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A novel formulation of the dynamic balancing of five-bar linkages with applications to link optimization

Djana Ilia · Rosario Sinatra

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Abstract In this report, we derive design equations and techniques for the dynamic balancing of five-bar linkage, using a novel and simplified approach. Firstly, in order to derive the dynamic equations of the mechanism we have applied the natural orthogonal complement method. Subsequently, an optimization method for the dynamic balancing of the linkage is proposed. The conditions of dynamic balancing of the five-bar linkage are expressed as seven equations and four inequalities, with twelve linkage parameters. The dynamic balancing of the mechanism is formulated and solved as an optimization problem under equality constraints. The application of the new approach is illustrated through a numerical example.

Keywords Dynamic balancing · Planar five-bar linkages · Optimum design

1 Introduction

Numerous researchers have investigated the problem of dynamic balancing. For example, Berkof and Lowen [1] introduced a general method, called *the method of linearly independent vectors that it is applicable to most planar linkages*, which provides the minimum required balancing equations. These authors showed that if the center of mass of the mechanism can be made stationary, then the shaking force vanishes.

Bagci [2] used the method of linearly independent vectors to derive the equations for some 4-, 6-, and 8-bar planar mechanisms. Idle loops were introduced for their complete force balancing. As distinct from the complex vector method of Berkof and Lowen [3],

D. Ilia
Department of Naval Engineering, Naval Faculty, Technological University of Vlora,
Ruga: Vlore-Skele, Vlore, Albania
e-mail: dianaba2001@yahoo.com

R. Sinatra (✉)
Dipartimento di Ingegneria Industriale e Meccanica, Università Degli Studi di Catania,
Viale A. Doria n. 6, 95125 Catania, Italy
e-mail: rsinatra@diim.unict.it

a new method for complete balancing of planar linkages was presented by Kochev [4], which applies ordinary vector algebra and produces the balancing equations in Cartesian form. Moreover, a dynamic-balancing software package, DAM 8, was reported by Thümmel [5].

Berkof [6] showed that for a particular case of in-line linkages, the use of inertia counterweight and a physical pendulum link cam provide a complete force and moment balance. What is more, the conditions for mass redistribution and an addition of supplementary masses that are needed to fully balance the inertia loads of the linkage were obtained.

Later, Bagci [7, 8] showed that idle loops could be added to more general four-bar linkages in order to achieve the same result. However, this method increases the individual forces that the linkage exerts on its supports. Angeles, Nahon and Thümmel [9] showed that this problem can be solved by force control under redundant actuation. In this way, the shaking moment due to these forces is eliminated, while the elimination of the shaking forces can be achieved through design. Furthermore, the active dynamic balancing of joint forces by redundant motors was considered by Thümmel [10] and applied to a four-bar linkage. An experimental single-degree of freedom mechanism with two motors was used for this proposal.

Many papers have been published on the dynamics balancing of the five-bar linkage. Ouyng, Li, and Zhang [11] proposed a novel approach for the force balancing and the facilitation of the design of controllers of RTC (real-time controllable) for the five-bar mechanism. This approach is called Adjusting Kinematic Parameter (AKP), but it only works for RTC mechanisms.

Shirinzadeh and Alici [12] have given a methodology for optimum dynamic balancing of planar parallel manipulators typified with a variable speed, 2 DOF five-bar linkage, with revolute joints. For the planar and spatial parallel mechanisms, many papers were done by Gosselin et al. In [13–15] the static balancing of planar parallel manipulator was discussed; in [16] was focused on the design of gravity-compensated of a six-degree-of-freedom parallel manipulator with revolute joints; in [17], the static balancing conditions are derived for the three-degree-of-freedom spatial parallel manipulator and in [18] similar conditions are obtained for the spatial four-degree-of-freedom parallel manipulator using two common methods, i.e., counterweights and springs. In [19], the dynamic balancing of multi-degree-of-freedom parallel mechanisms with multiple legs is presented.

Here, the dynamic balancing is formulated as an optimization problem such that a sum-squared value of bearing forces, driving torques, shaking moment, and the deviation of the angular momentum from its mean value area minimized throughout an operation range of the manipulator, provided that a set of balancing conditions, the size of some inertial and geometric parameters are satisfied.

This paper deals with dynamic balancing of the planar five-bar linkage, using a novel simplified approach, introduced by Angeles and Lee [20, 21]. A dynamically balanced mechanism requires that the shaking force and shaking moment, which are due to the moving inertia of the system, transmitted to the frame of the mechanism are zero.

To this end, we first derive the dynamic model of the foregoing mechanism by resorting to the method of the natural-orthogonal complement, and find then the shaking force acting in the frame. Then we come to the discussion of dynamic balancing, and derive equations of the complete dynamic balancing of the linkage. The conditions of dynamic balancing of the five-bar linkage are expressed by a system of seven equations and four inequality constraints, with 12 linkage parameters. The parameters will be solved as the optimum design problem under equality and inequality constraints.

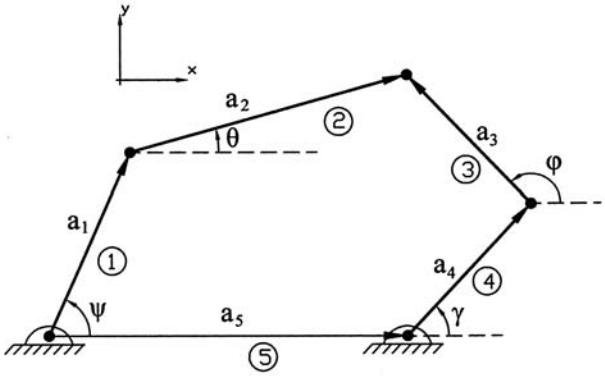


Fig. 1 Five-bar linkage

2 The kinematics analysis for the five-bar linkage mechanism

In this section, will be illustrated the kinematics of five-bar linkage mechanism shown in Fig. 1. For this mechanism, we assign ψ and γ as the angles that describe the input links 1 and 4, respectively, and φ and θ the angles that describe the coupler links 2 and 3, respectively.

For this mechanism, the loop closure equation is written as

$$\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_5 + \mathbf{a}_4 + \mathbf{a}_3. \tag{1}$$

With respect to x and y axis, we can write:

$$\begin{cases} a_1 \cos\psi + a_2 \cos\theta = a_3 \cos\varphi + a_4 \cos\gamma + a_5, \\ a_1 \sin\psi + a_2 \sin\theta = a_3 \sin\varphi + a_4 \sin\gamma. \end{cases} \tag{2}$$

If we solve (2) with respect to $\cos\theta$ and $\sin\theta$, we obtain

$$\cos\theta = \frac{a_3 \cos\varphi + a_4 \cos\gamma + a_5 - a_1 \cos\psi}{a_2}, \tag{3a}$$

$$\sin\theta = \frac{a_3 \sin\varphi + a_4 \sin\gamma - a_1 \sin\psi}{a_2}. \tag{3b}$$

Making the second power of (3a) and (3b) and adding both sides, we obtain the expression below

$$A \sin\varphi + B \cos\varphi = C, \tag{4}$$

where A, B and C are

$$A = a_1 \sin\psi - a_4 \sin\gamma, \tag{5a}$$

$$B = a_1 \cos\psi - a_4 \cos\gamma - a_5, \tag{5b}$$

$$C = \frac{a_1^2 + a_3^2 + a_4^2 + a_5^2 - a_2^2}{2a_3} + \frac{a_4 a_5}{a_3} \cos\gamma - \frac{a_1 a_5}{a_3} \cos\psi - \frac{a_1 a_4}{a_3} (\sin\gamma \sin\psi + \cos\gamma \cos\psi). \tag{5c}$$

