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Collision with friction; Part B: Poisson's and Stornge's hypotheses

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Abstract In Part B of this paper, planar collision theories, counterparts of the theory associated with Newton's hypotheses described in Part A, are developed in connection with Poisson's and Stronge's hypotheses. First, expressions for the normal and tangential impulses, the normal and tangential velocities of separation, and the change of the system mechanical energy are written for five types of collision. These together with Routh's semigraphical method and Coulomb's coefficient of friction are used to show that the algebraic signs of the four parameters introduced in Part A span the same five cases of system configuration of Part A. For each, α determines the type of collision which once found allows the evaluation of the normal and tangential impulses and ultimately the changes in the motion variables. The analysis of the indicated cases shows that for Poisson's hypothesis, a solution always exists which is unique, coherent and energy-consistent. The same applies to Stronge's hypothesis, however, for a narrower range of application. It is thus concluded that Poisson's hypothesis is superior as compared with Newton's and Stronge's hypotheses.

Keywords Collision · Collision with friction · Poison's hypothesis · Stronge's hypothesis · Routh's graph · Coulombs' coefficient of friction

1 Introduction

A one-point, planar collision theory based on Newton's hypothesis was developed in Part A of this paper. It was shown that the algebraic signs of 5 parameters, in conjunction with the ratio between the tangential and normal velocities of approach, called α, span eleven configuration-related cases of simple, nonholonomic systems undergoing collision and determine for each the associated types of collision. It was also shown that within the different cases, inconsistencies can occur by which the system mechanical energy increases. In Part B, Poisson's and Stronge's hypotheses are used to develop collision theories. With reference to Poisson's hypothesis, five types of collision are identified (authors generally defined more

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than five types of collision, e.g., [1–3], and [4]; however, it can be shown that the number of types of collision can be condensed to five, and that the additional types are special cases of the five). For each, explicit expressions are generated in Sect. 2 for the normal and tangential impulses and for components of the velocity of separation, and in Sect. 3 for changes in the system mechanical energy. The uniqueness, coherence, and energy-consistence of the solutions related to each of the cases are investigated in Sect. 3. Section 4 is dedicated to the exploration of relations between Stronge's and Poisson's hypotheses introducing a theory based on the former. An example in Sect. 5 and a brief comparison of the three theories in Sect. 6 conclude this work.

2 A collision theory with Poisson's hypothesis

In accordance with Poisson's hypothesis, the collision duration is regarded as comprising two phases, namely a compression phase, starting at t_1 and terminating at $\bar{t}$, the instant of maximum compression when

$$\bar{\mathbf{v}} \cdot \mathbf{n} = 0 \tag{1}$$

where

$$\bar{\mathbf{v}} \hat{=} \mathbf{v}^R(\bar{t}) \quad (t_1 < \bar{t} < t_2); \tag{2}$$

and a restitution phase, starting at $\bar{t}$ and terminating at t_2, when $\mathbf{R}(t_2) = 0$. The integrals defining I_n and I_t (see (A13))[1] can thus be divided as follows:

$$I_{nc} \hat{=} \left(\int_{t_1}^{\bar{t}} \mathbf{R}\, dt \right) \cdot \mathbf{n}, \qquad I_{nr} \hat{=} \left(\int_{\bar{t}}^{t_2} \mathbf{R}\, dt \right) \cdot \mathbf{n}, \tag{3}$$

$$I_{tc} \hat{=} \left(\int_{t_1}^{\bar{t}} \mathbf{R}\, dt \right) \cdot \mathbf{t}, \qquad I_{tr} \hat{=} \left(\int_{\bar{t}}^{t_2} \mathbf{R}\, dt \right) \cdot \mathbf{t}, \tag{4}$$

