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A formal procedure and invariants of a transition from conventional finite elements to the absolute nodal coordinate formulation

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Abstract In this paper, finite elements based on the absolute nodal coordinate formulation (ANCF) are studied. The formulation has been developed by various authors for the dynamical simulation of large-displacement and large-rotation problems in flexible multibody dynamics. This study introduces a procedure to track the general geometrical properties of ANCF elements back to their prototypes in the conventional finite-element method (FEM), which deals with small-displacement problems. In this study, it is shown that each known ANCF element can be derived from a conventional FEM using a universal transform. Moreover, some important static and dynamic properties of the elements in small-displacement problems are automatically preserved. In the past, the authors of each newly proposed ANCF element have made unnecessary efforts to show the consistency of the above mentioned properties.

Keywords Finite elements · Large displacements · Absolute nodal coordinates

1 Introduction

During the past few decades, a number of researchers have contributed to large deformation formulations for multibody applications. One of the remarkable approaches in this area, the absolute nodal coordinate formulation, has been introduced as a numerical tool for the design process of mechanical systems. The absolute nodal coordinate formulation is a finite element procedure which is capable of describing large rotations properly and which does not require the use of any incremental integration methods in order to satisfy the principle of virtual work and energy.

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In the ANCF, finite elements are defined without the use of any intermediate reference frame. Instead, the elements employ absolute position coordinates together with independent global slopes that are, in fact, partial derivatives of the position vector with respect to the element coordinates [1]. The use of global slope coordinates allows describing an arbitrary rigid body motion without using any rotation matrix but employing the matrix of global shape functions and the vector of global nodal coordinates. This unique feature leads to a linear representation of a position vector of an arbitrary material point and, as a consequence, to a constant mass matrix in both two- and three-dimensional cases [2, 3]. An additional advantage of the constant description of the mass matrix is that the vector of inertia forces that depends quadratically on velocities vanishes in the expression of the equations of motion.

Research on the absolute nodal coordinate formulation can be categorized into two large groups. In the first group, the formulation is used to describe conventional (or thin) beam and plate elements that cannot capture transverse shear deformation. The absolute nodal coordinate formulation is inherently developed for these types of elements. In this approach, an element is parameterized as a centerline of a beam or a mid-surface of a plate by employing global slope coordinates in the element longitudinal direction together with global position coordinates [1, 4]. In the second group of research, ANCF elements capable of describing transverse shear deformation are developed and utilized in practical applications [5, 6]. In this approach, the shear deformation can be accounted for by parameterization of the element as a volume by introducing additional slopes in the element transverse direction. This makes it straightforward to define elastic forces using a continuum mechanics approach [3]. The feature of a constant mass matrix remains in effect also in the case of shear deformable elements [7–10].

Analyzing the process of developing new ANCF elements, one can find a number of interesting features:

– A large number, if not all, of finite elements within the ANCF are developed "from scratch," i.e., without any relation to earlier findings of the finite element method. It is noteworthy that the ANCF has many similarities with the conventional finite element approach.
– After the authors have developed equations of motion of a new ANCF element, they usually accomplish the element validation with simple, even primitive tests, such as: (a) checking zero strain after a rigid-body rotation/translation; (b) comparing small-displacement solutions with known solutions and with known finite elements; or (c) comparing eigenfrequencies and eigenmodes.

However, an analysis presented in the current paper shows that many ANCF elements, at least those that use only longitudinal slopes, can be formally constructed from existing structural elements from a conventional finite element approach. Moreover, any structural element from a conventional FEM that uses only transverse displacements and slopes as a description of coordinates can be transformed into a corresponding ANCF element. After such a transition, some important static and dynamic properties of the new elements are preserved with respect to the original element. The authors assume that the shear deformable ANCF elements also possess such FEM-inherited properties. However, the validation of this claim is under way and will be a topic of future research.

The paper is organized as follows. In Sect. 2, known examples of generating ANCF elements from existing FEM elements are presented to introduce preliminary material for the formal definition of the general procedure on how to create new ANCF elements. In Sect. 3, this formal generalization procedure is formulated for all finite elements. Moreover,

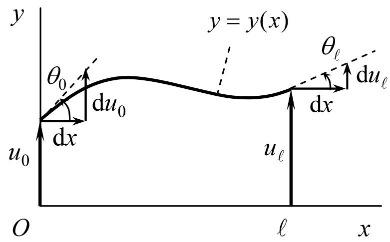


Fig. 1 Conventional FEM beam element

in the section, the transformation of terms of equations of motion after such a transition is thoroughly analyzed. Section 4 includes a discussion about properties that are preserved after the transformation is accomplished.

2 Examples of the transition from FEM to ANCF

In this section, examples of existing finite elements based on absolute nodal coordinates will be presented. In the examples, the sequence of element creation as well as a description of the element inherited root from conventional elements is discussed.

2.1 Beam elements

2.1.1 Original FEM beam element

The transformation from conventional elements to elements based on the absolute nodal coordinate formulation can be illustrated by giving a simple example of the conventional planar beam based on the Euler–Bernoulli theory, as depicted in Fig. 1, [11]. The element has two nodes. The length of the element in a straight undeformed configuration is denoted by ℓ. The arbitrary configuration of the element centerline can be defined using four variables: two degrees of freedom per each node:

– Transverse displacements u_0 and u_ℓ,
– Nodal slopes represented by derivatives $u'_0 = \frac{du_0}{dx} = \tan\theta_0$ and $u'_\ell = \frac{du_\ell}{dx} = \tan\theta_\ell$.

