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An optimization method for overdetermined kinematic problems formulated with natural coordinates

Sergio Ausejo · Ángel Suescun · Juan Celigüeta

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Abstract In this paper, we present an optimization method for solving the nonlinear constrained optimization problem arising from a motion reconstruction problem formulated with natural coordinates. A motion reconstruction problem consists in a kinematic analysis of a rigid multibody system whose motion is usually overdetermined by an excess of data. The method has been applied to the analysis of human motion which is a typical case of an overdetermined kinematic problem as a large number of markers are usually placed on a subject to capture its movement. The efficiency of the method has been tested both with computer-simulated and real experimental data using models that include open and closed kinematic loops.

Keywords Kinematic analysis · Motion reconstruction · Natural coordinates · Redundant constraint equations · Optimization

1 Introduction

The kinematic analysis of a mechanism has applications in fields like robotics, animation or ergonomics. In robotics, the kinematic analysis is performed for mechanisms with few degrees of freedom (DoFs) which are usually determined, i.e., the input data is enough to solve the problem. In animation, the mechanisms have more DoFs as a simplified human body is usually considered and the kinematic problem is frequently undetermined as the input data is not enough to estimate the posture of the subject [9]. In ergonomics and biomechanics, the kinematic human models have a large number of DoFs and the kinematic problem is overdetermined as more input data than necessary are available [5].

In ergonomics and biomechanics, the inputs to the kinematic analysis are usually the motion data measured with a motion capture system [21, 23]. In this context, the kinematic

S. Ausejo (✉) · Á. Suescun · J. Celigüeta
CEIT and TECNUN, University of Navarra, Manuel de Lardizábal 15, 20018 San Sebastián, Spain
e-mail: sausejo@ceit.es

analysis is usually called motion reconstruction. One of the most frequently used technologies for motion capture is the optoelectronic which measures the position of small balls, called markers, located on the skin of the subject typically at a frequency of 50 Hz.

In this paper, we will consider only kinematic methods for overdetermined problems. These methods can be classified according to Lu and O'Connor into three groups: direct methods (DMs), segmental optimization methods (SOMs) and global optimization methods (GOMs). In this context, GOM does not mean that a global minimizer is calculated. It means that measurement errors are minimized for the whole body at once instead of at segment level in SOMs.

DMs [4, 19] calculate the pose (position and orientation) of each body segment independently and directly from the coordinates of three non-collinear markers on each segment without taking into account the errors introduced by the skin movement artifact. Skin movement artifact is defined as the relative movement between markers and the underlying bone caused by passive and active soft tissues.

SOMs calculate the pose of each body segment independently but skin movement artifact is minimized in a least-squares sense at a body segment level [1, 10, 12, 29, 31]. SOMs estimate the optimal rigid body transformation by minimizing the deformation of the cluster of markers (minimum of three noncollinear markers) on the body segment in a least-squares sense. With DMs and SOMs joints may dislocate in the reconstructed motion as joint integrity is not guaranteed.

GOMs [7, 20, 24, 25, 27, 34] calculate the pose of all body segments at once by minimizing the global measurement error introduced by the skin movement artifact and guarantee joint integrity by introducing joint constraints. Usually the global measurement error is defined as the sum of squared distances between the measured and model-determined marker positions. In general, GOMs estimate body segment poses more accurately than DMs and SOMs [20, 25]. Current GOMs define skeletal models using relative coordinates and are formulated as nonlinear unconstrained optimization problems. However, a closed-loop system modeled with relative coordinates requires at least one nonlinear closing loop kinematic constraint, thus producing a nonlinear constrained optimization problem. Consequently, all the GOMs presented up to now are only valid for open-loop mechanisms. Furthermore, the existing GOMs are not valid for systems modeled with natural coordinates because these coordinates lead to a nonlinear constrained optimization problem with nonlinear equality constraints independently of the topology of the multibody system.

In this paper, we propose an optimization method for solving the nonlinear constrained optimization problem arising from an overdetermined kinematic problem formulated with natural coordinates [16, 17]. The method is valid for multibody models that include open and closed kinematic loops and it is able to converge to a local minimizer from any remote starting point. Natural coordinates are selected because the resultant optimization problem has a quadratic objective function and the equality constraints are linear or quadratic. Additionally, the optimization problem is the same for models with open and closed kinematic loops.

Natural coordinates have been used previously for inverse kinematic and inverse dynamic analysis to study sport performance [2, 11], in gait analysis [27, 28] or for motion reconstruction by means of a single camera [3]. Recently, Czaplicki [14] used natural coordinated for inverse dynamics, direct dynamics and static optimization analysis of a 2D lower limb model. However, natural coordinates have not been used to reconstruct the motion of a 3D whole body model with kinematic closed-loops and the motion reconstruction problem has not been formulated as the optimization problem described in Sect. 3.

