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The delay-range-dependent robust stability analysis for 2-D state-delayed systems with uncertainty

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Abstract This paper addresses the problems of delay-range-dependent stability and robust stability for uncertain two-dimensional (2-D) state-delayed systems in the Fornasini–Machesini second model, with the uncertainty assumed to be of norm bounded form. A generalized Lyapunov function candidate is introduced to prove the stability condition and some free-weighting matrices are used for less conservative conditions. The resulting stability and robust stability conditions in terms of linear matrix inequalities are delay-range-dependent. Some numerical examples are given to illustrate the method.

Keywords 2-D state-delayed systems · Robust stability · Linear matrix inequality · Delay-range-dependent

1 Introduction

As delay is encountered in many dynamic systems and is often a source of instability, much attention has been focused on the problem of stability analysis and controller design for one-dimensional (1-D) time-delay systems in the last decades, see e.g. (Zhu and Yang 2008; He et al. 2007; Zhang et al. 2007; Xu and Lam 2005; Xie et al. 2004). Two-dimensional (2-D) state-delayed systems have also been a topic of study for many years. The current available stability results for 2-D state-delayed systems fall into two groups: delay-independent stability conditions (Xu et al. 2007, 2008; Wu et al. 2007; Paszke et al. 2004) and delay-dependent ones (Paszke et al. 2006a,b; Xu and Yu 2009a; Peng and Guan 2009a,b; Xu and Yu 2009b; Chen and Fong 2007; Feng et al. 2010; Chen and Fong 2006). The former refers to the stability

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conditions which do not depend on delay; the latter contains information on the size of delay. Generally speaking, the delay-dependent stability condition is less conservative especially when the sizes of the delays are small.

Here are some existing delay-dependent results for 2-D state-delayed systems: Paszke et al. (2006a,b) considered the problems of delay-dependent robust stability of 2-D state-delayed linear systems in Roesser model; Chen and Fong (2006, 2007) discussed the delay-dependent H_∞ and robust H_∞ filtering for uncertain 2-D state-delayed systems in the FM second model; Peng and Guan (2009a,b) dealt with the delay-independent and delay-dependent output feedback H_∞ control and H_∞ filtering of 2-D discrete state-delayed systems in the FM second model; Xu and Yu (2009a,b) investigated the delay-dependent H_∞ control and guaranteed cost control for 2-D discrete state-delayed systems in the FM second model; Feng et al. (2010) discussed delay-dependent robust stability and stabilization of uncertain 2-D discrete systems with varying delays in the FM second model.

The existing delay-dependent conditions for 2-D systems in Paszke et al. (2006a,b); Xu and Yu (2009a); Peng and Guan (2009a,b); Xu and Yu (2009b); Chen and Fong (2007); Feng et al. (2010); Chen and Fong (2006) only dependent on the upper bound, that is, the range of delay considered in Paszke et al. (2006a,b); Xu and Yu (2009a); Peng and Guan (2009a,b); Xu and Yu (2009b); Chen and Fong (2007); Feng et al. (2010); Chen and Fong (2006) is from 0 to an upper bound. But actually the lower bound of delay in systems may not equal to 0. In this case, the existing delay-dependent stability conditions may be conservative. To improve this drawback, the delay-range-dependent stability and robust stability conditions are provided in this paper for 2-D discrete systems in the FM second model with delays.

By choosing a generalized Lyapunov function, this paper first presents a delay-range-dependent stability criterion for a nominal 2-D discrete state-delayed system described by the FM second model. The free-weighting matrix approach is adopted to lower the conservativeness of the delay-range-dependent stability condition, and an optimization procedure is used for computing the range of delays for which the system remains asymptotically stable. Then, the result is extended to the uncertain 2-D state-delayed systems with an unknown but norm-bounded parameter uncertainty. All the delay-range-dependent stability conditions are given in LMI. Two numerical examples are given to illustrate the effectiveness of the proposed method.

Throughout this paper, the zero matrix and the identity matrix with appropriate dimensions are denoted by 0 and I, respectively. The notation $X > 0\,(X \geq 0)$ represents that the matrix X is positive definite (semi-positive definite) for any real symmetric matrices X. Similarly, $X < 0\,(X \leq 0)$ denotes a real symmetric negative definite (semi-negative definite) matrix. And $*$ denotes the symmetric terms in symmetric matrix.

2 Preliminaries

Consider a 2-D discrete state-delayed system with uncertain parameters described by the following FM second model:

$$\begin{aligned} x(i+1, j+1) = &(A_1 + \Delta A_1)x(i+1, j) + (A_2 + \Delta A_2)x(i, j+1) \\ &+(A_{1d} + \Delta A_{1d})x(i+1, j-d_1) \\ &+(A_{2d} + \Delta A_{2d})x(i-d_2, j+1), \end{aligned} \tag{1}$$

where $x(i, j) \in \mathscr{R}^n$ is the state vector, the matrices A_1, A_2, A_{1d} and A_{2d} are known constant matrices; ΔA_1, ΔA_2, ΔA_{1d} and ΔA_{2d} are unknown matrices representing the parameter uncertainty in the system matrices and are assumed to be of the form

$$[\Delta A_1\ \Delta A_2\ \Delta A_{1d}\ \Delta A_{2d}] = DF(i,j)\,[E_1\ E_2\ E_{1d}\ E_{2d}], \tag{2}$$

where D, E_1, E_2, E_{1d} and E_{2d} are known real constant matrices and $F(i,j)$ is an unknown matrix satisfying

$$F^T(i,j)F(i,j) \le I. \tag{3}$$

The non-negative integers d_1 and d_2 are unknown but constant delays along the vertical direction and the horizontal direction, respectively, satisfying

$$h_{11} \le d_1 \le h_{12} < \infty;\ h_{21} \le d_2 \le h_{22} < \infty, \tag{4}$$

where h_{k2} and h_{k1} are non-negative integers with $0 \le h_{k1} < h_{k2} (k = 1, 2)$ and h_{k1} may not equal to 0. The boundary conditions for system (1) are specified as

$$\begin{cases} x(i,j) = \chi_{ij}, & \forall 0 \le i \le \mu_1;\ j = -h_{12}, -h_{12}+1, \cdots 1, 0, \\ x(i,j) = \phi_{ij}, & \forall 0 \le j \le \mu_2;\ i = -h_{22}, -h_{22}+1, \cdots 1, 0, \\ \chi_{00} = \phi_{00}, & \\ x(i,j) = 0, & \forall i > \mu_1;\ j = -h_{12}, -h_{12}+1, \cdots 1, 0, \\ x(i,j) = 0, & \forall j > \mu_2;\ i = -h_{22}, -h_{22}+1, \cdots 1, 0. \end{cases} \tag{5}$$

where μ_1 and μ_2 are given positive integers. To simplify the notation in the state-space model (1), define the following vector

$$x_{\alpha,\beta} = x(i+\alpha, j+\beta).$$

Then, system (1) can be rewritten as

$$x_{1,1} = (A_1+\Delta A_1)x_{1,0}+(A_2+\Delta A_2)x_{0,1}+(A_{1d}+\Delta A_{1d})x_{1,-d_1}+(A_{2d}+\Delta A_{2d})x_{-d_2,1}. \tag{6}$$

The following well-known results are used in the sequel.

