol{\psi}_1\boldsymbol{\xi}_4(i, j, l) \\ & - \sum_{l=i-\alpha(i)}^{i-\alpha_l-1} \boldsymbol{\xi}_4^T(i, j, l)\boldsymbol{\psi}_2\boldsymbol{\xi}_4(i, j, l) - \sum_{l=i-\alpha_h}^{i-\alpha(i)-1} \boldsymbol{\xi}_4^T(i, j, l)\boldsymbol{\psi}_3\boldsymbol{\xi}_4(i, j, l) \\ & - \sum_{l=j-\beta(j)}^{j-1} \boldsymbol{\xi}_5^T(i, j, l)\boldsymbol{\psi}_4\boldsymbol{\xi}_5(i, j, l) - \sum_{l=j-\beta(j)}^{j-\beta_l-1} \boldsymbol{\xi}_5^T(i, j, l)\boldsymbol{\psi}_5\boldsymbol{\xi}_5(i, j, l) \\ & - \sum_{l=j-\beta_h}^{j-\beta(j)-1} \boldsymbol{\xi}_5^T(i, j, l)\boldsymbol{\psi}_6\boldsymbol{\xi}_5(i, j, l) - 2\delta, \end{aligned} \tag{50a}$$

where

$$\begin{aligned} \delta &= \sum_{k=1}^{n} g_k[k_q y_k(i, j) - f_k(y_k(i, j))][f_k(y_k(i, j)) - k_o y_k(i, j)] \\ &= [k_q \boldsymbol{y}(i, j) - \boldsymbol{f}(\boldsymbol{y}(i, j))]^T \boldsymbol{G}[\boldsymbol{f}(\boldsymbol{y}(i, j)) - k_o \boldsymbol{y}(i, j)], \end{aligned} \tag{50b}$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{11} & -2k_qk_o\bar{A}_1^TG\bar{A}_2 & \mu_{13} & -2k_qk_o\bar{A}_1^TG\bar{A}_{d_2} & S_1 & -M_1 & 0 & 0 & (k_q+k_o)\bar{A}_1^TG-\bar{Z} \\ * & \mu_{22} & -2k_qk_o\bar{A}_2^TG\bar{A}_{d_1} & \mu_{24} & 0 & 0 & W_1 & -V_1 & (k_q+k_o)\bar{A}_2^TG-\hat{Z} \\ * & * & \mu_{33} & -2k_qk_o\bar{A}_{d_1}^TG\bar{A}_{d_2} & S_2 & -M_2 & 0 & 0 & (k_q+k_o)\bar{A}_{d_1}^TG \\ * & * & * & \mu_{44} & 0 & 0 & W_2 & -V_2 & (k_q+k_o)\bar{A}_{d_2}^TG \\ * & * & * & * & -Q_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q_4 & 0 & 0 \\ * & * & * & * & * & * & * & -Q_5 & 0 \\ * & * & * & * & * & * & * & * & \begin{matrix}(P_1+P_2\\ -2G+\bar{Z}+\hat{Z})\end{matrix} \end{bmatrix}, \tag{50c}$$

$$\mu_{11} = -P_1+Q_1+Q_2+(\alpha_h-\alpha_l+1)Q_3+\bar{Z}+N_1+N_1^T+\alpha_hX_{11}+(\alpha_h-\alpha_l)Y_{11}-2k_qk_o\bar{A}_1^TG\bar{A}_1, \tag{50d}$$

$$\mu_{13} = -N_1+N_2^T+M_1-S_1+\alpha_hX_{12}+(\alpha_h-\alpha_l)Y_{12}-2k_qk_o\bar{A}_1^TG\bar{A}_{d_1}, \tag{50e}$$

$$\mu_{22} = -P_2+Q_4+Q_5+(\beta_h-\beta_l+1)Q_6+\hat{Z}+U_1+U_1^T+\beta_hC_{11}+(\beta_h-\beta_l)D_{11}-2k_qk_o\bar{A}_2^TG\bar{A}_2, \tag{50f}$$

$$\mu_{24} = -U_1+U_2^T+V_1-W_1+\beta_hC_{12}+(\beta_h-\beta_l)D_{12}-2k_qk_o\bar{A}_2^TG\bar{A}_{d_2}, \tag{50g}$$

$$\mu_{33} = -Q_3-N_2-N_2^T+M_2+M_2^T-S_2-S_2^T+\alpha_hX_{22}+(\alpha_h-\alpha_l)Y_{22}-2k_qk_o\bar{A}_{d_1}^TG\bar{A}_{d_1}, \tag{50h}$$

$$\mu_{44} = -Q_6-U_2-U_2^T+V_2+V_2^T-W_2-W_2^T+\beta_hC_{22}+(\beta_h-\beta_l)D_{22}-2k_qk_o\bar{A}_{d_2}^TG\bar{A}_{d_2}, \tag{50i}$$

$$\bar{A}_1 = A_1+\Delta A_1,\quad \bar{A}_2 = A_2+\Delta A_2,\quad \bar{A}_{d_1} = A_{d_1}+\Delta A_{d_1},\quad \bar{A}_{d_2} = A_{d_2}+\Delta A_{d_2}, \tag{50j}$$