so that in light of (3), (4), and (A13),

$$I_n = I_{nc} + I_{nr}, \qquad I_t = I_{tc} + I_{tr}. \tag{5}$$

According to Poisson's hypothesis [5], the coefficient of restitution is defined

$$e \hat{=} I_{nr}/I_{nc}, \quad 0 \le e \le 1. \tag{6}$$

One can thus divide (A22) and (A23), namely,

$$\mathrm{v}_n^S - \mathrm{v}_n^A = m_{nn} I_n + m_{nt} I_t, \qquad \text{(A22) repeated}$$

$$\mathrm{v}_t^S - \mathrm{v}_t^A = m_{nt} I_n + m_{tt} I_t, \qquad \text{(A23) repeated}$$

into the equations

$$\bar{\mathbf{v}} \cdot \mathbf{n} - \mathbf{v}^A \cdot \mathbf{n} \underset{\text{(A22)}}{=} m_{nn} I_{nc} + m_{nt} I_{tc}, \tag{7}$$

[1] Equations numbers designated with A refer to equations of Part A numbered correspondingly.

$$\bar{\mathbf{v}} \cdot \mathbf{t} - \mathbf{v}^A \cdot \mathbf{t} \underset{(A23)}{=} m_{nt} I_{nc} + m_{tt} I_{tc}, \tag{8}$$

describing the compression phase and

$$\mathbf{v}^S \cdot \mathbf{n} - \bar{\mathbf{v}} \cdot \mathbf{n} \underset{(A22)}{=} m_{nn} I_{nr} + m_{nt} I_{tr}, \tag{9}$$

$$\mathbf{v}^S \cdot \mathbf{t} - \bar{\mathbf{v}} \cdot \mathbf{t} \underset{(A23)}{=} m_{nt} I_{nr} + m_{tt} I_{tr}, \tag{10}$$

describing the restitution phase. As before, the theory allows reverse sliding, sticking or forward sliding, and since sticking and reverse sliding follow forward sliding, a change of events takes place either during the compression phase at time t_c ($t_1 < t_c < \bar{t}$) or during the restitution phase at time t_r ($\bar{t} < t_r < t_2$). Thus, forward sliding between t_1 and t_c (or between t_1 and t_r) can be followed by sticking or by reverse sliding between t_c and t_2 (or between t_r and t_2). A detailed analysis of these possibilities requires further division of (7)–(10). For a change of events occurring during the compression phase,

$$I'_{nc} \hat{=} \left(\int_{t_1}^{t_c} \mathbf{R}\, dt \right) \cdot \mathbf{n}, \qquad I''_{nc} \hat{=} \left(\int_{t_c}^{\bar{t}} \mathbf{R}\, dt \right) \cdot \mathbf{n}; \quad I_{nc} = I'_{nc} + I''_{nc}, \tag{11}$$

$$I'_{tc} \hat{=} \left(\int_{t_1}^{t_c} \mathbf{R}\, dt \right) \cdot \mathbf{t}, \qquad I''_{tc} \hat{=} \left(\int_{t_c}^{\bar{t}} \mathbf{R}\, dt \right) \cdot \mathbf{t}; \quad I_{tc} = I'_{tc} + I''_{tc}, \tag{12}$$

and (7) and (8) become

$$\mathbf{v}^R(t_c) \cdot \mathbf{n} - \mathbf{v}^A \cdot \mathbf{n} = m_{nn} I'_{nc} + m_{nt} I'_{tc}, \tag{13}$$

$$\mathbf{v}^R(t_c) \cdot \mathbf{t} - \mathbf{v}^A \cdot \mathbf{t} = m_{nt} I'_{nc} + m_{tt} I'_{tc}, \tag{14}$$

and

$$\bar{\mathbf{v}} \cdot \mathbf{n} - \mathbf{v}^R(t_c) \cdot \mathbf{n} = m_{nn} I''_{nc} + m_{nt} I''_{tc}, \tag{15}$$

$$\bar{\mathbf{v}} \cdot \mathbf{t} - \mathbf{v}^R(t_c) \cdot \mathbf{t} = m_{nt} I''_{nc} + m_{tt} I''_{tc}. \tag{16}$$