The vertical displacement of an arbitrary point of the centreline given by a horizontal coordinate $x = 0 \ldots \ell$ can be computed using the interpolation technique as follows:

$$y(x) = s_1(x)u_0 + s_2(x)u'_0 + s_3(x)u_\ell + s_4(x)u'_\ell, \tag{1}$$

where $s_1, \ldots, s_4$ are the shape functions for beams that can be expressed as

$$\begin{aligned} s_1(x) &= 1 - 3\xi^2 + 2\xi^3, \qquad s_2(x) = \ell(\xi - 2\xi^2 + \xi^3), \\ s_3(x) &= 3\xi^2 - 2\xi^3, \qquad s_4(x) = \ell(\xi^3 - \xi^2), \end{aligned} \qquad \xi = x/\ell\,. \tag{2}$$

It is worth mentioning that the set of generalized coordinates contains transverse components of displacements only, and the coordinates cannot describe any horizontal displacements of the centerline. As a consequence, this particular model assumes small displacements and small deformations of any part of the centerline. This means, in practice, that all generalized coordinates, namely displacements u_0 and u_ℓ and slopes u'_0 and u'_ℓ, are assumed to be small.

One can rewrite (1) in the matrix form as

$$y(x) = \mathbf{s}(x) \cdot \mathbf{u}, \tag{3}$$

introducing the following row matrix of shape functions:

$$\mathbf{s} = \{s_1\ s_2\ s_3\ s_4\}, \tag{4}$$

and the corresponding column matrix of generalized coordinates

$$\mathbf{u} = \begin{Bmatrix} u_0 \\ u_0' \\ u_\ell \\ u_\ell' \end{Bmatrix} \quad \text{or} \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}. \tag{5}$$

Another notation for geometric relationship (1) can be expressed as follows:

$$y = \sum_{k=1}^{4} s_k u_k, \tag{6}$$

where variable u_k refers to the kth component of vector $\mathbf{u}$ in (5).

2.1.2 Planar ANCF beam element

In order to specify arbitrary displacements of a beam, the beam centerline can be parameterized using the arc parameter $p = 0 \ldots \ell$ while introducing two displacement fields for $x(p)$ and $y(p)$, as shown in Fig. 2. The displacement field can be expressed as follows:

$$\begin{aligned} x(p) &= s_1(p)x_0 + s_2(p)x_0' + s_3(p)x_\ell + s_4(p)x_\ell' \\ y(p) &= s_1(p)y_0 + s_2(p)y_0' + s_3(p)y_\ell + s_4(p)y_\ell'. \end{aligned} \tag{7}$$

Equation (7) can be written in matrix form as follows:

$$\mathbf{r}(p) = \mathbf{S}(p) \cdot \mathbf{q}, \tag{8}$$

where

$$\mathbf{r} = \begin{Bmatrix} x \\ y \end{Bmatrix}, \qquad \mathbf{S} = \left[\begin{array}{cc|cc|cc|cc} s_1 & 0 & s_2 & 0 & s_3 & 0 & s_4 & 0 \\ 0 & s_1 & 0 & s_2 & 0 & s_3 & 0 & s_4 \end{array}\right], \qquad \mathbf{q} = \left\{\begin{array}{c} x_0 \\ y_0 \\ \hline x_0' \\ y_0' \\ \hline x_\ell \\ y_\ell \\ \hline x_\ell' \\ y_\ell' \end{array}\right\}. \tag{9}$$

The shape functions introduced in (2) are also used in (7). It is important to note that the interpretation of generalized coordinates is similar: x_0, y_0 and x_ℓ, y_ℓ are x- and y- displacements of the two nodes; when values x_0', y_0', x_ℓ', and y_ℓ' are slopes of the x- and y-fields with respect to the p-axis.

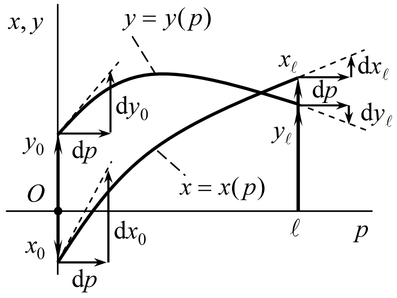


Fig. 2 Interpolation of x and y coordinates w.r.t. arc coordinate p in ANCF

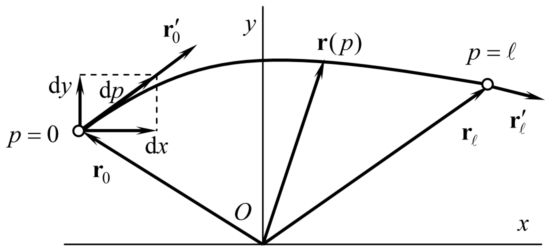


Fig. 3 Large-displacement and large-slope ANCF element [12]

Note that the slopes $x' = \mathrm{d}x/\mathrm{d}p$ and $y' = \mathrm{d}y/\mathrm{d}p$ are still tangents with respect to the p-axis, but they are also proportional to directing cosines to axes x and y, respectively. Then, introducing the displacement and slope vectors at the end points $\mathbf{r}_0 = \{x_0\ y_0\}^{\mathrm{T}}$, $\mathbf{r}'_0 = \{x'_0\ y'_0\}^{\mathrm{T}}$, $\mathbf{r}_\ell = \{x_\ell\ y_\ell\}^{\mathrm{T}}$, and $\mathbf{r}'_\ell = \{x'_\ell\ y'_\ell\}^{\mathrm{T}}$, the position of an arbitrary point of the beam can be obtained as shown in Fig. 3.

