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Contact mechanics for dynamical systems: a comprehensive review

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Abstract
This work reviews the main techniques to model dynamical systems with contact-impact events. Regularized and non-smooth formulations are considered, wherein the fundamental features associated with each approach are analyzed. A brief description of contact dynamics is presented, and an overview of the state-of-the-art of the main aspects related to the contact dynamics discipline is provided. This paper ends by identifying gaps in the current techniques and prospects for future research in the field of contact mechanics in multibody dynamics.

Keywords Contact mechanics · Dynamical systems · Multibody dynamics · Contact detection · Contact resolution · Regularized methods · Non-smooth formulations

1 Introduction

Many applications of multibody dynamics to real-world mechanical systems demand the analysis of contact scenarios [1]. Contact behavior depends on the matter at hand, material properties, and technique utilized to model the contact dynamics. In a simple and comprehensive manner, a contact dynamics formulation is a threefold problem, involving the determination of potential contact points between the colliding bodies within a multibody system, the evaluation of the contact-impact forces, and the establishment of the transition between contact and non-contact scenarios, and between different contact states [2].

Contact mechanics can be understood as the study of the deformation of solid bodies when they collide with each other. Frictional contact mechanics analyzes the interaction of colliding bodies in the presence of friction phenomena [3]. It is worth noting that contact mechanics is omnipresent in many multibody dynamics applications, and in many cases, the performance of the systems depends on the modeling process of the contact-impact events [4–11]. Contact dynamics, which deals with the motion analysis of multibody systems subjected to collisions, is still one the most challenging and complex areas in science and engineering [12–31].

Dedicated to Professor Manuel Seabra Pereira (1947-2016)

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When two bodies within a multibody system collide, the state of multibody system can change quite rapidly, resulting in jumps, or discontinuities, in the velocities, propagation of waves, noise and heat generation, high force levels, plastic deformation, energy conversion, among other mechanical phenomena [32–39].

Over the last four decades, the multibody dynamics community has exhibited an increasing interest in the resolution of the problems related to collisions between mechanical components [40–102]. Actual examples of mechanical systems in which contact-impact interactions play a key role are robotics and walking machines [103–109], railway systems [110–122], vehicle and crash models [123–133], biosystems and biomechatronics [134–147], machines and mechanisms [148–164], granular media and powder technologies [165–185], toys models [186–199], civil structures [200–218], sounds and musical instruments [219–234], fruit transport and handling [235–248], just to mention some examples under the umbrella of dynamical systems.

The process of modeling and simulating contact-impact events in multibody systems requires to determine the points of contact and to calculate the resulting reaction contact forces. In essence, the determination of the contact points, usually named as contact detection phase, evaluates when and which points of a pair of surfaces are in contact [249–257]. The corresponding reaction contact forces, associated with the contact resolution phase, are the result of the applied forces [258–261] or unilateral constraints [262–266].

In the contact of multibody systems, the interaction between two colliding bodies can be modeled using contact force-based approaches (continuous methods) [267–280], or techniques based on the geometric constraints (non-smooth formulations) [281–288]. In the former case, the transition from non-contact to contact situations is described by a continuous function, yielding in simple and efficient solutions. The force-based models can exhibit numerical difficulties due to bad conditioned system matrices and need of small time steps [76]. In the non-smooth formulations, the colliding bodies are considered to be absolutely rigid, and unilateral constraints are utilized to prevent the local interpenetration from occurring. Some numerical difficulties can also be associated with non-smooth approaches, such as undetermined systems, requiring special techniques to handle them.

This review analyzes the main aspects related to contact mechanics in dynamical systems. The emphasis of this work is on the regularized methods and non-smooth formulations, where the fundamental ingredients of each approach are highlighted to treat collisions. Discussion of the extensive literature on numerical schemes for contact-impact problems is beyond the scope of this paper, the interested reader is referred to the references [289–294]. In addition, methods to deal with rolling contacts, adhesive contacts, surfaces roughness, thermal effects, and other specific aspects associated with collisions are not within the objectives of this review. Good representation of these, and other phenomena, may be found in the works by Johnson [3], Kalker [295], Jean et al. [296], Goryacheva [297], Wriggers [298], Popov [299], Yastrebov [300], Rao et al. [301], Stronge [302], and Barber [303].

The structure of this paper is organized as follows. Section 2 includes a historical perspective of contact dynamics, where special emphasis is given to the achievements reached during the last decades. Section 3 discusses general aspects associated with contact dynamics under the framework of multibody systems methodologies. The main available techniques to treat contact-impact events in multibody dynamics are characterized in Sect. 4. Subsequently, Sect. 5 deals with the fundamental features related to the geometry of contact, namely in terms of the definition of contacting surfaces and a contact detection procedure. A comprehensive description of regularized contact force models, both for normal and tangential directions, is presented in Sect. 6. Techniques based on non-smooth formulations are presented in Sect. 7. Several demonstrative examples of application and corresponding results are provided in Sect. 8. Finally, Sect. 9 addresses concluding remarks, where future directions for research under the framework of contact mechanics for dynamical systems are highlighted.

2 A brief history of contact mechanics

The problem of studying collisions between bodies is a quite old domain that was initiated simultaneously with the development of the science of mechanics, and that has become an important branch in the field of multibody dynamics. In fact, the topic of contact-impact problems in dynamical systems has received a great deal of attention in the past decades and still remains an active area of research that led to the establishment of important works and even the publication of relevant textbooks devoted to this theme, such as the ones by Pfeiffer and Glocker [304], Glocker [263], Leine and Nijmeijer [305], Pfeiffer [306], Acary and Brogliato [291], and Flores and Lankarani [294]. Additionally, the interested reader is also referred to the following seminal works on contact problems [267, 284, 307, 308].

Over the last five centuries, a good number of researchers have investigated the contact problems. Friction has been studied for more than 500 years [309]. Leonardo da Vinci measured the friction action using blocks with different contact areas, but with same weight [310]. According to his findings, the friction force is proportional to the weight of the block and not dependent on the apparent area of contact. Associated results are often attributed to Guillaume Amontons [311], neglecting the contribution of Leonardo da Vinci. Charles-Augustin de Coulomb put those findings in a well-known formula referred to as Coulomb's friction law, stating that the tangential force is equal to the normal force times the coefficient of friction [312]. Coulomb conducted an experimental study of frictional phenomena. Leonard Euler, who introduced the symbol μ for the coefficient of friction, demonstrated that for a block on a slope the dynamic coefficient of friction has to be smaller than the static coefficient of friction [313].

Galileo Galilei, who was a pioneer in recognizing the concept of rigid body collision, stated that the impact forces can become unlimited [314]. Christiaan Huygens performed studies on completely elastic collisions between two-point masses [315]. His work that describes the relative velocities inversion during impact was extended and formulated by Isaac Newton, in 1687, by the coefficient of restitution in order to accommodate the energy dissipation during the impact process [316]. Newton introduced the concept of kinematic coefficient of restitution, which is still quite used nowadays and can be described as the quotient between final and initial relative impact velocities normal to the contacting surfaces. For most of the engineering applications, the coefficient of restitution varies with relative initial impact velocity [317].

Poisson [318] introduced the kinetic coefficient of restitution as the quotient between normal impulses at the contact point that takes place during the compression and restitution phases. The use of Poisson's hypothesis dates back to the nineteenth century, when Routh presented a graphical approach to obtain the resulting impulses for the impact between two bodies [319]. Whittaker [320] extended Newton's impact law to include friction effect. The introduction of the friction into contact problems is of great importance and brings major difficulties [321, 322]. It must be highlighted that Newton's and Poisson's impact theories are equivalent for direct collisions between rough bodies if the direction of slip does not change during the contact process.

Fourier [323] and Boltzmann [324] studied unilateral behavior, taking into account mechanical principles. The scientific problem of vibrations developed in elastic rods under longitudinal impacts was investigated by Young [325]. Goldsmith [315] demonstrated that the effect of waves could be neglected if the contact duration is long enough when compared with the lowest eigenfrequency of impacting bodies. For these cases, the contact problem can be solved using a quasi-static approach, such as the Hertzian theory [326]. The Hertz contact law describes the static compression of two isotropic elastic bodies, surfaces of which can be approximated by two paraboloids in the vicinity of contact point. Sears [327] employed the Hertz contact theory to study longitudinal collisions. Goldsmith [315] observed that Hertz's law provides good results for collisions between two spheres and for the impact of a sphere against a thick plate, only if the materials involved are hard and the initial impact velocities are low.

The subject of contact mechanics and its applications in multibody dynamics had not been developed until the last few decades. Wittenberg [328], Wehage [329], and Khulief et al. [330] utilized a piecewise approach to handle impact events in multibody systems. In this discontinuous technique, the resolution of the equations of motion is halted at the instant of collision, where an impulse-momentum balance is performed to obtain the rebound velocities. The resolution of the equations of motion is then resumed with the updated velocities until a new collision takes place. Wehage and Haug [331] utilized Newton's impact law together with piecewise contact approach to discuss contact problems in constrained multibody mechanical systems. Khulief and Shabana [332] formulated the generalized impulse-momentum balance equations to analyze impacts in multibody systems.

The problem of friction in multibody dynamics was investigated by Khulief [333]. Battle and Condomines [334] utilized a Lagrangian formulation and impulsive drivers to maintain the continuity of a set of generalized velocities during the impact process to model collisions in dynamical systems. A similar analysis was conducted by Lankarani and Nikravesh [44] to treat multibody systems with intermittent motion. These authors demonstrated that the numerical resolution of the canonical equations of motion is quite efficient and stable. Haug et al. [335] formulated and solved the equations of motion using the Lagrange multipliers technique. Newton's hypothesis and Coulomb's friction law were considered to represent the impacts. The problem was replicated by Wang and Kumar [336] and Anitescu et al. [337], solution of which was obtained as a quadratic programing problem.

Hunt and Crossley [40], Khulief and Shabana [41, 42], Lankarani and Nikravesh [43], and Flores et al. [77] utilized a continuous approach and effective mass to model contact-impact events in multibody systems. Kuwabara and Kono [165] presented a viscoelastic contact force model and compared it with experimental data resulting from collisions between two pendula. Their force model was capable to predict the velocity dependence of the coefficient of restitution for low dissipative conditions. Inspired by Dubowsky and Freudenstein investigations [338, 339] and Hunt and Crossley [40], Kraus and Kumar [340] proposed a compliant contact approach for rigid body collisions able to overcome the deficiencies associated with Newton's and Poisson's theories. The algorithm presented was appropriate to handle the different regimens of contact points, which was demonstrated in the peg-in-hole insertion problem.

Kane [341] pointed out an apparent paradox on the application of Newton's impact theory with Coulomb's friction to a problem of the collisions in a double pendulum with the ground, leading to an overestimation of energy in the system. Kane and Levinson [342] observed that the solution of rigid body impact based on Newton's approach produces energetically inconsistent data. Newton's hypothesis is not able to predict changes in the direction of slip, which is the source of overestimation of the rebound velocity as a result of an impact. Pereira and Nikravesh [343] also solved the double pendulum impact problem using Newton's impact law, establishing bounds on the coefficient of restitution to achieve the correct energy balance. Keller [344] presented a solution to Kane's paradox, which results in widespread interest in the contact dynamics research community [3, 302, 304, 351].

Based on Keller's work, Hurmuzlu and Marghitu [345] developed a differential-integral approach and used different models for the coefficient of friction. Their approach was applied to a contact-impact problem in planar mechanical systems. Zhang and Sharf [346] proposed an integrated form of Keller's solution to deal with rebound velocities. Han and Gilmore [347] used an algebraic formulation for the equations of motion together with Poisson's impact theory and Coulomb's friction law to define the tangential motion. Different motion regimens that characterize the dynamic response (sliding, sticking, and reverse sliding) were examined by analyzing velocities and accelerations at the contact points. These authors verified and compared their numerical results with experimental data for simple unconstrained systems with two and three bodies.

Wang and Mason [348], based on Routh's approach, compared the coefficient of restitution given by Newton and Poisson. They conducted studies able to eliminate the energy overestimation when Newton's impact theory is considered, their solution being applied to unconstrained impacting bodies. Wang and Mason [349] also used Routh's approach to discuss the contact problem between a moving body against the ground. Smith [350] presented an algebraic solution to the impact problem utilizing Newton's impact law. Brach [351] proposed another algebraic solution, revising Newton's impact theory and introducing impulse ratios, to characterize the dynamic behavior in tangential direction.

Pfeiffer [352] utilized contact and friction constraints to model and analyze the stick-slip phenomena. Dierassi [353, 354], considering a recursive summation technique, presented a study on one-step evaluation of impulse components during sticking and discontinuous sliding. Stronge [355, 356] showed energy inconsistencies in some solutions with Poisson's hypothesis when the coefficient of restitution is considered to be independent of the coefficient of friction. More recently, Stronge [357] presented a comprehensive investigation on the energetically consistent calculation for oblique impacts in unbalanced systems with friction. Najafabadi et al. [358] described a study on the energy dissipation during impact in a three-link constrained planar system using the energetic coefficient of restitution.