Similarly, for a change of events occurring during the restitution phase

$$I'_{nr} \hat{=} \left(\int_{\bar{t}}^{t_r} \mathbf{R}\, dt \right) \cdot \mathbf{n}, \qquad I''_{nr} \hat{=} \left(\int_{t_r}^{t_2} \mathbf{R}\, dt \right) \cdot \mathbf{n}; \quad I_{nr} = I'_{nr} + I''_{nr}, \tag{17}$$

$$I'_{tr} \hat{=} \left(\int_{\bar{t}}^{t_r} \mathbf{R}\, dt \right) \cdot \mathbf{t}, \qquad I''_{tr} \hat{=} \left(\int_{t_r}^{t_2} \mathbf{R}\, dt \right) \cdot \mathbf{t}; \quad I_{tr} = I'_{tr} + I''_{tr}, \tag{18}$$

and (9) and (10) become

$$\mathbf{v}^R(t_r) \cdot \mathbf{n} - \bar{\mathbf{v}} \cdot \mathbf{n} = m_{nn} I'_{nr} + m_{nt} I'_{tr}, \tag{19}$$

$$\mathbf{v}^R(t_r) \cdot \mathbf{t} - \bar{\mathbf{v}} \cdot \mathbf{t} = m_{nt} I'_{nr} + m_{tt} I'_{tr}, \tag{20}$$

and

$$\mathbf{v}^S \cdot \mathbf{n} - \mathbf{v}^R(t_r) \cdot \mathbf{n} = m_{nn} I''_{nr} + m_{nt} I''_{tr}, \tag{21}$$

$$\mathbf{v}^S \cdot \mathbf{t} - \mathbf{v}^R(t_r) \cdot \mathbf{t} = m_{nt} I''_{nr} + m_{tt} I''_{tr}. \tag{22}$$

Next, five types of collision, numbered as in [4], are defined and expressions for I_n, I_t, v_n^S and v_t^S (I_n and I_t in [4]) are generated for each as follows.

Type 1 *Sticking in compression*, comprising sliding ($t_1 \div t_c$) and sticking in compression ($t_c \div \bar{t}$); sticking in restitution ($\bar{t} \div t_2$). Type 1 is characterized by

$$I'_{tc} = -\mu I'_{nc}, \quad \mathbf{v}^R(t_c) \cdot \mathbf{t} = 0, \quad \bar{\mathbf{v}} \cdot \mathbf{t} = 0, \quad \mathbf{v}^S \cdot \mathbf{t} = 0. \tag{23}$$

When (23c) and (1) are substituted in (7) and (8), one has

$$-\mathrm{v}_n^A \underset{(1)}{=} m_{nn} I_{nc} + m_{nt} I_{tc}, \quad -\mathrm{v}_t^A \underset{(23\mathrm{c})}{=} m_{nt} I_{nc} + m_{tt} I_{tc}. \tag{24}$$

Equations (24) can be solved for I_{nc} and I_{tc}, yielding

$$I_{nc} = (m_{tt} + \alpha m_{nt})\left|\mathrm{v}_n^A\right|/\Delta, \qquad I_{tc} = -(m_{nt} + \alpha m_{nn})\left|\mathrm{v}_n^A\right|/\Delta. \tag{25}$$

Thus,

$$I_n \underset{(5),(6)}{=} (1+e) I_{nc}, \qquad I_t \underset{(\mathrm{A23}),(23\mathrm{d})}{=} -\left(m_{nt} I_n + \mathrm{v}_t^A\right)/m_{tt}, \tag{26}$$

and

$$\mathrm{v}_n^S \underset{(\mathrm{A22})}{=} \mathrm{v}_n^A + m_{nn} I_n + m_{nt} I_t \underset{(25),(26)}{=} e(m_{tt} + \alpha m_{nt})\left|\mathrm{v}_n^A\right|/m_{tt}. \tag{27}$$