Definition 1 (Paszke et al. 2004) Denote $X_r = \sup\{\|x(i,j)\| : i+j = r,\ i, j \in Z\}$. The 2-D discrete state-delayed systems (1) with any bounded boundary condition (5) is asymptotically stable if

$$\lim_{r\to\infty} X_r = 0.$$

Lemma 1 (Mahmoud 2000) (Schur complements) *For matrices* Σ_1, Σ_2 *and* Σ_3 *where* $\Sigma_1 > 0$ *and* $\Sigma_3 = \Sigma_3^T$ *then*

$$\Sigma_3 + \Sigma_2^T \Sigma_1^{-1} \Sigma_2 < 0,$$

if and only if

$$\begin{bmatrix} \Sigma_3 & \Sigma_2^T \\ \Sigma_2 & -\Sigma_1 \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -\Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_3 \end{bmatrix} < 0.$$

Lemma 2 (Petersen and Hollot 1986) *For given matrices* $Q = Q^T$, H *and* E *with appropriate dimensions*

$$Q + HFE + E^T F^T H^T < 0$$

holds for all F *satisfying* $F^T F \le I$, *if and only if there exists* $\varepsilon > 0$ *such that*

$$Q + \varepsilon^{-1} H H^T + \varepsilon E^T E < 0.$$

3 Main results

3.1 Stability analysis

When the uncertainty does not appear (i.e., the matrices ΔA_1, ΔA_2, ΔA_{1d} and ΔA_{2d} are zeroes), model (6) becomes

$$x_{1,1} = A_1 x_{1,0} + A_2 x_{0,1} + A_{1d} x_{1,-d_1} + A_{2d} x_{-d_2,1}. \tag{7}$$

In this part, we discuss the delay-range-dependent stability condition for nominal system (7).

Theorem 1 *The 2-D linear state-delayed system (7) with delays satisfying (4) and boundary conditions (5) is asymptotically stable if there exist matrices* $P_i > 0$, $R = diag\{R_1, R_2\} > 0$, $S = diag\{S_1, S_2\} > 0$, $N = diag\{N_1, N_2\} > 0$, $M_i = diag\{M_{1i}, M_{2i}\} > 0$, $\hat{N}_i = diag\{\hat{N}_{1i}, \hat{N}_{2i}\}$, $\hat{M}_i = diag\{\hat{M}_{1i}, \hat{M}_{2i}\}$ *and* $\hat{S}_i = diag\{\hat{S}_{1i}, \hat{S}_{2i}\} (i = 1, 2)$, *such that the following LMI is feasible*

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \hat{M}_1 & -\hat{S}_1 & H\hat{N}_1 & \bar{H}\hat{S}_1 & \bar{H}\hat{M}_1 & \bar{A}^T U & A^T(P_1+P_2) \\ * & \varphi_{22} & \hat{M}_2 & -\hat{S}_2 & H\hat{N}_2 & \bar{H}\hat{S}_2 & \bar{H}\hat{M}_2 & \bar{A}_d^T U & A_d{}^T(P_1+P_2) \\ * & * & -M_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -M_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -HS & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{H}(S+N) & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{H}N & 0 & 0 \\ * & * & * & * & * & * & * & -U & 0 \\ * & * & * & * & * & * & * & * & -(P_1+P_2) \end{bmatrix} < 0, \tag{8}$$

where $P_i,\ R_i,\ S_i,\ N_i,\ M_{ij},\ \hat{N}_{ij},\ \hat{M}_{ij},\ \hat{S}_{ij} \in \mathscr{R}^{n\times n}$ $(i = 1, 2,\ j = 1, 2)$,

$$\overline{P} = diag\{P_1, P_2\}, \varphi_{11} = -\overline{P}+R+M_1+M_2+\hat{N}_1+\hat{N}_1^T,\ \varphi_{12} = -\hat{N}_1+\hat{N}_2^T+\hat{S}_1-\hat{M}_1,$$
$$\varphi_{22} = -R-\hat{N}_2-\hat{N}_2^T+\hat{S}_2+\hat{S}_2^T-\hat{M}_2-\hat{M}_2^T,\ A = [A_1\ A_2],\ A_d = [A_{1d}\ A_{2d}],$$
$$\tilde{A}_1 = [A_1 - I\ A_2], \tilde{A}_2 = [A_1\ A_2 - I], \bar{A} = \left[\tilde{A}_1^T\ \tilde{A}_2^T\right]^T, \bar{A}_d = \left[A_d^T\ A_d^T\right]^T,$$
$$H = diag\{h_{12}I_n, h_{22}I_n\},\ \bar{H} = diag\{(h_{12}-h_{11})I_n, (h_{22}-h_{21})I_n\},\ U = HS+\bar{H}N.$$

Proof Introducing a generalized Lyapunov function $V_{\alpha,\beta}$ to express the energy stored in the point $x(i+\alpha, j+\beta)$:

$$V_{\alpha,\beta} = V(i+\alpha, j+\beta) = V^v_{\alpha,\beta} + V^h_{\alpha,\beta}, \tag{9}$$

with

$$V^v_{\alpha,\beta} = x^T_{\alpha,\beta} P_1 x_{\alpha,\beta} + \sum_{l=-d_1}^{-1} x^T_{\alpha,\beta+l} R_1 x_{\alpha,\beta+l} + \sum_{\theta=-h_{12}}^{-1} \sum_{l=\theta}^{-1} y^T_{\alpha,\beta+l} S_1 y_{\alpha,\beta+l}$$
$$+ \sum_{\theta=-h_{12}}^{-h_{11}-1} \sum_{l=\theta}^{-1} y^T_{\alpha,\beta+l} N_1 y_{\alpha,\beta+l} + \sum_{m=1}^{2} \sum_{l=-h_{1m}}^{-1} x^T_{\alpha,\beta+l} M_{1m} x_{\alpha,\beta+l},$$

$$V^h_{\alpha,\beta} = x^T_{\alpha,\beta} P_2 x_{\alpha,\beta} + \sum_{l=-d_2}^{-1} x^T_{\alpha+l,\beta} R_2 x_{\alpha+l,\beta} + \sum_{\theta=-h_{22}}^{-1} \sum_{l=\theta}^{-1} z^T_{\alpha+l,\beta} S_2 z_{\alpha+l,\beta}$$

$$+\sum_{\theta=-h_{22}}^{-h_{21}-1}\sum_{l=\theta}^{-1} z^T_{\alpha+l,\beta}N_2 z_{\alpha+l,\beta}+\sum_{m=1}^{2}\sum_{l=-h_{2m}}^{-1} x^T_{\alpha+l,\beta}M_{2m}x_{\alpha+l,\beta},$$

$$y_{\alpha,\beta+l}=x_{\alpha,\beta+l+1}-x_{\alpha,\beta+l};\ \ z_{\alpha+l,\beta}=x_{\alpha+l+1,\beta}-x_{\alpha+l,\beta},\ \ \alpha,\beta\in Z, \tag{10}$$

where $P_i>0,\ R_i>0,\ S_i>0,\ N_i>0,\ M_{ij}>0\ (i=1,2,\ j=1,2)$. Calculate the increments $\Delta V^h_{1,1}$ and $\Delta V^v_{1,1}$ respectively:

$$\begin{aligned}
\Delta V^v_{1,1} &= V^v_{1,1}-V^v_{1,0}\\
&= x^T_{1,1}P_1x_{1,1}+\sum_{l=-d_1}^{-1} x^T_{1,l+1}R_1x_{1,l+1}+\sum_{\theta=-h_{12}}^{-1}\sum_{l=\theta}^{-1} y^T_{1,l+1}S_1y_{1,l+1}\\
&\quad+\sum_{\theta=-h_{12}}^{-h_{11}-1}\sum_{l=\theta}^{-1} y^T_{1,l+1}N_1y_{1,l+1}+\sum_{m=1}^{2}\sum_{l=-h_{1m}}^{-1} x^T_{1,l+1}M_{1m}x_{1,l+1}\\
&\quad-x^T_{1,0}P_1x_{1,0}-\sum_{l=-d_1}^{-1} x^T_{1,l}R_1x_{1,l}-\sum_{\theta=-h_{12}}^{-1}\sum_{l=\theta}^{-1} y^T_{1,l}S_1y_{1,l}\\
&\quad-\sum_{\theta=-h_{12}}^{-h_{11}-1}\sum_{l=\theta}^{-1} y^T_{1,l}N_1y_{1,l}-\sum_{m=1}^{2}\sum_{l=-h_{1m}}^{-1} x^T_{1,l}M_{1m}x_{1,l}\\
&= x^T_{1,1}P_1x_{1,1}-x^T_{1,0}P_1x_{1,0}+x^T_{1,0}R_1x_{1,0}-x^T_{1,-d_1}R_1x_{1,-d_1}+h_{12}y^T_{1,0}S_1y_{1,0}\\
&\quad-\sum_{l=-h_{12}}^{-1} y^T_{1,l}S_1y_{1,l}+(h_{12}-h_{11})y^T_{1,0}N_1y_{1,0}-\sum_{l=-h_{12}}^{-h_{11}-1} y^T_{1,l}N_1y_{1,l}\\
&\quad+\sum_{m=1}^{2}\left(x^T_{1,0}M_{1m}x_{1,0}-x^T_{1,-h_{1m}}M_{1m}x_{1,-h_{1i}}\right),\\
\Delta V^h_{1,1} &= V^h_{1,1}-V^h_{0,1}\\
&= x^T_{1,1}P_2x_{1,1}+\sum_{l=-d_2}^{-1} x^T_{l+1,1}R_2x_{l+1,1}+\sum_{\theta=-h_{22}}^{-1}\sum_{l=\theta}^{-1} z^T_{l+1,1}S_2z_{l+1,1}\\
&\quad+\sum_{\theta=-h_{22}}^{-h_{21}-1}\sum_{l=\theta}^{-1} z^T_{l+1,1}N_2z_{l+1,1}+\sum_{m=1}^{2}\sum_{l=-h_{2m}}^{-1} x^T_{l+1,1}M_{2m}x_{l+1,1}\\
&\quad-x^T_{0,1}P_2x_{0,1}-\sum_{l=-d_2}^{-1} x^T_{l,1}R_2x_{l,1}-\sum_{\theta=-h_{22}}^{-1}\sum_{l=\theta}^{-1} z^T_{l,1}S_2z_{l,1}\\
&\quad-\sum_{\theta=-h_{22}}^{-h_{21}-1}\sum_{l=\theta}^{-1} z^T_{l,1}N_2z_{l,1}-\sum_{m=1}^{2}\sum_{l=-h_{2m}}^{-1} x^T_{l,1}M_{2m}x_{l,1}\\
&= x^T_{1,1}P_2x_{1,1}-x^T_{0,1}P_2x_{0,1}+x^T_{0,1}R_2x_{0,1}-x^T_{-d_2,1}R_2x_{-d_2,1}+h_{22}z^T_{0,1}S_2z_{0,1}
\end{aligned}$$

$$- \sum_{l=-h_{22}}^{-1} z_{l,1}^T S_2 z_{l,1} + (h_{22} - h_{21}) z_{0,1}^T N_2 z_{0,1} - \sum_{l=-h_{22}}^{-h_{21}-1} z_{l,1}^T N_2 z_{l,1}$$
$$+ \sum_{m=1}^{2} \left(x_{0,1}^T M_{2m} x_{0,1} - x_{-h_{2m},1}^T M_{2m} x_{-h_{2m},1} \right),$$

Denote $\Delta V_{1,1} = \Delta V_{1,1}^v + \Delta V_{1,1}^h$, then

$$\begin{aligned}
\Delta V_{1,1} &= x_{1,1}^T (P_1 + P_2) x_{1,1} - x_{1,0}^T P_1 x_{1,0} - x_{0,1}^T P_2 x_{0,1} \\
&\quad + x_{1,0}^T R_1 x_{1,0} + x_{0,1}^T R_2 x_{0,1} - x_{1,-d_1}^T R_1 x_{1,-d_1} - x_{-d_2,1}^T R_2 x_{-d_2,1} \\
&\quad + \sum_{m=1}^{2} \left(x_{1,0}^T M_{1m} x_{1,0} - x_{1,-h_{1m}}^T M_{1m} x_{1,-h_{1m}} \right) \\
&\quad + \sum_{m=1}^{2} \left(x_{0,1}^T M_{2m} x_{0,1} - x_{-h_{2m},1}^T M_{2m} x_{-h_{2m},1} \right) \\
&\quad + h_{12} y_{1,0}^T S_1 y_{1,0} + h_{22} z_{0,1}^T S_2 z_{0,1} + (h_{12} - h_{11}) y_{1,0}^T N_1 y_{1,0} + (h_{22} - h_{21}) z_{0,1}^T N_2 z_{0,1} \\
&\quad - \sum_{l=-h_{12}}^{-1} y_{1,l}^T S_1 y_{1,l} - \sum_{l=-h_{22}}^{-1} z_{l,1}^T S_2 z_{l,1} - \sum_{l=-h_{12}}^{-h_{11}-1} y_{1,l}^T N_1 y_{1,l} - \sum_{l=-h_{22}}^{-h_{21}-1} z_{l,1}^T N_2 z_{l,1} \\
&= (Ax + A_d x_d)^T (P_1 + P_2)(Ax + A_d x_d) - x^T \overline{P} x + x^T R x - x_d^T R x_d \\
&\quad + x^T (M_1 + M_2) x - x_{h_1}^T M_1 x_{h_1} - x_{h_2}^T M_2 x_{h_2} + h_{12} y_{1,0}^T S_1 y_{1,0} + h_{22} z_{0,1}^T S_2 z_{0,1} \\
&\quad + (h_{12} - h_{11}) y_{1,0}^T N_1 y_{1,0} + (h_{22} - h_{21}) z_{0,1}^T N_2 z_{0,1} \\
&\quad - \sum_{l=-d_1}^{-1} y_{1,l}^T S_1 y_{1,l} - \sum_{l=-h_{12}}^{-d_1-1} y_{1,l}^T (S_1 + N_1) y_{1,l} \\
&\quad - \sum_{l=-d_1}^{-h_{11}-1} y_{1,l}^T N_1 y_{1,l} - \sum_{l=-d_2}^{-1} z_{l,1}^T S_2 z_{l,1} \\
&\quad - \sum_{l=-h_{22}}^{-d_2-1} z_{l,1}^T (S_2 + N_2) z_{l,1} - \sum_{l=-d_2}^{-h_{21}-1} z_{l,1}^T N_2 z_{l,1},
\end{aligned}$$

where

$$x = \left[x_{1,0}^T \; x_{0,1}^T \right]^T, x_d = \left[x_{1,-d_1}^T \; x_{-d_2,1}^T \right]^T, x_{h_1} = \left[x_{1,-h_{11}}^T \; x_{-h_{21},1}^T \right]^T,$$
$$x_{h_2} = \left[x_{1,-h_{12}}^T \; x_{-h_{22},1}^T \right]^T.$$