In terms of the vector of generalized coordinates, $\mathbf{q}$ in (8) can be redefined as follows:

$$\mathbf{q} = \begin{Bmatrix} \mathbf{r}_0 \\ \mathbf{r}'_0 \\ \mathbf{r}_\ell \\ \mathbf{r}'_\ell \end{Bmatrix} \quad \text{or} \quad \mathbf{q} = \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \mathbf{q}_4 \end{Bmatrix}, \tag{10}$$

and the expression for the radius vector of an arbitrary point on the beam centerline can be defined as

$$\mathbf{r} = \sum_{k=1}^{4} s_k \mathbf{q}_k. \tag{11}$$

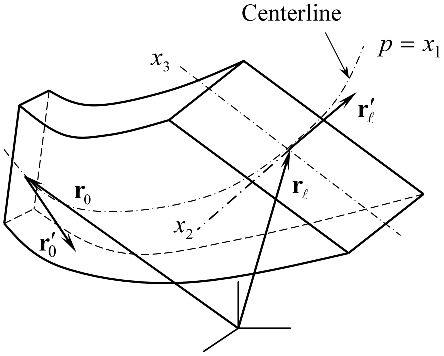
2.1.3 Spatial ANCF beam element

One can note that the set of (1)–(6) resembles relations (7)–(11) in many details. This circumstance can be extended by giving another example of the spatial beam element proposed in [13] and [14].

Spatial implementation of the beam centreline can be described using three independent fields resembling (1) and (7) as:

$$x(p) = s_1(p)x_0 + s_2(p)x'_0 + s_3(p)x_\ell + s_4(p)x'_\ell,$$
$$y(p) = s_1(p)y_0 + s_2(p)y'_0 + s_3(p)y_\ell + s_4(p)y'_\ell,$$

Fig. 4 Three-dimensional model of the centerline of a beam



$$z(p) = s_1(p)z_0 + s_2(p)z'_0 + s_3(p)z_\ell + s_4(p)z'_\ell.$$

The above equation can also be represented using (8) and following the shape function matrix and vector of nodal coordinates:

$$\mathbf{r} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}, \quad \mathbf{S} = \left[\begin{array}{ccc|ccc|ccc|ccc} s_1 & 0 & 0 & s_2 & 0 & 0 & s_3 & 0 & 0 & s_4 & 0 & 0 \\ 0 & s_1 & 0 & 0 & s_2 & 0 & 0 & s_3 & 0 & 0 & s_4 & 0 \\ 0 & 0 & s_1 & 0 & 0 & s_2 & 0 & 0 & s_3 & 0 & 0 & s_4 \end{array}\right], \quad \mathbf{q} = \left\{\begin{array}{c} x_0 \\ y_0 \\ z_0 \\ \hline x'_0 \\ y'_0 \\ z'_0 \\ \hline x_\ell \\ y_\ell \\ z_\ell \\ \hline x'_\ell \\ y'_\ell \\ z'_\ell \end{array}\right\}.$$

This 3D version of vector $\mathbf{q}$ has the same form as (10), while the only difference is brought by the number of components in nodal vectors $\mathbf{r}_0 = \{x_0\ y_0\ z_0\}^{\mathrm{T}}$, $\mathbf{r}'_0 = \{x'_0\ y'_0\ z'_0\}^{\mathrm{T}}$, $\mathbf{r}_\ell = \{x_\ell\ y_\ell\ z_\ell\}^{\mathrm{T}}$, and $\mathbf{r}'_\ell = \{x'_\ell\ y'_\ell\ z'_\ell\}^{\mathrm{T}}$. These vectors and the structure of the finite element are visualized in Fig. 4.

To conclude this section, it can be said that (8)–(11) are general in the sense that they can be implemented in two- and three-dimensional beam elements based on the absolute nodal coordinate formulation.

Note that the three-dimensional ANCF beam implementation represents the geometry of the beam centreline only, without describing the orientation of its cross-section. That means that, in fact, this implementation applies to cable elements only; see a successful implementation in [14]. Based on this simple representation, various authors, e.g., [12] and [15], have proposed ANCF modifications of the Euler–Bernoulli beam elements accounting for cross-section inertia. Relevant publications about beam elements can be found in [16–18].

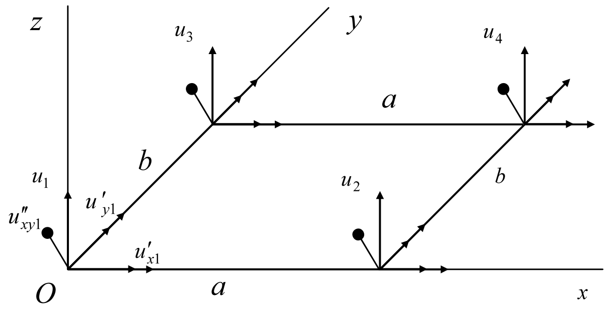


Fig. 5 Hermitean plate bending element having 16 d.o.f.

2.2 Plate elements

In this section, the procedure of generating new elements in the ANCF is implemented for a plate bending element. The plate element based on the ANFC uses the Kirchhoff theory of thin plates.

2.2.1 Original FEM element

The plate element considered in this section is a 16-degree-of-freedom Hermitean rectangle element of length a and width b shown in Fig. 5. Each of its four nodes introduces four degrees of freedom: e.g., it has a vertical displacement u_1 for node 1, two slopes $u'_{x1} = (\partial u/\partial x)_1$ and $u'_{y1} = (\partial u/\partial y)_1$, and a second-order slope $u''_{xy1} = (\partial^2 u/\partial x \partial y)_1$. This implementation of the bending plate element is not the simplest one possible, but it is chosen in this study to emphasize the possibility to manage the second-order slopes when generating ANCF elements.