The rest of the paper is organized as follows. Section 2 presents how human skeletal models are modeled using natural coordinates, Sect. 3 defines mathematically a motion reconstruction problem for a GOM, Sect. 4 presents the optimization method for overdetermined kinematic problems proposed in this paper, Sect. 5 shows the numerical results, and Sect. 6 presents conclusions and future work.

2 Human skeletal model

We have used a multibody approach to define human skeletal models. A model is defined using *natural coordinates* [16, 17] which describe the position and orientation of bodies through the Cartesian components of points and vectors located at the mechanism joints. Natural coordinates have been used due to their potential advantages for the optimization problem arising from the motion reconstruction problem. These advantages are discussed in the next section.

Natural coordinates are not independent coordinates but interrelated through certain equations known as *constraint equations*. The number of constraint equations is equal to the difference between the number of coordinates and the number of degrees of freedom (DoFs). Natural coordinates do not include relative coordinates (joint angles and relative distances) but they can be added to the model together with the corresponding constraint equations. There are four types of constraint equations: *rigid body constraints* that originate from the rigid body conditions for each element; *joint constraints* that originate from the kinematic pairs; *relative coordinate constraints* that originate from the additional relative coordinates; and *driving constraints* that are used to define the motion of the multibody system.

Rigid body constraints, joint constraints and relative coordinate constraints are referred to as *kinematic constraints*. The set of m kinematic constraints that define a model can be expressed in matrix form as follows:

$$\Phi(\mathbf{q}) = 0$$

where $\mathbf{q}$ is the column vector of n dependent coordinates. In practice, there are situations where an excess of kinematic constraints is obtained [17]. This means that some of the kinematic constraints, which are called *redundant constraints*, are not independent from the remaining ones.

The coordinates whose motion is defined by driving constraints are called *driven coordinates*. The r driven coordinates are a subset of $\mathbf{q}$ and they can be expressed in matrix form as follows:

$$\mathbf{z} = \mathbf{S}\mathbf{q}$$

where $\mathbf{z}$ is the column vector of r driven coordinates, which can include Cartesian coordinates or relative coordinates. $\mathbf{S}$ is an $r \times n$ matrix and the elements of its ith row are all zero except the one corresponding to the coordinate of $\mathbf{q}$ selected as the ith driven coordinate. The set of r driving constraints can be expressed in matrix form as follows:

$$\Psi(\mathbf{q}, t) = \mathbf{S}\mathbf{q} - \mathbf{d}(t) = 0$$

where $\mathbf{d}$ is a column vector of size $r \times 1$ whose ith element is a given function of time $g_i(t)$ that gives the value of corresponding driven coordinate. The driving constraints can be used to define the motion of any dependent coordinate of the model, e.g., joint angle, point coordinate or vector coordinate. However, when an optoelectronic motion capture system is used to record the motion of the subject, $\mathbf{z}$ usually contains the model-marker coordinates and $\mathbf{d}$ contains only the measured-marker coordinates.

The outputs of motion reconstruction are usually the joint angles of the model. Joint angles can be added to the vector $\mathbf{q}$ which requires the addition to $\Phi(\mathbf{q})$ of additional constraint equations. However, it is more efficient to calculate joint angles from the natural coordinates using a fast and simple post-processing.

3 Motion reconstruction problem

In this paper, we propose a new GOM to solve the motion reconstruction problem for a human skeletal model formulated with natural coordinates. Therefore, the pose of all body segments of the multibody model have to be calculated at once by minimizing the global measurement error and the joint integrity has to be guaranteed. We define the global measurement error as the sum of squared distances between the measured and model-determined marker positions.

In order to fulfill the previous conditions, the kinematic constraints of the model have to be exactly satisfied and the quadratic error of the driving constraints has to be minimized. Mathematically, this is a nonlinear constrained optimization problem or nonlinear programming (NLP) problem:

$$\begin{aligned} &\underset{\mathbf{q}\in\Re^n}{\text{minimize}} \quad f(\mathbf{q}) = \frac{1}{2}\Psi^{\mathrm{T}}(\mathbf{q},t)\Psi(\mathbf{q},t) \\ &\text{subject to:} \quad \Phi(\mathbf{q}) = 0 \end{aligned} \tag{1}$$

The objective function does not depend on the time variable because the NLP problem (1) is solved for a given time, i.e., for each frame recorded with the motion capture system. The NLP problem (1) has a quadratic objective function, and the equality constraints are always a linear or quadratic function of the natural coordinates [17]. As indicated previously, there are situations where the equality constraints may contain redundant constraints. Additionally, the Jacobian matrix of the equality constraints is sparse with linear or constant terms. This means that efficient algorithms for sparse matrix factorization can be used.