$$\xi_3(i,j) = \Big[x^T(i,j+1)\ x^T(i+1,j)\ x^T(i-\alpha(i),j+1)\ x^T(i+1,j-\beta(j)) \\ x^T(i-\alpha_l,j+1)\ x^T(i-\alpha_h,j+1)\ x^T(i+1,j-\beta_l)\ x^T(i+1,j-\beta_h)\ f^T(y(i,j))\Big]^T, \tag{50k}$$

$$\xi_4(i,j,l) = \Big[x^T(i,j+1)\ x^T(i-\alpha(i),j+1)\ \eta_1^T(l,j+1)\Big]^T, \tag{50l}$$

$$\xi_5(i,j,l) = \Big[x^T(i+1,j)\ x^T(i+1,j-\beta(j))\ \eta_2^T(i+1,l)\Big]^T. \tag{50m}$$

It may be observed that the identities (38–43) and (46–49) help to impose tighter bounding on $\Delta V(x(i,j))$. Further, for the nonlinearities given by (2), the quantity δ (see (50b)) is nonnegative (Kandanvli and Kar 2010; Kar and Singh 2001a, 2004).

From (50a), it is clear that $\Delta V(x(i,j)) < 0$ for $\xi_3(i,j) \neq 0$ if $\mu < 0$, (8) and (9) hold true and $\Delta V(x(i,j))= 0$ only when $\xi_3(i,j) = 0$.

To complete the proof of the theorem, it now remains to show that for any initial conditions satisfying (4), $x(i,j) \to 0$ as $i \to \infty$ and/or $j \to \infty$. It follows from (29) and $\Delta V(x(i,j)) \le 0$ that

$$\bar{V}(x(i+1,j+1)) + \tilde{V}(x(i+1,j+1)) \le \bar{V}(x(i,j+1)) + \tilde{V}(x(i+1,j)). \tag{51}$$

Let $D(r)$ denote the set defined by

$$D(r) \overset{\Delta}{=} \{(i,j) : i+j = r, \quad i \ge 0, \quad j \ge 0\}. \tag{52}$$

For any nonnegative integer $r \ge \max\{K, L\}$, it follows from (51) and the initial condition (4) that

$$\begin{aligned}\sum_{(i+j)\in D(r+1)} V(x(i,j)) &= \sum_{(i+j)\in D(r+1)} \Big[\bar{V}(x(i,j)) + \tilde{V}(x(i,j))\Big] \\ &= \bar{V}(x(r+1,0)) + \bar{V}(x(r,1)) + \bar{V}(x(r-1,2)) + \cdots + \bar{V}(x(2,r-1)) \\ &\quad + \bar{V}(x(1,r)) + \bar{V}(x(0,r+1)) + \tilde{V}(x(r+1,0)) + \tilde{V}(x(r,1)) \\ &\quad + \tilde{V}(x(r-1,2)) + \cdots + \tilde{V}(x(2,r-1)) + \tilde{V}(x(1,r)) + \tilde{V}(x(0,r+1)) \\ &\le \bar{V}(x(r+1,0)) + \bar{V}(x(r-1,1)) + \bar{V}(x(r-2,2)) + \cdots + \bar{V}(x(1,r-1)) \\ &\quad + \bar{V}(x(0,r)) + \bar{V}(x(0,r+1)) + \tilde{V}(x(r+1,0)) + \tilde{V}(x(r,0)) + \tilde{V}(x(r-1,1))\end{aligned}$$

$$
\begin{aligned}
&+\cdots+\tilde{V}(\boldsymbol{x}(2\,,\,r-2))+\tilde{V}(\boldsymbol{x}(1\,,\,r-1))+\tilde{V}(\boldsymbol{x}(0\,,\,r+1))+\bar{V}(\boldsymbol{x}(r,\,0))\\
&+\tilde{V}(\boldsymbol{x}(0,\,r))-\bar{V}(\boldsymbol{x}(r,\,0))-\tilde{V}(\boldsymbol{x}(0,\,r))\\
=&\sum_{(i+j)\in D(r)} V(\boldsymbol{x}(i,j)),
\end{aligned}
\tag{53}
$$

where the equality sign holds only when

$$
\sum_{(i+j)\in D(r)} V(\boldsymbol{x}(i,j))=0. \tag{54}
$$

In the above derivation, the fact that $\boldsymbol{x}(i,\ r+1)=\mathbf{0}, i=-\alpha_h,-\alpha_h+1,\ \ldots,0,\boldsymbol{x}(r+1,\ j)=\mathbf{0}, j=-\beta_h,-\beta_h+1,\ \ldots,0,\boldsymbol{x}(r,\ 0)=\mathbf{0},\boldsymbol{x}(0,\ r)=\mathbf{0}$, and the positive definiteness of the function $V(\boldsymbol{x}(i,j))$ have been used. Denote $h=\max\{\alpha_h,\ \beta_h\}$. Inequality (53) implies that the energy stored at all points along the $D(r+1)$ to $D(r-h+1)$ (i.e., all points in $D(r+1)\cup\cdots\cup D(r-h+1)$) is less than the energy stored at the points along the $D(r)$ to $D(r-h)$ (i.e., all points in $D(r)\cup\cdots\cup D(r-h)$). From (53), we obtain