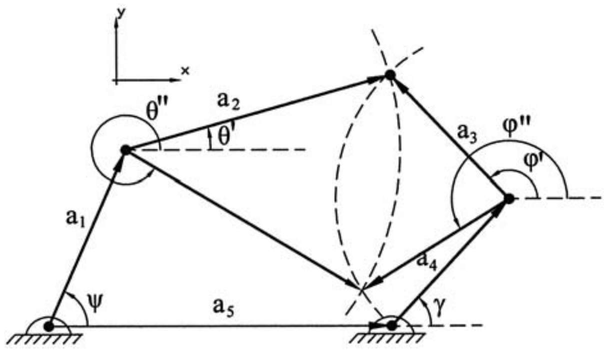


Fig. 2 Two possible linkage configurations

In (4), we substitute $\sin\varphi = 2T/1+T^2$ and $\cos\varphi = 1-T^2/1+T^2$ where $T = \mathrm{tg}(\varphi/2)$, and hence we have a quadratic equation in T. Then φ is readily computed as follows:

$$\varphi = 2\,\mathrm{arctg}\left(\frac{A \pm \sqrt{A^2+B^2-C^2}}{B+C}\right). \tag{6}$$

The two signs (+), (−) correspond to two possible linkage configurations which are called conjugate, shown in Fig. 2.

If we substitute (6) in the (3b), we obtain

$$\theta = \arcsin\left(\frac{a_3 \sin\varphi + a_4 \sin\gamma - a_1 \sin\psi}{a_2}\right). \tag{7}$$

Now we attempt to find the velocity ratios and their derivatives with respect to time. At first, we derive the velocity ratios, r_φ and r_θ, defined as

$$r_\varphi = \frac{\dot\varphi}{\dot\psi} = \frac{d\varphi}{d\psi}, \tag{8a}$$

$$r_\theta = \frac{\dot\theta}{\dot\psi} = \frac{d\theta}{d\psi}. \tag{8b}$$

In order to do this, we will resort to complex number algebra where the vectors in Fig. 1 are represented as complex numbers

$$\mathbf{a}_1 = a_1 e^{i\psi}, \qquad \mathbf{a}_2 = a_2 e^{i\theta}, \qquad \mathbf{a}_3 = a_3 e^{i\varphi}, \qquad \mathbf{a}_4 = a_4 e^{i\gamma}, \qquad \mathbf{a}_5 = a_5 e^{i0}. \tag{9}$$

Now we solve (1) for $\mathbf{a}_2$,

$$\mathbf{a}_2 = \mathbf{a}_5 + \mathbf{a}_4 + \mathbf{a}_3 - \mathbf{a}_1. \tag{10}$$

The conjugate equation of (11) is

$$\overline{\mathbf{a}_2} = \overline{\mathbf{a}_5} + \overline{\mathbf{a}_4} + \overline{\mathbf{a}_3} - \overline{\mathbf{a}_1}. \tag{11}$$

Multiplying both sides of (10) and (11) and bearing in mind that

$$\mathbf{a}_j\overline{\mathbf{a}_j} = \mathbf{a}_j^2, \qquad \mathbf{a}_j\overline{\mathbf{a}_k} + \mathbf{a}_k\overline{\mathbf{a}_j} = 2\mathbf{a}_j\mathbf{a}_k\cos(\alpha_j - \alpha_k). \tag{12}$$

We obtain the expression below

$$f(\varphi,\psi,\gamma)=k_1+k_2\cos(\gamma-\varphi)+k_3\cos\varphi-k_4\cos(\varphi-\psi)$$
$$+k_5\cos\gamma-k_6\cos(\gamma-\psi)-k_7\cos\psi=0, \tag{13}$$

where

$$k_1=a_1^2-a_2^2+a_3^2+a_4^2+a_5^2,\qquad k_2=2a_4a_3,\qquad k_3=2a_5a_3, \tag{14a}$$
$$k_4=2a_1a_3,\qquad k_5=2a_4a_5,\qquad k_6=2a_1a_4,\qquad k_7=2a_1a_5. \tag{14b}$$

Upon differentiation of both sides of (13) with respect to time, we obtain

$$\frac{df}{dt}=\frac{\partial f}{\partial\psi}\dot{\psi}+\frac{\partial f}{\partial\gamma}\dot{\gamma}+\frac{\partial f}{\partial\varphi}\dot{\varphi}=0 \tag{15}$$

which can be rewritten,

$$\frac{\partial f}{\partial\varphi}\frac{\dot{\varphi}}{\dot{\psi}}=-\frac{\partial f}{\partial\psi}-\frac{\partial f}{\partial\gamma}\frac{\dot{\gamma}}{\dot{\psi}}. \tag{16}$$

From (16), we can obtain r_φ as

$$r_\varphi=-\left(\frac{\frac{\partial f}{\partial\psi}+\frac{\partial f}{\partial\gamma}r_\gamma}{\frac{\partial f}{\partial\varphi}}\right), \tag{17}$$

where $r_\gamma=\frac{\dot{\gamma}}{\dot{\psi}}=\frac{d\gamma}{d\psi}$.

The partial derivatives appearing in (17) are

$$\frac{\partial f}{\partial\psi}=k_4\sin(\varphi-\psi)+k_6\sin(\gamma-\psi)+k_7\sin\psi, \tag{18a}$$
$$\frac{\partial f}{\partial\gamma}=-k_2\sin(\gamma-\varphi)-k_5\sin\gamma+k_6\sin(\gamma-\psi), \tag{18b}$$
$$\frac{\partial f}{\partial\varphi}=-k_2\sin(\gamma-\varphi)-k_3\sin\varphi+k_4\sin(\varphi-\psi). \tag{18c}$$

So, r_φ is a function of ψ, γ, φ, $\dot{\psi}$, and $\dot{\gamma}$.

Once the velocity ratio r_φ is known, the $\dot{\varphi}$ can be determined as

$$\dot{\varphi}=r_\varphi\dot{\psi}. \tag{19}$$

Moreover, upon differentiation r_φ with respect to time, we obtain

$$\dot{r}_\varphi=\frac{dr_\varphi}{dt}=\frac{\partial r_\varphi}{\partial\gamma}\dot{\gamma}+\frac{\partial r_\varphi}{\partial\varphi}\dot{\varphi}+\frac{\partial r_\varphi}{\partial\psi}\dot{\psi}+\frac{\partial r_\varphi}{\partial\dot{\gamma}}\ddot{\gamma}+\frac{\partial r_\varphi}{\partial\dot{\psi}}\ddot{\psi}. \tag{20}$$

In the same way, we calculate r_θ and its derivative (see Appendix).

3 Dynamic modeling using the natural orthogonal complement

In this section, the dynamic analysis of five-bar linkage is presented, shown in Fig. 3 based on the method of natural orthogonal complement, as first introduced by Angeles and Lee [13]. In the case of the planar mechanism, the twist $\mathbf{t}_i$ of the ith link, the wrench, $\mathbf{w}_i$ exerted onto the same link and the extended mass matrix $\mathbf{M}$ are defined as

$$\mathbf{t}_i = \begin{bmatrix} \omega_i \\ \dot{\mathbf{c}}_i \end{bmatrix}, \qquad \mathbf{w}_i^E = \begin{bmatrix} T_i \\ \mathbf{f}_i \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} I_i & \mathbf{0}^T \\ \mathbf{0} & m_i \mathbf{1} \end{bmatrix}, \tag{21}$$

where ω_i, T_i and I_i are the scalar angular velocity, the torque of the ith link and the polar moment of inertia of the ith link about its mass center, respectively. The vectors $\dot{\mathbf{c}}_i$ and $\mathbf{f}_i$ are the 2-dimensional vectors representing the velocity of the mass center of the ith link and the force acting at that mass center, and $\mathbf{0}$ and $\mathbf{1}$ are the 2-dimensional zero vector, and the 2×2 identity matrix, respectively.