The displacement field is expressed in terms of element nodal coordinates and shape functions as follows:

$$z = \mathbf{s}(x, y) \cdot \mathbf{u}, \tag{12}$$

$$\mathbf{s} = \{s_{11}\ s_{12}\ s_{13}\ s_{14} \quad s_{21}\ s_{22}\ s_{23}\ s_{24} \quad s_{31}\ s_{32}\ s_{33}\ s_{34} \quad s_{41}\ s_{42}\ s_{43}\ s_{44}\}, \tag{13}$$

where two-dimensional shape functions $s_{ij} = s_i(x)s_j(y)$ are composed of the shape functions (2) for beams. The vector of nodal coordinates can be written as

$$\mathbf{u} = \left\{u_1\ u'_{y1}\ u_4\ u'_{y4} \quad u'_{x1}\ u''_{xy1}\ u'_{x4}\ u''_{xy4} \quad u_2\ u'_{y2}\ u_3\ u'_{y3} \quad u'_{x2}\ u''_{xy2}\ u'_{x3}\ u''_{xy3}\right\}^{\mathrm{T}}.$$

2.2.2 ANCF plate element

The first thin plate element in the ANCF has been proposed in [4]. It resembles the Hermitean plate element described above and is parameterized using the mid-surface $\mathbf{r}(p_1, p_2)$ as follows:

$$\mathbf{r} = \mathbf{S}(p_1, p_2) \cdot \mathbf{q}, \tag{14}$$

where the shape function matrix can be written as

$$\mathbf{S} = [s_{11}\mathbf{I} \quad s_{12}\mathbf{I} \quad s_{13}\mathbf{I} \quad s_{14}\mathbf{I} \quad \ldots \quad s_{41}\mathbf{I} \quad s_{42}\mathbf{I} \quad s_{43}\mathbf{I} \quad s_{44}\mathbf{I}]. \tag{15}$$

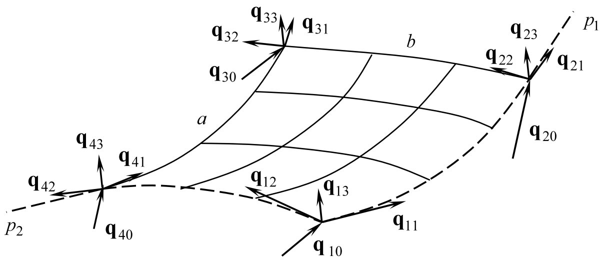


Fig. 6 ANCF plate bending element having 48 d.o.f. [4]

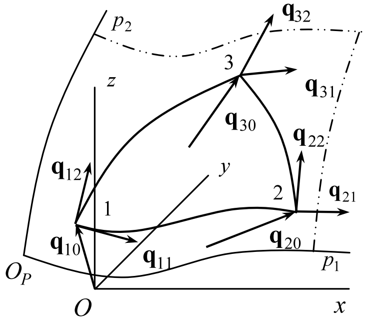


Fig. 7 Triangular plate bending element in ANCF having 27 d.o.f. [19]

The shape function matrix is composed of the same shape functions as the original plate element from the previous section. The vector of nodal coordinates can be written as

$$\mathbf{q} = \left\{ \mathbf{r}_1^{\mathrm{T}}\, \mathbf{r}_{1;2}^{\mathrm{T}\prime}\, \mathbf{r}_4^{\mathrm{T}}\, \mathbf{r}_{4;2}^{\mathrm{T}\prime} \quad \mathbf{r}_{1;1}^{\mathrm{T}\prime}\, \mathbf{r}_{1;}^{\mathrm{T}\prime\prime}\, \mathbf{r}_{4;1}^{\mathrm{T}\prime}\, \mathbf{r}_{4;}^{\mathrm{T}\prime\prime} \quad \mathbf{r}_2^{\mathrm{T}}\, \mathbf{r}_{2;2}^{\mathrm{T}\prime}\, \mathbf{r}_3^{\mathrm{T}}\, \mathbf{r}_{3;2}^{\mathrm{T}\prime} \quad \mathbf{r}_{2;1}^{\mathrm{T}\prime}\, \mathbf{r}_{2;}^{\mathrm{T}\prime\prime}\, \mathbf{r}_{3;1}^{\mathrm{T}\prime}\, \mathbf{r}_{3;}^{\mathrm{T}\prime\prime} \right\}^{\mathrm{T}},$$

which consists of nodal displacement vectors $\mathbf{r}_k = (\mathbf{r})_{\mathrm{node}\,k}$, slope vectors $\mathbf{r}'_{k;l} = (\partial^2 \mathbf{r}/\partial p_l)_k$, and of second-order slope vectors $\mathbf{r}''_{k;} = (\partial^2 \mathbf{r}/\partial p_1 \partial p_2)_k$, see Fig. 6, in which the vectors are denoted by $\mathbf{q} = \{\mathbf{q}_{10}^{\mathrm{T}}\, \mathbf{q}_{11}^{\mathrm{T}}\, \mathbf{q}_{12}^{\mathrm{T}}\, \mathbf{q}_{13}^{\mathrm{T}} \cdots \mathbf{q}_{40}^{\mathrm{T}}\, \mathbf{q}_{41}^{\mathrm{T}}\, \mathbf{q}_{42}^{\mathrm{T}}\, \mathbf{q}_{43}^{\mathrm{T}}\}^{\mathrm{T}}$.

This example is important in demonstrating that nodal coordinates as second derivatives (or second slopes) can also be adopted for the absolute nodal coordinate formulation.

It is noteworthy that the relation in (14) is applied to ANCF finite elements in this paper. It is also important to note that matrix $\mathbf{S}$ does not depend on coordinates $\mathbf{q}$.

Besides this implementation of plate bending elements in the ANCF, there exist a number of other implementations including rectangular and triangular plate elements, see Fig. 7. It is not the focus of this paper to present details of these implementations, but it is worth pointing out general features of all such elements.

3 The formal procedure of transition from FEM to ANCF

This section is devoted to the generalization of the procedure of transformation of any existing finite elements which use small transverse displacements and slopes to a new ANCF element using arbitrary large position vectors and slopes as nodal vectors of degrees of freedom. Particular examples of such a transformation have been shown in Sect. 2.

Firstly, the general geometrical aspects of the transformation procedure are considered. Secondly, detailed modifications in terms of the equations of motion are studied. And finally, a list of known implementations of the procedure with references is pointed out.

3.1 Transformation of geometry

Among the geometrical properties, such quantities as the expression of an arbitrary point of the centerline of a beam or of the mid-surface of a plate must be considered. Also, the transformation of shape functions and nodal degrees of freedom needs to be accounted for.