$$
\lim_{r\to\infty}\sum_{(i+j)\in D(r)} V(\boldsymbol{x}(i,j))=0. \tag{55}
$$

Consequently,

$$
\lim_{i+j\to\infty}\|\boldsymbol{x}(i,j)\|=\mathbf{0}. \tag{56}
$$

Thus, by Definition 1, the conditions $\boldsymbol{\mu}<\mathbf{0}$, (8) and (9) are sufficient conditions for the global asymptotic stability of system (1–4). Using Schur's complement, the condition $\boldsymbol{\mu}<\mathbf{0}$ is equivalent to

$$
\begin{bmatrix}
\tilde{\boldsymbol{\mu}}_{11} & \mathbf{0} & \tilde{\boldsymbol{\mu}}_{13} & \mathbf{0} & \boldsymbol{S}_1 & -\boldsymbol{M}_1 & \mathbf{0} & \mathbf{0} & k_q\bar{\boldsymbol{A}}_1^T\boldsymbol{G}-\bar{\boldsymbol{Z}} & -k_q\sqrt{-2k_o}\bar{\boldsymbol{A}}_1^T\boldsymbol{G}\\
* & \tilde{\boldsymbol{\mu}}_{22} & \mathbf{0} & \tilde{\boldsymbol{\mu}}_{24} & \mathbf{0} & \mathbf{0} & \boldsymbol{W}_1 & -\boldsymbol{V}_1 & k_q\bar{\boldsymbol{A}}_2^T\boldsymbol{G}-\hat{\boldsymbol{Z}} & -k_q\sqrt{-2k_o}\bar{\boldsymbol{A}}_2^T\boldsymbol{G}\\
* & * & \tilde{\boldsymbol{\mu}}_{33} & \mathbf{0} & \boldsymbol{S}_2 & -\boldsymbol{M}_2 & \mathbf{0} & \mathbf{0} & k_q\bar{\boldsymbol{A}}_{\mathrm{d}_1}^T\boldsymbol{G} & -k_q\sqrt{-2k_o}\bar{\boldsymbol{A}}_{\mathrm{d}_1}^T\boldsymbol{G}\\
* & * & * & \tilde{\boldsymbol{\mu}}_{44} & \mathbf{0} & \mathbf{0} & \boldsymbol{W}_2 & -\boldsymbol{V}_2 & k_q\bar{\boldsymbol{A}}_{\mathrm{d}_2}^T\boldsymbol{G} & -k_q\sqrt{-2k_o}\bar{\boldsymbol{A}}_{\mathrm{d}_2}^T\boldsymbol{G}\\
* & * & * & * & -\boldsymbol{Q}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & -\boldsymbol{Q}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & -\boldsymbol{Q}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & -\boldsymbol{Q}_5 & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & \begin{matrix}[\{(k_o/2k_q)-2\}\boldsymbol{G}\\ +\boldsymbol{P}_1+\boldsymbol{P}_2+\bar{\boldsymbol{Z}}+\hat{\boldsymbol{Z}}]\end{matrix} & \sqrt{-k_o/2}\boldsymbol{G}\\
* & * & * & * & * & * & * & * & * & -k_q\boldsymbol{G}
\end{bmatrix} < \mathbf{0}, \tag{57a}
$$

where

$$
\tilde{\boldsymbol{\mu}}_{11}=-\boldsymbol{P}_1+\boldsymbol{Q}_1+\boldsymbol{Q}_2+(\alpha_h-\alpha_l+1)\boldsymbol{Q}_3+\bar{\boldsymbol{Z}}+\boldsymbol{N}_1+\boldsymbol{N}_1^T+\alpha_h\boldsymbol{X}_{11}+(\alpha_h-\alpha_l)\boldsymbol{Y}_{11}, \tag{57b}
$$

$$
\tilde{\boldsymbol{\mu}}_{13}=-\boldsymbol{N}_1+\boldsymbol{N}_2^T+\boldsymbol{M}_1-\boldsymbol{S}_1+\alpha_h\boldsymbol{X}_{12}+(\alpha_h-\alpha_l)\boldsymbol{Y}_{12}, \tag{57c}
$$

$$
\tilde{\boldsymbol{\mu}}_{22}=-\boldsymbol{P}_2+\boldsymbol{Q}_4+\boldsymbol{Q}_5+(\beta_h-\beta_l+1)\boldsymbol{Q}_6+\hat{\boldsymbol{Z}}+\boldsymbol{U}_1+\boldsymbol{U}_1^T+\beta_h\boldsymbol{C}_{11}+(\beta_h-\beta_l)\boldsymbol{D}_{11}, \tag{57d}
$$