Several optimization algorithms exist for solving the NLP problem (1) when there are not redundant constraints, e.g., Sequential Quadratic Programming methods [8] or Interior-point methods [15, 32]. However, when the equality constraints contain redundant constraints, as is the case in our problem, only a few optimization algorithms are available. Wright [33] presented an algorithm valid for redundant equality constraints and large optimization problems, which is based on the SQP method. Izmailov and Solodov [18] presented an algorithm valid for redundant equality constraints which is not practical for large optimization problems like our NLP problem (1) because it requires a singular value decomposition (SVD) of the Jacobian matrix of the equality constraints. The SVD yields a dense matrix in general even when the Jacobian matrix of the constraints is sparse and the time required for computing the SVD is high for large optimization problems.

4 Optimal tracking method

In this paper, we propose an optimization method, which is called Optimal Tracking Method (OTM), to solve the NLP problem (1) with or without redundant equality constraints. OTM is an iterative method that at each iteration step solves a quadratic programming (QP) subproblem. The linear equality constraints of the QP subproblem at an intermediate iteration point $\mathbf{q}^k$ are the linearized equality constraints of the NLP problem (1):

$$\Phi_{\mathbf{q}}^k \Delta\mathbf{q} = -\Phi^k \tag{2}$$

where $\Delta\mathbf{q}$ is the increment in the coordinates, Φ^k is the vector of kinematic constraints evaluated at the point $\mathbf{q}^k$ and $\Phi_{\mathbf{q}}^k$ is the Jacobian matrix of the kinematic constraints evaluated at the point $\mathbf{q}^k$. The ith row of $\Phi_{\mathbf{q}}^k$ is the transposed gradient vector ($\nabla\phi_i^{\mathrm{T}}$) of the ith kinematic constraint.

The objective function of the QP subproblem is the same objective function of the NLP problem (1) because it is already quadratic. This objective function can be written at an intermediate iteration point $\mathbf{q}^k + \Delta\mathbf{q}$ using its Hessian matrix $\mathbf{H}^k$ and gradient vector $\mathbf{g}^k$ as follows:

$$h(\Delta\mathbf{q}) = f\left(\mathbf{q}^k + \Delta\mathbf{q}\right) = \frac{1}{2}\Delta\mathbf{q}^{\mathrm{T}}\mathbf{H}^k\Delta\mathbf{q} + \mathbf{g}^{k^{\mathrm{T}}}\Delta\mathbf{q} + f^k$$

where

$$\mathbf{g}^k = \nabla f^k = \mathbf{S}^{\mathrm{T}}\left(\mathbf{S}\mathbf{q}^k - \mathbf{d}\right) = \mathbf{S}^{\mathrm{T}}\Psi^k, \tag{3}$$

$$\mathbf{H}^k = \nabla^2 f^k = \mathbf{S}^{\mathrm{T}}\mathbf{S} \tag{4}$$

The matrix $\mathbf{H}^k$ is a constant diagonal matrix with 0s on the main diagonal except for the positions corresponding to driven coordinates which contain 1s. Therefore, $\mathbf{H}^k$ is $n \times n$ and positive semidefinite, because the number of driven coordinates is always less than the number of dependent coordinates. From now on, $\mathbf{H}^k$ is denoted simply as $\mathbf{H}$ because it is constant and does not depend on $\mathbf{q}^k$.

Consider a generic motion reconstruction problem with m nonlinear kinematic constraints (Φ), n dependent coordinates ($\mathbf{q}$), r driving constraints (Ψ), or equivalently, r driven coordinates, and s DoFs. Then, the QP subproblem that has to be solved at each iteration step can be written as

$$\begin{aligned} &\underset{\Delta\mathbf{q}\in\Re^n}{\text{minimize}}\ h(\Delta\mathbf{q}) = \frac{1}{2}\Delta\mathbf{q}^{\mathrm{T}}\mathbf{H}\Delta\mathbf{q} + \mathbf{g}^{k^{\mathrm{T}}}\Delta\mathbf{q} + f^k \\ &\text{subject to} \quad \Phi_{\mathbf{q}}^k\Delta\mathbf{q} = -\Phi^k \end{aligned} \tag{5}$$

When the m nonlinear kinematic constraints are independent, the QP subproblem can be solved using standard QP algorithms. However, when there are redundant constraints, some problems arise.