Type 2 *Sticking in restitution*, comprising sliding in compression ($t_1 \div \bar{t}$), sliding ($\bar{t} \div t_r$) and sticking in restitution ($t_r \div t_2$). Type 2 is characterized by

$$I_{tc} = -\mu I_{nc}, \quad I'_{tr} = -\mu I'_{nr}, \quad \mathbf{v}^R(t_r) \cdot \mathbf{t} = 0, \quad \mathbf{v}^S \cdot \mathbf{t} = 0. \tag{28}$$

Substitutions from (1) and (28a) in (7) lead to $-\mathrm{v}_n^A = m_{nn} I_{nc} - \mu m_{nt} I_{nc}$, and with (5) and (6)

$$I_n = (1+e)\left|\mathrm{v}_n^A\right|/(m_{nn} - \mu m_{nt}), \qquad I_t = -\left(m_{nt} I_n + \mathrm{v}_t^A\right)/m_{tt}, \tag{29}$$

where I_t is obtained from (A23) in view of (28d). The use of (1) and (5) in (9) yields $\mathrm{v}_n^S = m_{nn} e I_{nc} + m_{nt}(I_t - I_{tc})$, which becomes with the aid of (28a) and (29b), $\mathrm{v}_n^S = \Delta e I_{nc}/m_{tt} - m_{nt}/m_{tt}\mathrm{v}_t^A + m_{nt}(\mu m_{tt} - m_{nt}) I_{nc}/m_{tt}$ or after rearrangements with $\alpha \underset{(\mathrm{A34})}{\hat{=}} \mathrm{v}_t^A/|\mathrm{v}_n^A|$,

$$\mathrm{v}_n^S = (1+e)\Delta\left|\mathrm{v}_n^A\right|/[(m_{nn} - \mu m_{nt})m_{tt}] - [1 + (m_{nt}/m_{tt})\alpha]\left|\mathrm{v}_n^A\right|. \tag{30}$$

Type 3 *Reverse sliding in compression*, comprising sliding ($t_1 \div t_c$) and reverse sliding in compression ($t_c \div \bar{t}$), reverse sliding in restitution ($\bar{t} \div t_2$). Type 3 is characterized by

$$I'_{tc} = -\mu I'_{nc}, \quad I''_{tc} = \mu I''_{nc}, \quad \mathbf{v}^R(t_c) \cdot \mathbf{t} = 0, \quad I_{tr} = \mu I_{nr}. \tag{31}$$

Substitutions from (31c) and (31a) in (14) yield

$$\mathrm{v}_t^A = (\mu m_{tt} - m_{nt}) I'_{nc}. \tag{32}$$

By (7)

$$-\mathrm{v}_n^A = m_{nn} I_{nc} + m_{nt}[-\mu I'_{nc} + \mu(I_{nc} - I'_{nc})] \tag{33}$$

(see (31a), (31b), and (11c)). Eliminating I'_{nc} from (32)–(33) one has

$$I_{nc} = [1 + 2\mu m_{nt}\alpha/(\mu m_{tt} - m_{nt})]\left|\mathrm{v}_n^A\right|/(m_{nn} + \mu m_{nt}). \tag{34}$$

Next,

$$\begin{aligned} I_{tc} &\underset{(12\mathrm{c})}{=} I'_{tc} + I''_{tc} \underset{(31\mathrm{a}),(31\mathrm{b})}{=} \mu(I''_{nc} - I'_{nc}) \underset{(11\mathrm{c})}{=} \mu(I_{nc} - 2I'_{nc}) \\ &\underset{(32)}{=} \mu\left[I_{nc} - 2\mathrm{v}_t^A/(\mu m_{tt} - m_{nt})\right]; \qquad I_{tr} \underset{(31\mathrm{d}),(6)}{=} \mu e I_{nc} \end{aligned} \tag{35}$$

and using (35) in (5b) one gets

$$I_n = (1 + e)I_{nc}, \qquad I_t = \mu I_n - 2\mu \mathrm{v}_t^A/(\mu m_{tt} - m_{nt}). \tag{36}$$