Recalling the relation (10), the following equations hold for any matrices $\hat{N}_{ij}, \; \hat{S}_{ij}, \; \hat{M}_{ij}, \; i = 1, 2, \; j = 1, 2$ with appropriate dimensions

$$0 = 2 \left[x_{1,0}^T \hat{N}_{11} + x_{1,-d_1}^T \hat{N}_{12} \right] \left[x_{1,0} - x_{1,-d_1} - \sum_{l=-d_1}^{-1} y_{1,l} \right], \qquad (11)$$

$$0 = 2\left[x_{0,1}^T\hat{N}_{21} + x_{-d_2,1}^T\hat{N}_{22}\right]\left[x_{0,1} - x_{-d_2,1} - \sum_{l=-d_2}^{-1} z_{l,1}\right], \tag{12}$$

$$0 = 2\left[x_{1,0}^T\hat{S}_{11} + x_{1,-d_1}^T\hat{S}_{12}\right]\left[x_{1,-d_1} - x_{1,-h_{12}} - \sum_{l=-h_{12}}^{-d_1-1} y_{1,l}\right], \tag{13}$$

$$0 = 2\left[x_{0,1}^T\hat{S}_{21} + x_{-d_2,1}^T\hat{S}_{22}\right]\left[x_{-d_2,1} - x_{-h_{22},1} - \sum_{l=-h_{22}}^{-d_2-1} z_{l,1}\right], \tag{14}$$

$$0 = 2\left[x_{1,0}^T\hat{M}_{11} + x_{1,-d_1}^T\hat{M}_{12}\right]\left[x_{1,-h_{11}} - x_{1,-d_1} - \sum_{l=-d_1}^{-h_{11}-1} y_{1,l}\right], \tag{15}$$

$$0 = 2\left[x_{0,1}^T\hat{M}_{21} + x_{-d_2,1}^T\hat{M}_{22}\right]\left[x_{-h_{21},1} - x_{-d_2,1} - \sum_{l=-d_2}^{-h_{21}-1} z_{l,1}\right]. \tag{16}$$

Adding the right hand sides of Eqs. (11)–(16) to $\Delta V_{1,1}$ allows us to rewrite $\Delta V_{1,1}$ as

$$\begin{aligned}
\Delta V_{1,1} = {} & \varsigma^T\left\{\Gamma + \begin{bmatrix} A^T \\ A_d^T \\ 0 \\ 0 \end{bmatrix}(P_1+P_2)\,[A\ A_d\ 0\ 0] + \begin{bmatrix} \bar{A}^T \\ \bar{A}_d^T \\ 0 \\ 0 \end{bmatrix}(HS+\bar{H}N)\left[\bar{A}\ \bar{A}_d\ 0\ 0\right]\right\}\varsigma \\
& - \sum_{l=-d_1}^{-1} y_{1,l}^T S_1 y_{1,l} - \sum_{l=-h_{12}}^{-d_1-1} y_{1,l}^T(S_1+N_1)y_{1,l} - \sum_{l=-d_1}^{-h_{11}-1} y_{1,l}^T N_1 y_{1,l} \\
& - \sum_{l=-d_2}^{-1} z_{l,1}^T S_2 z_{l,1} - \sum_{l=-h_{22}}^{-d_2-1} z_{l,1}^T(S_2+N_2)z_{l,1} - \sum_{l=-d_2}^{-h_{21}-1} z_{l,1}^T N_2 z_{l,1} \\
& - \sum_{l=-d_1}^{-1}\left(\zeta^T\tilde{N}_1 y_{1,l} + y_{1,l}^T\tilde{N}_1^T\zeta\right) - \sum_{l=-d_2}^{-1}\left(\xi^T\tilde{N}_2 z_{l,1} + z_{l,1}^T\tilde{N}_2^T\xi\right) \\
& - \sum_{l=-h_{12}}^{-d_1-1}\left(\zeta^T\tilde{S}_1 y_{1,l} + y_{1,l}^T\tilde{S}_1^T\zeta\right) - \sum_{l=-h_{22}}^{-d_2-1}\left(\xi^T\tilde{S}_2 z_{l,1} + z_{l,1}^T\tilde{S}_2^T\xi\right) \\
& - \sum_{l=-d_1}^{-h_{11}-1}\left(\zeta^T\tilde{M}_1 y_{1,l} + y_{1,l}^T\tilde{M}_1^T\zeta\right) - \sum_{l=-d_2}^{-h_{21}-1}\left(\xi^T\tilde{M}_2 z_{l,1} + z_{l,1}^T\tilde{M}_2^T\xi\right) \\
\le {} & \varsigma^T\left\{\Gamma + [A\ A_d\ 0\ 0]^T(P_1+P_2)[A\ A_d\ 0\ 0] + \left[\bar{A}\ \bar{A}_d\ 0\ 0\right]^T(HS+\bar{H}N)\right. \\
& \left.\left[\bar{A}\ \bar{A}_d\ 0\ 0\right] + \hat{N}HS^{-1}\hat{N}^T + \hat{S}\bar{H}(S+N)^{-1}\hat{S}^T + \hat{M}\bar{H}N^{-1}\hat{M}^T\right\}\varsigma \\
& - \sum_{l=-d_1}^{-1}\left[\zeta^T\tilde{N}_1 + y_{1,l}^T S_1\right]S_1^{-1}\left[\tilde{N}_1^T\zeta + S_1 y_{1,l}\right] \\
& - \sum_{l=-d_2}^{-1}\left[\xi^T\tilde{N}_2 + z_{l,1}^T S_2\right]S_2^{-1}\left[\tilde{N}_2^T\xi + S_2 z_{l,1}\right]
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{l=-h_{12}}^{-d_1-1}\left[\zeta^T\tilde{S}_1+y_{1,l}^T(S_1+N_1)\right](S_1+N_1)^{-1}\left[\tilde{S}_1^T\zeta+(S_1+N_1)\,y_{1,l}\right]\\
&-\sum_{l=-h_{22}}^{-d_2-1}\left[\xi^T\tilde{S}_2+z_{l,1}^T(S_2+N_2)\right](S_2+N_2)^{-1}\left[\tilde{S}_2^T\xi+(S_2+N_2)\,z_{l,1}\right]\\
&-\sum_{l=-d_1}^{-h_{11}-1}\left[\zeta^T\tilde{M}_1+y_{1,l}^TN_1\right]N_1^{-1}\left[\tilde{M}_1^T\zeta+N_1y_{1,l}\right]\\
&-\sum_{l=-d_2}^{-h_{21}-1}\left[\xi^T\tilde{M}_2+z_{l,1}^TN_2\right]N_2^{-1}\left[\tilde{M}_2^T\xi+N_2z_{l,1}\right],
\end{aligned}\tag{17}
$$

where

$$
\varsigma=\begin{bmatrix}x\\x_d\\x_{h_1}\\x_{h_2}\end{bmatrix},\ \zeta=\begin{bmatrix}x_{1,0}\\x_{1,-d_1}\\x_{1,-h_{11}}\\x_{1,-h_{12}}\end{bmatrix},\ \xi=\begin{bmatrix}x_{0,1}\\x_{-d_2,1}\\x_{-h_{21},1}\\x_{-h_{22},1}\end{bmatrix},\ \Gamma=\begin{bmatrix}\varphi_{11}&\varphi_{12}&\hat{M}_1&-\hat{S}_1\\ *&\varphi_{22}&\hat{M}_2&-\hat{S}_2\\ *&*&-M_1&0\\ *&*&*&-M_2\end{bmatrix},
$$