4.1 Incompatibility of the linearized kinematic constraints

When redundant constraints exist within the m nonlinear kinematic constraints in (1), some problems arise with the linearized kinematic constraints in the QP subproblem (5). At a solution point $\mathbf{q}^*$ of the NLP problem (1), the linear system of (2) is compatible underdetermined; the number of linearly independent constraints in $\Phi_{\mathbf{q}}^k$ is $(n - s)$, and there are $m - (n - s)$ linearly dependent constraints coming from the $m - (n - s)$ nonlinear redundant constraints in $\Phi(\mathbf{q})$. But the problem is that at an intermediate iteration point $\mathbf{q}^k$, the redundant constraints can induce in (2) more linearly independent constraints than required [17]. In this situation, there are three potential difficulties with the QP subproblem (5):

1. The linear system of equations (2) could become incompatible. Therefore, the QP subproblem does not have a solution because a feasible region does not exist.
2. The system of equations (2) could be compatible determined. This means that the number of linearly independent equations induced by the redundant equations in $\Phi_{\mathbf{q}}^{k}$ is s. Therefore, the feasible region of the QP subproblem is a single point.
3. The system of equations (2) could be compatible underdetermined with the rank of $\Phi_{\mathbf{q}}^{k}$ greater than $(n-s)$ but less than n. Then, there is an excess of linear constraints and the multibody loses some DoFs but the QP subproblem (5) can be solved.

OTM deals with the incompatibility of linear system of (2) by modifying the linearized kinematic constraints. Instead of (2) consider the following linear equality constraints

$$\mathbf{A}^{k}\Delta\mathbf{q}=\mathbf{b}^{k} \tag{6}$$

where

$$\mathbf{A}^{k}=\Phi_{\mathbf{q}}^{k\,\mathrm{T}}\Phi_{\mathbf{q}}^{k}, \tag{7}$$

$$\mathbf{b}^{k}=-\Phi_{\mathbf{q}}^{\mathrm{k}\,\mathrm{T}}\Phi^{k} \tag{8}$$

Equation (6) corresponds to the normal equations of (2). The linear system of (6) is always compatible. Thus, the problem of incompatible linear constraints coming from the nonlinear redundant constraints is eliminated. $\mathbf{A}^{k}$ is a $n\times n$ sparse symmetric positive semidefinite matrix. Its rank at the solution is $(n-s)$ but at intermediate iteration points it is greater or equal to $(n-s)$ and less or equal to n.

It can be argued that (2) is preferable to (6) because $\Phi_{\mathbf{q}}^{k}$ is in general smaller, has less nonzero elements and has better conditioning than $\mathbf{A}^{k}$. Unfortunately, as mentioned above, (2) could become incompatible. Then, the QP subproblem (5) does not have a solution, the iterative process cannot find the next iteration point and consequently, it fails to find a solution.

Instead of the QP subproblem (5) OTM solves the following QP subproblem with the new linear constraint equations (6):

$$\begin{aligned}&\underset{\Delta\mathbf{q}\in\Re^{\mathrm{n}}}{\text{minimize}}\;\; h(\Delta\mathbf{q})=\frac{1}{2}\Delta\mathbf{q}^{\mathrm{T}}\mathbf{H}\Delta\mathbf{q}+\mathbf{g}^{k\,\mathrm{T}}\Delta\mathbf{q}+f^{k}\\&\text{subject to}\quad \mathbf{A}^{k}\Delta\mathbf{q}=\mathbf{b}^{k}\end{aligned} \tag{9}$$

The question of the linearly dependent constraints induced by the redundant constraints in the linear system of (6) has not been addressed, yet. Two possible approaches to handle linearly dependent constraints are: first, eliminate the linearly dependent constraints in $\mathbf{A}^{k}$ and then use a standard QP algorithm; and second, let the optimization algorithm deal directly with the linearly dependent constraints.

The first approach can be applied, but it has to be performed at each iteration step because the number of linearly dependent constraints could change at each iteration. OTM uses the second approach, which is more efficient because the factorization of $\mathbf{A}^{k}$ is not required at each iteration in order to detect and eliminate the linearly dependent constraints. Instead, the linearly dependent constraints are eliminated directly (see Sect. 4.3) during the solution of the QP subproblem.