Now the Newton–Euler equations for the ith link moving in the plane, as depicted in Fig. 3, if m_i is mass of the ith link, can be written, for $i = 1, 2, 3, 4$, as

$$\mathbf{r}_i^T \mathbf{E} \mathbf{f}_{i-1,i} + (\mathbf{r}_{i,i+1} - \mathbf{r}_i)^T \mathbf{E} \mathbf{f}_{i,i+1} + T_i = I_i \dot{\omega}_i, \tag{22a}$$

$$\mathbf{f}_{i-1,i} - \mathbf{f}_{i,i+1} + \mathbf{f}_i = m_i \ddot{\mathbf{c}}_i, \tag{22b}$$

where the cross product has been represented in a 2-dimensional form using the 2×2 orthogonal matrix $\mathbf{E}$ representing a rotation in the plane through an angle of 90°, namely,

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{23}$$

Upon combining (22a) and (22b), we obtain

$$\mathbf{M}_i \dot{\mathbf{t}}_i = \mathbf{w}_i^E + \mathbf{B}_i \boldsymbol{\varphi}_i, \quad i = 1, 2, 3, 4, \tag{24}$$

where the 3×4 matrix $\mathbf{B}_i$ and the 4-dimensional vector $\boldsymbol{\varphi}_i$ are defined as

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{r}_i^T \mathbf{E} & \mathbf{b}_i^T \mathbf{E} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varphi}_i = \begin{bmatrix} \mathbf{f}_{i-1,i} \\ \mathbf{f}_{i,i+1} \end{bmatrix} \tag{25}$$

for, $i = 1, 2, 3, 4$ and $\mathbf{b}_i = \mathbf{r}_{i,i+1} - \mathbf{r}_i$.

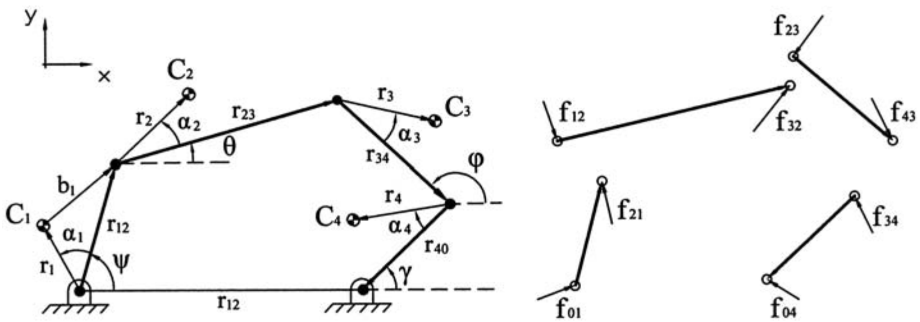


Fig. 3 Five-bar linkage

Hence, (24) can be assembled in the form

$$\mathbf{M}\dot{\mathbf{t}} = \mathbf{w}^E + \mathbf{B}\boldsymbol{\varphi}, \tag{26}$$

where 12×12 matrix $\mathbf{M}$ of extended mass and the 12×10 matrix $\mathbf{B}$ are given below as

$$\mathbf{M} \equiv \mathrm{diag}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4), \tag{27}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{r}_1^T\mathbf{E} & \mathbf{b}_1^T\mathbf{E} & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{1} & -\mathbf{1} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{0}^T & \mathbf{r}_2^T\mathbf{E} & \mathbf{b}_2^T\mathbf{E} & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{O} & \mathbf{1} & -\mathbf{1} & \mathbf{O} & \mathbf{O} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{r}_3^T\mathbf{E} & \mathbf{b}_3^T\mathbf{E} & \mathbf{0}^T \\ \mathbf{O} & \mathbf{O} & \mathbf{1} & -\mathbf{1} & \mathbf{O} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{r}_4^T\mathbf{E} & \mathbf{b}_4^T\mathbf{E} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{1} & -\mathbf{1} \end{bmatrix}. \tag{28}$$

Moreover, the 12-dimensional vector of external wrenches $\mathbf{w}^E$ and the 10-dimensional vector of joint forces $\boldsymbol{\varphi}$ are defined as

$$\mathbf{w}^E \equiv \left[\left(\mathbf{w}_1^E\right)^T, \left(\mathbf{w}_2^E\right)^T, \left(\mathbf{w}_3^E\right)^T, \left(\mathbf{w}_4^E\right)^T\right], \tag{29a}$$

$$\boldsymbol{\varphi} \equiv \left[\boldsymbol{\varphi}_1^T, \mathbf{f}_{23}^T, \boldsymbol{\varphi}_4^T\right]^T. \tag{29b}$$

Referring to Fig. 4, we can write the kinematics constraint equations of the four-bar linkage as

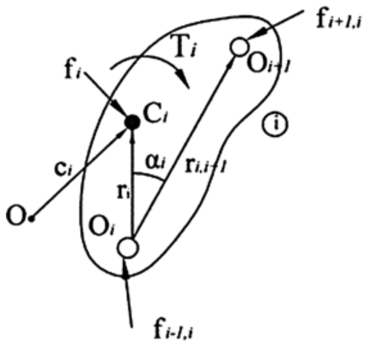
$$\omega_i \equiv \omega_{i-1} + \dot{\theta}_i, \tag{30}$$

$$\mathbf{c}_i \equiv \mathbf{c}_{i-1} - \mathbf{r}_{i-1} + \mathbf{r}_{i-1,i} + \mathbf{r}_i. \tag{31}$$

Furthermore, by differentiating both sides of (31) with respect to time, one obtains

$$\dot{\mathbf{c}}_i \equiv \dot{\mathbf{c}}_{i-1} + \omega_{i-1}\mathbf{E}\mathbf{b}_{i-1} + \omega_i\mathbf{E}\mathbf{r}_i. \tag{32}$$

Fig. 4 Free-body diagram of a general link



Now (32) can be written in the form

$$\mathbf{A}_{i,i-1}\mathbf{t}_{i-1} + \mathbf{A}_{i,i}\mathbf{t}_i = \mathbf{0}, \quad i = 1, 2, 3, 4. \tag{33}$$

In the above equations, the 2×3 matrices $\mathbf{A}_{i,i-1}$ and $\mathbf{A}_{i,i}$ for $i = 1, 2, 3, 4$, are given as

$$\mathbf{A}_{i,i-1} \equiv [\mathbf{E}\mathbf{b}_{i-1} \quad \mathbf{1}], \qquad \mathbf{A}_{i,i} \equiv [-\mathbf{E}\mathbf{r}_i \quad \mathbf{1}]. \tag{34}$$

On the other hand, (33) are equivalent to the linear homogeneous system below

$$\mathbf{A}\mathbf{t} = \mathbf{0}, \tag{35}$$

where $\mathbf{t} = [\mathbf{t}_1^T, \mathbf{t}_2^T, \mathbf{t}_3^T, \mathbf{t}_4^T]^T$ and $\mathbf{A}$ take on the form displayed below for the mechanism under study

$$\mathbf{A} \equiv \begin{bmatrix} -\mathbf{E}\mathbf{r}_1 & \mathbf{1} & \mathbf{0} & \mathbf{O} & \mathbf{1} & \mathbf{O} & \mathbf{0} & \mathbf{O} \\ -\mathbf{E}\mathbf{b}_1 & -\mathbf{1} & -\mathbf{E}\mathbf{r}_2 & \mathbf{1} & \mathbf{0} & \mathbf{O} & \mathbf{0} & \mathbf{O} \\ \mathbf{0} & \mathbf{O} & -\mathbf{E}\mathbf{b}_2 & -\mathbf{1} & -\mathbf{E}\mathbf{r}_3 & \mathbf{1} & \mathbf{0} & \mathbf{O} \\ \mathbf{0} & \mathbf{O} & \mathbf{0} & \mathbf{O} & -\mathbf{E}\mathbf{b}_3 & -\mathbf{1} & -\mathbf{E}\mathbf{r}_4 & \mathbf{1} \\ \mathbf{0} & \mathbf{O} & \mathbf{0} & \mathbf{O} & \mathbf{0} & \mathbf{O} & -\mathbf{E}\mathbf{b}_4 & -\mathbf{1} \end{bmatrix} \tag{36}$$

in which $\mathbf{O}$ is the 2×2 zero matrix.