Generalizing the examples given in Sect. 2, one can point out the general diagram on how to accomplish the transformation:

$$\begin{array}{c} y(x) = \mathbf{s}(x) \cdot \mathbf{u} \quad \text{(original FEM element)} \\ \downarrow \\ \mathbf{r}(p) = \mathbf{S}(p) \cdot \mathbf{q} \quad \text{(new ANCF element)} \end{array} \tag{16}$$

The formal transition from the set of shape functions $\mathbf{s}$ from (3) to the matrix of shape functions $\mathbf{S}$ from (8) can be written as

$$\begin{array}{c} \mathbf{s} = \{s_1\ s_2\ \dots\ s_N\} \\ \downarrow \\ \mathbf{S} = [\,s_1\mathbf{I}\ s_2\mathbf{I}\ \dots\ s_N\mathbf{I}\,] \end{array} \tag{17}$$

and the transition from the set of conventional nodal coordinates to the vector of ANCF nodal coordinates can be written as

$$\mathbf{u} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_N \end{array} \right\} \rightarrow \mathbf{q} = \left\{ \begin{array}{c} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \vdots \\ \mathbf{q}_N \end{array} \right\}. \tag{18}$$

Relations (16)–(18) can be summarized as follows:

$$\begin{aligned} y &\rightarrow \mathbf{r} = y \otimes \Re, \\ \mathbf{s} &\rightarrow \mathbf{S} = \mathbf{s} \otimes \mathbf{I}, \\ \mathbf{u} &\rightarrow \mathbf{q} = \mathbf{u} \otimes \Re, \end{aligned} \tag{19}$$

where the symbol $\otimes$ represents the Kronecker product, and $\Re$ describes the m-dimensional space. In (19), $\mathbf{I}$ is the unity matrix of size m; the value of m is either 2 or 3.

In short form, the formal procedure of the transformation of the FEM element to an ANC element can be described by the following diagram:

$$\begin{array}{ccccccl} y & = & \mathbf{s} & \cdot & \mathbf{u} & & \text{(original FEM element)} \\ \otimes & & \otimes & & \otimes & & \\ \Re & & \mathbf{I} & & \Re & & \text{(mapping of the terms)} \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{r} & = & \mathbf{S} & \cdot & \mathbf{q} & & \text{(new ANCF element)} \end{array} \tag{20}$$

In index notation, transformation (16) takes the following form:

$$
\begin{array}{c}
y = \sum_{k=1}^{N} s_k u_k \\
\downarrow \\
\mathbf{r} = \sum_{k=1}^{N} s_k \mathbf{q}_k
\end{array}
\tag{21}
$$

Equation (21) explicitly shows that the shape functions remain unchanged during the transformation. As a matter of fact, only small-valued nodal degrees of freedom u_k are replaced by the nodal vectors $\mathbf{q}_k$ that may take large values.

3.2 Transformation of equations of motion

After the transition procedure from a conventional FEM element to an ANCF element described above, the equations of motion of the element needs also to be modified in a way which allows shedding a light on the static and dynamic properties of the ANCF element.

The dynamics of a mechanical system can be expressed using the Lagrange equation as follows:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\mathbf{x}}} - \frac{\partial T}{\partial \mathbf{x}} = \frac{\partial W}{\partial \mathbf{x}} - \frac{\partial U}{\partial \mathbf{x}},
\tag{22}
$$

where T and U are the kinetic and strain energy of the element, while W is the gravity force potential. The vector of generalized coordinates $\mathbf{x}$ represents either vector $\mathbf{u}$ in the case of the conventional finite element, or vector $\mathbf{q}$ in the case of the generated ANCF element.

In (22), the transition procedure for terms based on energies T and W are easy to carry out, particularly in comparison to the transition procedure of the term which depends on the strain energy U. For this reason, corresponding terms are considered separately in the following sections.

3.2.1 Transformation of constant terms: mass matrix and gravity forces

Kinetic and virtual energy can be computed in the traditional way as an integral over volume V of the finite element:

$$
\begin{aligned}
T &= \frac{1}{2}\iiint_V \mathbf{v}\cdot\mathbf{v}\,\mathrm{d}m, \\
W &= \iiint_V \mathbf{r}\cdot\mathbf{g}\,\mathrm{d}m,
\end{aligned}
\tag{23}
$$

where $\mathrm{d}m = \rho\,\mathrm{d}V$ is the infinitesimal mass element of the body, while ρ and $\mathrm{d}V$ are the element mass density and volume, respectively. Vector $\mathbf{g}$ represents the gravity acceleration. The position and velocity of a particle of the body in (23) can be computed as follows:

$$
\begin{aligned}
\mathbf{r} &= \mathbf{S}\cdot\mathbf{x}, \\
\mathbf{v} &= \mathbf{S}\cdot\dot{\mathbf{x}},
\end{aligned}
\tag{24}
$$

where $\mathbf{S}$ is either $\mathbf{s}_{\mathrm{FEM}}$ or $\mathbf{S}_{\mathrm{ANC}}$, and $\dot{\mathbf{x}}$ denotes either $\dot{\mathbf{u}}$ or $\dot{\mathbf{q}}$, as explained above.