The problem of rigid body collisions with multiple contact points was analyzed by Marghitu and Hurmuzlu [359]. A detailed analysis of energy dissipation within rigid body impacts was addressed by Chatterjee [360] and Batlle [361]. Glocker [362, 363] presented two comprehensive and detailed treatises on the energetic consistencies for standard impacts together with several applications.

It should be highlighted that most of the investigations described above are limited to unconstrained and planar systems. Papastavridis [364] presented an analytical dynamics formulation of constrained multibody mechanical systems of rigid bodies with impulse constraints. For simple multibody systems, Stronge [302] utilized the piecewise approach and the energetic coefficient of restitution to treat impact events. Glocker and Pfeiffer [48], based on the unilateral technique proposed by Moreau [365], used Poisson's impact theory together with a complementarity approach to obtain the normal and tangential impulses at the contact points in the context of dynamics of multibody systems. Johansson and Klarbring [366] developed an approach based on the impenetrability condition and Coulomb's friction law, where the equations of motion were formulated in terms of velocities and impulses rather than accelerations and forces [367].

Ahmed et al. [368] proposed a joint-coordinate Poisson-based canonical formulation for the treatment of frictional impact problems in constrained multibody mechanical systems. Subsequently, Lankarani and Pereira [2] presented a general formulation to model impacts with friction in open and closed multibody systems, where the kinetic coefficient of restitution defined by Poisson's hypothesis was considered. This methodology was able to correctly predict different regimens of tangential contacts, namely sliding, sticking, and reverse sliding. Lankarani [57] and Stoenescu and Marghitu [369] studied kinematic systems with impacts. Pereira and her co-workers [82, 83, 370], in a series of interesting papers, presented a complete and critical analysis of cylindrical contact force models for multibody dynamics. Their regularized approach was compared with FEM data and applied to dynamic modeling of chain drives [371]. Boos and McPhee [81] developed a volumetric contact force model, which combines both elastic and dissipative terms expressed as function of the volume indentation. Their force model was validated with experimental data. Several contact force models for different types of applications have been proposed in the same research group over the last years [86, 87, 92].

Uchida et al. [372] presented a general formulation able to model simultaneous frictional impacts in spatial multibody systems. Their impact approach, named as PLUS (acronym for Poisson, Lankarani, Uchida, and Sherman), was effective in capturing the main features in 3D contact events. Several other researchers have considered the problem of studying three-dimensional systems with frictional contacts over the last years [373–381]. Kleinert et al. [382] applied the non-smooth approach to study large-scale problems of granular matter, using differential variational inequality (DVI) to formulate and solve collisions. The DVI approach has been identified as a powerful tool to deal with multiple contact problems in dynamical systems [383–385]. Pazouki and his co-authors [386] presented a comparative analysis of regularized and non-smooth formulation in the context of granular media dynamics.

3 Fundamental issues in contact dynamics

The key features associated with both normal and tangential contact problems in dynamical systems are revisited in this section, since they constitute the main ingredients necessary for the process of modeling contact-impact events in multibody systems [43, 263].

Figure 1 represents the behavior of the one-dimensional central collision between two solid and isotropic spheres. For the sake of simplicity, let us consider that the spheres are moving with constant velocities and without any external forces. Before the collision, the velocity of sphere 1 is higher than the velocity of sphere 2, meaning that sphere 1 will collide with sphere 2. After the impact, the velocity of sphere 2 is higher than the velocity of the sphere 1, this implies that the two spheres separate from each other when the collision ends [387].

During the impact between the two spheres, local deformation or pseudo-penetration occurs, resulting in reaction normal contact forces that act over the contact period. Figure 1 also shows the evolution of the deformation and normal contact force at the impact duration, as well as the velocities and accelerations of each sphere before, during, and after the collision. In these diagrams, $t^{(-)}$ represents the instant just before the impact, $t^{(+)}$ denotes the instant immediately after the impact, and Δt is the duration of the impact, which is considered to be finite for illustrative purpose. In the collision represented in Fig. 1, the relative approaching velocity and the relative separating velocity are defined as, respectively,

$$\dot{\delta}^{(-)} = v_1^{(-)} - v_2^{(-)}, \tag{1}$$

$$\dot{\delta}^{(+)} = v_1^{(+)} - v_2^{(+)}. \tag{2}$$

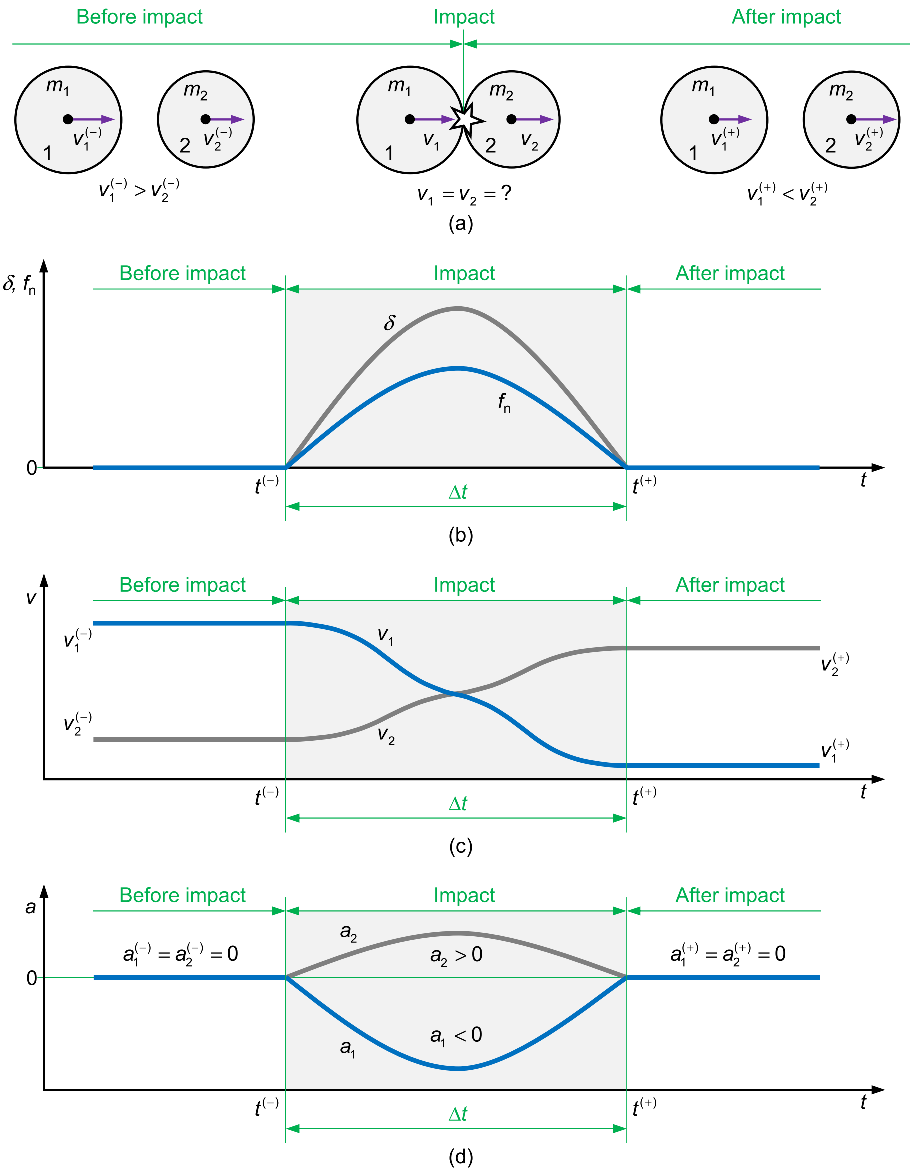


Fig. 1 (**a**) One-dimensional central collision between two solid spheres; (**b**) Deformation and normal contact force evolutions during the collision; (**c**) Velocities of the spheres before, during, and after the collision; (**d**) Accelerations of the spheres before, during, and after the collision

A contact-impact event happens during the collision of two or more bodies that may be external or belong to a multibody system [1, 43]. Poisson [318] divided the collision process into two distinct and complementary phases, namely the approaching (loading or compression) period and the separating (unloading or restitution) period. During the approaching phase, the colliding bodies deform in the normal direction of the contact, and the relative velocity of the colliding points is gradually reduced to zero. The end of the approaching phase is referred to as the instant of maximum deformation. The separation phase of the contact starts at this instant and finishes when the colliding bodies separate from each other. During the contact process, part of the kinetic energy of the system is dissipated due to propagation of waves, viscoelastic material behavior, and noise and heat generation [47, 52]. From the macro mechanical point of view, the several different ways by which kinetic energy dissipation happens are collectively condensed in the coefficient of restitution [388, 389].

The coefficient of restitution constitutes the foundation of the impact models of mechanical systems [307, 372]. For a fully elastic collision, this parameter is equal to unity, while for a fully inelastic collision, the coefficient of restitution is null. The most general and predominant type of collision involves a coefficient of restitution, value of which varies between 0 and 1 [390]. In rigid multibody systems the coefficient of restitution can be established at three different levels, namely kinematic, kinetic, and energetic, which correspond to Newton, Poisson, and Stronge hypotheses, respectively.

The kinematic coefficient of restitution, which is based on Newton's impact theory [316], can be established as the quotient between the relative normal velocities of the colliding bodies just after and just before the impact [315]. Newton's hypothesis can be written as

$$c_{\mathrm{r}} = -\frac{\dot{\delta}^{(+)}}{\dot{\delta}^{(-)}}. \tag{3}$$

The kinetic coefficient of restitution, which is based on Poisson's impact theory [391], is equal to the quotient between the accumulated normal impulses corresponding to the restitution and compression phase [315]. Poisson's hypothesis can be expressed by

$$c_{\mathrm{r}} = -\frac{p_{\mathrm{f}} - p_{\mathrm{c}}}{p_{\mathrm{c}}}, \tag{4}$$

in which p_{f} denotes the total normal impulse, or the final impulse, after the restitution phase, and p_{c} represents the normal impulse at which the relative normal velocity is null. It is clear that p_{f} is the accumulated impulse during the compression and restitution phases, while p_{c} is the impulse for the compression phase.

Finally, the energetic coefficient of restitution, which is based on Stronge's impact theory [391, 392], is equal to the square root of the negative of the ratio of elastic strain energy released during restitution to the internal energy of deformation absorbed during compression. Stronge's hypothesis can be written as

$$c_{\mathrm{r}} = \sqrt{-\frac{W(p_{\mathrm{f}}) - W(p_{\mathrm{c}})}{W(p_{\mathrm{c}})}}, \tag{5}$$

where $W(p_{\mathrm{f}})$ is the work done by the normal impulse during impact, and $W(p_{\mathrm{c}})$ represents the work done by the normal impulse during compression phase.

Frictional contact problems are fascinating in multibody dynamics not only due to their ubiquity, but also because of their complex nature [2, 263, 393]. By and large, friction happens when two contacting bodies have relative motion [1, 277–279]. In fact, two contacting bodies with no null relative tangential velocity develop friction forces acting in the opposite direction to the local relative motion.

Figure 2 represents the interaction of a solid block and the ground. In a stationary situation, the weight of the block f_{g} is balanced by the normal reaction force f_{n} as it can be observed in the diagram of Fig. 2a. It is clear that the block remains in stationary regimen if an external applied force f is not sufficient to move the block, as it is the case illustrated

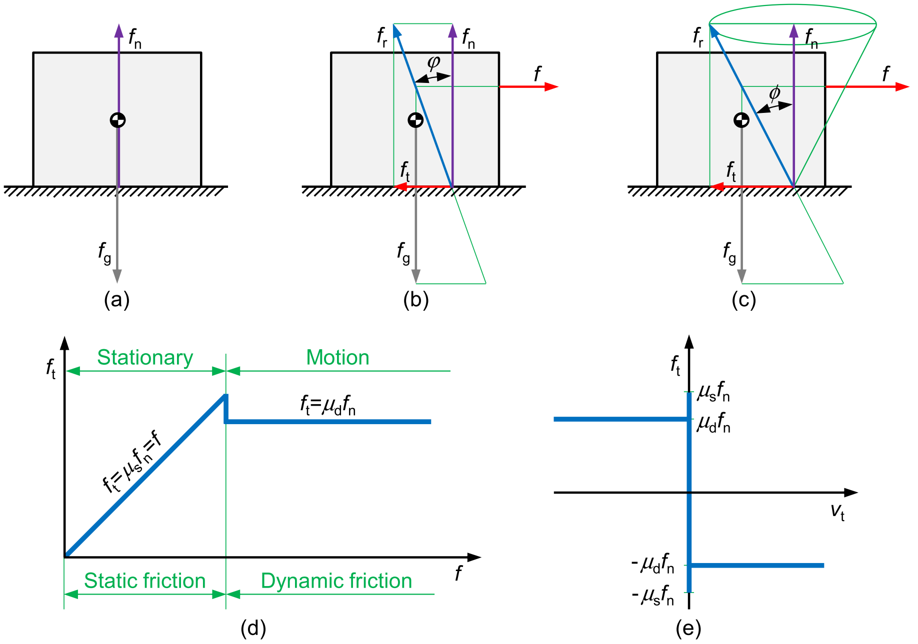


Fig. 2 (**a**) Stationary block on the ground without an external applied force; (**b**) Stationary block when a small external force is applied; (**c**) Block in motion due to a large external applied force; (**d**) Friction force evolution for the different regimens of the block; **(e)** Coulomb's friction law

in Fig. 2b. In this static situation, the friction force prevents the motion of the block from occurring. It can be verified that the friction force and the applied force cancel out each other, and the block keeps its stationary phase. Thus, when the external applied force acts on the block and it is maintained in a stationary regimen, the weight of the bock and the applied force are equilibrated by the oblique reaction force f_r as Fig. 2b depicts. The angle φ, represented in Fig. 2b, is the adhesion angle. For the stationary regimen, the static friction can be expressed by Coulomb's friction law [312]

$$f_t = \mu_s f_n, \tag{6}$$

where μ_s represents the static friction coefficient, and f_n denotes the normal reaction force.