$$
\tilde{\boldsymbol{\mu}}_{24}=-\boldsymbol{U}_1+\boldsymbol{U}_2^T+\boldsymbol{V}_1-\boldsymbol{W}_1+\beta_h\boldsymbol{C}_{12}+(\beta_h-\beta_l)\boldsymbol{D}_{12}, \tag{57e}
$$

$$
\tilde{\boldsymbol{\mu}}_{33}=-\boldsymbol{Q}_3-\boldsymbol{N}_2-\boldsymbol{N}_2^T+\boldsymbol{M}_2+\boldsymbol{M}_2^T-\boldsymbol{S}_2-\boldsymbol{S}_2^T+\alpha_h\boldsymbol{X}_{22}+(\alpha_h-\alpha_l)\boldsymbol{Y}_{22}, \tag{57f}
$$

$$\tilde{\mu}_{44} = -Q_6 - U_2 - U_2^T + V_2 + V_2^T - W_2 - W_2^T + \beta_h C_{22} + (\beta_h - \beta_l) D_{22}. \tag{57g}$$

Now, using (3a), (57) can be rewritten in the following form:

$$M + \bar{H} F \bar{E} + \bar{E}^T F^T \bar{H}^T < 0, \tag{58a}$$

where

$$M = \begin{bmatrix} \tilde{\mu}_{11} & 0 & \tilde{\mu}_{13} & 0 & S_1 & -M_1 & 0 & 0 & k_q A_1^T G - \bar{Z} & -k_q\sqrt{-2k_o} A_1^T G \\ * & \tilde{\mu}_{22} & 0 & \tilde{\mu}_{24} & 0 & 0 & W_1 & -V_1 & k_q A_2^T G - \hat{Z} & -k_q\sqrt{-2k_o} A_2^T G \\ * & * & \tilde{\mu}_{33} & 0 & S_2 & -M_2 & 0 & 0 & k_q A_{d_1}^T G & -k_q\sqrt{-2k_o} A_{d_1}^T G \\ * & * & * & \tilde{\mu}_{44} & 0 & 0 & W_2 & -V_2 & k_q A_{d_2}^T G & -k_q\sqrt{-2k_o} A_{d_2}^T G \\ * & * & * & * & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q_4 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -Q_5 & 0 & 0 \\ * & * & * & * & * & * & * & * & \begin{matrix}[\{(k_o/2k_q) - 2\}G \\ +P_1 + P_2 + \bar{Z} + \hat{Z}]\end{matrix} & \sqrt{-k_o/2}G \\ * & * & * & * & * & * & * & * & * & -k_q G \end{bmatrix}, \tag{58b}$$

$$\bar{H}^T = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0\,0\; k_q H^T G & -k_q\sqrt{-2k_o} H^T G \end{bmatrix}, \tag{58c}$$

$$\bar{E} = \begin{bmatrix} E_1\; E_2\; E_{d_1}\; E_{d_2}\; 0\,0\,0\,0\,0\,0 \end{bmatrix}. \tag{58d}$$

It may be noted that the matrix inequality (58a) along with (8) and (9) can, in principle, be solved using the MATLAB toolbox (Boyd et al. 1994; Gahinet et al. 1995). However, it would be tedious to satisfy (58a) for all admissible uncertainties. To avoid this problem, using Lemma 1, we express (58a) as

$$M + \varepsilon^{-1} \bar{H} \bar{H}^T + \varepsilon \bar{E}^T \bar{E} < 0, \tag{59}$$

where $\varepsilon > 0$. The equivalence of (59) and (7) follows trivially from Schur's complement. This completes the proof of Theorem 1.

Remark 1 Theorem 1 provides a delay-dependent global asymptotic stability condition for the 2-D system described by (1–4). The conditions in Theorem 1 are in LMI framework and can easily be solved using Matlab LMI Toolbox (Boyd et al. 1994; Gahinet et al. 1995).

Remark 2 In the proof of Theorem 1, the utilization of (38–43) and (46–49) enables to estimate the upper bound on the forward difference of the Lyapunov function in a better way, without a need for using bounding inequalities (Gao et al. 2004; Jiang et al. 2005; Liu et al. 2006; Zhu and Yang 2008). The matrix variables ***N, M, S, U, V, W,X,Y,C*** and ***D*** in Theorem 1 are the degrees of freedom which are beneficial in the reduction of conservatism of the stability condition. It may be mentioned that the method based on introducing such slack matrix variables has been extensively used in the derivation of delay-dependent results for time-delay systems (Feng et al. 2010; He et al. 2008; Kandanvli and Kar 2010).

Remark 3 Note that (7) is dependent on the values of k_o and k_q. For a given 2-D system described by (1–4), it may happen that the system is globally asymptotically stable for a set of values of k_o and k_q, while the system may show unstable behavior for another set of values of k_o and k_q. Theorem 1 may also be helpful to determine the values of k_o and k_q that would be required to guarantee the global asymptotic stability of the class of systems described by (1–4).