4.2 Lagrange theorem

Applying the Lagrange's theorem, we obtain that the solution to the QP subproblem (9) must satisfy (10) and (11):

$$\begin{bmatrix} \mathbf{H} & \mathbf{A}^k \\ \mathbf{A}^k & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q} \\ -\lambda \end{bmatrix} = \begin{bmatrix} -\mathbf{g}^k \\ \mathbf{b}^k \end{bmatrix}, \tag{10}$$

$$\lambda_{ld} = 0 \tag{11}$$

where λ_{ld} is the column vector of the Lagrange multipliers associated to the dependent columns of $\mathbf{A}^k$ and λ is the column vector of all the Lagrange multipliers. The linear (10) and (11) are only necessary conditions for a minimum of the QP subproblem (9). A maximum and a saddle point also satisfy these equations. However, under certain conditions, the solution of the linear (10) and (11) is unique. Furthermore, this unique solution is a global isolated minimum of the QP subproblem (9) at any iteration point $\mathbf{q}^k$. Suppose that Z is a basis for the nullspace of $\mathbf{A}^k$ and assume that the column vectors $\Delta\mathbf{q}^*$ and λ^* satisfy (10) and (11) and the reduced-Hessian matrix

$$\mathbf{Z}^{\mathrm{T}}\mathbf{H}\mathbf{Z}$$

is positive definite, then $\Delta\mathbf{q}^*$ is a global isolated minimum of the QP subproblem (9). The positive definiteness of the reduced-Hessian matrix $\mathbf{Z}^T\mathbf{H}\mathbf{Z}$ is a sufficient condition for a global isolated minimum of the QP subproblem (9). This can be proved in three steps:

1. Firstly, if the number of driven coordinates is enough to define completely all DoFs of the human skeletal model, then the reduced-Hessian matrix $\mathbf{Z}^{\mathrm{T}}\mathbf{H}\mathbf{Z}$ is positive definite at any iteration point $\mathbf{q}^k$ (see Appendix for details).
2. Secondly, suppose that the reduced-Hessian matrix $\mathbf{Z}^{\mathrm{T}}\mathbf{H}\mathbf{Z}$ is positive definite, then the solution of (10) and (11) is unique (for example, see [22]).
3. Thirdly, suppose that the reduced-Hessian matrix $\mathbf{Z}^{\mathrm{T}}\mathbf{H}\mathbf{Z}$ is positive definite, then the unique solution of (10) and (11) is a global isolated minimum of the QP subproblem (9) (for example, see [22]).

4.3 Numerical method for OTM

At each iteration step OTM has to solve (10) and (11). The coefficient matrix in (10) is composed of matrices $\mathbf{A}^k$ and $\mathbf{H}$. Recall that $\mathbf{H}$ is an $n \times n$ constant positive semidefinite diagonal matrix and $\mathbf{A}^k$ is a $n \times n$ sparse symmetric positive semidefinite matrix with rank p where $(n - s) \leq p \leq n$. Then, the coefficient matrix in (10) is sparse, symmetric and, in general, singular. Therefore, a stable numerical method is required in order to detect the linearly dependent constraints, set their associated variables λ_{ld} to zero and solve the equations.

A numerical method based on the QR decomposition has been developed. The method consists in performing a QR decomposition of the coefficient matrix in (10). Then, the linearly dependent constraints can be detected in $\mathbf{R}$ using a threshold and their associated variables λ_{ld} can be set to zero. Finally, the vectors $\Delta\mathbf{q}$ and λ_{li} can be obtained by a back-substitution. The QR decomposition can be computed efficiently because we only need the product of $\mathbf{Q}$ by the vector of independent terms and we do not need to compute $\mathbf{Q}$ explicitly.

4.4 Global convergence to a local minimizer

OTM is based on a sequence of QP subproblems which are expected to converge to the solution of the NLP problem (1). To ensure global convergence OTM is equipped with two global convergence strategies: merit function and maxmin. Notice that the term global convergence does not mean that a global minimizer of (1) is calculated. An algorithm is said to be globally convergent if, under suitable conditions, it will converge to some local minimizer from any remote starting point [8].

The merit function strategy employs a merit function, which is a measure of progress towards a local minimizer, for achieving global convergence. The maxmin strategy does not require the definition of a merit function. It calculates the two different values of the step-length parameter that minimize the value of the objective function and the 2-norm of Φ. From the two values, the maximum is selected as the step-length parameter. The convergence properties of OTM have been studied by means of numerical experiments that show that OTM is globally convergent.

4.5 Weighted OTM

OTM can be enhanced by allowing different weighting factors for each driving constraint. The weighting factors allow assigning to each driving constraint a different weight in the solution. For an optoelectronic motion capture system this means that different weighting factors can be assigned to each marker. If a marker is noisier than others, then we can assign to this marker a smaller weighting factor. For this purpose, a weighting matrix $\mathbf{W}$ is included in the objective function, and the NLP problem (1) is reformulated as follows

$$\begin{aligned} &\underset{\mathbf{q}\in\Re^n}{\text{minimize}} \quad f(\mathbf{q}) = \frac{1}{2}\Psi^{\mathrm{T}}(\mathbf{q},t)\mathbf{W}(t)\Psi(\mathbf{q},t) \\ &\text{subject to:} \quad \Phi(\mathbf{q}) = 0 \end{aligned} \tag{12}$$

where $\mathbf{W}$ is a weighting diagonal matrix of size $r \times r$ with positive or zero weighting factors on the main diagonal.