The maximum friction force takes place at the end of the stationary phase, as it is represented in Fig. 2d, meaning that the motion of the block is in the eminence to be initiated. When block starts its motion, the magnitude of the friction force is reduced. This scenario is visible by the discontinuity of the plot in Fig. 2d.

When the external applied force f is large enough to move the block, the static friction changes to the dynamic friction. In this regimen, the oblique reaction force f_r has two components, namely the tangential and the normal forces, as Fig. 2c shows. It is clear that the tangential force f_t represents the friction force that opposes the relative motion. During the motion of the block on the ground, the friction force is given by Coulomb's law [312]

$$f_t = \mu_d f_n, \tag{7}$$

where μ_{d} represents the dynamic coefficient of friction, and f_{n} denotes the normal reaction force. It must be noticed that, in general, μ_{s} is greater than μ_{d} [394]. In the dynamic friction regimen, the angle ϕ is called friction angle. The cone illustrated in Fig. 2c is often referred to as the friction cone [347]. It is clear that when the reaction force f_{r} is situated inside this cone, there is no sliding of the block. The tangent of the friction angle is, by definition, the coefficient of the friction, that is,

$$\mu_{\mathrm{d}} = \tan\phi = \frac{f_{\mathrm{t}}}{f_{\mathrm{n}}} \tag{8}$$

which represents, in fact, Coulomb's friction law.

Figure 2e shows the representation of the Coulomb, or dry, friction force model, which states that the friction force opposes the relative motion of bodies and is proportional to the normal reaction force. Coulomb's friction model is dependent on the relative tangential velocity v_{t} except for the case of null velocity, where the friction force is a multivalued function of the external tangential force [263, 304].

4 Techniques to model contacts in multibody dynamics

The problem of modeling contact-impact events in multibody dynamics embraces two main tasks, namely the evaluation of the geometry of contact, and the resolution of the contact interaction. The first task incorporates the definition of the contacting surfaces and the contact detection procedure. For the determination of the contact points, gap functions are usually utilized, for which the point of minimum distance between the surfaces is used as the potential contact point [193, 395]. This procedure can be performed analytically or numerically, depending on the surfaces level of complexity. Furthermore, the contact detection step can be implemented independently of the contact resolution solver module [396, 397]. In turn, the resolution of the contact itself includes the calculation of the normal and tangential contact forces developed at the contact [33, 43, 77], as well as the application of the contact forces in the multibody system equations of motion under analysis. The technique selected to perform the contact resolution task must be able to handle the transition between different regimens at the contact points [57, 76, 304, 372].

There are two main techniques to solve contact dynamic problems, specifically: the regularized approaches (continuous methods) and the non-smooth formulations (piecewise methods) [2, 57, 395]. In the former techniques, also known as compliance or elastic methods, the contacting bodies are considered to be deformable at the contact zone, and the contact forces can be expressed as a continuous function of the local deformation between the contacting surfaces. In turn, in the non-smooth formulations, also called instantaneous or rigid methods, the contacting bodies are assumed to be truly rigid, and the contact dynamics is resolved by applying unilateral constraints in order to avoid the penetration from occurring [45, 191, 304].

The regularized approaches are quite important in the context of multibody dynamics because of their good computational efficiency and extreme simplicity to be implemented. However, in some circumstances, numerical problems can arise, resulting from bad conditioned system matrices [76, 398]. With the regularized methods there are no impulses at the impact process, hence there is no need for impulse dynamics computations. Therefore, the transition between contact and non-contact situations can easily be handled from the system configuration and contact kinematics [43, 61]. With these methods, the contact forces include spring-damper elements to prevent interpenetration from occurring, and no explicit kinematic constraints are utilized, but simply contact reaction forces are considered instead.

In the regularized approaches, the location of the contact point does not coincide in the contacting bodies, and a large number of potential, or candidate, contact points exist, the actual contact point being the one associated with the maximum indentation. Thus, relative pseudo-penetration between contacting bodies is permitted to occur, reason why the regularized methods are often called elastic approaches [395, 398]. The pseudo-penetration plays a key role as it is utilized to calculate the contact reaction forces according to an appropriate constitutive law [77]. In general, the contact force models can include viscoelastic and plastic terms [47, 49], as well as contact kinematics and geometric properties of the contacting surfaces [52, 399]. The existence of friction in the continuous methods can easily be incorporated by considering any regularized friction force model [106, 393, 400].

An inconvenience associated with regularized approaches deals with the estimation of the contact parameters, in particular when the contact geometry is of complex nature [104, 401]. A second difficulty, or limitation, of the regularized methods is the introduction of high-frequency dynamics into the system due to the existence of contact related spring-damper elements in the contacting surfaces. Thus, when the dynamics requires the integration scheme to take small time steps, the computational efficiency can be penalized [76]. In the methods based on non-smooth formulations, the contact points on both colliding bodies are necessarily coincident due to the unilateral constraints introduced into the system. In these methods, the relative interpenetration between the colliding bodies is not allowed, since the bodies are considered to be entirely rigid at the contact zone [284, 402, 403].

Assuming that the contacting bodies are absolutely rigid, as opposed to locally deformable bodies as in the regularized approaches, the non-smooth formulations resolve the contact-impact problems using unilateral constraints to determine impulses to avoid penetration from occurring. At the core of non-smooth methods is an explicit formulation of the unilateral constraints between colliding rigid bodies [45, 404].

The central idea of the non-smooth formulations is the non-penetration condition that only prevents bodies from moving toward each other and not apart, reason why this approach is called unilateral constraint [405, 406]. For this purpose, usually, a complementarity formulation is utilized to describe the relation between the contact force and the gap distance at the contact point. Such a unilateral constraint does not permit the interpenetration of the two colliding bodies and ensures that either the contact force or the gap distance is null. This means that, when the gap distance is positive (open or inactive contact), the corresponding contact force is null. Conversely, when the contact force is positive (closed or active contact), the gap distance is null [304]. Thus, this formulation leads to a complementarity problem, which constitutes the rule that permits to treat multibody systems with unilateral constraints [407, 408].

The numerical problems related to the regularized approaches do not appear in the non-smooth methods, but they lead to other difficulties and requirements [292, 395]. For instance, the existence of a unique solution is not guaranteed, because in some cases the system can be undetermined or have multiple solutions [291, 409–411]. In general, commercial multibody codes with collision and dry friction features deal with the non-smooth nature of the problem by an ad hoc regularized solution, using continuous models to avoid undesired interpenetration between bodies, which can ultimately lead to some numerical and computational difficulties.

Figure 3 shows the graphical representation of the normal and tangential contact forces for the regularized approaches and non-smooth formulations. In essence, the regularized approaches and the non-smooth methods, utilized to handle contact-impact events under the

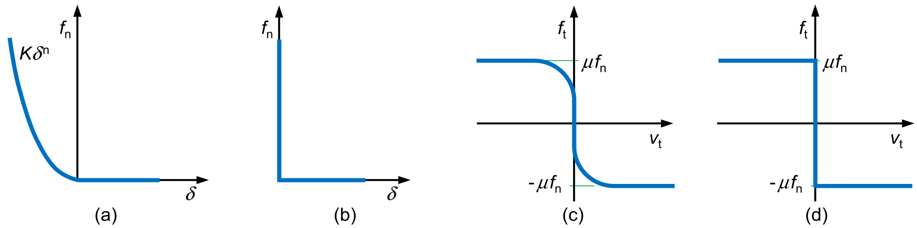


Fig. 3 (**a**) Regularized normal contact force model; (**b**) Non-smooth normal contact force model; (**c**) Regularized tangential contact force model; (**d**) Non-smooth tangential contact force model

Table 1 Summary of the main features associated with regularized and non-smooth techniques

Regularized approaches	Non-smooth formulations
Bodies can locally deform	Bodies are strictly rigid
Pseudo-penetration is allowed	Impenetrability condition is utilized
Contact forces are continuous	Impulse-momentum is applied
Can cause high-frequency	Are robust and stable
Small time steps are required	Large time steps can be used
Local properties can be difficult to establish	Local properties are simple to identify
Multiple contacts are easy to handle	Difficult for multiple contacts
Differential equations are stiff	Undetermined and multiple solution can arise
Easy to implement	Not easy to generalize

framework of multibody dynamics, inevitably have advantages and disadvantages. Anyway, none of these techniques briefly characterized above can be identified as superior. In fact, a particular multibody mechanical system with collisions might easily be described by one method; nevertheless, this does not automatically imply a general predominance of that formulation in all multibody applications [80, 175, 340, 412].

Table 1 presents some of key features associated with the regularized and non-smooth techniques, which allows for a simple and quick comparison. From the accuracy and fidelity of the results obtained, one critical issue related to frictional contact problems deals with the discretization, or modeling process, of the mechanical system under analysis. If the problem is well discretized, in general, both regularized and non-smooth techniques are effective to treat any frictional problem. In any case, the evaluation of the geometry of contact (contact detection) is the same regardless of the choice of the technique selected to model the contact interaction between the colliding bodies (contact resolution) whether the regularized approaches or non-smooth formulations are being used [260, 413].

5 Geometry of contact in multibody dynamics

The geometry of contact in multibody dynamics encompasses three fundamental aspects, namely: (*i*) the geometric description of the contacting surfaces; (*ii*) the identification of the potential contact points; (*iii*) the evaluation of the contact kinematics. These three features characterize the preparation phase of the contact modeling process in dynamical systems. The computational accuracy and efficiency of the preparation of the contact problems in

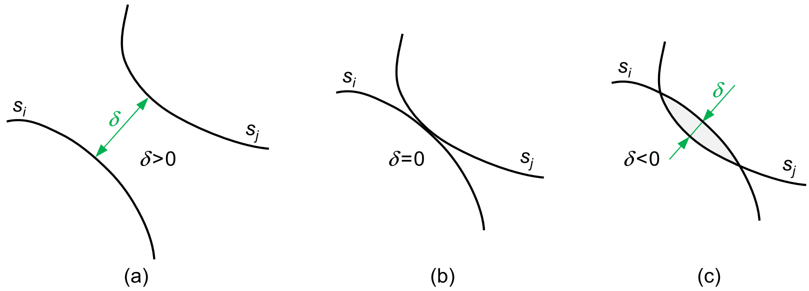


Fig. 4 (**a**) Open or inactive contact relative to a non-contact situation, $\delta > 0$; (**b**) Instant of the beginning of contact, $\delta = 0$; (**c**) Closed or active contact relative to a contact scenario, $\delta < 0$

multibody dynamics strongly depends on the level of complexity of the contacting surfaces [414, 415], the number of potential colliding elements [19, 384], and kinematics of the bodies [416, 417].

In most of the practical applications, the contact locus is considered to be punctual due to convex boundary nature of the surfaces where contact points might occur. The contacting surfaces of the colliding bodies can be defined by straight lines [418, 419], circles [134, 141], spheres [420, 421], planes [422, 423], polygonal meshes [251, 259, 261, 396], superquadric elements [75, 424, 425], superellipsoidal surfaces [19, 426–430], freeform surfaces [257, 415], etc. No matter how the contacting surfaces are established, it is required to search for the potential contact points in the moving bodies. A demanding task in the contact detection step in multibody dynamics is to check whether the potential, or candidate, contact points are in contact or not. For that, the point of minimum distance between the contacting surfaces is utilized as the potential contact point, employing gap distances [63]. Figure 4 shows three different scenarios between two generic contacting surfaces, where the gap distance δ assumes three distinct values [431], which allows for the identification of active (closed) and inactive (open) contacts.

It has been recognized that most of the time consumed in modeling and analyzing impact problems is spent in the contact detection task. For simple geometries, such as in revolute clearance joints [52] and granular media [26], the contact detection step can be performed analytically. In these cases, the location of the contact points is given explicitly by functions of the coordinates of the contacting bodies. Surfaces of complex nature, such as in human articulations [431] and rail-wheel systems [257], the identification of the potential contact points must be done considering numerical procedures [76]. The geometry of contact in multibody dynamics contemplates as input the geometry and kinematics of the simulated systems, and produces outputs according to the queries if, where, when, and which points are in contact. In fact, the geometry and kinematics of contacting surfaces constitute the fundamental ingredients to formulate and analyze contact-impact events in dynamical systems [403].