$$
\tilde{N}_i=\begin{bmatrix}\hat{N}_{i1}\\\hat{N}_{i2}\\0\\0\end{bmatrix},\ \tilde{S}_i=\begin{bmatrix}\hat{S}_{i1}\\\hat{S}_{i2}\\0\\0\end{bmatrix},\ \tilde{M}_i=\begin{bmatrix}\hat{M}_{i1}\\\hat{M}_{i2}\\0\\0\end{bmatrix},\ i=1,2,\ \hat{N}=\begin{bmatrix}\hat{N}_1\\\hat{N}_2\\0\\0\end{bmatrix},\ \hat{S}=\begin{bmatrix}\hat{S}_1\\\hat{S}_2\\0\\0\end{bmatrix},
$$

$$
\hat{M}=\begin{bmatrix}\hat{M}_1\\\hat{M}_2\\0\\0\end{bmatrix}.
$$

Since $S_i>0,\ N_i>0,\ i=1,2$, from (17) we obtain that

$$
\begin{aligned}
\Delta V_{1,1}\le\varsigma^T&\Big\{\Gamma+[A\quad A_d\quad 0\quad 0]^T(P_1+P_2)[A\quad A_d\quad 0\quad 0]+\left[\bar{A}\quad\bar{A}_d\quad 0\quad 0\right]^T(HS+\bar{H}N)\\
&\left[\bar{A}\quad\bar{A}_d\quad 0\quad 0\right]+NHS^{-1}\hat{N}^T+\hat{S}\bar{H}(S+N)^{-1}\hat{S}^T+\hat{M}\bar{H}N^{-1}\hat{M}^T\Big\}\,\varsigma.
\end{aligned}
$$

Now we prove that if condition (8) holds, then $\Delta V_{1,1}\le 0$, By Lemma 1 (Schur complement), condition (8) is equivalent to

$$
\begin{aligned}
&\Gamma+[A\quad A_d\quad 0\quad 0]^T(P_1+P_2)[A\quad A_d\quad 0\quad 0]+\left[\bar{A}\quad\bar{A}_d\quad 0\quad 0\right]^T(HS+\bar{H}N)\\
&\quad\times\left[\bar{A}\quad\bar{A}_d\quad 0\quad 0\right]+\hat{N}HS^{-1}\hat{N}^T+\hat{S}\bar{H}(S+N)^{-1}\hat{S}^T+\hat{M}\bar{H}N^{-1}\hat{M}^T<0,
\end{aligned}\tag{18}
$$

which guarantees

$$
\Delta V_{1,1}\le 0,\tag{19}
$$

where the equality holds only when

$$
\begin{aligned}
&x=0,\ x_d=0,\ x_{h_1}=0,\ x_{h_2}=0,\\
&y_{1,l}=0,\ l=-h_{12},-h_{12}+1,\ldots,-1;\ \ z_{l,1}=0,\ l=-h_{22},-h_{22}+1,\ldots,-1,
\end{aligned}\tag{20}
$$

that is,

$$
x_{1,l}=0,\ l=-h_{12},-h_{12}+1,\ldots,0;\ \ x_{l,1}=0,\ l=-h_{22},-h_{22}+1,\ldots,0.\tag{21}
$$

In this case system (7) is certainly stable. When $\Delta V_{1,1} < 0$, that is,

$$V^v_{1,1} + V^h_{1,1} < V^v_{1,0} + V^h_{0,1}, \tag{22}$$

For any integer $k > \max\{\mu_1, \mu_2\}$, $V^h(k, 0) = V^h(0, k) = V^v(k, 0) = V^v(0, k) = 0$, from (22) and the boundary conditions (5) it follows that

$$\begin{aligned}
\sum_{i+j=k+1} V(i, j) &= \sum_{i+j=k+1} [V^v(i, j) + V^h(i, j)] \\
&= V^v(k, 1) + V^h(k, 1) + V^v(k-1, 2) + V^h(k-1, 2) + \cdots \\
&\quad + V^v(1, k) + V^h(1, k) \\
&< V^v(k, 0) + V^h(k-1, 1) + V^v(k-1, 1) + V^h(k-2, 2) \\
&\quad\quad + \cdots + V^v(1, k-1) + V^h(0, k) \\
&= V^v(k, 0) + V^h(0, k) + V^v(k-1, 1) + V^h(k-1, 1) \\
&\quad\quad + \cdots + V^v(1, k-1) + V^h(1, k-1) + V^v(0, k) + V^h(k, 0) \\
&= \sum_{i+j=k} [V^v(i, j) + V^h(i, j)] = \sum_{i+j=k} V(i, j),
\end{aligned} \tag{23}$$

Let $D_k = \{(i, j) : i + j = k\}$ Paszke et al. (2006b), $h = max\{h_{12}, h_{22}\}$. Inequality (23) implies that the energy stored at all points along the D_{k+1} to D_{k-h+1} (i.e. all points in $D_{k+1} \bigcup \cdots \bigcup D_{k-h+1}$) is less than the energy stored at the points along the D_k to D_{k-h} (i.e. all points in $D_k \bigcup \cdots \bigcup D_{k-h}$). From Hinamoto (1989), we obtain

$$\lim_{i+j\to\infty} V(i, j) = 0. \tag{24}$$

This implies that $\lim\limits_{i+j\to\infty} \parallel x_{0,0} \parallel = \lim\limits_{i+j\to\infty} \parallel x(i, j) \parallel = 0$. That is,

$$\lim_{r\to\infty} X_r = 0. \tag{25}$$

By Definition 1, system (7) is asymptotically stable. This completes the proof. □

Remark 1 Theorem 1 provides a sufficient condition for the delay-range-dependent stability for 2-D state-delayed system where the upper bound h_{12}, h_{22} and the lower bound h_{11}, h_{21} are all given. If only one bound is not known, we can utilize the following optimization procedure to compute the size of delay range for stability condition. For example, when bounds h_{11}, h_{21} and h_{12} are fixed, the maximization of the bound h_{22} satisfying stability condition can be cast into an optimization problem. We can maximize $h_{22} = 1/\delta$ by solving a generalized eigenvalue problem (GEVP) given by:

$$\min_{P_1>0, P_2>0, R>0, S>0, N>0, M_1>0, M_2>0} \delta \; s.t.(8).$$

Similarly, if the bounds h_{12}, h_{21} and h_{22} are fixed, we can also get the optimal bound h_{11} for stability condition by solving the following convex optimization problem:

$$\min_{P_1>0, P_2>0, R>0, S>0, N>0, M_1>0, M_2>0} h_{11} \; s.t.(8).$$

Remark 2 The delay-independent stability criterion given by Theorem 3 of Paszke et al. (2004) is a special case of Theorem 1. Actually, if we set $S = 0$, $N = 0$, $M_1 = 0$, $M_2 = 0$

in the function $V(i, j)$ and do not introduce any free-weighting matrices, we can see that the LMI (8) reduces to

$$\begin{bmatrix} -P_1+R_1 & 0 & 0 & 0 & A_1^T(P_1+P_2) \\ * & -P_2+R_2 & 0 & 0 & A_2^T(P_1+P_2) \\ * & * & R_1 & 0 & A_{1d}^T(P_1+P_2) \\ * & * & * & R_2 & A_{2d}^T(P_1+P_2) \\ * & * & * & * & -(P_1+P_2) \end{bmatrix} < 0, \tag{26}$$

Let $P_1 = P - Q - R_2,\ P_2 = Q + R_2,\ R_1 = Q_1,\ R_2 = Q_2$, LMI (26) is rewritten as

$$\begin{bmatrix} Q+Q_1+Q_2-P & 0 & 0 & 0 & A_1^T(P_1+P_2) \\ * & -Q & 0 & 0 & A_2^T(P_1+P_2) \\ * & * & Q_1 & 0 & A_{1d}^T(P_1+P_2) \\ * & * & * & Q_2 & A_{2d}^T(P_1+P_2) \\ * & * & * & * & -(P_1+P_2) \end{bmatrix} < 0, \tag{27}$$

By Schur complement, LMI (27) is equivalent to LMI (6) in Theorem 3 of Paszke et al. (2004).