It is apparent that $\mathbf{A}^T = \mathbf{B}$, which is not haphazard. This is a result of the reciprocity between the feasible twists of the constrained mechanical system and the constraint wrenches exerted by the joints. In fact, upon rewriting (26) in terms of $\mathbf{A}$,

$$\mathbf{M}\dot{\mathbf{t}} = \mathbf{w}^E + \mathbf{A}^T\boldsymbol{\varphi}. \tag{37}$$

It becomes apparent that the internal wrench, $\boldsymbol{\varphi}$, is no more than the vector of Lagrange multipliers of classical dynamics.

Now the twist of the individual links are given as

$$\mathbf{t}_1 = \dot{\psi}\begin{bmatrix} 1 \\ \mathbf{E}\mathbf{r}_1 \end{bmatrix}, \qquad \mathbf{t}_2 = \dot{\psi}\begin{bmatrix} 0 \\ \mathbf{E}\mathbf{r}_{12} \end{bmatrix} + \dot{\theta}\begin{bmatrix} 1 \\ \mathbf{E}\mathbf{r}_2 \end{bmatrix},$$
$$\mathbf{t}_3 = \dot{\gamma}\begin{bmatrix} 0 \\ \mathbf{r}_{40}^T\mathbf{E} \end{bmatrix} + \dot{\varphi}\begin{bmatrix} 1 \\ \mathbf{b}_3^T\mathbf{E} \end{bmatrix}, \qquad \mathbf{t}_4 = \dot{\gamma}\begin{bmatrix} 1 \\ \mathbf{b}_4^T\mathbf{E} \end{bmatrix}. \tag{38}$$

Now we define the vector of generalized angular velocity as

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\psi} \\ \dot{\gamma} \end{bmatrix} \tag{39}$$

and so we can write the twist as a linear transformation of $\dot{\mathbf{q}}$:

$$\mathbf{t} = \mathbf{u}\dot{\mathbf{q}}, \tag{40}$$

where

$$\mathbf{u} = \begin{bmatrix} 1 & -\mathbf{r}_1^T\mathbf{E} & r_\theta & -(\mathbf{r}_{12} + \mathbf{r}_\theta\mathbf{r}_2)^T\mathbf{E} & r_\varphi & r_\varphi\mathbf{b}_3^T\mathbf{E} & 0 & \mathbf{0}^T \\ 0 & \mathbf{0}^T & 0 & \mathbf{0}^T & 0 & \mathbf{r}_{40}^T\mathbf{E} & 1 & \mathbf{b}_4^T\mathbf{E} \end{bmatrix}^T. \tag{41}$$

If we substitute $\mathbf{t}$ in (35), we obtain

$$\mathbf{A}\mathbf{u}\dot{\mathbf{q}} = \mathbf{0} \tag{42}$$

which must hold true for any $\dot{\mathbf{q}}$,

$$\mathbf{A}\mathbf{u} = \mathbf{0}. \tag{43}$$

That means that $\mathbf{u}$ is an orthogonal complement of $\mathbf{A}$; $\mathbf{u}$ lies in the nullspace of $\mathbf{A}$.

So, after multiplication of both sides of (37) by the transpose of $\mathbf{u}$, the vector of non-working constraint wrenches is eliminated from the said equations, and hence we obtain

$$\mathbf{u}^T\mathbf{M}\dot{\mathbf{t}} = \mathbf{u}^T\mathbf{w}^E. \tag{44}$$

Then by differentiating both sides of (40) with respect to time, we have

$$\dot{\mathbf{t}} = \mathbf{u}\ddot{\mathbf{q}} + \dot{\mathbf{u}}\dot{\mathbf{q}}. \tag{45}$$

Upon substituting (45) in (44), the dynamical equations of the system are obtained as

$$\mathbf{I}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{C}(\dot{\mathbf{q}}, \mathbf{q})\dot{\mathbf{q}} + \boldsymbol{\tau}. \tag{46}$$

This is the mathematical model governing the dynamics of this mechanical system, where

$\mathbf{I}(\mathbf{q}) \equiv \mathbf{u}^T\mathbf{M}\mathbf{u}$: *generalized inertia matrix* of 2×2-dimensions;
$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \equiv -\mathbf{u}^T\mathbf{M}\dot{\mathbf{u}}\dot{\mathbf{q}}$: 2-dimensional vector of *generalized centrifugal and coriolis force* terms;
$\boldsymbol{\tau}(\mathbf{q}) \equiv \mathbf{u}^T\mathbf{w}^E$: 2-dimensional vector of *generalized external force* arising from actuation, gravity, and dissipation effects.

4 Dynamic balancing

In this section, the dynamic balancing of the five-bar linkage, which contains links with arbitrary mass distributions is investigated using a novel simplified approach. As depicted in Fig. 4, the center of the ith joint and the mass center of the ith link are denoted by points O_i and C_i, for $i = 0, 1, 2, 3, 4$, respectively.

We now introduce the definitions below:

$$m = \sum_{i=1}^{4} m_i, \tag{47a}$$

$$m\mathbf{c} = \sum_{i=1}^{4} m_i\mathbf{c}_i, \tag{47b}$$

where $\mathbf{c}_i$ is the position vector of the mass center C_i of the same link, m is the total mass of the linkage, and $\mathbf{c}$ is the position vector of the mass center C of the linkage.

Upon differentiation of both sides of (47b) with respect to time, we obtain

$$m\dot{\mathbf{c}} = \sum_{i=1}^{4} m_i\dot{\mathbf{c}}_i. \tag{48}$$

The dynamic force balancing of the linkage is achieved if the mass center of the linkage is stationary, namely

$$m\dot{\mathbf{c}} = \mathbf{0}. \tag{49}$$

Hence, the condition for the total force balancing of the five-bar linkage can be stated as

$$m_1\dot{\mathbf{c}}_1 + m_2\dot{\mathbf{c}}_2 + m_3\dot{\mathbf{c}}_3 + m_4\dot{\mathbf{c}}_4 = \mathbf{0}. \tag{50}$$

Now using (40), the velocity vectors of the center of mass of each link are derived as

$$\dot{\mathbf{c}}_1 = \dot{\psi}\mathbf{E}\mathbf{r}_1, \qquad \dot{\mathbf{c}}_2 = \dot{\psi}\mathbf{E}(\mathbf{r}_{12} + r_\theta\mathbf{r}_2), \tag{51a}$$

$$\dot{\mathbf{c}}_3 = \dot{\psi}\mathbf{E}(r_\varphi\mathbf{r}_3 - r_\varphi\mathbf{r}_{34} - r_\gamma\mathbf{r}_{40}), \qquad \dot{\mathbf{c}}_4 = \dot{\psi}\mathbf{E}r_\gamma(\mathbf{r}_4 - \mathbf{r}_{40}). \tag{51b}$$