There are situations where some of the driving constraints are required to be satisfied exactly. For example, suppose that the shoe sole must remain exactly parallel to the ground. This implies that the motion of a foot vector perpendicular to the sole has to be forced to remain perpendicular to the ground at every time. A possible solution is to include this type of driving constraints together with the kinematic constraints, making them to be satisfied exactly.

In order to consider this requirement, OTM can be enhanced by dividing the driving constraints into two groups: driving constraint Ψ_m included in the objective function, whose errors are minimized; and driving constraints Ψ_s included in the equality constraints, which are satisfied exactly. Then, the weighted NLP problem (12) can be rewritten as

$$\begin{aligned} &\underset{\mathbf{q}\in\Re^n}{\text{minimize}} \quad f(\mathbf{q}) = \frac{1}{2}\Psi_m^{\mathrm{T}}(\mathbf{q},t)\mathbf{W}_m(t)\Psi_m(\mathbf{q},t) \\ &\text{subject to:} \quad \begin{bmatrix} \Phi(\mathbf{q}) \\ \mathbf{W}_s(t)\Psi_s(\mathbf{q},t) \end{bmatrix} = 0 \end{aligned} \tag{13}$$

where $\mathbf{W}_m$ and $\mathbf{W}_s$ are weighting diagonal matrices associated with Ψ_m and Ψ_s, respectively. $\mathbf{W}_m$ is similar to the weighting matrix presented previously. However, the weighting factors of $\mathbf{W}_s$ can be only 0s or 1s. When a weighting factor is 0, the associated driving constraint is inactive. When a weighting factor is 1, the associated driving constraint is active.

The new QP subproblem that has to be solved at each iteration step is similar to the QP subproblem (9), but the Hessian matrix $\mathbf{H}$ and the gradient vector $\mathbf{g}^k$ must be substituted by the weighted Hessian matrix $\mathbf{H}_w$ and the weighted gradient vector $\mathbf{g}_w^k$ which are defined as:

$$\mathbf{g}_w^k = \mathbf{S}^{\mathrm{T}}\mathbf{W}(\mathbf{S}\mathbf{q}^k - \mathbf{d}) = \mathbf{S}^{\mathrm{T}}\mathbf{W}\Psi^k,$$

$$\mathbf{H}_w = \mathbf{S}^{\mathrm{T}}\mathbf{W}\mathbf{S}$$

The benefits of using a weighted reconstruction and the strategies for using weighting factors can be found in [6].

5 Results

OTM has been tested on two different motion reconstruction problems: 10 generic reach movements and 12 steering movement. The performance of OTM was evaluated using two parameters: mean time per frame and convergence rate. The convergence rate is the percentage of frames that converged to the solution for a given tolerance value. OTM accepts an iteration point as the solution when the 2-norm of $\Phi(\mathbf{q})$ is less than a predefined tolerance. However, it may happen that the maximum number of iterations (*nMaxIter*) defined by the user is reached. In this work, *nMaxIter* was set to 25 for all motions.

The convergence rate for a given tolerance is usually very close to 100%. However, there are at least two causes that prevent the convergence rate from reaching 100%:

1. If the initial approximation for the first frame is very far from to the solution, the *nMaxIter* defined by the user can be reached before achieving the desired tolerance. This may happen during a few initial frames.
2. The global convergence strategy (see Sect. 4.4) selects the value of the step-length parameter α such that acceptable progress towards the solution is made. However, this is not always true and the global convergence strategy can select an excessively short step-length and *nMaxIter* can be reached before achieving the desired tolerance.

All the motions were reconstructed with two different solvers, QRf and QRx. Both solvers use the numeric method described in Sect. 4.3 and differ only in the global convergence strategy. QRf uses the merit function strategy and QRx used the maxmin strategy. Both solvers and the skeletal model were implemented in Matlab® and the CPU used was a Pentium IV 3.00 GHz.

5.1 Generic reach movements

For the generic reach movements, a whole body model based on the RAMSIS model was used to reconstruct the 10 motions. The model was tailored to each subject under investigation using the software PCMAN [26]. The skeletal kinematic model has 27 rigid bodies connected by 10 spherical joints, 7 revolute joints, 6 universal joints, and 1 floating joint (3 translations and 3 rotations) between ground and pelvis. The model has not kinematic closed-loops and is defined with 402 coordinates, 138 driving constraints and 389 kinematic constraints (21 of them redundant). Therefore, the coefficient matrix in (10) has size 804×804.