At this stage, it must be noticed that the contact detection step requires, in general, a tremendous computational effort due to the iterative nature of the numerical procedure utilized. This aspect plays a crucial role in complex surfaces and in problems with multiple and simultaneous contacts. Several authors have employed lookup-table-based techniques with the aim of improving the computational efficiency when dealing with collisions [415, 432–437]. In turn, problems with hundreds of simultaneous contacts have been simulated with GPU parallelization in order to distribute the computational cost associated with search of contact [438–442]. The computational accuracy and efficiency of modeling and

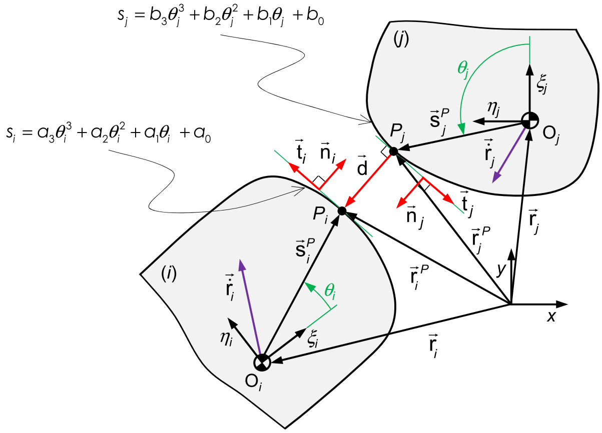


Fig. 5 Representation of the contact between two generic freeform profiles, where the gap distance is exaggerated with the purpose to include of the necessary geometric elements

analysis of dynamical systems with contact-impact events are features of central importance in computer games, virtual reality, and real-time simulation scenarios, where realistic and effective responses of the collisions are required [414, 443–446]. Another approach to reduce the time consumed during the contact detection step consists of building bounding objects of simple geometric nature, such as spheres or boxes. Thus, plausible contact scenarios are considered instead of taking into consideration all the possible contacts. Some of the most popular contact detection algorithms are the axis-aligned bounding box (AABB) trees, oriented bounding box (OBB) trees, binary space partitioning (BSP) trees, and inner sphere trees (IST) [259, 396, 415, 447–457].

In what follows, a general and straightforward procedure to treat the geometry of contact in multibody dynamics is described. Figure 5 depicts two generic contacting surfaces of two colliding bodies, which are represented by a collection of points. This type of freeform profile is branded by three key features, chiefly: (i) the spatial position; (ii) the sense of orientation; (*iii*) the measure of proximity, or distance, between bodies. Thus, the central issue is how to compute such representations in the context of multibody systems methodologies.

Firstly, let us consider that the potential contact points on bodies i and j are represented by P_i and P_j, respectively. Further, the contacting surfaces are defined by two cubic spline functions as [431]

$$s_i = a_3\theta_i^3 + a_2\theta_i^2 + a_1\theta_i + a_0, \tag{9}$$

$$s_j = b_3\theta_j^3 + b_2\theta_j^2 + b_1\theta_j + b_0, \tag{10}$$

in which a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3 are the cubic spline polynomial coefficients, and θ_i and θ_j represent the profile of the curve polar parameters that define the splines considered [458].

The distance function between potential contact points, P_i and P_j, of the two freeform profiles represented in Fig. 5, can be written as

$$\mathbf{d} = \mathbf{r}_j^P - \mathbf{r}_i^P, \tag{11}$$

where $\mathbf{r}_i^P$ and $\mathbf{r}_j^P$ are the global coordinates with respect to the inertial reference frame [459]

$$\mathbf{r}_k^P = \mathbf{r}_k + \mathbf{A}_k\mathbf{s}_k'^P \quad (k = i, j), \tag{12}$$

in which $\mathbf{r}_i$ and $\mathbf{r}_j$ are the global position vectors of bodies i and j, and $\mathbf{s}_i'^P$ and $\mathbf{s}_j'^P$ represent the local components of the two potential contact points. In turn, $\mathbf{A}_i$ and $\mathbf{A}_j$ denote the rotational transformation matrices [459].

The normal vector to the plane of collision can be defined as

$$\mathbf{n} = \frac{\mathbf{d}}{d}, \tag{13}$$

in which the magnitude of the vector $\mathbf{d}$ is given by

$$d = \delta = \sqrt{\mathbf{d}^{\mathrm{T}}\mathbf{d}}. \tag{14}$$

The tangential vector $\mathbf{t}$ can be obtained by rotating the vector $\mathbf{n}$ in the counter-clockwise direction by 90°, as shown in Fig. 5.

The first condition for the potential contact points P_i and P_j to be satisfied is that those points belong to the contacting surfaces of the colliding bodies i and j. The second condition corresponds to the minimum distance given by Eq. (11). Nevertheless, this equation is not sufficient to find the possible contact points between the two generic freeform profiles represented in Fig. 5, because it does not cover all the scenarios that can happen in a contact problem in multibody dynamics. Thus, the actual contact points are established as those that correspond to maximum penetration, that is, points of maximum indentation measured along the normal direction. It is worth noting that a normal contact direction in the contact detection process is not known beforehand and usually needs to be determined iteratively [460].

In a simple manner, potential contact points P_i and P_j of the two contacting profiles illustrated in Fig. 5 must fulfil the following four conditions: (i) the points belong to contacting surfaces of the bodies i and j; (ii) the distance between the candidate contact points, given by Eq. (11), corresponds to the minimum distance; (iii) the vectors $\mathbf{d}$ and $\mathbf{n}_i$ are collinear; (iv) the normal vectors $\mathbf{n}_i$ and $\mathbf{n}_j$ are collinear. Conditions (iii) and (iv) can be expressed as

$$\mathbf{d} \times \mathbf{n}_i = \mathbf{0}, \tag{15}$$

$$\mathbf{n}_j \times \mathbf{n}_i = \mathbf{0}. \tag{16}$$

It must be noted that in the current case, Eqs. (15) and (16) form a system of two nonlinear equations with two unknowns that can be solved numerically, employing, for instance, the Newton–Raphson iterative procedure [461]. This nonlinear problem has to be solved at every time step of resolution of the equations of motion of the multibody system under analysis. The obtained solutions correspond to the effective location of the potential contact points. Subsequently, the value of pseudo-penetration can be evaluated using Eq. (14).

The remaining information relative to the contact kinematics can be established based on the computation of the velocities of the contact points, which are expressed as [459]

$$\dot{\mathbf{r}}_k^P = \dot{\mathbf{r}}_k + \dot{\mathbf{A}}_k \mathbf{s}_k'^P \quad (k = i, j), \tag{17}$$

where the dot represents the derivative with respect to time. Then, the relative velocity of the contact points must be projected onto the normal and tangential directions of the contacting surfaces since they play a key role in the determination of this kind of contact dynamics problem. The scalar normal and tangential velocities are given by

$$v_{\mathrm{n}} = \dot{\delta} = \left(\dot{\mathbf{r}}_j^P - \dot{\mathbf{r}}_i^P\right)^{\mathrm{T}} \mathbf{n}, \tag{18}$$

$$v_{\mathrm{t}} = \left(\dot{\mathbf{r}}_j^P - \dot{\mathbf{r}}_i^P\right)^{\mathrm{T}} \mathbf{t}. \tag{19}$$

It is clear that the normal relative velocity defines whether the colliding bodies are approaching or separating. In turn, the tangential relative velocity establishes whether the contacting bodies are sliding or sticking, which are of paramount importance in the friction analysis in multibody dynamics [462–464].

In summary, the geometry of contact between two bodies in multibody dynamics is represented as the seven-tuple

$$C = \left\{ i \quad j \quad P_i \quad P_j \quad \delta \quad v_{\mathrm{n}} \quad v_{\mathrm{t}} \right\}^{\mathrm{T}}, \tag{20}$$

where i and j are the colliding bodies, P_i and P_j denote the contact points, δ is the pseudo-penetration, v_{n} represents the normal relative velocity, and v_{t} is the tangential relative velocity.

6 Regularized methods for dynamical systems

Following the formulation proposed by Nikravesh [459], the kinematic constraints in multibody systems can be described by algebraic equations in a compact form as

$$\boldsymbol{\Phi}(\mathbf{q}, t) = \mathbf{0}, \tag{21}$$

in which $\mathbf{q}$ represents the vector of generalized coordinates, and t denotes the time variable.

Based on the Lagrange multipliers technique, Nikravesh [459] presented the translational and rotational equations of motion for constrained multi-rigid-body systems as

$$\mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \boldsymbol{\lambda} = \mathbf{g}, \tag{22}$$

where $\mathbf{M}$ represents the generalized mass matrix, $\ddot{\mathbf{q}}$ denotes the vector of generalized accelerations, $\boldsymbol{\Phi}_{\mathbf{q}}$ is the Jacobian matrix, $\boldsymbol{\lambda}$ contains the Lagrange multipliers associated with the system's kinematic constraints, and $\mathbf{g}$ is the vector of generalized forces that includes all the external applied forces, such as those that result from contact-impact events.

In order to have a proper solution for the dynamic response of multibody systems, it is necessary to add the algebraic constraint Eqs. (21) to the equations of motion (22), resulting in a set of differential algebraic equations (DAE) of index 3. With the purpose to avoid this type of equations, which present some numerical difficulties, the acceleration constraint equations must be considered instead of using Eq. (21). Taking the second time derivative of Eq. (21) yields

$$\boldsymbol{\Phi}_{\mathbf{q}} \ddot{\mathbf{q}} = \boldsymbol{\gamma}, \tag{23}$$

where $\boldsymbol{\gamma}$ represents the right-hand side of acceleration constraint equations, which contains the terms exclusively function of position, velocity, and time.

Combining Eqs. (22) and (23), the equations of motion for a constrained multibody mechanical system can be written in the matrix form [459]

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \\ \boldsymbol{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{Bmatrix}. \tag{24}$$

This equation is a system of DAE of index 1 that is solved for accelerations and Lagrange multipliers. Then, the accelerations and velocities are integrated in time to determine the new velocities and positions. This numerical procedure is repeated until the final time of simulation is reached. Different strategies exist in the thematic literature to handle, for instance, the constraints violation and to ensure good computational accuracy and efficiency [160, 459, 465–471].

When two bodies come into contact with each other, both normal and tangential contact forces are applied and removed in a very short time interval, demanding for special attention in terms of the integrator scheme utilized in the resolution of the equations of motion (24). In general, integration algorithms with both variable time step and order are preferable [76]. The resolution of a collision problem in the context of multibody systems embraces two main tasks, namely the evaluation of contact forces and the introduction of those forces into the equations of motion. The contributions of the forces and moments that result from collisions to the vector of generalized forces **g** are determined by projecting the normal and tangential forces onto the x and y directions. The contact forces, which act at the contact points (see Fig. 6), are transferred to the center of mass of the colliding bodies, and the corresponding transport moments are also applied to each body. Thus, with regard to Fig. 6, the resulting forces and moments acting on the center of mass of colliding body i are computed as follows [459]:

$$\mathbf{f}_i = \mathbf{f}_\mathrm{n} + \mathbf{f}_\mathrm{t}, \tag{25}$$

$$\boldsymbol{\tau}_i = \mathbf{s}_i^P \times \mathbf{f}_i. \tag{26}$$

The corresponding forces and moments that act on colliding body j are defined as follows:

$$\mathbf{f}_j = -\mathbf{f}_i, \tag{27}$$

$$\boldsymbol{\tau}_j = -\mathbf{s}_j^P \times \mathbf{f}_i. \tag{28}$$

The contact forces in multibody dynamics, modeled with regularized methods, can be evaluated using appropriate constitutive laws, in which the forces vary in a continuous manner. In other words, when two bodies collide, the velocities are continuous during the impact duration (see Fig. 1d), as the bodies undergo a local deformation, or indentation. The regularized force models must account for energy store and energy dissipation processes during the contact period, which are typically modeled as spring and damper elements [340]. In most of the common applications, in the context of multibody systems, the normal and tangential contact forces are based on Hertz's law [326] and Coulomb's law [312], respectively.

The oldest and simplest contact force model is the one associated with Hooke's theory, which can be applied when a contact is active. This regularized force model considers a linear spring to mimic the contact interaction and can be expressed as [472]

$$f_\mathrm{n} = k\delta, \tag{29}$$

where k represents the spring stiffness related to the contact materials, and δ is the penetration between the contacting surfaces (14). The contact stiffness can be determined analytically, numerically, or experimentally. Figure 7a shows a generic representation of the force-penetration evolution for the linear Hooke contact force model. This approach is quite simple but does not account for any kind of energy dissipation during the contact process. In fact, Hooke's law is valid for collisions involving extremely low impact velocities [315].