The substitution of relations (24) into the expression of kinetic energy in (23) leads to the following representation:

$$T = \frac{1}{2}\dot{\mathbf{x}} \cdot \left[\iiint_V \mathbf{S}^{\mathrm{T}} \cdot \mathbf{S}\, \mathrm{d}m \right] \cdot \dot{\mathbf{x}}. \tag{25}$$

The expression in the brackets of (25) represents the mass matrix. The mass matrix depends on the shape function matrix only, and for this reason, the matrix remains constant. Computing the mass matrix in terms of original shape functions represented by a row $\mathbf{s}$ and in terms of ANCF functions given by matrix $\mathbf{S}$, one can find the transformation of the mass matrix as follows:

$$\mathbf{M}_{\mathrm{FEM}} = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ & \ddots & \vdots \\ \mathrm{sym} & & m_{nn} \end{bmatrix} \to \mathbf{M}_{\mathrm{ANC}} = \begin{bmatrix} m_{11}\mathbf{I} & \cdots & m_{1n}\mathbf{I} \\ & \ddots & \vdots \\ \mathbf{sym} & & m_{nn}\mathbf{I} \end{bmatrix} = \mathbf{M}_{\mathrm{FEM}} \otimes \mathbf{I}. \tag{26}$$

In (26), the same inertia coefficients $m_{ij} = \iiint_V s_i s_j\, \mathrm{d}m$ appear in both matrices.

The potential function W needed in (23) can be computed as follows:

$$W = \iiint_V \mathbf{x} \cdot \mathbf{S}^{\mathrm{T}} \cdot \mathbf{g}\, \mathrm{d}m. \tag{27}$$

Generalized gravity forces can be defined as follows:

$$\mathbf{Q}^{\mathrm{grav}} = \frac{\partial W}{\partial \mathbf{x}} = \iiint_V \mathbf{S}^{\mathrm{T}} \cdot \mathbf{g}\, \mathrm{d}m.$$

The corresponding transformation rule from the original FEM element to the ANCF element can be expressed as

$$\mathbf{Q}^{\mathrm{grav}}_{\mathrm{FEM}} = \begin{Bmatrix} Q^{\mathrm{g}}_1 \\ \vdots \\ Q^{\mathrm{g}}_n \end{Bmatrix} \to \mathbf{Q}^{\mathrm{grav}}_{\mathrm{ANC}} = \begin{Bmatrix} Q^{\mathrm{g}}_1 \mathbf{k} \\ \vdots \\ Q^{\mathrm{g}}_n \mathbf{k} \end{Bmatrix} = \mathbf{Q}^{\mathrm{grav}}_{\mathrm{FEM}} \otimes \mathbf{k}, \tag{28}$$

where $\mathbf{k}$ denotes the unit vector in the vertical direction, namely $\mathbf{k} = \{0\ 1\}^{\mathrm{T}}$ in the planar case or $\mathbf{k} = \{0\ 0\ 1\}^{\mathrm{T}}$ in the spatial case. The numerical coefficients $Q^{\mathrm{g}}_k = g \iiint_V s_k\, \mathrm{d}m$ are the same for the original and the ANCF element.

3.2.2 Elastic forces

The last term in the equations of motion in (22) differs from the previous ones as it is based on the strain energy of deformation U, which is differently defined in the original finite element method and in the absolute nodal coordinate formulation.

Without loss of generality, the expression for the strain energy is given using the simplest case of the Euler–Bernoulli beam element. In the conventional finite element approach, the strain energy is usually expressed in the form

$$U = \frac{1}{2} EI \int_0^{\ell} y''^2\, \mathrm{d}x, \tag{29}$$

where EI is the bending rigidity of the beam, y'' is the second derivative of the centerline deflection, and the integration is performed through the beam arc coordinate x.

Obtaining the shape functions and nodal coordinates as expressed in (3), one can derive the expression $y'' = \mathbf{s}'' \cdot \mathbf{u}$ and finally get the value of elastic forces as a gradient of the strain energy expressed in (29) as follows:

$$\mathbf{Q}_{\text{FEM}}^{\text{elast}} = \frac{\partial U}{\partial \mathbf{u}} = EI \int_0^\ell \frac{\partial y''}{\partial \mathbf{u}} y'' \,\mathrm{d}x = \left[EI \int_0^\ell \mathbf{s}''\mathbf{s}'' \,\mathrm{d}x \right] \cdot \mathbf{u} = \mathbf{K}_{\text{FEM}} \cdot \mathbf{u}, \tag{30}$$

where the expression in the bracket represents the constant stiffness matrix $\mathbf{K}_{\text{FEM}}$. It is worth saying here that the stiffness matrix includes bending rigidity only, since the introduced Euler–Bernoulli beam does not have longitudinal degrees of freedom.

In the absolute nodal coordinate formulation, the expression for strain energy is more complicated, as it includes a component due to longitudinal deformations. This is due to the absolute nature of the coordinates. Thus, the commonly used expression for strain energy can be written as

$$U = \frac{1}{2} EA \int_0^\ell \varepsilon^2 \,\mathrm{d}p + \frac{1}{2} EI \int_0^\ell \kappa^2 \,\mathrm{d}p, \tag{31}$$

where ε is the longitudinal deformation of the beam centerline (EA is the tensional rigidity),

$$\varepsilon = \frac{1}{2}\left(\mathbf{r}' \cdot \mathbf{r}' - 1\right), \tag{32}$$

and κ is the transverse curvature of the beam defined as follows:

$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}. \tag{33}$$

In the latter formulas (32) and (33), primes denote differentiation with respect to the arc coordinate of the beam. The corresponding vectors can be computed in a straightforward way as follows:

$$\mathbf{r}' = \mathbf{S}' \cdot \mathbf{q},$$
$$\mathbf{r}'' = \mathbf{S}'' \cdot \mathbf{q}.$$

However, the explicit expression for the strain energy to be computed via (31) is considerably more complicated than the above expression (29) in the case of the FEM element. As a result, the vector of generalized elastic forces in ANCF is highly nonlinear:

$$\mathbf{Q}_{\text{ANC}}^{\text{elast}} = \frac{\partial U}{\partial \mathbf{q}} = \mathbf{K}_{\text{ANC}}(\mathbf{q}) \cdot \mathbf{q}, \tag{34}$$

where $\mathbf{K}_{\text{ANC}}(\mathbf{q})$ is the nonlinear stiffness matrix, which has, in a general case of arbitrary large displacements, few common terms with the constant stiffness matrix $\mathbf{K}_{\text{FEM}}$ of the linear FEM element. However, assuming the displacements to be small, it is possible to find that nonzero components of the nonlinear stiffness matrix $\mathbf{K}_{\text{ANC}}$ correspond to elements of the linear stiffness matrix $\mathbf{K}_{\text{FEM}}$; see Sect. 4.2.