Also,

$$\begin{aligned} \mathrm{v}_n^S &\underset{(1),(9),(6),(35\mathrm{b})}{=} e(m_{nn} + \mu m_{nt})I_{nc} \\ &\underset{(34)}{=} e[1 + 2\mu m_{nt}\alpha/(\mu m_{tt} - m_{nt})]\left|\mathrm{v}_n^A\right|. \end{aligned} \tag{37}$$

Finally, the use of (A23) together with (36) leads after rearrangements to

$$\mathrm{v}_t^S = (\mu m_{tt} + m_{nt})\left[I_n - \mathrm{v}_t^A/(\mu m_{tt} - m_{nt})\right]. \tag{38}$$

Type 4 *Reverse sliding in restitution*, comprising sliding in compression ($t_1 \div \bar{t}$), sliding ($\bar{t} \div t_r$) and reverse sliding in restitution ($t_r \div t_2$). Type 4 is characterized by

$$I_{tc} = -\mu I_{nc}, \quad I'_{tr} = -\mu I'_{nr}, \quad \mathbf{v}^R(t_r)\cdot\mathbf{t} = 0, \quad I''_{tr} = \mu I''_{nr}. \tag{39}$$

The use of (7) leads to

$$-\mathrm{v}_n^A \underset{(39\mathrm{a})}{=} (m_{nn} - \mu m_{nt})I_{nc}. \tag{40}$$

Now,

$$-\bar{\mathbf{v}}\cdot\mathbf{t} \underset{(20),(39\mathrm{c}),(39\mathrm{b})}{=} -(\mu m_{tt} - m_{nt})I'_{nr} \tag{41}$$

and using (40), (41), (39a), and (8) in (17c), one has

$$I''_{nr} = (1 + e)I_{nc} - \mathrm{v}_t^A/(\mu m_{tt} - m_{nt}). \tag{42}$$

Thus,

$$\begin{aligned} I_t &\underset{(5\mathrm{b}),(18\mathrm{c})}{=} I_{tc} + I'_{tr} + I''_{tr} \underset{(39\mathrm{a}),(39\mathrm{b}),(39\mathrm{d})}{=} -\mu I_{nc} - \mu I'_{nr} + \mu I''_{nr} \\ &\underset{(17\mathrm{c})}{=} -\mu I_{nc} - \mu(eI_{nc} - I''_{nr}) + \mu I''_{nr} = -\mu(1 + e)I_{nc} + 2\mu I''_{nr} \end{aligned} \tag{43}$$

and the use of (43) leads to equations identical with (36), namely,

$$I_n \underset{(40)}{=} (1 + e)I_{nc}, \qquad I_t \underset{(42),(43)}{=} \mu I_n - 2\mu \mathrm{v}_t^A/(\mu m_{tt} - m_{nt}). \tag{44}$$

Equation (9), in conjunction with (1), leads in view of (17c) and (42) to

$$\mathrm{v}_n^S = e\left|\mathrm{v}_n^A\right| + 2\mu m_{nt}\left|\mathrm{v}_n^A\right|/(\mu m_{tt} - m_{nt})\left[(1+e)r_m - \alpha\right]. \tag{45}$$

Last, v_t^S can be obtained from (22) with the aid of (42), (39c), and (39d), i.e.,

$$\mathrm{v}_t^S = (\mu m_{tt} + m_{nt})\left[(1+e)r_m - \alpha\right]\left|\mathrm{v}_n^A\right|/(\mu m_{tt} - m_{nt}). \tag{46}$$