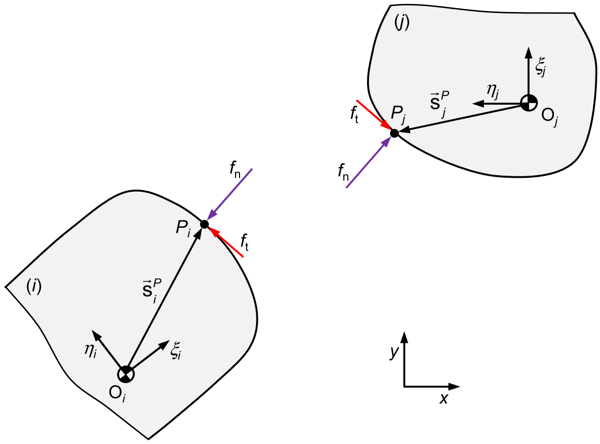


Fig. 6 Normal and tangential contact forces generated during a collision between bodies i and j

A more advanced contact force model was developed by Hertz. It considers a nonlinear relation between force and penetration as [326]

$$f_{\mathrm{n}} = K\delta^{n}, \tag{30}$$

where the nonlinear exponent n is typically equal to 3/2. The contact stiffness K can be determined analytically as a function of material properties and the geometry of contacting surfaces [294]. Figure 7b depicts the force-penetration relation for nonlinear Hertz's law. In a similar manner to Hooke's law, the Hertz contact force model is unable to predict any energy dissipation associated with the contact-impact events.

The first contact force model that accommodates energy dissipation in collisions is the Kelvin–Voigt approach. This model combines a linear spring with a linear damper to represent the contact forces as [315]

$$f_{\mathrm{n}} = K\delta + D\dot{\delta}, \tag{31}$$

where the first parcel is the elastic force term, and the second parcel denotes the dissipative force component, in which D represents the damping coefficient, and $\dot{\delta}$ is the normal relative velocity of the contacting bodies (18). Figure 7c shows the force-penetration relation for the linear Kelvin–Voigt contact force model. It is worth noting that this approach exhibits discontinuities at the beginning and ending of the contact process. In fact, the damping term originates finite forces when the penetration is null, which is not acceptable from a physical point of view. Furthermore, at the end of contact, the Kelvin–Voigt force model produces negative forces that are not correct because the bodies involved in the collision cannot attract each other.

Hunt and Crossley [40], in their seminal work, presented a contact force model that associates a nonlinear spring with a nonlinear damper in parallel to mimic the contact interaction. This force model can be expressed as

$$f_{\mathrm{n}} = K\delta^{n}\left[1 + \frac{3(1-c_{\mathrm{r}})}{2}\frac{\dot{\delta}}{\dot{\delta}^{(-)}}\right], \tag{32}$$

where the first term represents nonlinear elastic Hertz's law, and the second term is the dissipative parcel, c_{r} being the coefficient, and $\dot{\delta}^{(-)}$ is the normal contact velocity at the

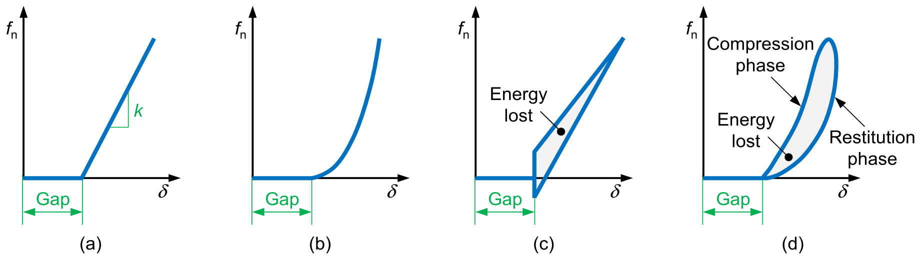


Fig. 7 Force-penetration relations for different contact force models: (**a**) Hooke's law; (**b**) Hertz's law; (**c**) Kelvin–Voigt approach; (**d**) Hunt and Crossley contact force model

initial instant of impact. Figure 7d illustrates the force-penetration evolution for the Hunt and Crossley contact force model, in which the compression and restitution phases of an impact can be identified. In this diagram, the area of the hysteresis loop represents the amount of energy lost during the impact process. The Hunt and Crossley force model does not present any discontinuity at the beginning or ending of the collision.

The most popular contact force model in the multibody dynamics community is the one proposed by Lankarani and Nikravesh [43], which was developed based on the Hertzian contact theory and on the damping approach by Hunt and Crossley. The contact force model presented by Lankarani and Nikravesh can be written as

$$f_{\mathrm{n}} = K\delta^{n}\left[1 + \frac{3(1-c_{\mathrm{r}}^{2})}{4}\frac{\dot{\delta}}{\dot{\delta}^{(-)}}\right], \tag{33}$$

which is valid for collisions with high values of the coefficient of restitution [43], that is, this model is applicable to elastic impacts [47]. The contact force model presented by Lankarani and Nikravesh has been utilized in many areas of science and engineering [473–489].

More recently, Flores et al. [77] described a contact force model applicable to the entire domain of possible values for the coefficient of restitution, which is given by

$$f_{\mathrm{n}} = K\delta^{n}\left[1 + \frac{8(1-c_{\mathrm{r}})}{5c_{\mathrm{r}}}\frac{\dot{\delta}}{\dot{\delta}^{(-)}}\right]. \tag{34}$$

The use of contact force models (33) and (34) provides a similar evolution of the force-penetration diagram as for the case of the Hunt and Crossley approach (see Fig. 7d). For low values of the coefficient of restitution, the hysteresis loop for the Flores et al. contact force model is larger [294]. It must be noticed that the contact force models (32)–(34) can exhibit some limitations when the contacts are too long, and when the velocity ratio $\dot{\delta}/\dot{\delta}^{(-)}$ becomes significantly less than 1 [19, 120, 435]. Over the last years, a good number of contact force models have been presented in the literature, the reader interested in detailed information is referred to the following references [29, 37, 61, 92, 97–100, 268, 270, 272, 275, 490, 491].

When two bodies collide with each other, besides the normal contact forces, tangential or friction forces are also generated. In fact, two contacting bodies with no null tangential relative velocity develop friction forces that act in the opposite direction to the local relative velocity. Haug et al. [335] directly solved the differential equations of motion by using the Lagrange multipliers technique. Newton's impact law was utilized for normal contact, while Coulomb's friction law was considered for the tangential contact. More recently, Haug revisited the problem of modeling friction in multibody systems [492, 493].

The most well-known friction force model is, undoubtedly, the one represented by Coulomb's law, which can be expressed as [312]

$$f_t = \begin{cases} \leq \mu_s f_n & \text{if } v_t = 0 \\ \mu_d f_n \operatorname{sgn}(v_t) & \text{if } v_t \neq 0 \end{cases} \tag{35}$$

with

$$\operatorname{sgn}(v_t) = \begin{cases} 0 & \text{if } \|\mathbf{v}_t\| = 0 \\ \frac{v_t}{\|\mathbf{v}_t\|} & \text{if } \|\mathbf{v}_t\| \neq 0 \end{cases}, \tag{36}$$

in which μ_s and μ_d represent the static and dynamic coefficients of friction, respectively, f_n denotes the normal contact force, and v_t is the tangential relative velocity of contacting elements (19). Figure 8a shows the graphical representation of Coulomb's friction force model. It must be noticed that this friction law exhibits some numerical difficulties in terms of computational implementation in multibody systems simulations because it does not give any specific value when the tangential relative velocity is null [256, 277–280, 435, 494–496]. This issue was well described by Glocker when stated that "*With that friction law, one has chosen one of the most complicated force laws that occur in application problems. It seems to be so easy and so clear at a first view, however, when trying to apply it, or even when just trying to write it down as a mathematical expression, one immediately encounters a lot of serious and not expected problems of very different nature*" [263].

Threlfall [497] proposed a regularized friction force model that does not present discontinuities, as it can be observed from the diagram of Fig. 8b. The Threlfall friction force model can be written as

$$f_t = \begin{cases} \mu_d f_n \left(1 - e^{-\frac{3v_t}{v_0}}\right) \operatorname{sgn}(v_t) & \text{if } v_t \leq v_0 \\ 0.95 \mu_d f_n & \text{if } v_t > v_0 \end{cases}, \tag{37}$$

where v_0 is a threshold velocity.

Bengisu and Akay [498] presented an alternative friction force model as

$$f_t = \begin{cases} \left[-\frac{\mu_s f_n}{v_0} (\|\mathbf{v}_t\| - v_0)^2 + \mu_s f_n\right] \operatorname{sgn}(v_t) & \text{if } v_t \leq v_0 \\ \left[\mu_d f_n + (\mu_s f_n - \mu_d f_n)\, e^{-\kappa(\|\mathbf{v}_t\| - v_0)}\right] \operatorname{sgn}(v_t) & \text{if } v_t > v_0 \end{cases}, \tag{38}$$

where κ is a positive parameter that represents the negative slope of the sliding state. Figure 8c depicts the evolution of the Bengisu and Akay friction force model.

Ambrósio [499] proposed another regularized approach for Coulomb's law that includes a ramp to avoid numerical difficulties. This friction force model can be written as

$$f_t = c_d \mu_d f_n \operatorname{sgn}(v_t) \tag{39}$$

with

$$c_d = \begin{cases} 0 & \text{if } v_t < v_0 \\ \frac{v_t - v_0}{v_1 - v_0} & \text{if } v_0 \leq v_t \leq v_1 \\ 1 & \text{if } v_t > v_1 \end{cases}, \tag{40}$$

in which the dynamic correction factor c_d prevents that the friction force changes direction for almost null values of the tangential relative velocity. Figure 8d shows Ambrósio's friction law.

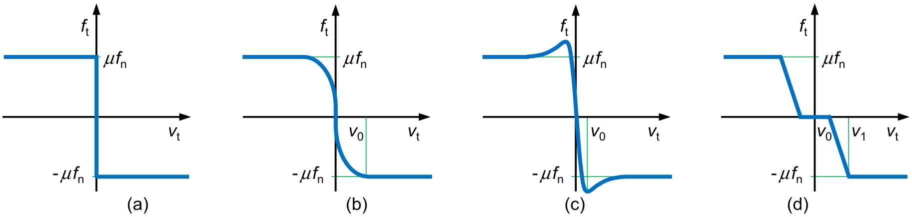


Fig. 8 Graphical representation of several friction force models: (**a**) Coulomb's friction law; (**b**) Threlfall friction force model; (**c**) Bengisu and Akay friction force model; (**d**) Ambrósio friction force model

The use of friction models (37), (38), and (39) has the advantage of allowing the numerical stabilization of the integration algorithm used during the resolution of equations of motion for constrained multibody systems. However, these approaches do not consider the stiction; thus, several alternative friction force models have been proposed over last decades, the interested reader is referred to the following references [61, 86, 276, 309, 340, 393, 394, 492–515].

7 Non-smooth formulations for dynamical systems

Non-smooth dynamics is characterized by discontinuities, or jumps, in the system's kinematic quantities, namely at the velocity level, which are the result of collisions [516]. Non-smooth theory has its roots in the work by Moreau [365], who established the foundations of this powerful formulation. Panagiotopoulos [517] expanded this methodology by introducing inequalities with regard to non-convex features. Pfeiffer and Glocker, in a series of cornerstone publications, developed and applied the non-smooth formulation to the case of multibody systems with contact-impact events [66–68, 186, 281–284, 306, 402]. In the non-smooth approach, the colliding bodies are considered to be rigid, that is, the contact zone does not deform in a classic sense. A fundamental law with respect to this concept is the complementarity rule often called Signorini's law [407]. This rule states that in contact dynamics either relative kinematic quantities are zero and the corresponding constraint force are not zero, or vice-versa [406].

The equations of motion of a multibody system with frictional unilateral contacts can be expressed, at the acceleration level, as [304]

$$\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} - \mathbf{w}_{\mathrm{N}}\boldsymbol{\lambda}_{\mathrm{N}} - \mathbf{w}_{\mathrm{T}}\boldsymbol{\lambda}_{\mathrm{T}} = \mathbf{0} \tag{41}$$

$$\dot{\mathbf{q}} = \mathbf{u} \quad \forall t, \tag{42}$$

where $\mathbf{M}$ is the positive-definite and symmetric mass matrix, $\dot{\mathbf{u}}$ represents the vector that contains the system accelerations, $\mathbf{h}$ denotes the vector of all external and gyroscopic forces acting in the system, $\mathbf{w}_{\mathrm{N}}$ and $\mathbf{w}_{\mathrm{T}}$ are the generalized normal and tangential forces directions, $\boldsymbol{\lambda}_{\mathrm{N}}$ and $\boldsymbol{\lambda}_{\mathrm{T}}$ are the normal and tangential contact forces, $\mathbf{q}$ represents the vector of generalized coordinates, and $\mathbf{u}$ is the system generalized velocities.

The solution of the equations of motion (41) requires incorporation of appropriate constitutive laws for the normal and tangential contact forces, such as the set-valued law of Signorini's rule and the set-valued law of Coulomb's friction model [263]. Non-smooth systems cannot be described solely by equations of motion (41) when impulsive forces exist.