If the lower bounds $h_{k1} = 0$ $(k = 1, 2)$, i.e.

$$0 \le d_k \le h_{k2} < \infty \quad (k = 1,\ 2), \tag{28}$$

we can obtain a corollary from Theorem 1.

Corollary 1 *The 2-D linear state-delayed system (7) with delays satisfying (28) and boundary conditions (5) is asymptotically stable if there exist matrices* $P_i > 0$, $R = diag\{R_1, R_2\} > 0$, $S = diag\{S_1, S_2\} > 0$, $M_2 = diag\{M_{12}, M_{22}\} > 0$, $\hat{N}_i = diag\{\hat{N}_{1i}, \hat{N}_{2i}\}$ *and* $\hat{S}_i = diag\{\hat{S}_{1i}, \hat{S}_{2i}\}$, $(i = 1, 2)$, *such that the following LMI is feasible*

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & -\hat{S}_1 & H\hat{N}_1 & H\hat{S}_1 & \bar{A}^THS & A^T(P_1+P_2) \\ * & \Psi_{22} & -\hat{S}_2 & H\hat{N}_2 & H\hat{S}_2 & \bar{A}_d^THS & {A_d}^T(P_1+P_2) \\ * & * & -M_2 & 0 & 0 & 0 & 0 \\ * & * & * & -HS & 0 & 0 & 0 \\ * & * & * & * & -HS & 0 & 0 \\ * & * & * & * & * & -HS & 0 \\ * & * & * & * & * & * & -(P_1+P_2) \end{bmatrix} < 0, \tag{29}$$

where $P_i,\ R_i,\ S_i,\ N_i,\ M_{i2},\ \hat{N}_{ij},\ \hat{S}_{i,j} \in \mathscr{R}^{n\times n}$ $(i = 1, 2,\ j = 1, 2)$,

$$\Psi_{11} = -\bar{P} + R + M_2 + \hat{N}_1 + \hat{N}_1^T,\ \Psi_{12} = -\hat{N}_1 + \hat{N}_2^T + \hat{S}_1,$$
$$\Psi_{22} = -R - \hat{N}_2 - \hat{N}_2^T + \hat{S}_2 + \hat{S}_2^T,$$

$\overline{P},\ A,\ A_d,\ \tilde{A}_1,\ \tilde{A}_2,\ \bar{A},\ \bar{A}_d$ *and* H *are defined in Theorem 1.*

3.2 Robust stability

When we consider 2-D discrete state-delayed system (6) with uncertain parameters, the robust stability condition can be deduced from Theorem 1.

Theorem 2 *The 2-D linear state-delayed system (6) with parameter uncertainties (2)–(3), delays (4), and boundary conditions (5) is robust stable if there exist matrices* $P_i > 0$, $R = diag\{R_1, R_2\} > 0$, $S = diag\{S_1, S_2\} > 0$, $N = diag\{N_1, N_2\} > 0$, $M_i =$

$diag\{M_{1i}, M_{2i}\} > 0$, $\hat{N}_i = diag\{\hat{N}_{1i}, \hat{N}_{2i}\}$, $\hat{M}_i = diag\{\hat{M}_{1i}, \hat{M}_{2i}\}$, $\hat{S}_i = diag\{\hat{S}_{1i}, \hat{S}_{2i}\}$, and scalars $\varepsilon_i > 0 (i = 1, 2)$, such that the following LMI is feasible

$$\begin{bmatrix} \bar{\varphi}_{11} & \bar{\varphi}_{12} & \hat{M}_1 & -\hat{S}_1 & H\hat{N}_1 & \bar{H}\hat{S}_1 & \bar{H}\hat{M}_1 & \bar{A}^TU & A^T(P_1+P_2) & 0 & 0 \\ * & \bar{\varphi}_{22} & \hat{M}_2 & -\hat{S}_2 & H\hat{N}_2 & \bar{H}\hat{S}_2 & \bar{H}\hat{M}_2 & \bar{A}_d^TU & A_d^T(P_1+P_2) & 0 & 0 \\ * & * & -M_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -M_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -HS & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{H}(S+N) & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{H}N & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -U & 0 & 0 & U\bar{D} \\ * & * & * & * & * & * & * & * & -(P_1+P_2) & (P_1+P_2)D & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix}$$
$$< 0, \tag{30}$$

where $P_i, R_i, S_i, N_i, M_{ij}, \hat{N}_{ij}, \hat{M}_{ij}, \hat{S}_{ij} \in \mathscr{R}^{n\times n} (i = 1, 2,\ j = 1, 2)$,

$$E = [E_1 \quad E_2],\ E_d = [E_{1d} \quad E_{2d}],\ \bar{D} = diag\{D, D\},$$
$$\bar{\varphi}_{11} = \varphi_{11} + (\varepsilon_1 + 2\varepsilon_2)E^TE,\ \bar{\varphi}_{12} = \varphi_{12} + (\varepsilon_1 + 2\varepsilon_2)E^TE_d,$$
$$\bar{\varphi}_{22} = \varphi_{22} + (\varepsilon_1 + 2\varepsilon_2)E_d^TE_d,$$

$\overline{P}$, A, A_d, $\tilde{A}_1$, $\tilde{A}_2$, $\bar{A}$, $\bar{A}_d$, H, $\bar{H}$, U, φ_{11}, φ_{12} and φ_{22} are defined in Theorem 1.

Proof Note the differences between the coefficient matrices in systems (6) and (7). Replacing A_k and A_{kd} in (8) by $A_k + DFE_k$, and $A_{kd} + DFE_{kd}$, respectively, yields

$$\varphi + \Lambda_1^TF\Lambda_2 + \Lambda_2^TF^T\Lambda_1 + \Lambda_3^T\bar{F}\Lambda_4 + \Lambda_4^T\bar{F}^T\Lambda_3 < 0, \tag{31}$$

where

$$\bar{E} = \begin{bmatrix} E^T & E^T \end{bmatrix}^T,\ \bar{E}_d = \begin{bmatrix} E_d^T & E_d^T \end{bmatrix}^T,\ \bar{F} = diag\{F, F\},$$
$$\Lambda_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D^T(P_1+P_2) \end{bmatrix},$$
$$\Lambda_2 = [E \quad E_d \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$
$$\Lambda_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{D}^TU & 0 \end{bmatrix},$$
$$\Lambda_4 = \begin{bmatrix} \bar{E} & \bar{E}_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If condition (31) holds, system (6) is robust stable. By Lemma 2, (31) holds if and only if there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\varphi + \varepsilon_1^{-1}\Lambda_1^T\Lambda_1 + \varepsilon_1\Lambda_2^T\Lambda_2 + \varepsilon_2^{-1}\Lambda_3^T\Lambda_3 + \varepsilon_2\Lambda_4^T\Lambda_4 < 0. \tag{32}$$

Then by Lemma 1 (Schur complement), it is easy to see that (32) is equivalent to (30). The equivalence of conditions (30) and (31) implies that if condition (30) holds, system (6) is robust stable. The proof is completed.