Remark 4 Theorem 1 solves the stability problem for time-varying delay in a range given by (1c) with $0 < \alpha_l \leq \alpha_h$ and $0 < \beta_l \leq \beta_h$. In practice, the time-varying delay often lies in a range, in which the lower bound is not zero. The results pertaining to the situation where $0 \leq \alpha_l \leq \alpha_h$ and $0 \leq \beta_l \leq \beta_h$ can be worked out by employing a 2-D Lyapunov function of the form (20–28) with $\boldsymbol{Z}_2 = \boldsymbol{Q}_1 = \boldsymbol{Z}_4 = \boldsymbol{Q}_4 = \boldsymbol{0}$ and setting $\boldsymbol{S} = \boldsymbol{W} = \boldsymbol{0}$, $\boldsymbol{Y} = \boldsymbol{D} = \boldsymbol{0}$.

For the constant delay case, the lower and upper delay bounds in (1c) become identical (i.e., $\alpha_l = \alpha_h = \alpha$ and $\beta_l = \beta_h = \beta$). In this case, as a direct consequence of Theorem 1, we have the following result.

Corollary 1 *The system (1–4) with $0 < \alpha(i) = \alpha$ and $0 < \beta(j) = \beta$ is globally asymptotically stable if there exist appropriately dimensioned matrices $\boldsymbol{P}_i = \boldsymbol{P}_i^T > \boldsymbol{0}$ $(i = 1, 2)$, $\boldsymbol{Q} = \boldsymbol{Q}^T > \boldsymbol{0}$, $\bar{\boldsymbol{Q}} = \bar{\boldsymbol{Q}}^T > \boldsymbol{0}$, a diagonal matrix $\boldsymbol{G} > \boldsymbol{0}$ and a positive scalar ε satisfying*

$$
\begin{bmatrix}
\begin{matrix}(-\boldsymbol{P}_1 + \boldsymbol{Q} \\ +\varepsilon \boldsymbol{E}_1^T \boldsymbol{E}_1)\end{matrix} & \varepsilon \boldsymbol{E}_1^T \boldsymbol{E}_2 & \varepsilon \boldsymbol{E}_1^T \boldsymbol{E}_{\mathrm{d}_1} & \varepsilon \boldsymbol{E}_1^T \boldsymbol{E}_{\mathrm{d}_2} & k_q \boldsymbol{A}_1^T \boldsymbol{G} & -k_q \sqrt{-2k_o} \boldsymbol{A}_1^T \boldsymbol{G} & \boldsymbol{0} \\
* & \begin{matrix}(-\boldsymbol{P}_2 + \bar{\boldsymbol{Q}} \\ +\varepsilon \boldsymbol{E}_2^T \boldsymbol{E}_2)\end{matrix} & \varepsilon \boldsymbol{E}_2^T \boldsymbol{E}_{\mathrm{d}_1} & \varepsilon \boldsymbol{E}_2^T \boldsymbol{E}_{\mathrm{d}_2} & k_q \boldsymbol{A}_2^T \boldsymbol{G} & -k_q \sqrt{-2k_o} \boldsymbol{A}_2^T \boldsymbol{G} & \boldsymbol{0} \\
* & * & -\boldsymbol{Q} + \varepsilon \boldsymbol{E}_{\mathrm{d}_1}^T \boldsymbol{E}_{\mathrm{d}_1} & \varepsilon \boldsymbol{E}_{\mathrm{d}_1}^T \boldsymbol{E}_{\mathrm{d}_2} & k_q \boldsymbol{A}_{\mathrm{d}_1}^T \boldsymbol{G} & -k_q \sqrt{-2k_o} \boldsymbol{A}_{\mathrm{d}_1}^T \boldsymbol{G} & \boldsymbol{0} \\
* & * & * & -\bar{\boldsymbol{Q}} + \varepsilon \boldsymbol{E}_{\mathrm{d}_2}^T \boldsymbol{E}_{\mathrm{d}_2} & k_q \boldsymbol{A}_{\mathrm{d}_2}^T \boldsymbol{G} & -k_q \sqrt{-2k_o} \boldsymbol{A}_{\mathrm{d}_2}^T \boldsymbol{G} & \boldsymbol{0} \\
* & * & * & * & \begin{matrix}[\{(k_o/2k_q) - 2\}\boldsymbol{G} \\ +\boldsymbol{P}_1 + \boldsymbol{P}_2]\end{matrix} & \sqrt{-k_o/2}\boldsymbol{G} & k_q \boldsymbol{G}\boldsymbol{H} \\
* & * & * & * & * & -k_q \boldsymbol{G} & -k_q \sqrt{-2k_o} \boldsymbol{G}\boldsymbol{H} \\
* & * & * & * & * & * & -\varepsilon \boldsymbol{I}
\end{bmatrix},
$$

$$< \boldsymbol{0}. \tag{60}$$

Proof Taking $\alpha_l = \alpha_h = \alpha$, $\beta_l = \beta_h = \beta$ and choosing