Equalities of measures provide an elegant and effective way to obtain a valid and comprehensive description of non-smooth systems with impacts. Thus, when the equations of motion for impacts are integrated over a singleton in time, it yields

$$\mathbf{M}\left(\mathbf{u}^{+}-\mathbf{u}^{-}\right)-\mathbf{w}_{\mathrm{N}}\boldsymbol{\Lambda}_{\mathrm{N}}-\mathbf{w}_{\mathrm{T}}\boldsymbol{\Lambda}_{\mathrm{T}}=\mathbf{0} \tag{43}$$

$$\dot{\mathbf{q}}=\mathbf{u} \quad \text{a.e.,} \tag{44}$$

in which $\mathbf{u}^{-}$ and $\mathbf{u}^{+}$ represent pre- and post-impact velocities, and $\boldsymbol{\Lambda}_{\mathrm{N}}$ and $\boldsymbol{\Lambda}_{\mathrm{T}}$ denote the normal and tangential impulsive forces, which are well defined in the case of impacts.

In order to be able to use together the equations of motion without impacts (41) and the equations of motion with impacts (43), let us multiply these two set of equations by dt and dη, respectively, yielding

$$\mathbf{M}\dot{\mathbf{u}}\mathrm{d}t-\mathbf{h}\mathrm{d}t-\mathbf{w}_{\mathrm{N}}\boldsymbol{\lambda}_{\mathrm{N}}\mathrm{d}t-\mathbf{w}_{\mathrm{T}}\boldsymbol{\lambda}_{\mathrm{T}}\mathrm{d}t=\mathbf{0}, \tag{45}$$

$$\mathbf{M}\left(\mathbf{u}^{+}-\mathbf{u}^{-}\right)\mathrm{d}\eta-\mathbf{w}_{\mathrm{N}}\boldsymbol{\Lambda}_{\mathrm{N}}\mathrm{d}\eta-\mathbf{w}_{\mathrm{T}}\boldsymbol{\Lambda}_{\mathrm{T}}\mathrm{d}\eta=\mathbf{0}. \tag{46}$$

Thus, adding Eqs. (45) and (46) results in

$$\mathbf{M}\mathrm{d}\mathbf{u}-\mathbf{h}\mathrm{d}t-\mathbf{w}_{\mathrm{N}}\mathrm{d}\mathbf{P}_{\mathrm{N}}-\mathbf{w}_{\mathrm{T}}\mathrm{d}\mathbf{P}_{\mathrm{T}}=\mathbf{0}, \tag{47}$$

where the Lebesgue measure is represented by dt, and dη denotes the sum of the Dirac impulsive measures at the impacts. The measure for the velocities $\mathrm{d}\mathbf{u}=\dot{\mathbf{u}}\mathrm{d}t+(\mathbf{u}^{+}-\mathbf{u}^{-})\mathrm{d}\eta$ is split in Lebesgue measurable part $\dot{\mathbf{u}}\mathrm{d}t$, which is continuous, and the atomic part which occurs at the discontinuity points with the left and right limits $\mathbf{u}^{-}$ and $\mathbf{u}^{+}$, and the Dirac point measure dη. For impact free motion it holds that $\mathrm{d}\mathbf{u}=\dot{\mathbf{u}}\mathrm{d}t$. Similarly, the measure for the so-called percussions corresponds to a Lagrangian multiplier, which gathers both finite contact forces $\boldsymbol{\lambda}$ and impulsive contact forces $\boldsymbol{\Lambda}$, that is, $\mathrm{d}\mathbf{P}=\boldsymbol{\lambda}\mathrm{d}t+\boldsymbol{\Lambda}\mathrm{d}\eta$ [63, 66]. In the case of non-impulsive motion, all measures dη vanish and a formal division by dt yields the equations of motion (41).

It must be noticed that the system's kinematics, which are required to evaluate the contact and impulsive forces, can be expressed as [263]

$$\gamma_{\mathrm{N}}=\mathbf{w}_{\mathrm{N}}^{\mathrm{T}}\mathbf{u}+\tilde{w}_{\mathrm{N}} \tag{48}$$

$$\gamma_{\mathrm{T}}=\mathbf{w}_{\mathrm{T}}^{\mathrm{T}}\mathbf{u}+\tilde{w}_{\mathrm{T}} \tag{49}$$

that represent the normal and tangential relative velocities of the potential contact points, where $\mathbf{w}_{\mathrm{N}}$ and $\mathbf{w}_{\mathrm{T}}$ represent the generalized normal and tangential forces directions, and $\tilde{w}_{\mathrm{N}}$ and $\tilde{w}_{\mathrm{T}}$ are the Jacobian terms that represent the rheonomic constraints [66].

The equations of motion (47) can be complemented with appropriate laws for normal and tangential contact-impact forces. For this purpose, a unilateral version of Newton's impact hypothesis is utilized for the normal direction with coefficient of restitution ϵ_{N}. In turn, Coulomb's friction law is considered for the tangential direction with coefficient of friction μ which is complemented by a tangential coefficient of restitution ϵ_{T}. Thus, the normal and tangential contact-impact laws can be written as inclusions in the form

$$-\mathrm{d}\mathbf{P}_{\mathrm{N}}\in\mathrm{Upr}\left(\xi_{\mathrm{N}}\right), \tag{50}$$

$$-\mathrm{d}\mathbf{P}_{\mathrm{T}}\in\mu\mathrm{d}\mathbf{P}_{\mathrm{N}}\mathrm{Sgn}\left(\xi_{\mathrm{T}}\right), \tag{51}$$

with

$$\xi_{\mathrm{N}} := \gamma_{\mathrm{N}}^{+} + \varepsilon_{\mathrm{N}} \gamma_{\mathrm{N}}^{-}, \tag{52}$$

$$\xi_{\mathrm{T}} := \gamma_{\mathrm{T}}^{+} + \varepsilon_{\mathrm{T}} \gamma_{\mathrm{T}}^{-}, \tag{53}$$

where

$$\left(\gamma_{\mathrm{N}}^{+}, \gamma_{\mathrm{T}}^{-}\right) := (\gamma_{\mathrm{N}}, \gamma_{\mathrm{T}}) \left(\mathbf{u}^{\pm}\right). \tag{54}$$

Finally, the complete description of the dynamics of non-smooth systems, which accounts for both contact and impact phases, is given by Eqs. (47)–(54). This problem can be solved by using Moreau's time-stepping method as a linear complementarity problem (LCP) [518] or as an augmented Lagrangian approach [367].

At this stage, it is opportune to revisit the concepts associated with Eqs. (50) and (51), namely the unilateral primitive and the Sgn-multifunction [63, 263, 519]. The set-valued map unilateral primitive is a maximal monotone set-value map related to complementarity problems, which can be written as

$$\mathrm{Upr}(x) := \begin{cases} \{0\} & x > 0 \\ (-\infty, 0] & x = 0 \\ \emptyset & x < 0 \end{cases}. \tag{55}$$

The graphical representation of the unilateral primitive map is presented in Fig. 9a. It is clear that each complementarity condition of an LCP can be expressed as one unilateral primitive (Upr) inclusion as [63]

$$-y \in \mathrm{Upr}(x) \quad \Leftrightarrow \quad y \geq 0, \quad x \geq 0, \quad xy = 0. \tag{56}$$

The second maximal monotone set-valued map is the filled-in relay function Sgn-multifunction, which can be written as [63, 263, 519]

$$\mathrm{Sgn}(x) := \begin{cases} \{+1\} & x > 0 \\ [-1, +1] & x = 0 \\ \{-1\} & x < 0 \end{cases}. \tag{57}$$

It is important to highlight that, while the classical Sgn-function is defined with $\mathrm{Sgn}(0) = 0$, the Sgn-multifunction is set-valued at $x = 0$. The graphical representation of the Sgn-multifunction is depicted in Fig. 9b. An inclusion in the Sgn-multifunction can always be represented by two inclusions involving the unilateral primitive [63]. This decomposition, illustrated in Fig. 9c, can be expressed as

$$-y \in \mathrm{Sgn}(x) \quad \Leftrightarrow \quad \exists x_{\mathrm{R}}, x_{\mathrm{L}} \,\mathrm{s.t.} \begin{cases} -y \in +\mathrm{Upr}(x_{\mathrm{R}}) + 1 \\ -y \in -\mathrm{Upr}(x_{\mathrm{L}}) - 1 \\ x = x_{\mathrm{R}} - x_{\mathrm{L}} \end{cases} \tag{58}$$

8 Examples of application

This section comprises several examples of application that are utilized to illustrate the key role played by the modeling process of contact-impact events in dynamical systems.

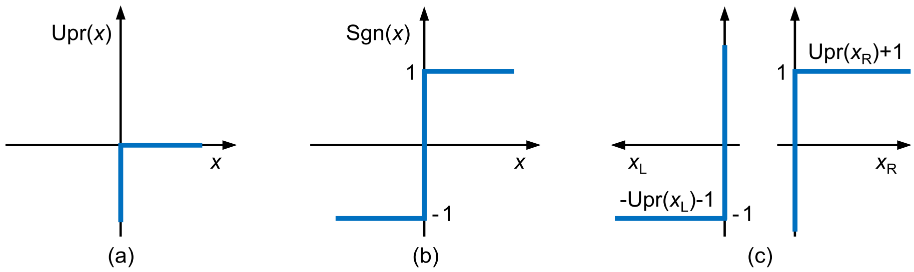


Fig. 9 (**a**) The map $x \to \mathrm{Upr}(x)$; (**b**) The map $x \to \mathrm{Sgn}(x)$; (**c**) The decomposition $\mathrm{Sgn}(x)$ into $\mathrm{Upr}(x)$

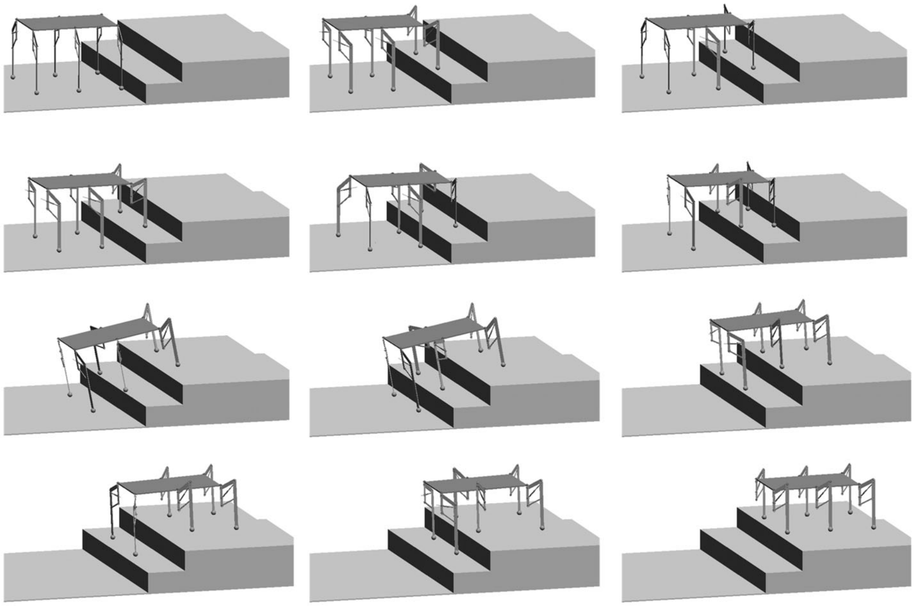


Fig. 10 Snapshots of the hexapod robotic system of a standard set of stairs climbing dynamic simulation

1. ***Hexapod robotic system*** **[520]** The first example is a hexapod walking machine that involves normal and tangential contact phenomena between the feet and ground surfaces and stairs. Figure 10 shows a three-dimensional multibody model of the hexapod robotic system analyzed, which is composed of a mainframe and six similar and symmetrically distributed legs. Each leg is comprised ofa four-bar linkage connected to the main body by means of a revolute joint. The hexapod system operates by six rotational motors and six linear actuators, which accomplish traction and elevation motions, respectively. A spherical foot is rigidly attached to each leg, the normal and tangential contact interactions with ground and stairs are modeled with regularized approaches. Two representative computational simulations have been performed, which allows to assess the dynamic behavior of the hexapod system. In the first simulation, a straight path on a planar horizontal surface is considered, while the second scenario deals with climbing a standard set of stairs. Figure 10 depicts an animation sequence of the computational simulation relative to the stairs climbing case.

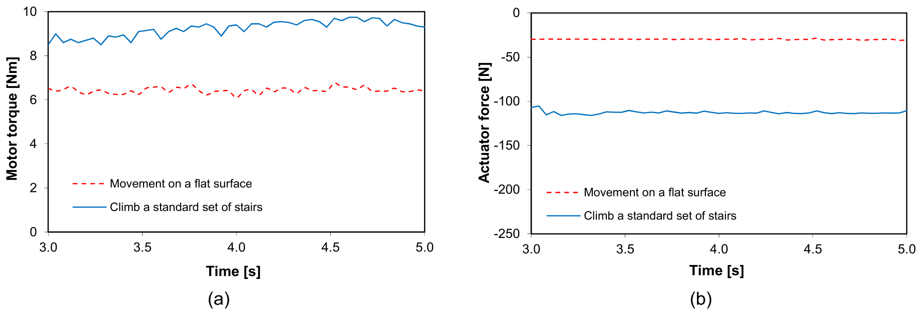


Fig. 11 (**a**) Torque developed in the rotational motor during the hexapod traction motion; (**b**) Force generated in the linear motor on the front leg during the traction motion

Fig. 12 Planar slider-crank mechanism with a revolute clearance joint between the connecting-rod and slider

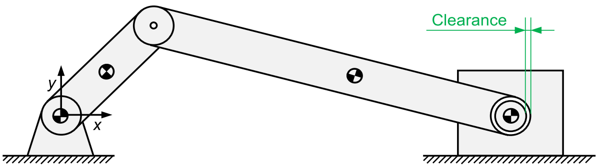


Figure 11 illustrates the time evolution of the torque and force developed in the rotational and linear drivers of a front leg for the motion on a flat surface and stairs climbing. The worst scenario in terms of mechanical load on the machine components occurs in the stairs climbing case. Overall, this study permits to examine how critical the contact process is for the success of hexapod motion simulations. In particular, the contact detection procedure adopted, as well as the smooth transition between different contact regimens, is of paramount importance to ensure dynamic stability of the hexapod robotic system.