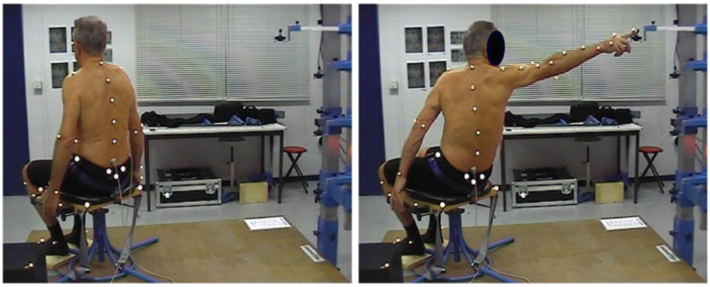

Fig. 1 Generic reach movement: (left) starting posture; (right) posture when the target is reached

Table 1 Mean time per frame for generic reach movements (in seconds)

Numeric method	Tolerance for accepting a solution			
	10^{-4}	10^{-5}	10^{-6}	10^{-7}
QRf	0.53	0.83	1.08	1.25
QRx	0.64	0.79	1.02	1.25

Table 2 Convergence rate (in %) of the generic reach movements

Numeric method	Tolerance for accepting a solution			
	10^{-4}	10^{-5}	10^{-6}	10^{-7}
QRf	99.56	98.14	97.60	97.49
QRx	98.74	98.58	97.93	97.32

The subjects were asked to push a toggle switch from a standardized initial posture and to return back to the initial posture (Fig. 1). The trajectories of 39 surface markers were recorded with the VICON optoelectronic motion capture system using nine cameras operating at 50 Hz. Marker trajectories were preprocessed by filtering them using a Butterworth filter with a cut-off frequency of 4 Hz.

The mean time per frame required to reconstruct the motions depends obviously on the tolerance for accepting the solution (Table 1). However, the global convergence strategy has not a significant influence on the mean time per frame. The convergence rate is always above 97% for every tolerance (Table 2). However, 100% is not reached is due to a combination of the two causes presented above.

The marker distance errors of a representative trial of the generic arm reaching motions are shown in Fig. 2 using a boxplot. The marker distance errors are defined as the distances between the measured and model-determined marker at each frame and they are quite similar for all trials within an experiment. These distance errors were calculated from a motion reconstructed with a tolerance of 10^{-7} and the QRf numeric method. The medians of the marker distance errors are mostly between 5 and 25 mm, which is considered a typical value in motion reconstruction.

5.2 Steering movements

The computer-simulated steering maneuver consisted in turning the steering wheel right and left. The driver held the steering wheel at the ten-to-two position during the whole

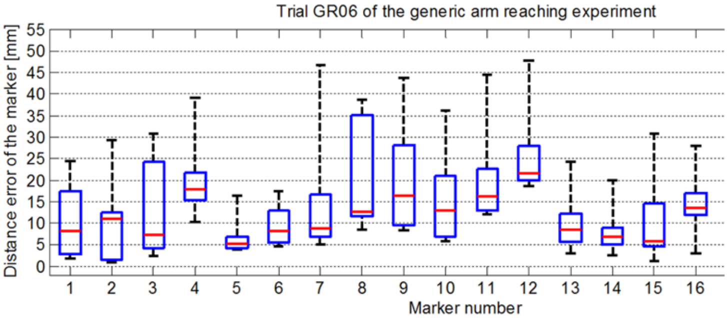

Fig. 2 Boxplot of the distance error of representative markers of a generic arm reaching motion

Table 3 Mean time per frame for steering movements (in seconds)

Numeric method	Tolerance for accepting a solution			
	10^{-4}	10^{-5}	10^{-6}	10^{-7}
QRf	0.43	0.52	0.64	0.71
QRx	0.44	0.48	0.60	0.67

Table 4 Convergence rate (in %) of the steering movements

Numeric method	Tolerance for accepting a solution			
	10^{-4}	10^{-5}	10^{-6}	10^{-7}
QRf	99.56	99.34	99.34	99.34
QRx	99.84	99.72	99.67	99.67

motion. The marker trajectories obtained from the computer-simulated motions were free from measurement errors. Then, artificial noise [13, 20] was added to the trajectories of the 28 markers used in order to generate realistic motion data.

The computer-generated motions were reconstructed using an upper body model with 12 rigid bodies and 29 DoFs. The model includes 2 closed-loops corresponding to each shoulder girdle, which were modeled similarly to van der Helm [30]. The model has 306 coordinates, 96 driving constraints and 292 kinematic constraints (15 of them redundant). Therefore, the coefficient matrix in (10) has size 612×612.