$$
\begin{aligned}
&\boldsymbol{Q}_1 = \sigma_1 \boldsymbol{I}, \quad \boldsymbol{Q}_2 = \sigma_2 \boldsymbol{I}, \quad \boldsymbol{Q}_3 = \boldsymbol{Q}, \quad \boldsymbol{Q}_4 = \sigma_3 \boldsymbol{I}, \quad \boldsymbol{Q}_5 = \sigma_4 \boldsymbol{I}, \quad \boldsymbol{Q}_6 = \bar{\boldsymbol{Q}}, \quad \boldsymbol{Z}_1 = \sigma_5 \boldsymbol{I}/\alpha, \\
&\boldsymbol{Z}_2 = \sigma_6 \boldsymbol{I}/\alpha, \quad \boldsymbol{Z}_3 = \sigma_7 \boldsymbol{I}/\beta, \quad \boldsymbol{Z}_4 = \sigma_8 \boldsymbol{I}/\beta, \boldsymbol{X}_{11} = \sigma_9 \boldsymbol{I}/\alpha, \boldsymbol{X}_{22} = \sigma_{10} \boldsymbol{I}/\alpha, \boldsymbol{Y}_{11} = \sigma_{11} \boldsymbol{I}/\alpha, \\
&\boldsymbol{Y}_{22} = \sigma_{12} \boldsymbol{I}/\alpha, \quad \boldsymbol{C}_{11} = \sigma_{13} \boldsymbol{I}/\beta, \quad \boldsymbol{C}_{22} = \sigma_{14} \boldsymbol{I}/\beta, \quad \boldsymbol{D}_{11} = \sigma_{15} \boldsymbol{I}/\beta, \quad \boldsymbol{D}_{22} = \sigma_{16} \boldsymbol{I}/\beta, \\
&\boldsymbol{X}_{12} = \boldsymbol{0}, \ \boldsymbol{Y}_{12} = \boldsymbol{0}, \ \boldsymbol{C}_{12} = \boldsymbol{0}, \ \boldsymbol{D}_{12} = \boldsymbol{0}, \ \boldsymbol{N} = \boldsymbol{0}, \ \boldsymbol{M} = \boldsymbol{0}, \ \boldsymbol{S} = \boldsymbol{0}, \ \boldsymbol{U} = \boldsymbol{0}, \ \boldsymbol{V} = \boldsymbol{0}, \ \boldsymbol{W} = \boldsymbol{0},
\end{aligned}
\tag{61}
$$

for sufficiently small positive scalars σ_i $(i = 1, 2, \ldots, 16)$, the conditions in Theorem 1 reduces to (60). This completes the proof. □

Remark 5 Corollary 1 provides a delay-independent condition for the global asymptotic stability of a class of 2-D state-delayed uncertain discrete systems involving both overflow and quantisation nonlinearities. It is worth mentioning that the limit cycle-free realizability condition provided by Corollary 1 pertaining to 2-D state-delayed uncertain FMSLSS model (under various combinations of overflow and quantisation) has not been mentioned, to the best of authors' knowledge, in any previous works.

Remark 6 A result analogous to Theorem 1 (or Corollary 1) also holds true for the other class of 2-D systems described by the Roesser (1975) model.

4 Illustrative example

In this section, we shall demonstrate the application of our proposed criterion (Theorem 1) for the stability analysis of thermal processes in chemical reactors, heat exchangers and pipe furnaces, which can be expressed in the following partial differential equation with time-delays (Xu and Yu 2009a,b)

$$\frac{\partial\theta(x,t)}{\partial x} = -\frac{\partial\theta(x,t)}{\partial t} - a_0\theta(x,t) - a_1\theta(x,t-\tau), \tag{62}$$

where $\theta(x,t)$ is the temperature at space $x \in [0, x_f]$ and time $t \in [0,\infty)$, τ is the time delay, a_0 and a_1 are real coefficients. Taking

$$\theta(i,j) = \theta(i\Delta x, j\Delta t) \tag{63}$$

and applying

$$\frac{\partial\theta(x,t)}{\partial x} \cong \frac{\theta(i,j)-\theta(i-1,j)}{\Delta x}, \quad \frac{\partial\theta(x,t)}{\partial t} \cong \frac{\theta(i,j+1)-\theta(i,j)}{\Delta t} \tag{64}$$

for both derivatives in (64), it is easy to verify that (62) can be expressed in the following discrete form:

$$\theta(i,j+1) = \left(1 - \frac{\Delta t}{\Delta x} - a_0\Delta t\right)\theta(i,j) + \frac{\Delta t}{\Delta x}\theta(i-1,j) - a_1\Delta t\theta(i,j-\beta(j)), \tag{65}$$

where $\beta(j) = \text{int}(\tau/\Delta t + 1)$, $\text{int}(\cdot)$ is the integer function.