2. *Revolute joint with clearance* [521] The existence of a gap, or clearance, in actual joints is necessary for the functionality of the mechanical systems. A revolute clearance joint, the so-called journal-bearing, can be modeled by contact-impact forces generated between the journal and bearing surfaces. For that, regularized methods are utilized to evaluate the normal and tangential contact forces. Figure 12 shows a planar slider-crank mechanism that includes a revolute joint with clearance, namely the one located at the slider body. In this type of joint, the journal can freely move inside the bearing. The remaining joints of the slider-crank multibody model are considered to be ideal joints, that is, they are modeled with kinematic constraints.

The dynamic behavior of the slider-crank multibody model is displayed in Fig. 13, where the torque acting on the crank body and the journal center trajectory inside the bearing boundaries are plotted. The results are relative to two complete crank rotations after the steady-state has been reached, and they are plotted against those obtained with an ideal joint. The crank torque diagram presents high peaks that are associated with impacts between the journal and bearing surfaces, as depicted in Fig. 13a. Moreover, the smooth evolution of the crank torque indicates that the journal and bearing surfaces are in continuous contact regimen, meaning that the journal follows the bearing wall. These scenarios can also be observed in the plot of Fig. 13b, where the different types of relative motion between the journal and bearing elements are visible, namely the free flight motion, the continuous contact mode, and the impacts followed by rebounds. The points plotted outside the clearance

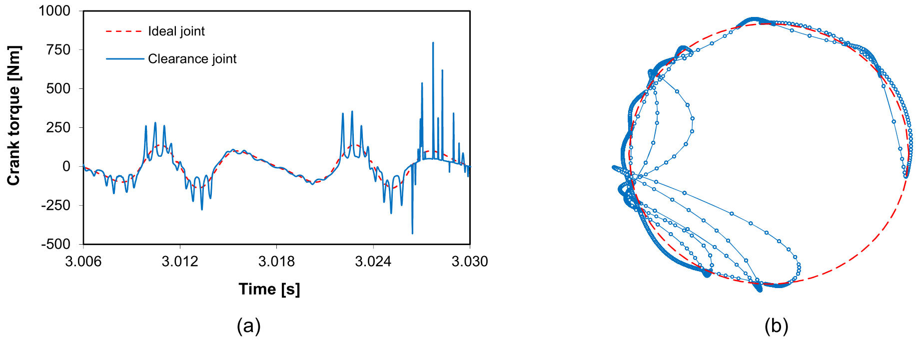


Fig. 13 (**a**) Crank torque; (**b**) Journal center trajectory relative to the bearing

circle represent the penetration between the journal and the bearing. In addition, during the free flight motion, the distance between two consecutive markers is larger, which means that the integration scheme is able to adjust the time step for the different scenarios. Thus, for the first impact after the free flight motion, the integrator decreases the time step to ensure that the first penetration depth does not exceed the one physically acceptable for the materials involved in the impact process. This procedure permits to demonstrate the importance of using integration schemes with both variable order and step when simulating multibody systems with contact-impact problems [76].

3 ***Spherical joint with clearance*** **[421]** Spherical joints with clearance can be employed, for instance, on car suspensions and human hip articulations models. Figure 14 depicts a generic representation of a spherical clearance joint within a multibody system, which is composed of a ball that can freely move inside the socket. A spherical joint with clearance does not impose any kinematic constraint to the system, but it can be modeled by intra-joint contact forces that are the result of collisions between ball and socket surfaces. Thus, the regularized models described above are utilized to determine the intra-joint normal and tangential contact forces.

Figure 15 shows the path of the ball center within the socket boundaries for the dynamic response of a spatial four-bar mechanism that includes a spherical clearance joint. It should be noted that the gray half-spherical surface represents the radial clearance size, while the small spheres inside it denote the ball center trajectory. The first six impacts between the ball and socket elements are illustrated in Fig. 15a, which are immediately followed by rebounds. The free flight motions of the ball are represented by clear spheres, whilst the impacts are illustrated by dark (red) spheres. Figure 15b shows the ball and the socket continuous contact motion, after steady state has been reached, meaning that the ball remains in contact with the socket wall.

4. ***Translational joint with clearance*** **[522]** A translational clearance joint is composed of a prismatic guide that holds a slider block. Figure 16 illustrates a planar slider-crank mechanism with a translational clearance joint between the ground and slider elements. The presence of the clearance joint introduces two extra degrees of freedom and permits free motion of the slider inside the guide limits. Figure 16 also shows four possible configurations of the slider with respect to the upper and lower guide surfaces. The modeling process of translational joints with clearance involves the precise contact detection and transition between

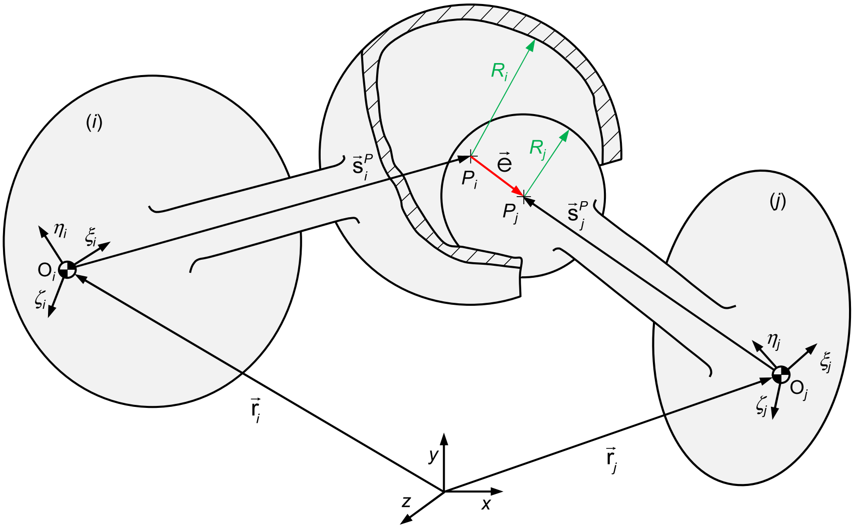


Fig. 14 Representation of a generic spherical joint with clearance connecting bodies i and j

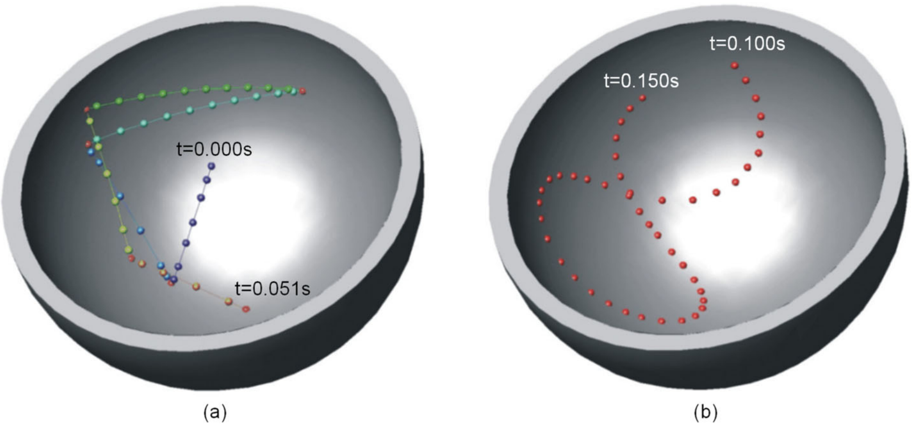


Fig. 15 Ball center trajectory inside the socket limits: (**a**) First instants of simulation where free flight motion and impacts followed by rebounds are visible; (**b**) Continuous, or permanent, contact between the ball and socket wall (Color figure online)

these four different scenarios. The problem of handling translational clearance joints under the framework of multibody dynamics was solved using both regularized methods [523] and non-smooth formulations [522].

Figure 17 depicts the phase space portrait of the connecting rod and the dimensionless motion of the slider inside the guide for two full crank rotations after the steady-state has been reached. These diagrams allow for the identification of the different types of slider motion inside the guide, namely impacts followed by rebounds, which are visible in the discontinuities at the velocities. Additionally, periods of continuous, or permanent, contact between the slider and the guide walls can also be observed in the plots of Fig. 17.

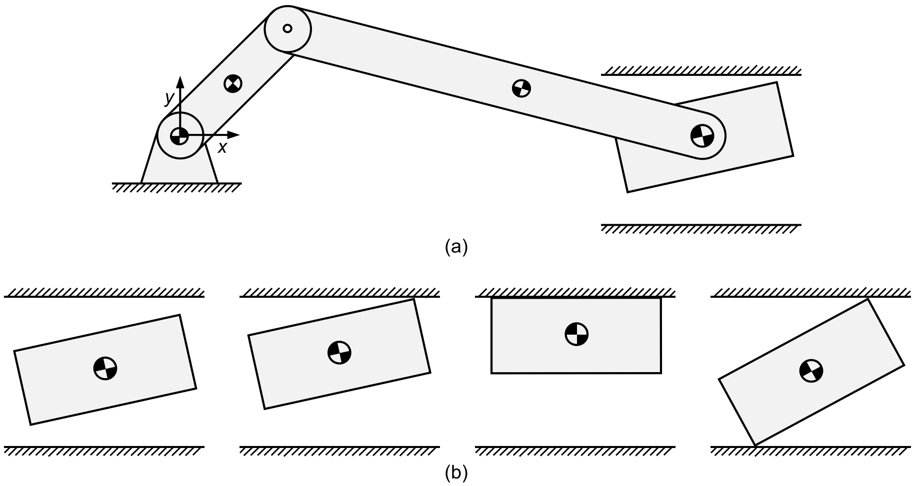


Fig. 16 (**a**) Planar slider-crank mechanism with a translational joint with clearance; (**b**) Four different scenarios for the slider position with respect to guide limits

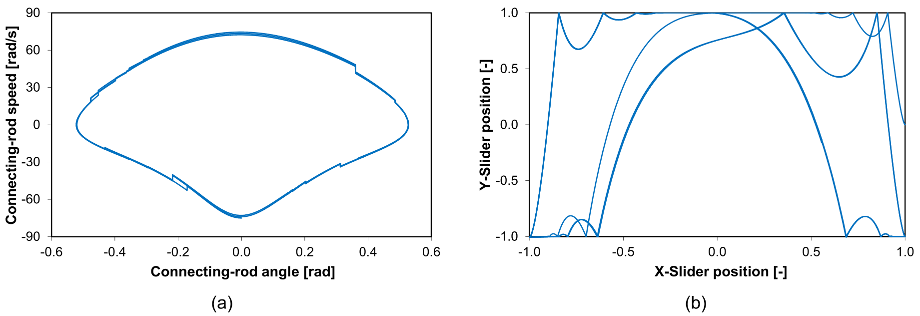


Fig. 17 (**a**) Phase space portrait of the connecting rod; (**b**) Dimensionless motion of the slider inside the guide

5. *Cam-follower mechanism* [193] A cam-follower system of an industrial cutting file machine was considered as a demonstrative example of application of the regularized and nonsmooth techniques to handle contact-impact events [524–526]. Figure 18 shows a picture of a machine-tool used to produce files, as well as the cam-follower mechanism responsible for the motion of the cutting beater. The contact-impact phenomena that occur between the cam and the follower must be precisely modeled since they strongly affect the quality of the files produced. The cam-follower mechanism operates with high loads and high speeds, the cam being composed of six rebounds that produce a small follower displacement. Therefore, these ingredients make the numerical process of modeling the collisions between the cam and the follower surfaces quite demanding in terms of both the computational accuracy and the efficiency points of view.