Upon substitution of (51a) and (51b) into (50), we obtain

$$\mu_1\mathbf{r}_1 + \mu_2(\mathbf{r}_{12} + r_\theta\mathbf{r}_2) + \mu_3(r_\varphi\mathbf{r}_3 - r_\varphi\mathbf{r}_{34} - r_\gamma\mathbf{r}_{40}) + \mu_4 r_\gamma(\mathbf{r}_4 - \mathbf{r}_{40}) = \mathbf{0} \tag{52}$$

with μ_i defined as the ith mass ratio, namely

$$\mu_i = \frac{m_i}{m}. \tag{53}$$

In general, vector $\mathbf{r}_i$, results from a mapping of vector $\mathbf{r}_{i,i+1}$, by an arbitrary 2×2 matrix $\mathbf{L}_i$, i.e.,

$$\mathbf{r}_i = \mathbf{L}_i\mathbf{r}_{i,i+1}, \quad i = 1, 2, 3, 4, \tag{54}$$

where the 2×2 matrix $\mathbf{L}_i$ is

$$\mathbf{L}_i = (\rho_i\cos\alpha_i)\mathbf{1} + (\rho_i\sin\alpha_i)\mathbf{E}, \quad i = 1, 2, 3, 4 \tag{55}$$

and can be regarded as a linear transformation that rotates vector $\mathbf{r}_{i,i+1}$ through an angle α_i, while changing its magnitude by a factor ρ_i given by

$$\rho_i = \frac{\|\mathbf{r}_i\|}{\|\mathbf{r}_{i,i+1}\|}. \tag{56}$$

The (54) can be written

$$\mathbf{r}_i = \rho_i\left[(\cos\alpha_i)\mathbf{r}_{i,i+1} + (\sin\alpha_i)\mathbf{E}\mathbf{r}_{i,i+1}\right], \quad i = 1, 2, 3, 4. \tag{57}$$

Furthermore, using (57), we can rewrite (52) as

$$\begin{aligned}&\left[\mu_1\rho_1(\cos\alpha_1) + \mu_2\right]\mathbf{r}_{12} + \mu_1\rho_1(\sin\alpha_1)\mathbf{E}\mathbf{r}_{12} + \mu_2\rho_2(\cos\alpha_2)r_\theta\mathbf{r}_{23}\\ &\quad + \mu_2\rho_2(\sin\alpha_2)r_\theta\mathbf{E}\mathbf{r}_{23} + \left[\mu_3\rho_3(\cos\alpha_3) - \mu_3\right]r_\varphi\mathbf{r}_{34} + \mu_3\rho_3(\sin\alpha_3)r_\varphi\mathbf{E}\mathbf{r}_{34}\\ &\quad + \left[\mu_4\rho_4(\cos\alpha_4) - \mu_3 - \mu_4\right]r_\gamma\mathbf{r}_{40} + \mu_4\rho_4(\sin\alpha_4)r_\gamma\mathbf{E}\mathbf{r}_{40} = \mathbf{0}.\end{aligned} \tag{58}$$

After transformation, we obtain:

$$\begin{aligned}&\left[\mu_2\rho_2(\cos\alpha_2) - \mu_1\rho_1(\cos\alpha_1) - \mu_2\right]r_\theta\mathbf{r}_{23} + \left[\mu_2\rho_2(\sin\alpha_2) - \mu_1\rho_1(\sin\alpha_1)\right]r_\theta\mathbf{E}\mathbf{r}_{23}\\ &\quad + \left[(\mu_3\rho_3(\cos\alpha_3) - \mu_3) - \mu_1\rho_1(\cos\alpha_1) - \mu_2\right]r_\varphi\mathbf{r}_{34}\end{aligned}$$

$$+\big[\mu_3\rho_3(\sin\alpha_3)-\mu_1\rho_1(\sin\alpha_1)\big]r_\varphi\mathbf{E}\mathbf{r}_{34}$$
$$+\big[(\mu_4\rho_4(\cos\alpha_4)-\mu_4-\mu_3)-\mu_1\rho_1(\cos\alpha_1)-\mu_2\big]r_\gamma\mathbf{r}_{40}$$
$$+\big[\mu_4\rho_4(\sin\alpha_4)-\mu_1\rho_1(\sin\alpha_1)\big]r_\gamma\mathbf{E}\mathbf{r}_{40}=\mathbf{0}. \tag{59}$$

Note that the coefficients of $r_\theta\mathbf{r}_{23}$, $r_\varphi\mathbf{r}_{34}$, and $r_\gamma\mathbf{r}_{40}$ above would be equal, were it not for their first terms. Also, the coefficients of $r_\theta\mathbf{E}\mathbf{r}_{23}$, $r_\varphi\mathbf{E}\mathbf{r}_{34}$, and $r_\gamma\mathbf{E}\mathbf{r}_{40}$ would be equal, if it were not for their first terms. In order to obtain the foregoing equalities, then we impose the relation below:

$$\mu_2\rho_2(\cos\alpha_2)=\mu_3\rho_3(\cos\alpha_3)-\mu_3, \tag{60}$$
$$\mu_2\rho_2(\cos\alpha_2)=\mu_4\rho_4(\cos\alpha_4)-\mu_4-\mu_3, \tag{61}$$
$$\mu_2\rho_2(\sin\alpha_2)=\mu_3\rho_3(\sin\alpha_3), \tag{62}$$
$$\mu_2\rho_2(\sin\alpha_2)=\mu_4\rho_4(\sin\alpha_4). \tag{63}$$

Under the above conditions, (59) can be simplified as

$$\big[\mu_2\rho_2(\cos\alpha_2)-\mu_1\rho_1(\cos\alpha_1)-\mu_2\big](r_\theta\mathbf{r}_{23}+r_\varphi\mathbf{r}_{34}+r_\gamma\mathbf{r}_{40})$$
$$+\big[\mu_2\rho_2(\sin\alpha_2)-\mu_1\rho_1(\sin\alpha_1)\big]\mathbf{E}(r_\theta\mathbf{r}_{23}+r_\varphi\mathbf{r}_{34}+r_\gamma\mathbf{r}_{40})=\mathbf{0}. \tag{64}$$

Upon multiplying both sides of (64) by $(r_\theta\mathbf{r}_{23}+r_\varphi\mathbf{r}_{34}+r_\gamma\mathbf{r}_{40})^T$, one obtains

$$\big[\mu_2\rho_2(\cos\alpha_2)-\mu_1\rho_1(\cos\alpha_1)-\mu_2\big]\cdot\|r_\theta\mathbf{r}_{23}+r_\varphi\mathbf{r}_{34}+r_\gamma\mathbf{r}_{40}\|^2=0 \tag{65}$$

by virtue of the identity

$$(r_\theta\mathbf{r}_{23}+r_\varphi\mathbf{r}_{34}+r_\gamma\mathbf{r}_{40})^T\mathbf{E}(r_\theta\mathbf{r}_{23}+r_\varphi\mathbf{r}_{34}+r_\gamma\mathbf{r}_{40})=0. \tag{66}$$

In fact, $\mathbf{a}^T\mathbf{E}\mathbf{a}=0$ for any 2D vector $\mathbf{a}$. Likewise, upon multiplying both sides of the same (64), by $[\mathbf{E}(r_\theta\mathbf{r}_{23}+r_\varphi\mathbf{r}_{34}+r_\gamma\mathbf{r}_{40})]^T$, one obtains

$$\big[\mu_2\rho_2(\sin\alpha_2)-\mu_1\rho_1(\sin\alpha_1)\big]\cdot\|r_\theta\mathbf{r}_{23}+r_\varphi\mathbf{r}_{34}+r_\gamma\mathbf{r}_{40}\|^2=0. \tag{67}$$

Now the norm of the vector, and indeed the square of this norm, cannot vanish unless the vector itself vanishes. However, vector $(r_\theta\mathbf{r}_{23}+r_\varphi\mathbf{r}_{34}+r_\gamma\mathbf{r}_{40})$ cannot vanish for arbitrary configurations. Indeed, for this vector sum to vanish, its two vector terms must be parallel. That is not our case. Thus, (65) and (67) imply the two conditions below:

$$\mu_2\rho_2(\cos\alpha_2)-\mu_1\rho_1(\cos\alpha_1)-\mu_2=0, \tag{68}$$
$$\mu_2\rho_2(\sin\alpha_2)-\mu_1\rho_1(\sin\alpha_1)=0. \tag{69}$$

Moreover, from the definition of μ_i, these parameters are not independent, but must satisfy the conditions

$$\mu_1+\mu_2+\mu_3+\mu_4=1, \tag{70}$$
$$\mu_1>0,\quad \mu_2>0,\quad \mu_3>0,\quad \mu_4>0. \tag{71}$$

Hence, the *dynamic balancing conditions* sought are expressed by a system of seven (60, 61, 62, 63, 68, 69, and 70) and four inequalities (71).