By setting $\boldsymbol{x}^T(i,j) = \begin{bmatrix}\theta^T(i-1,j) & \theta^T(i,j)\end{bmatrix}$, (65) can be converted into the following FMSLSS model:

$$\begin{aligned}\boldsymbol{x}(i+1,j+1) = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\boldsymbol{x}(i,j+1) + \begin{bmatrix}0 & 0\\ \frac{\Delta t}{\Delta x} & \left(1-\frac{\Delta t}{\Delta x} - a_0\Delta t\right)\end{bmatrix}\boldsymbol{x}(i+1,j)\\ + \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\boldsymbol{x}(i-\alpha(i),j+1) + \begin{bmatrix}0 & 0\\ 0 & -a_1\Delta t\end{bmatrix}\boldsymbol{x}(i+1,j-\beta(j)).\end{aligned} \tag{66}$$

In the presence of nonlinearities and uncertainties, system (66) fits into the format of (1–4). Let $a_0 = 5, a_1 = 1.2, \Delta x = 0 \cdot 4, \Delta t = 0 \cdot 1, 3 \le \alpha(i) \le 7$ and $3 \le \beta(j) \le 7$. Assume that the present system is subjected to parameter uncertainties of the form (3) with $\boldsymbol{H} = \begin{bmatrix}0\\ 0.1\end{bmatrix}, \boldsymbol{E}_1 = \boldsymbol{E}_2 = \begin{bmatrix}0.01 & 0\end{bmatrix}$ and $\boldsymbol{E}_{d_1} = \boldsymbol{E}_{d_2} = \begin{bmatrix}0 & 0.01\end{bmatrix}$. Let the nonlinearities belong to the sector $[k_o, k_q] = [-1, 1]$ which includes two's complement, saturation, zeroing, triangular, magnitude truncation, combination of two's complement and magnitude truncation, combination of saturation and magnitude truncation, combination of zeroing and magnitude truncation, combination of triangular and magnitude truncation, etc. We wish to determine whether the system under consideration is globally asymptotically stable. Using the Matlab LMI Toolbox (Boyd et al. 1994; Gahinet et al. 1995) it turns out that (7–9) are feasible for the following values of unknown parameters

$$\boldsymbol{P}_1 = \begin{bmatrix} 1.6200 & -0.5808 \\ -0.5808 & 7.9670 \end{bmatrix}, \quad \boldsymbol{P}_2 = \begin{bmatrix} 3.0991 & 0.9011 \\ 0.9011 & 6.8801 \end{bmatrix}, \quad \boldsymbol{Q}_1 = \begin{bmatrix} 0.1787 & -0.0416 \\ -0.0416 & 0.2426 \end{bmatrix},$$
$$\boldsymbol{Q}_2 = \begin{bmatrix} 0.1819 & -0.0439 \\ -0.0439 & 0.2505 \end{bmatrix}, \quad \boldsymbol{Q}_3 = \begin{bmatrix} 0.0744 & -0.0372 \\ -0.0372 & 0.1420 \end{bmatrix}, \quad \boldsymbol{Q}_4 = \begin{bmatrix} 0.1753 & -0.0406 \\ -0.0406 & 0.2434 \end{bmatrix},$$
$$\boldsymbol{Q}_5 = \begin{bmatrix} 0.1782 & -0.0435 \\ -0.0435 & 0.2557 \end{bmatrix}, \quad \boldsymbol{Q}_6 = \begin{bmatrix} 0.0626 & -0.0540 \\ -0.0540 & 0.6891 \end{bmatrix}, \quad \boldsymbol{Z}_1 = \begin{bmatrix} 0.0101 & -0.0044 \\ -0.0044 & 0.0174 \end{bmatrix},$$
$$\boldsymbol{Z}_2 = \begin{bmatrix} 0.0244 & -0.0101 \\ -0.0101 & 0.0420 \end{bmatrix}, \quad \boldsymbol{Z}_3 = \begin{bmatrix} 0.0101 & -0.0040 \\ -0.0040 & 0.0187 \end{bmatrix}, \quad \boldsymbol{Z}_4 = \begin{bmatrix} 0.0245 & -0.0094 \\ -0.0094 & 0.0450 \end{bmatrix},$$
$$\boldsymbol{X} = \begin{bmatrix} 0.0360 & -0.0111 & -0.0007 & 0.0003 \\ -0.0111 & 0.0525 & 0.0003 & -0.0012 \\ -0.0007 & 0.0003 & 0.0073 & -0.0035 \\ 0.0003 & -0.0012 & -0.0035 & 0.0134 \end{bmatrix},$$
$$\boldsymbol{Y} = \begin{bmatrix} 0.0671 & -0.0191 & -0.0003 & 0.0001 \\ -0.0191 & 0.0958 & 0.0001 & -0.0005 \\ -0.0003 & 0.0001 & 0.0194 & -0.0080 \\ 0.0001 & -0.0005 & -0.0080 & 0.0336 \end{bmatrix},$$
$$\boldsymbol{C} = \begin{bmatrix} 0.0663 & 0.0200 & -0.0007 & 0.0315 \\ 0.0200 & 0.0833 & 0.0002 & 0.0305 \\ -0.0007 & 0.0002 & 0.0067 & -0.0044 \\ 0.0315 & 0.0305 & -0.0044 & 0.0328 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 0.1176 & 0.0314 & -0.0001 & 0.0535 \\ 0.0314 & 0.1445 & 0.0003 & 0.0522 \\ -0.0001 & 0.0003 & 0.0181 & -0.0093 \\ 0.0535 & 0.0522 & -0.0093 & 0.0643 \end{bmatrix},$$
$$\boldsymbol{N} = \begin{bmatrix} -0.0029 & 0.0012 \\ 0.0012 & -0.0049 \\ 0.0033 & -0.0014 \\ -0.0014 & 0.0056 \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} 0.0003 & -0.0001 \\ -0.0001 & 0.0004 \\ -0.0208 & 0.0083 \\ 0.0083 & -0.0354 \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} 0.00004 & -0.00004 \\ -0.00004 & 0.0001 \\ 0.0142 & -0.0050 \\ -0.0050 & 0.0229 \end{bmatrix},$$
$$\boldsymbol{U} = \begin{bmatrix} -0.0027 & 0.0011 \\ 0.0013 & -0.0051 \\ 0.0033 & -0.0013 \\ -0.0011 & 0.0058 \end{bmatrix}, \quad \boldsymbol{V} = \begin{bmatrix} -0.0003 & 0.0031 \\ -0.0005 & 0.0036 \\ -0.0206 & 0.0082 \\ 0.0076 & -0.0346 \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} 0.0003 & -0.0019 \\ 0.0002 & -0.0019 \\ 0.0144 & -0.0044 \\ -0.0044 & 0.0221 \end{bmatrix},$$
$$\boldsymbol{G} = \begin{bmatrix} 2.7003 & 0 \\ 0 & 8.0467 \end{bmatrix}, \quad \varepsilon = 2.2916. \tag{67}$$