Figure 19 illustrates an animation sequence of the dynamic computational simulation of the cam-follower motion during the first instants after the follower reaches the up dead point until the cam and the follower experience a new contact. It must be noted that this time

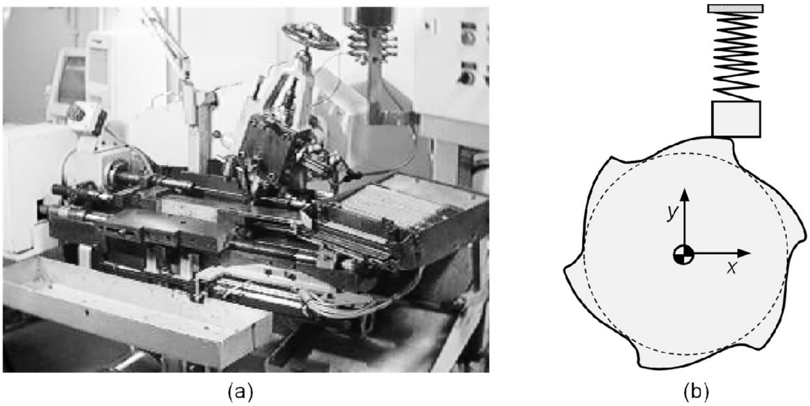


Fig. 18 (**a**) Picture of an industrial cutting-file machine; (**b**) Cam-follower mechanism used in the machine-tool

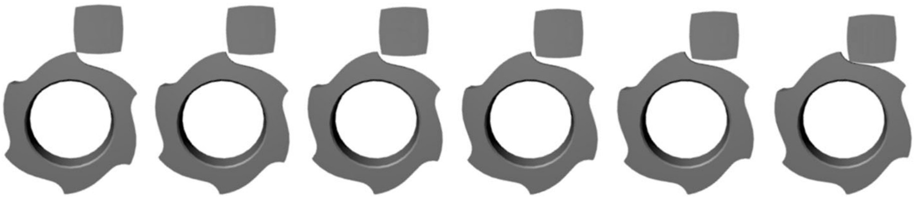


Fig. 19 Animation sequence of the virtual simulation of the cam-follower movement during the first instants after the follower reaches the up dead point

interval is of paramount and crucial importance to ensure that the machine-tool produces files with appropriate quality [524–526].

6. *Woodpecker toy* [367] The woodpecker toy is one of the most popular multibody benchmarks in the field of systems working with frictional contacts. Figure 20 shows a picture of the woodpecker toy and the corresponding multibody mechanical system. This model is composed of a pole, a sleeve that operates with some amount of clearance, a helical spring, and the woodpecker itself. The motion of the woodpecker is simple and intuitive, being acted by the gravity effect only. During the descend motion of the woodpecker, several frictional contacts can be activated, namely the contact between the beak and the pole, and the contact interaction between the sleeve and the pole. This last case can be seen as a translational joint with clearance.

The frictional contacts that may occur in the dynamics of the woodpecker have been simulated with regularized methods [527] and non-smooth approaches [193]. Figure 21 displays an animation sequence of the global motion produced by the woodpecker during, approximately, one period. These representative diagrams permit to identify the dynamic behavior of the toy as well as the different motion phases. In particular, the contact-impact events in terms of sliding and locking of the woodpecker are visible since the woodpecker motion is stable with a regular solution with a period equal to 0.146 s [48, 186, 187, 190, 367].

7. *Human knee articulation* [431] The natural and healthy human knee articulation is a synovial joint that connects the distal condylar surfaces of the femur, the proximal condylar

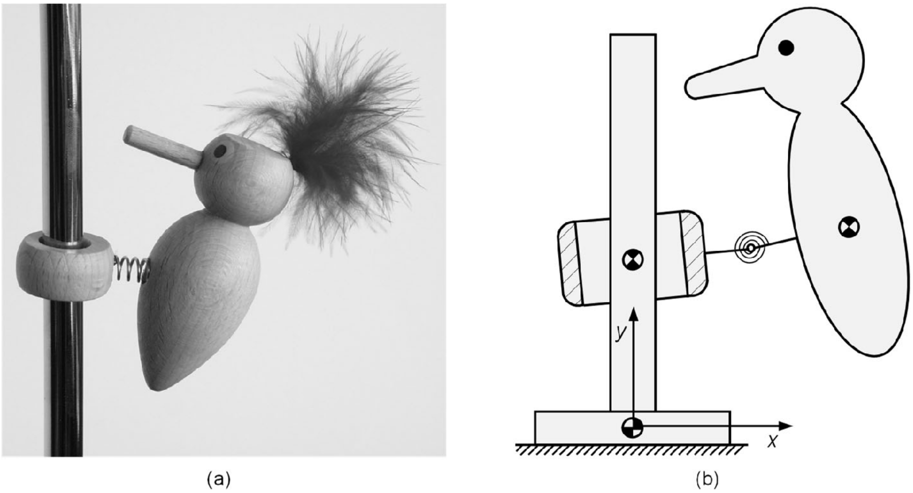


Fig. 20 (**a**) Picture of a woodpecker toy; (**b**) Equivalent multibody mechanical model

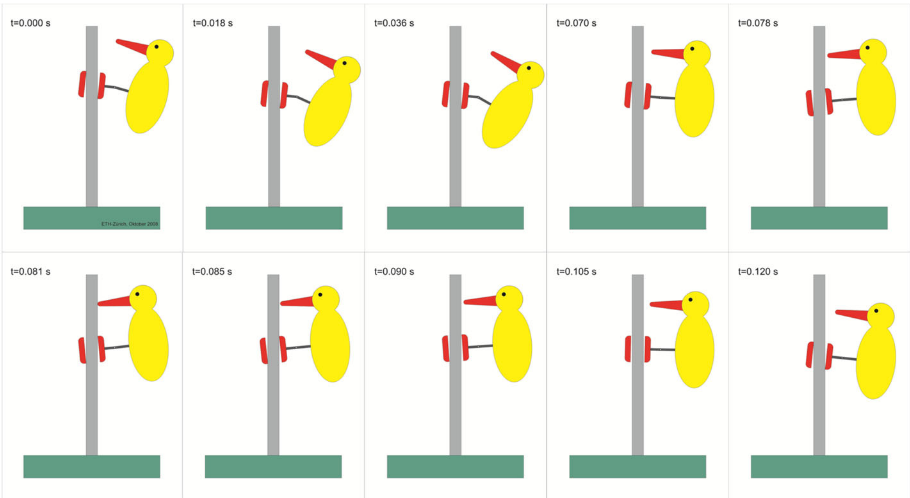
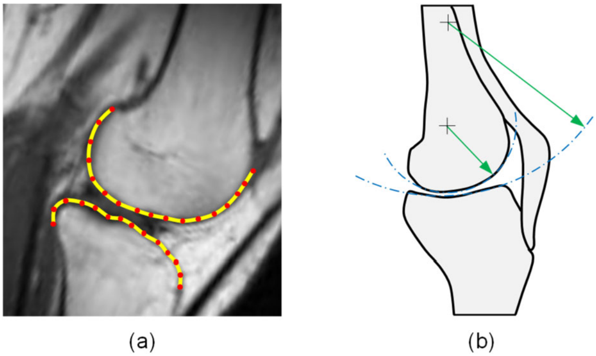


Fig. 21 Snapshots of the woodpecker descend motion computational simulation

surfaces of the tibia, and the posterior surface of the patella. Figure 22a shows the outlines of the femur and the tibia free form profiles of the knee joint in the sagittal plane. The collection of points can be described by cubic splines in order to define a free form contact pair, for which the contact detection approach described in Sect. 5 can be applied. The geometric definition of the femur and the tibia can also be done using revolute joints with clearance, as Fig. 22b depicts [78].

Figure 23a presents a two-dimensional multibody knee model that is composed of tibia and femur segments, which are connected by four ligaments, namely the two cruciates and the two collaterals, modeled as nonlinear elastic springs [431]. This biomechanical multibody system has been solved using the regularized approach presented above in what con-

Fig. 22 (**a**) Magnetic resonance image of the human knee articulation in the sagittal plane; (**b**) Femur and tibia profiles defined as a revolute joint with clearance



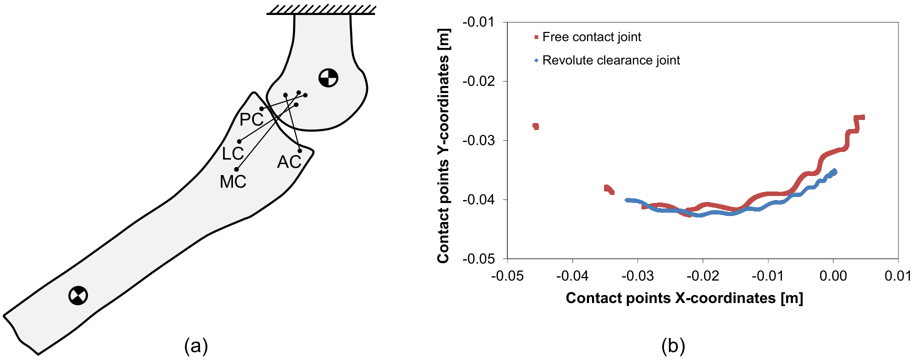


Fig. 23 (**a**) Multibody knee joint model composed of femur and tibia components and four primary ligaments; (**b**) Tibia contact points for human knee articulation modeled as a free contact joint and as a revolute clearance joint

cerns the contact detection and resolution phases. Figure 23b shows that the free form approach exhibits a knee flexion larger than that for the case of revolute joint with clearance formulation. It can be observed that the contact between the femur and tibia surfaces is essentially continuous, meaning that the two anatomical segments are in permanent contact. This is sound because the human knee experiences contacts at low impact velocities.

8. *Human foot-ground interaction* [528] Figure 24 displays a two-dimensional biomechanical foot system utilized to simulate the contact interaction with ground. This multibody model encompasses three rigid bodies that represent the shank, the main foot segment, and the toes. The ground is a fourth body which is considered to be rigid, flat, and smooth. Two ideal revolute joints connect the shank and toes to the main foot part. A torsional springer-damper element is attached at the metatarsal-phalangeal articulations. In order for the foot-ground interaction to be modeled, the plantar surface of the foot is represented by a set of spherical surfaces with adjustable radii and locations [294, 529, 530].

The behavior of the foot-ground multibody system is performed based on a forward dynamic analysis using experimental data to establish the necessary prescribed kinematic guide elements. Furthermore, the contact interaction between the foot plantar surface and the ground is modeled by considering regularized approaches for the normal and tangential contact-impact forces. The contact detection procedure comprehends simple geometries,

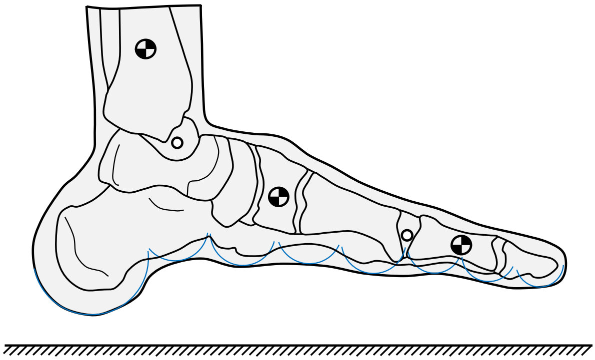


Fig. 24 Planar biomechanical multibody foot-ground model

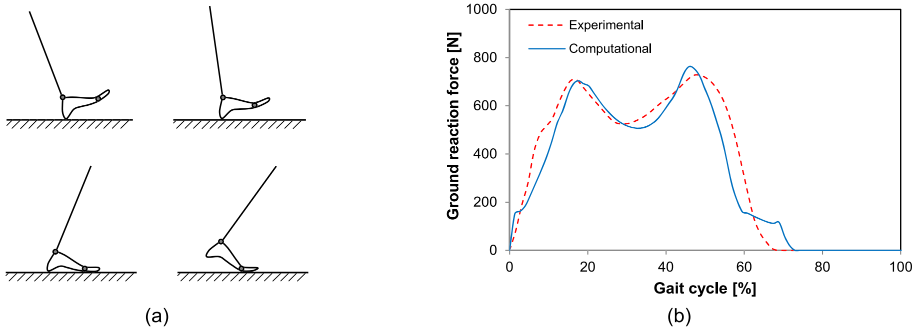


Fig. 25 (**a**) Animation sequence of the human gait cycle of the biomechanical foot model; (**b**) Vertical ground reaction force for a complete gait cycle

namely spheres and planes. Figure 25a illustrates an animation sequence of the global motion that results from the computational simulation, which allows for the identification of different human gait phases, namely the swing and the stance periods. Figure 25b shows the vertical ground reaction force diagrams for both experimental and computational approaches, which are in accordance.

9 Concluding remarks

A comprehensive review of contact mechanics for dynamical systems has been presented in this paper. For that, the regularized methods (continuous or contact force-based approaches) and non-smooth formulations (piecewise or geometric based approaches) have been compared as the main available techniques to treat contact-impact events in multibody systems. In the sequel of this process, the principal features associated with the definition of the contacting surfaces and the contact detection procedures have been analyzed. Several demonstrative examples of applications in the umbrella of multibody systems methodologies have been discussed, which allowed to highlight the key aspects related to the process of modeling contact-impact events in dynamical systems. Future directions for research under the framework of contact mechanics in multibody dynamics may include the following: the identification and estimation of the contact parameters for complex scenarios; the development of benchmark problems to assess the suitability of the existing techniques to handle contact-impact events; the analysis of contact problems with very large contact areas; the study of contacts with very flexible bodies; the development of techniques to accelerate the contact detection with multiple potential contacts.

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