5 Example

In this section, we present an example to illustrate the foregoing method of balancing. The problem is solved in the MATLAB environment. We consider the five-bar linkage shown in Fig. 3, whose dimensions and inertial parameters are reported in Table 1. The input motions of links 1 and 4 are the *cycloidal* motion and the harmonic motion and are plotted in Fig. 5 and Fig. 6.

From (44), we can evaluate the driving torque

$$\boldsymbol{\tau} \equiv \mathbf{u}^T \mathbf{w}^E = \mathbf{u}^T \mathbf{M}\dot{\mathbf{t}}, \tag{72}$$

where $\mathbf{u}^T$, $\mathbf{M}$, $\dot{\mathbf{t}}$ are given in Sect. 3.

Fig. 5 Motion of the input link 1

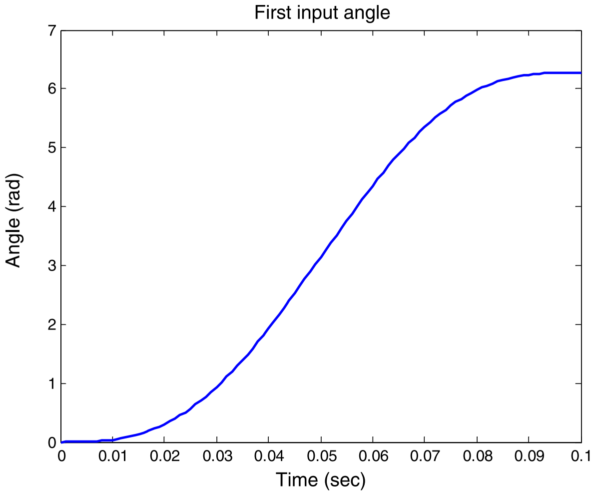
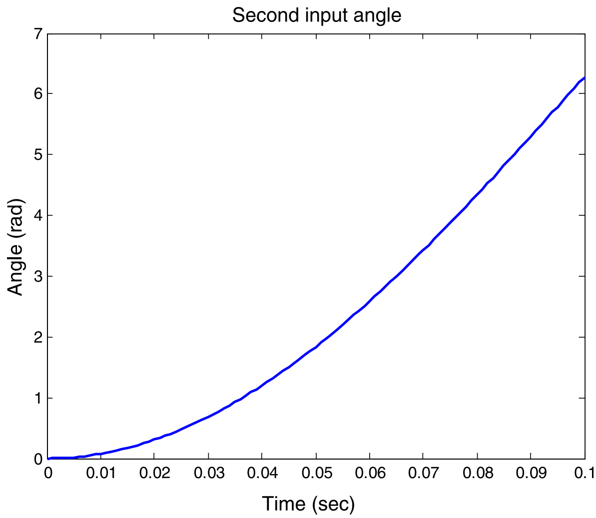


Fig. 6 Motion of the input link 4



The driving torques for unbalanced mechanisms are shown in Figs. 7, 8, 10, and 11.

Then knowing $\boldsymbol{\tau}$, we can find the external wrench $\mathbf{w}^E$ exerted on the system, solving the matrix equation:

$$\boldsymbol{\tau} \equiv \mathbf{u}^T \mathbf{w}^E. \tag{73}$$

In (26), we substitute external wrench $\mathbf{w}^E$, and then solve it for $\boldsymbol{\varphi}$. The vector of joints forces $\boldsymbol{\varphi}$ is given as

$$\boldsymbol{\varphi} = \begin{bmatrix} f_{01x} & f_{01y} & f_{12x} & f_{12y} & f_{23x} & f_{23y} & f_{34x} & f_{34y} & f_{40x} & f_{40y} \end{bmatrix}^T. \tag{74}$$

Now we can evaluate the components of shaking force exerted in the frame as:

$$F_x = f_{01x} + f_{40x}, \tag{75a}$$

$$F_y = f_{01y} + f_{40y}. \tag{75b}$$

The diagrams of shaking force components for the unbalanced mechanisms are shown in Figs. 9 and 12.

The mechanisms are then balanced using the conditions derived in Sect. 4. To this end, we aim to render the links as inertially symmetric as possible. Here, we define a binary link as one which has its mass center between two joint centers, and exactly in mid-position between them; so, $\alpha_i = 0$ and $\rho_i = 1/2$. For the moment, we are not interested in optimizing the mass ratios μ_1, μ_2, μ_3, and μ_4. So, we can select these values prior to the optimization, and these values must satisfy the conditions (70) and (71).

Hence, we solve the foregoing problem as one optimum design, with design variables:

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4, \quad \rho_1, \quad \rho_2, \quad \rho_3, \quad \rho_4. \tag{76}$$

The objective function to minimize is

$$\text{objfun} = \sum_{i=1}^{4} \left[w_\alpha \alpha_i^2 + w_\rho \left(\rho_i - \frac{1}{2} \right)^2 \right], \tag{77}$$

where w_α and w_ρ are the weighting coefficients, which must satisfy the conditions

$$0 < w_\alpha, w_\rho < 1 \quad \text{and} \quad w_\alpha + w_\rho = 1. \tag{78}$$

The condition function contains the conditions of balancing

$$\text{confun} = [\text{Equations (60), (61), (62), (63), (68), (69)}]. \tag{79}$$

The initial guess used is a vector that contains the guess values of variables and is given

$$x_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}. \tag{80}$$

The optimization is solved as nonlinear constrained optimization of a function. Finally, we obtain the values of eight variables, which are given in Table 1.

With the new geometric parameters, after dynamic balancing, the dynamic analysis has been repeated. In the new results, the torques acting in the input links and the x- and y-components of the shaking force acting in the support of the balanced mechanism, are negligible and need not be plotted.

Fig. 7 Unbalanced linkage for driving torque in link 1

First case:

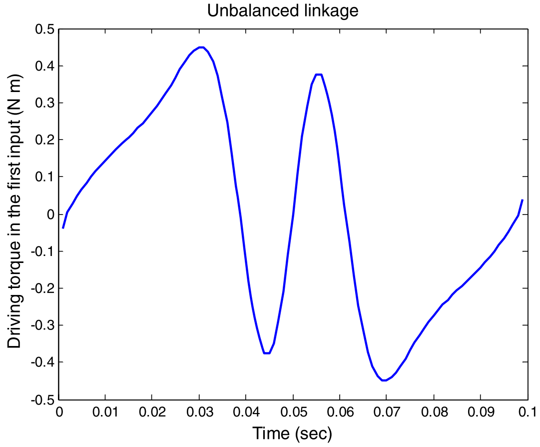


Fig. 8 Unbalanced linkage for driving torque in link 4

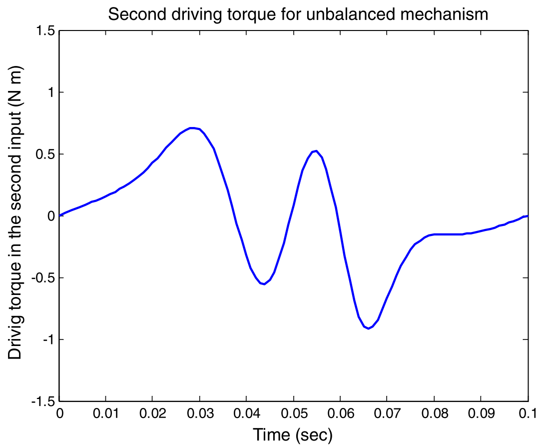


Fig. 9 Unbalanced linkage for shaking forces

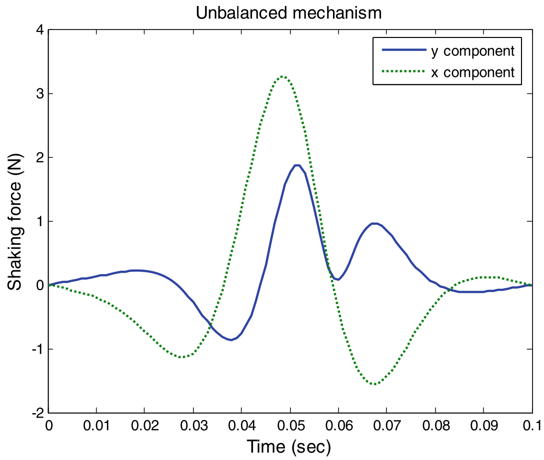


Fig. 10 Unbalanced linkage for driving torque in link 1

Second case:

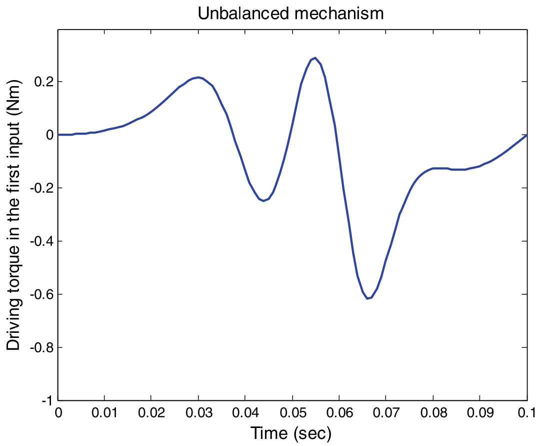


Fig. 11 Unbalanced linkage for driving torque in link 4

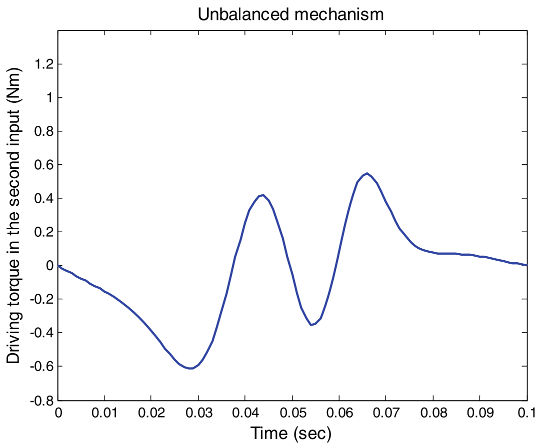


Fig. 12 Unbalanced linkage for shaking forces

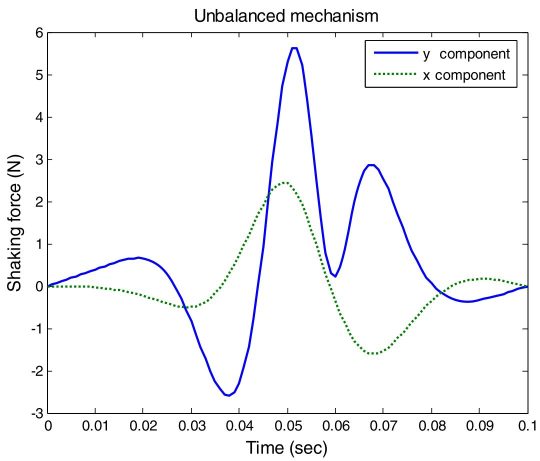


Table 1 Five-bar linkage parameters

Parameter	Case I		Case II	
	Unbalanced linkage	Balanced linkage	Unbalanced linkage	Balanced linkage
$\|\mathbf{r}_{12}\|$ (m)	0.02	0.02	0.035	0.035
$\|\mathbf{r}_{23}\|$ (m)	0.1	0.1	0.08	0.08
$\|\mathbf{r}_{34}\|$ (m)	0.1	0.1	0.1	0.1
$\|\mathbf{r}_{40}\|$ (m)	0.04	0.04	0.03	0.03
$\|\mathbf{r}_{01}\|$ (m)	0.04	0.04	0.05	0.05
m_1 (kg)	0.03	0.04	0.02	0.03
m_2 (kg)	0.15	0.14	0.2	0.024
m_3 (kg)	0.15	0.14	0.25	0.031
m_4 (kg)	0.06	0.08	0.05	0.075
I_1 (kg·m^2)	1×10^{-6}	1.333×10^{-6}	2.041×10^{-6}	1.6×10^{-5}
I_2 (kg·m^2)	1.25×10^{-4}	2.1666×10^{-4}	1.06×10^{-4}	1.69×10^{-4}
I_3 (kg·m^2)	1.25×10^{-4}	1.6166×10^{-4}	2.08×10^{-4}	3.6×10^{-4}
I_4 (kg·m^2)	8×10^{-6}	1.066×10^{-5}	3.75×10^{-6}	1.72×10^{-5}
α_1 (rad)	0	0	0	0
α_2 (rad)	0	0	0	0.012
α_3 (rad)	0	0	0	0.065
α_4 (rad)	0	0	0	0
ρ_1	0.5	0.22	0.5	0.16
ρ_2	0.5	0.8372	0.5	0.0753
ρ_3	0.5	0.5872	0.5	0.0627
ρ_4	0.5	1.64	0.5	1.373

6 Conclusion

The paper describes a method of designing a general five-bar linkage such that is capable of keeping dynamic balance at any configuration. The conditions of balancing are derived starting from the fact that the mass center of the balance linkage must be stationary. These conditions are the seven equations and four inequalities and will be satisfied by twelve design variables of the balanced linkage. The forgoing design variables are obtained from an optimization procedure as one to minimize the sum of the weighed norms of the individual design variables. The numerical example presented here demonstrates that the shaking force of balanced linkage is negligible compared to the linkage before it was optimized.

By the example, we see that the computation of the shaking force exerted to the frame has been achieved very easily with the aid of the dynamic modeling with natural orthogonal complement.

The negative value of the ρ_i means that the mass center of link 1 is not located within the line segment O_1O_2 but on the extension of the line O_1O_2.

The optimal design procedure proposed here does not guarantee a reduction of inertia moment of the linkage, so in general for the balanced mechanism we expect the increasing of the applied torques of the two driving joints.