Table 1 Mapping of the structural finite elements of beams and plates from conventional FEM to ANCF

	Conventional FEM elements			Elements adopted for ANCF	
	Year, Author(s)	N	m	$\aleph$	Year, Author(s)
–	Standard beam element	4	2	8	1996 Shabana [12]
			3	12	2006 Dmitrochenko, Yoo et al. [14]
–	1974 Narayanaswami, Adelman [20]	4	2	8	2007 Mikkola, Dmitrochenko [21]
□	1966 Bogner, Fox et al. [22]	16	3	48	2003 Dmitrochenko, Pogorelov [4]
□	1963 Melosh [23]	12	3	36	2005 Dufva and Shabana [24]
Δ	1988 Specht [25]	9	3	27	2007 Dmitrochenko, Mikkola [19]
Δ	1971 Morley [26]	6	3	18	2007 Dmitrochenko, Mikkola [19]
◊	1968 Clough & Felippa [27]	12	3	36	2007 Schwab, Gerstmayr et al. [28]

3.2.3 *Transformation of equations of motion: summary*

In the preceding paragraphs, it has been shown how some of the important terms of the equations of motion are changed when one applies the formal transformation rule to obtain an ANCF element from an existing conventional FEM element. The results obtained above can be summarized in the following diagram:

$$\begin{array}{ccccccccccl}
\mathbf{M}_{\mathrm{FEM}} & \cdot & \ddot{\mathbf{u}} & + & \mathbf{K}_{\mathrm{FEM}} & \cdot & \mathbf{u} & = & \mathbf{Q}^{\mathrm{grav}}_{\mathrm{FEM}} & & \text{(FEM, small displacements } \mathbf{u}) \\
\otimes & & \otimes & & \downarrow & & \otimes & & \otimes & & \\
\mathbf{I} & & \Re & & \downarrow & & \Re & & \mathbf{I} & & \text{(mapping of terms)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{M}_{\mathrm{ANC}} & \cdot & \ddot{\mathbf{q}} & + & \mathbf{K}_{\mathrm{ANC}}(\mathbf{q}) & \cdot & \mathbf{q} & = & \mathbf{Q}^{\mathrm{grav}}_{\mathrm{ANC}} & & \text{(ANC, large position vector } \mathbf{q})
\end{array}$$

As one can see, most terms of the equations are transformed in a simple way: namely, their size is multiplied by 2 or 3 as a result of applying Kronecker's product $\otimes$. The only exception is the stiffness matrix, which is transformed in a rather complicated way using a nonlinear mapping $\mathbf{K}_{\mathrm{FEM}} \to \mathbf{K}_{\mathrm{ANC}}(\mathbf{q})$.

3.3 Known implementations for thin structural elements

The procedure described above can be considered as the immersion of 1- or 2-manifold into a two-dimensional space, the number of degrees of freedom N of the source element is multiplied by $m = 2$, so that the number of degrees of freedom is $\aleph = N \cdot m$. The immersion into a three-dimensional space will multiply it by $m = 3$. The full list of known implementations of structural ANCF elements is given in Table 1, right-hand column. The corresponding conventional elements are presented in the left-hand column. This list can be extended by adding any existing structural element to the left-hand side. The interpretation of the symbols in the first column of the table is as follows: – stands for beam elements, while □, Δ, and ◊ stand for plate elements such as rectangular, triangular, and quadrilateral ones, accordingly.

4 Invariants of the transformation

If the transformation procedure described above has been applied to a structural element from an FEM library to obtain a new element using ANCF, it can be shown that some important static and dynamic properties are preserved automatically.

4.1 Strains during an arbitrary rigid-body rotation

Let the position and deformation of an ANCF element (the beam element for simplicity) be given by a set of nodal vectors $\mathbf{q}_k$ and scalar representation of shape functions s_k according to (11) as follows:

$$\mathbf{r} = \sum_{k=1}^{4} s_k \mathbf{q}_k .$$

This configuration can be considered as the initial reference configuration. One of the measures of deformation of the element in this configuration is the longitudinal deformation (32), which can be computed as follows:

$$\varepsilon = \frac{1}{2}\left(\mathbf{r}' \cdot \mathbf{r}' - 1\right) = \frac{1}{2}\left(\sum_{k,l=1}^{4} s'_k \mathbf{q}_k \cdot s'_l \mathbf{q}_l - 1\right). \tag{35}$$

Another configuration which is considered here is the large rigid-body rotation of the whole element. This can be expressed by the vector of nodal coordinates as follows:

$$\mathbf{r}^* = \sum_{k=1}^{4} s_k \mathbf{q}_k^*, \qquad \mathbf{q}_k^* = \mathbf{R} \cdot \mathbf{q}_k .$$

Each nodal coordinate vector $\mathbf{q}_k$ which is either a position vector or a slope vector is multiplied by the rotation matrix $\mathbf{R}$. After the rotation, it is easy to demonstrate that the value of longitudinal deformation is preserved:

$$\varepsilon_* = \frac{1}{2}\left(\mathbf{r}'_* \cdot \mathbf{r}'_* - 1\right) = \frac{1}{2}\left(\sum_{k,l=1}^{4} s'_k s'_l \underbrace{\mathbf{q}_k^* \cdot \mathbf{q}_l^*}_{\mathbf{q}_k \cdot \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} \cdot \mathbf{q}_l} - 1\right) = \frac{1}{2}\left(\sum_{k,l=1}^{4} s'_k s'_l \mathbf{q}_k \cdot \mathbf{q}_l - 1\right) = \varepsilon .$$

Note that scalars s'_l are commutative with respect to vectors $\mathbf{q}_k^*$, and the rotation matrix is orthogonal: $\mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$.