The mean time per frame required to reconstruct the 12 motions does not depend significantly on the global convergence strategy (Table 3) although the maxmin strategy requires slightly less time for most of the tolerance values. The convergence rate reached almost 100% for all the tolerance values (Table 4).

6 Conclusions and future work

In this paper, we presented an optimization method for solving the nonlinear constrained optimization problem arising from a kinematic analysis of a rigid multibody system whose motion is overdetermined. This means that there is an excess of motion data to determine the motion of the multibody system. A typical case for overdetermined kinematic problems appears in human motion reconstruction where a large number of markers are usually placed on a subject to capture its motion.

A mayor contribution of this paper is the optimization method, called Optimal Tracking Method (OTM), which has been specially designed to handle redundant equality constraints. Additionally, to ensure convergence to a local minimizer from any remote starting point, it is equipped with a global convergence strategy. OTM also allows assigning different weighting factors for each driving constraint.

The optimization problem arising from an overdetermined kinematic problem formulated with relative coordinates requires fewer variables than the equivalent problem formulated with natural coordinates. However, relative coordinates give objective functions and equality constraints (only for models with closed-loops), which are highly nonlinear. Using natural coordinates the objective function is quadratic and the equality constraints are linear or quadratic. Additionally, the Jacobian matrix of the equality constraints is sparse with linear or constant terms. This means that the optimization problem can be solved very efficiently using the algorithm for sparse matrix factorization presented in this paper.

OTM has been tested with computer-simulated data and real experimental data giving satisfactory results. Almost 100% of all the frames were successfully reconstructed within the desired tolerance in a reasonable time.

A comparative study between motion reconstruction methods formulated with relative coordinates and OTM is suggested as a topic for further research. A fair comparison should include skeletal models with open- and closed-loops. Another possible path for future work is the addition of inequality constraints. They can be used to include for example joint limits or additional constraints from the environment (e.g., collision avoidance).

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Appendix

Suppose a motion reconstruction problem without redundant constraints that has $m\ (=n-s)$ independent nonlinear kinematic constraints Φ and $r\ (=s)$ driving constraints Ψ, such that the motion of the s DoFs of the skeletal model is defined. Note that the Jacobian matrix of the driving constraints is constant

$$\Psi_{\mathbf{q}}^{k} = \mathbf{S}.$$

The motion reconstruction problem can be solved using the Newton–Raphson method, which is an iterative method:

$$\begin{bmatrix} \Phi_{\mathbf{q}}^{k} \\ \mathbf{S} \end{bmatrix} \Delta\mathbf{q}^{k} = -\begin{bmatrix} \Phi^{k} \\ \Psi^{k} \end{bmatrix}, \tag{14}$$

$$\mathbf{q}^{k+1} = \mathbf{q}^{k} + \alpha\Delta\mathbf{q}^{k}$$

Equation (14) is compatible determined at any iteration point. This means that the matrix

$$\mathbf{B} = \begin{bmatrix} \Phi_{\mathbf{q}}^{k} \\ \mathbf{S} \end{bmatrix}$$

has full rank, otherwise (14) would be underdetermined, which is contrary to the hypotheses of our problem. The rank of $\Phi_{\mathbf{q}}^{k}$ is $(n-s)$ and the rank of $\mathbf{S}$ is always $r(=s)$. Hence, $\mathbf{S}$

must be such that it complements the rank of $\Phi_{\mathbf{q}}^k$ to n. From a physical point of view this means that the driven coordinates must be chosen in such a way that the motion of all DoFs is defined.

If Φ has some redundant constraints, the matrix **B** has some additional dependent rows but its column rank is n. Additionally, if the number of driving constraints is greater than s, the matrix **B** has also some additional dependent rows but its column rank is still n. Therefore, when a motion reconstruction problem, with or without redundant constraint, has $r(\geq s)$ driving constraints that define the motion of all DoFs of the skeletal model, the rank of matrix **B** is always n at any iteration point. Then, the matrix $\mathbf{B}^{\mathrm{T}}\mathbf{B}$ has also rank n and is positive definite.

Taking into account (4) and (7), the matrix $(\mathbf{A}^k + \mathbf{H})$ can be written as follows

$$(\mathbf{A}^k + \mathbf{H}) = \mathbf{B}^{\mathrm{T}}\mathbf{B}$$

Therefore, $(\mathbf{A}^k + \mathbf{H})$ is positive definite. Finally, if the matrix $(\mathbf{A}^k + \mathbf{H})$ is positive definite, it can be proved straightforwardly that the matrix $\mathbf{Z}^{\mathrm{T}}\mathbf{H}\mathbf{Z}$ is positive definite.

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