When the lower bounds $h_{k1} = 0$ $(k = 1, 2)$, we obtain the following corollary from Theorem 2. □

Corollary 2 *The 2-D linear state-delayed system (6) with parameter uncertainties (2)–(3), delays (4), and boundary conditions (5) is robust stable, if there exist matrices $P_i >$*

$0, R = diag\{R_1, R_2\} > 0, S = diag\{S_1, S_2\} > 0, M_2 = diag\{M_{12}, M_{22}\} > 0, \hat{N}_i = diag\{\hat{N}_{1i}, \hat{N}_{2i}\}, \hat{S}_i = diag\{\hat{S}_{1i}, \hat{S}_{2i}\}$ *and scalars* $\varepsilon_i > 0 (i = 1, 2)$, *such that the following LMI is feasible*

$$\begin{bmatrix} \bar{\Psi}_{11} & \bar{\Psi}_{12} & -\hat{S}_1 & H\hat{N}_1 & H\hat{S}_1 & \bar{A}^T HS & A^T(P_1+P_2) & 0 & 0 \\ * & \bar{\Psi}_{22} & -\hat{S}_2 & H\hat{N}_2 & H\hat{S}_2 & \bar{A}_d^T HS & A_d{}^T(P_1+P_2) & 0 & 0 \\ * & * & -M_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -HS & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -HS & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -HS & 0 & 0 & HS\bar{D} \\ * & * & * & * & * & * & -(P_1+P_2) & (P_1+P_2)D & 0 \\ * & * & * & * & * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0,$$

where $P_i,\ R_i,\ S_i,\ N_i,\ M_{i2},\ \hat{N}_{ij},\ \hat{S}_{i,j} \in \mathscr{R}^{n\times n}\ (i = 1, 2,\ j = 1, 2)$,

$$\bar{\Psi}_{11} = \Psi_{11} + (\varepsilon_1 + 2\varepsilon_2)E^T E,\ \bar{\Psi}_{12} = (\varepsilon_1 + 2\varepsilon_2)E^T E_d,$$
$$\bar{\Psi}_{22} = \Psi_{22} + (\varepsilon_1 + 2\varepsilon_2)E_d^T E_d,$$

$\overline{P},\ A,\ A_d,\ \tilde{A}_1,\ \tilde{A}_2,\ \bar{A},\ \bar{A}_d$ *and* H *are defined in Theorem 1,* $\Psi_{11},\ \Psi_{12},\ \Psi_{22},\ E$ *and* E_d *are defined in Theorem 2.*

4 Numerical examples

In this section, we illustrate the new results via two examples. Example 1 will show the benefits of result in nominal systems, and Example 2 will demonstrate the effectiveness of the proposed method in uncertain systems.

Example 1 In the illustrative example of Xu and Yu (2009a), a thermal process expressed in a partial differential equation with time delays is modeled into the 2-D state-delayed FM second model (1). Here we consider the asymptotical stability of system (1) (without parameter uncertainty) with the following coefficient matrices given in Xu and Yu (2009a)

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0.25 & 0.65 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_{1d} = \begin{bmatrix} 0 & 0 \\ 0 & -0.12 \end{bmatrix},\ A_{2d} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It was shown in Xu and Yu (2009a) that for any constant delay d_1 satisfying $0 < d_1 \leq 5$, system (1) is asymptotically stable. However, by using Corollary 1 and Remark 1, we can show that the system is asymptotically stable for $0 \leq d_1 \leq 13$. This means that the upper bound obtained in this paper is greater than that given in Xu and Yu (2009a).

Note that 2-D state-delayed Roesser Model is a special case of 2-D state-delayed FM second model, and the above system can actually be described by 2-D state-delayed Roesser Model. Therefore, we can also use Theorem 1 in Paszke et al. (2006b), which presents a constant delay-dependent stability condition for 2-D state-delayed Roesser Model, to study the stability of the above system. Direct computation shows that this system is asymptotically stable for any constant delay d satisfying $0 \leq d \leq 13$. This means that Corollary 1 in this paper is as good as Theorem 1 in Paszke et al. (2006b)for the above system described by 2-D state-delayed Roesser Model. The trajectories of the two state variables of the system (1) with $d_1 = 13$ are shown in Fig. 1. It shows that system (1) is asymptotically stable with $d_1 = 13$.

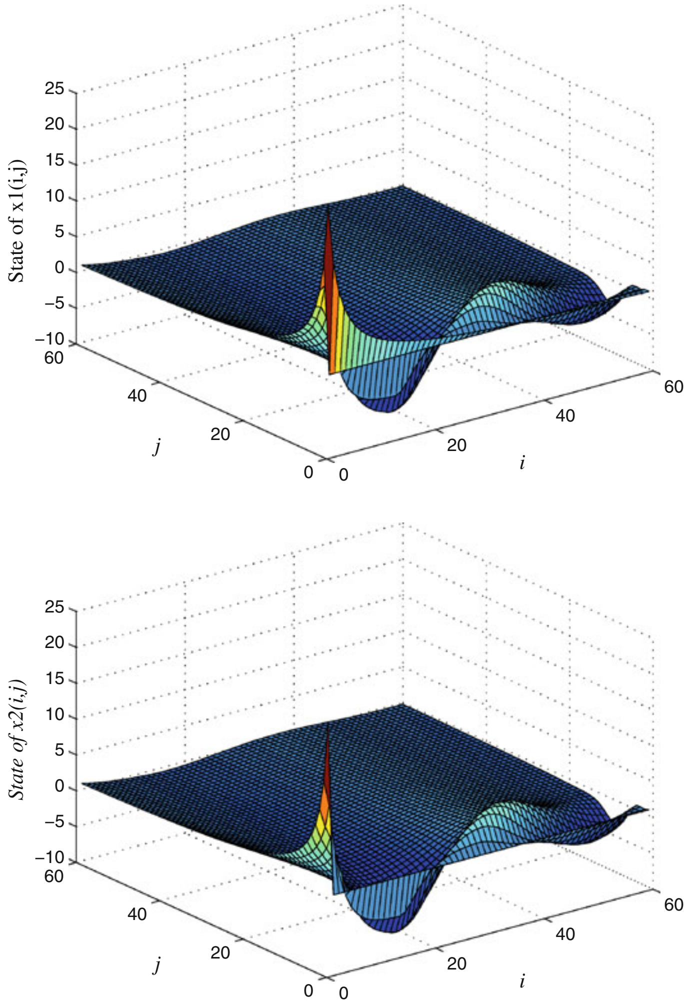

Fig. 1 State trajectories of system given in Example 1 with $d_1 = 13$

Example 2 Consider system (1) with coefficient matrices

$$A_1 = \begin{bmatrix} -0.2450 & 0.0307 \\ -0.1444 & 0.0008 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.2860 & 0.1800 \\ -0.1435 & -0.2601 \end{bmatrix},$$

$$A_{1d} = \begin{bmatrix} 0.1448 & 0.1489 \\ 0.0808 & 0.0541 \end{bmatrix}, \; A_{2d} = \begin{bmatrix} 0.0881 & 0.1220 \\ 0.1872 & 0.0430 \end{bmatrix},$$

uncertainty matrices

$$D = [0.4768 \; 0.0219]^T, \; E_1 = \begin{bmatrix} 0.0272 & 0.3127 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0.0129 & 0.3840 \end{bmatrix},$$
$$E_{1d} = \begin{bmatrix} 0.1366 & 0.0186 \end{bmatrix}, \; E_{2d} = \begin{bmatrix} 0.0071 & 0.1225 \end{bmatrix},$$

and delay range $2 \leq d_1 \leq 10$; $6 \leq d_2 \leq 8$. In this case, we solve the LMI (30) using Matlab LMI Toolbox, the solutions are as follows:

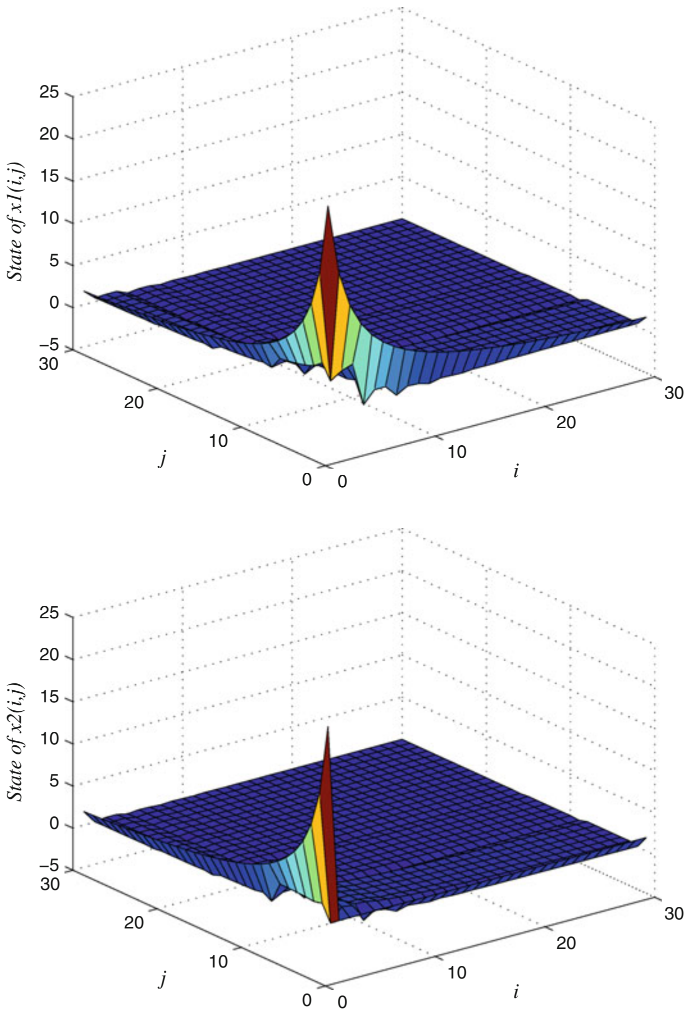

Fig. 2 State trajectories of system given in Example 2

$$P_1 = \begin{bmatrix} 21.0296 & -1.0278 \\ -1.0278 & 29.9637 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 16.9526 & 3.9937 \\ 3.9937 & 44.3551 \end{bmatrix},$$

$$R = \begin{bmatrix} 6.8560 & 5.3649 & 0 & 0 \\ 5.3649 & 8.3323 & 0 & 0 \\ 0 & 0 & 7.8197 & 2.2945 \\ 0 & 0 & 2.2945 & 12.5821 \end{bmatrix}, \quad S = \begin{bmatrix} 0.0040 & -0.0057 & 0 & 0 \\ -0.0057 & 0.0229 & 0 & 0 \\ 0 & 0 & 0.0653 & -0.0255 \\ 0 & 0 & -0.0255 & 0.0295 \end{bmatrix},$$

$$N = \begin{bmatrix} 0.0051 & -0.0075 & 0 & 0 \\ -0.0075 & 0.0300 & 0 & 0 \\ 0 & 0 & 0.4469 & -0.1662 \\ 0 & 0 & -0.1662 & 0.1748 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0.0916 & -0.1194 & 0 & 0 \\ -0.1194 & 0.5275 & 0 & 0 \\ 0 & 0 & 0.1412 & -0.1354 \\ 0 & 0 & -0.1354 & 0.5379 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.0916 & -0.1195 & 0 & 0 \\ -0.1195 & 0.5275 & 0 & 0 \\ 0 & 0 & 0.1468 & -0.1380 \\ 0 & 0 & -0.1380 & 0.5405 \end{bmatrix}, \ \hat{N}_1 = \begin{bmatrix} 0.0008 & -0.0029 & 0 & 0 \\ -0.0008 & 0.0013 & 0 & 0 \\ 0 & 0 & -0.0050 & 0.0019 \\ 0 & 0 & -0.0071 & 0.0037 \end{bmatrix},$$

$$\hat{N}_2 = \begin{bmatrix} 0.0013 & -0.0027 & 0 & 0 \\ 0.0001 & -0.0004 & 0 & 0 \\ 0 & 0 & 0.0039 & -0.0002 \\ 0 & 0 & 0.0008 & -0.0016 \end{bmatrix}, \ \hat{M}_1 = \begin{bmatrix} 0.0004 & -0.0015 & 0 & 0 \\ -0.0005 & 0.0007 & 0 & 0 \\ 0 & 0 & -0.0004 & 0.0155 \\ 0 & 0 & -0.0455 & 0.0416 \end{bmatrix},$$

$$\hat{M}_2 = \begin{bmatrix} 0.0011 & -0.0015 & 0 & 0 \\ -0.0005 & 0.0014 & 0 & 0 \\ 0 & 0 & 0.0622 & -0.0187 \\ 0 & 0 & 0.0183 & -0.0200 \end{bmatrix}, \ \hat{S}_1 = \begin{bmatrix} -0.0006 & 0.0028 & 0 & 0 \\ 0.0004 & -0.0014 & 0 & 0 \\ 0 & 0 & 0.0007 & -0.0178 \\ 0 & 0 & 0.0495 & -0.0465 \end{bmatrix},$$

$$\hat{S}_2 = \begin{bmatrix} -0.0016 & 0.0028 & 0 & 0 \\ 0.0009 & -0.0025 & 0 & 0 \\ 0 & 0 & -0.0671 & 0.0204 \\ 0 & 0 & -0.0200 & 0.0221 \end{bmatrix}, \ \varepsilon_1 = 48.4912, \varepsilon_2 = 0.6214.$$

The obtained solutions guarantee that system (1) with parameter uncertainties satisfying (2)–(3) is asymptotically stable for $d_1 = 3$; $d_2 = 6$. Figure 2 shows that system (1) given in this example is robust stable.

5 Conclusion

This paper has presented sufficient stability and robust stability conditions for 2-D state-delayed systems described by the FM second model. These stability criteria are less conservative for two reasons: one is that they are delay-range-dependent; another is the introduction of some free-weighting matrices. Two numerical examples have been given to demonstrate the effectiveness of the proposed methods. In the future we will study how to apply these robust stability criteria to the robust control synthesis problems.

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