In a similar way, it is possible to demonstrate that the transverse curvature defined by (33) is also preserved during an arbitrary rigid-body rotation. That means that the expression for the strain energy given by (31) remains unchanged after the rotation, as does the vector of elastic forces (34).

4.2 Small displacements

In this section, a special test case of loading is considered. In the test, an ANCF element in the initial undeformed configuration is loaded by its own weight. Again, for the sake of simplicity, the beam element described in Sect. 2.1.2 is studied.

The geometry of the undeformed straight-line configuration of the beam can be expressed by the following relations:

$$\|\mathbf{r}'\| = 1, \qquad \|\mathbf{r}''\| = 0.$$

As a consequence, the longitudinal deformation and the transverse curvature vanish:

$$\begin{aligned} \varepsilon &= \frac{1}{2}\left(\mathbf{r}' \cdot \mathbf{r}' - 1\right) = 0, \\ \kappa &= \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = 0. \end{aligned} \tag{36}$$

However, the asymptotic behavior of the two deformations is different when the configuration of the element is in a nearly straight-line position. Accordingly, for small vertical deflections, the quantity $\|\mathbf{r}'\|$ remains close to 1, $\|\mathbf{r}'\| \approx 1$, while the quantity $\|\mathbf{r}''\|$ changes more rapidly. Note that the condition $\|\mathbf{r}'\| = \mathbf{r}' \cdot \mathbf{r}' \approx 1$ implies $\mathbf{r}' \cdot \mathbf{r}'' \approx 0$, or vectors $\mathbf{r}'$ and $\mathbf{r}''$ are nearly orthogonal. For these reasons, the estimation of the behavior of the deformation quantities from (36) can be written as

$$\varepsilon \approx 0, \kappa = \frac{\|\mathbf{r}'\| \|\mathbf{r}''\| \sin\angle(\mathbf{r}', \mathbf{r}'')}{\|\mathbf{r}'\|^3} \approx \|\mathbf{r}''\|,$$

and their squares are approximated as

$$\begin{aligned} \varepsilon^2 &\approx 0, \\ \kappa^2 &\approx \mathbf{r}'' \cdot \mathbf{r}''. \end{aligned}$$

Furthermore, the approximate expression for the strain energy takes the form

$$U \approx \underbrace{\frac{1}{2} EA \int_0^\ell \underbrace{\varepsilon^2}_{\approx 0} \mathrm{d}p}_{\approx 0} + \frac{1}{2} EI \int_0^\ell \mathbf{r}'' \cdot \mathbf{r}'' \,\mathrm{d}p,$$

which allows us to compute the estimated value of the vector of the elastic forces:

$$\mathbf{Q}_{\mathrm{ANC}}^{\mathrm{elast}} = \frac{\partial U}{\partial \mathbf{q}} \approx EI \int_0^\ell \underbrace{\frac{\partial \mathbf{r}''^{\mathrm{T}}}{\partial \mathbf{q}} \cdot \mathbf{r}''}_{\mathbf{S}''^{\mathrm{T}} \cdot \mathbf{S}'' \cdot \mathbf{q}} \mathrm{d}p = \left[EI \int_0^\ell \mathbf{S}''^{\mathrm{T}} \cdot \mathbf{S}'' \,\mathrm{d}p \right] \cdot \mathbf{q} = \mathbf{K}_{\mathrm{ANC}}^{\mathrm{lin}} \cdot \mathbf{q}.$$

The expression in brackets corresponds to the approximated constant stiffness matrix $\mathbf{K}_{\mathrm{ANC}}^{\mathrm{lin}}$ obtained by assuming the displacement to be small. Apparently, this stiffness matrix is closely related to the bending stiffness matrix of the original finite element:

$$\mathbf{K}_{\mathrm{ANC}}^{\mathrm{lin}} = \mathbf{K}_{\mathrm{FEM}} \otimes \mathbf{I}.$$

Thus, in the case of small vertical displacements, the stiffness coefficients in both the original FEM element and in the ANCF element remain unchanged. Furthermore, since the generalized gravity forces have the same numerical values in both formulations (see Sect. 3.2.1), small vertical displacements in the linear static problem are preserved after the transformation of the element.

It should be noted that in this section, vertical displacements only are taken into account since in the original structural elements of beams and plates considered in Sect. 2, only the bending behavior of the elements is accounted for.

4.3 Inertia properties and natural frequencies at initial configuration

In Sect. 3.2.1, it was pointed out that mass matrices of the original finite element and the ANC-transformed element use the same inertia coefficients. On the other hand, in the previous Sect. 4.2 it is shown that also stiffness coefficients are the same if the displacements are small around the undeformed straight-line configuration. Combining these two ideas, one can conclude that the natural frequencies of oscillations of the two elements near this initial configuration will be same.

It should be noted that in similar way to the previous section, only the transverse oscillations and their frequencies are compared here since the original structural elements of beams and plates simply do not have longitudinal degrees of freedom, and the corresponding longitudinal frequencies are absent.

5 Conclusions

In this paper, a study is conducted on relationships between conventional FEM and recently introduced ANCF structural elements that use the Euler–Bernoulli or Kirchhoff theory for beams and plates. It is shown that elements in FEM that use only transverse displacements and slopes as generalized coordinates can be transformed into a corresponding ANCF element. After such a transition, some important geometric, static, and dynamic properties of the obtained elements are preserved.

It is also expected that similar relationships exist for fully parameterized shear deformable ANCF elements that have been recently proposed in [2, 5, 29–31]. These elements employ the Timoshenko beam theory or the Reissner–Mindlin theory. However, during this research, the authors failed to find conventional elements that could correspond to the latter references. The research in this direction is not finished yet, and it will be a subject of subsequent research.

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