Appendix

In the same way as r_φ, we calculate r_θ and its derivative. For this safe, we solve (1) for $\mathbf{a}_3$

$$\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_4 - \mathbf{a}_5. \tag{A1}$$

The equation of conjugates of vectors in (A1):

$$\overline{\mathbf{a}_3} = \overline{\mathbf{a}_1} + \overline{\mathbf{a}_2} - \overline{\mathbf{a}_4} - \overline{\mathbf{a}_5}. \tag{A2}$$

Multiplying both side of (A1) and (A2), we obtain

$$\begin{aligned}&\mathbf{a}_1\overline{\mathbf{a}_1} + \mathbf{a}_2\overline{\mathbf{a}_1} - \mathbf{a}_4\overline{\mathbf{a}_1} - \mathbf{a}_5\overline{\mathbf{a}_1} + \mathbf{a}_1\overline{\mathbf{a}_2} + \mathbf{a}_2\overline{\mathbf{a}_2} - \mathbf{a}_4\overline{\mathbf{a}_2} - \mathbf{a}_5\overline{\mathbf{a}_2} - \mathbf{a}_1\overline{\mathbf{a}_4} - \mathbf{a}_2\overline{\mathbf{a}_4}\\ &\quad + \mathbf{a}_4\overline{\mathbf{a}_4} + \mathbf{a}_5\overline{\mathbf{a}_4} - \mathbf{a}_1\overline{\mathbf{a}_5} - \mathbf{a}_2\overline{\mathbf{a}_5} + \mathbf{a}_4\overline{\mathbf{a}_5} + \mathbf{a}_5\overline{\mathbf{a}_5} = \mathbf{a}_3\overline{\mathbf{a}_3}\end{aligned} \tag{A3}$$

which can be rewritten as

$$\begin{aligned}&a_1^2 + a_2^2 + a_4^2 + a_5^2 - a_3^2 + 2a_1a_2\cos(\psi - \theta) - 2a_1a_4\cos(\psi - \gamma) - 2a_5a_2\cos\theta\\ &\quad - 2a_2a_4\cos(\gamma - \theta) + 2a_5a_4\cos\gamma - 2a_1a_5\cos\psi = 0.\end{aligned} \tag{A4}$$

In the simple form, it can be rewritten as

$$\begin{aligned}f(\psi,\gamma,\theta) = k_1 &+ k_2\cos(\psi - \theta) - k_3\cos(\psi - \gamma) - k_4\cos\theta\\ &- k_5\cos(\gamma - \theta) + k_6\cos\gamma - k_7\cos\psi = 0,\end{aligned} \tag{A5}$$

where

$$k_1 = a_1^2 + a_2^2 + a_4^2 + a_5^2 - a_3^2, \qquad k_2 = 2a_1a_2, \qquad k_3 = 2a_1a_4, \tag{A6a}$$

$$k_4 = 2a_5a_2, \qquad k_5 = 2a_2a_4, \qquad k_6 = 2a_5a_4, \qquad k_7 = 2a_1a_5. \tag{A6b}$$

In order to obtain r_θ, we will derivate with respect to time the (A5) as

$$\frac{df}{dt} = \frac{\partial f}{\partial\psi}\dot{\psi} + \frac{\partial f}{\partial\theta}\dot{\theta} + \frac{\partial f}{\partial\gamma}\dot{\gamma} = 0 \tag{A7}$$

which can be rewritten as

$$\frac{\partial f}{\partial\theta}\dot{\theta} = -\frac{\partial f}{\partial\psi}\dot{\psi} - \frac{\partial f}{\partial\gamma}\dot{\gamma}. \tag{A8}$$

From (A8), we obtain r_θ as

$$r_\theta = \frac{\dot{\theta}}{\dot{\psi}} = -\frac{\frac{\partial f}{\partial\psi} + \frac{\partial f}{\partial\gamma}\frac{\dot{\gamma}}{\dot{\psi}}}{\frac{\partial f}{\partial\theta}}, \tag{A9}$$

where

$$r_\gamma = \frac{\dot{\gamma}}{\dot{\psi}}.$$

The partial derivative into (35) is calculated as

$$\frac{\partial f}{\partial \psi} = -k_2 \sin(\psi - \theta) + k_3 \sin(\psi - \gamma) + k_7 \sin\psi, \tag{A10a}$$

$$\frac{\partial f}{\partial \gamma} = k_3 \sin(\psi - \gamma) + k_5 \sin(\gamma - \theta) - k_6 \sin\gamma, \tag{A10b}$$

$$\frac{\partial f}{\partial \theta} = -k_2 \sin(\psi - \theta) + k_4 \sin\theta + k_5 \sin(\gamma - \theta), \tag{A10c}$$

and

$$r_\theta = \frac{k_3 \sin(\psi - \gamma) + k_7 \sin\psi - k_2 \sin(\psi - \theta) + (k_3 \sin(\psi - \gamma) + k_5 \sin(\gamma - \theta) - k_6 \sin\gamma)\dot{\gamma}/\dot{\psi}}{k_2 \sin(\psi - \theta) - k_4 \sin\theta - k_5 \sin(\gamma - \theta)}, \tag{A11}$$

where r_θ is a function of ψ, γ, θ, $\dot{\psi}$, and $\dot{\gamma}$.

Once the velocity ratio r_θ is known, the angular velocity $\dot{\theta}$ can be calculated as

$$\dot{\theta} = r_\theta \dot{\psi}. \tag{A12}$$

Upon differentiation of both sides of (A11), with respect to time, we can obtain

$$\frac{dr_\theta(\psi, \gamma, \theta, \dot{\psi}, \dot{\gamma})}{dt} = \frac{\partial r_\theta}{\partial \psi}\dot{\psi} + \frac{\partial r_\theta}{\partial \gamma}\dot{\gamma} + \frac{\partial r_\theta}{\partial \theta}\dot{\theta} + \frac{\partial r_\theta}{\partial \dot{\psi}}\ddot{\psi} + \frac{\partial r_\theta}{\partial \dot{\gamma}}\ddot{\gamma}. \tag{A13}$$

If we suppose

$$\begin{aligned} m = {} & k_3 \sin(\psi - \gamma) + k_7 \sin\psi - k_2 \sin(\psi - \theta) \\ & + \big(k_3 \sin(\psi - \gamma) + k_5 \sin(\gamma - \theta) - k_6 \sin\gamma\big)\dot{\gamma}/\dot{\psi}, \end{aligned} \tag{A14a}$$

$$n = k_2 \sin(\psi - \theta) - k_4 \sin\theta - k_5 \sin(\gamma - \theta). \tag{A14b}$$

The partial derivations into (A13) can be simplified as

$$\frac{\partial r_\theta}{\partial \psi} = \frac{n(k_3 \cos(\psi - \gamma) + k_7 \cos\psi - k_2 \cos(\psi - \theta) + r_\gamma k_3 \cos(\psi - \gamma)) - mk_2 \cos(\psi - \theta)}{n^2}, \tag{A15a}$$

$$\frac{\partial r_\theta}{\partial \gamma} = \frac{n(k_3 \cos(\psi - \gamma) + r_\gamma k_3 \cos(\psi - \gamma) + r_\gamma k_5 \cos(\gamma - \theta) - r_\gamma k_6 \cos\gamma) + mk_5 \cos(\gamma - \theta)}{n^2}, \tag{A15b}$$

$$\frac{\partial r_\theta}{\partial \theta} = \frac{n(-k_2 \cos(\psi - \theta) + r_\gamma k_5 \cos(\gamma - \theta)) - m(k_2 \cos(\psi - \theta) - k_4 \cos\theta) - k_5 \cos(\gamma - \theta)}{n^2}, \tag{A15c}$$

$$\frac{\partial r_\theta}{\partial \dot{\psi}} = -\frac{\dot{\gamma}\ddot{\psi}(k_3 \sin(\psi - \gamma) + k_5 \sin(\gamma - \theta) - k_6 \sin\gamma)}{n\dot{\psi}^2}, \tag{A15d}$$

$$\frac{\partial r_\theta}{\partial \dot{\gamma}} = \frac{\ddot{\gamma}(k_3 \sin(\psi - \gamma) + k_5 \sin(\gamma - \theta) - k_6 \sin\gamma)}{n\dot{\psi}}. \tag{A15e}$$

In the (A13), $\dot{\psi}$, $\dot{\gamma}$, $\ddot{\psi}$, and $\ddot{\gamma}$ are obtained from derivations of input motions and $\dot{\theta}$ is given in (A12).

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