Thus, according to Theorem 1, the 2-D system under consideration is globally asymptotically stable.

5 Concluding remarks

A delay-dependent criterion (Theorem 1) for the global asymptotic stability of uncertain 2-D discrete systems described by the FMSLSS model with interval-like time-varying state-delays under various combinations of quantisation and overflow nonlinearities is proposed. The proposed stability conditions are given in a numerically efficient LMI framework. An example demonstrating the effectiveness of the presented method is given. The 2-D results discussed in this paper can easily be extended to m-D ($m > 2$) systems.

It is known that the extension of the 1-D Lyapunov function to the 2-D case can be broadly classified into two different types: the 2-D Lyapunov function whose coefficients are constant (Agathoklis et al. 1989; Anderson et al. 1986; El-Agizi and Fahmy 1979; Xiao et al. 1997) and the 1-D Lyapunov function whose coefficients are functions of a complex variable (Agathoklis 1987; Fornasini and Marchesini 1980; Lu and Lee 1985; Sendaula 1986). The use of constant Lyapunov functions, in general, causes conservativeness of the stability results. The approach in Lu and Lee (1985) permits to reduce the problem of establishing stability of 2-D linear Roesser (1975) model to the existence of positive definite Hermitian solution of a 1-D Lyapunov equation with a complex parameter. However, one should expect noticeable complications to carry out the stability analysis of 2-D dynamics in presence of parameter uncertainties, time-varying delays, quantisation/overflow nonlinearities using frequency dependent Lyapunov functions. The strategy adopted in this paper is to exploit quadratic Lyapunov function with constant coefficients. The computation of the evolution of the Lyapunov function along the trajectories of the system leads to computationally efficient stability conditions. In this context, it may be mentioned that the 2-D constant coefficient Lyapunov functions have been widely used for the stability analysis of 2-D systems and many significant results have been obtained (Chen 2009, 2010a,b; Chen and Fong 2006, 2007; Du and Xie 1999; El-Agizi and Fahmy 1979; Feng et al. 2010; Hinamoto 1993, 1997; Kar 2008, 2010, 2012; Kar and Singh 1997, 1999, 2000, 2001a,b, 2004, 2005; Leclerc and Bauer 1994; Liu 1998; Liu and Michel 1994; Lu 1994a,b, 1995; Ooba 2000; Paszke et al. 2004, 2006; Peng and Guan 2009a; Peng and Guan, 2009b; Singh 2005; Tzafestas et al. 1992; Wang and Liu 2003; Xiao and Hill 1996; Xu et al. 2005; Xu and Yu 2006, 2009a,b). Using constant coefficient Lyapunov functions, it has been established in El-Agizi and Fahmy (1979) that normal form 2-D digital filters described by Roesser (1975) model are free from overflow oscillations. Normal form structures of digital filters represent an important class of systems having low sensitivities to tolerances in the coefficients and low output roundoff noise. The necessary and sufficient conditions for the asymptotic stability of the positive 2-D linear systems have been established successfully in Chu and Liu (2007); Kaczorek (2002, 2007); Kaczorek (2009) using constant coefficient Lyapunov functions.

It is expected that Theorem 1 can be applied to some useful classes of realistic 2-D systems as a global asymptotic stability test. The presented stability results can be improved further by making use of frequency dependent Lyapunov functions together with more precise characterisation of uncertainties, nonlinearities and delays. Further investigation is required to reduce the gap between ‘sufficiency’ and ‘necessity’ for a 2-D system to be globally asymptotically stable, which occurs in the present approach.

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