Type 5 *Forward sliding* $(t_1 \div t_2)$. Type 5 is characterized by

$$I_{tc} = -\mu I_{nc}, \quad I_{tr} = -\mu I_{nr}. \tag{47}$$

Substitution from (1) and (47a) in (7) yields

$$I_{nc} = \left|\mathrm{v}_n^A\right|/(m_{nn} - \mu m_{nt}), \tag{48}$$

thus

$$I_n = (1+e)I_{nc}, \quad I_t = -\mu I_n. \tag{49}$$

Substitutions from (1), (6), and (48) in (9) leads to

$$\mathrm{v}_n^S = (m_{nn} - \mu m_{nt})eI_{nc} \underset{(48)}{=} e\left|\mathrm{v}_n^A\right|; \tag{50}$$

and (A23) yields with (47)–(49),

$$\mathrm{v}_t^S = \mathrm{v}_t^A - (\mu m_{tt} - m_{nt})(1+e)I_{nc} = \left[\alpha - (1+e)r_m\right]\left|\mathrm{v}_n^A\right|. \tag{51}$$

I_n and I_t are thus available from (26), (29), (36), (44), or (49) depending on type of collision, which must now be uncovered. This can be accomplished with the aid of additional information derived from Routh's graph [6] as follows.

3 Routh-based semi-graphical method

3.1 Routh's graph

Let $\tilde{I}_n$ and $\tilde{I}_t$ be the normal and tangential impulses defined in (A54), and recall the three lines in the $\tilde{I}_t - \tilde{I}_n$ plane introduced in Part A, namely,

forward sliding line $\quad L_{FS} \Rightarrow \tilde{I}_t = -\mu\tilde{I}_n,$ (A55) repeated

sticking line $\quad L_{ST} \Rightarrow \tilde{I}_t = -\left(\tilde{I}_n m_{nt} + \mathrm{v}_t^A\right)/m_{tt},$ (A56) repeated

reverse sliding line $\quad L_{RS} \Rightarrow \tilde{I}_t = \mu\tilde{I}_n - 2\mu\mathrm{v}_t^A/(\mu m_{tt} - m_{nt}).$ (A57) repeated

These lines are shown in Fig. 1, which is similar to Fig. 2 in Part A except for the maximum compression line

$$L_{MC} \Rightarrow \tilde{I}_t \underset{(A22)}{=} -\left(\tilde{I}_n m_{nn} + \mathrm{v}_n^A\right)/m_{nt} \tag{52}$$

drawn in two positions designated $L_{MC}^{(1)}$ and $L_{MC}^{(2)}$ on both sides of point G, where

$$\tilde{I}_n(G) = \mathrm{v}_t^A/(\mu m_{tt} - m_{nt}). \qquad \text{(A58) repeated}$$

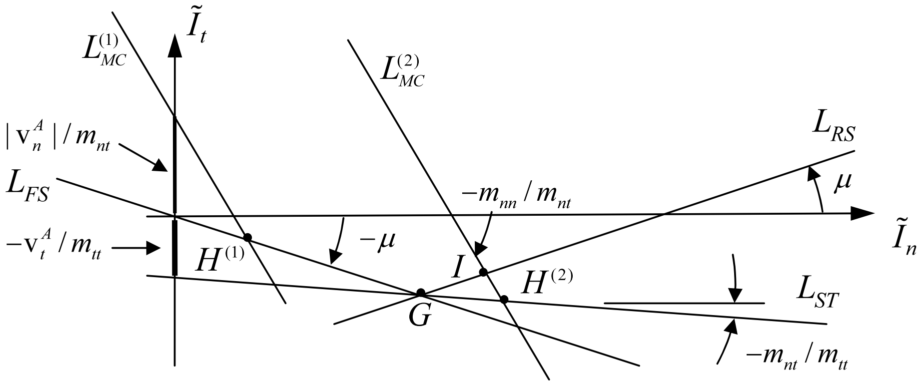
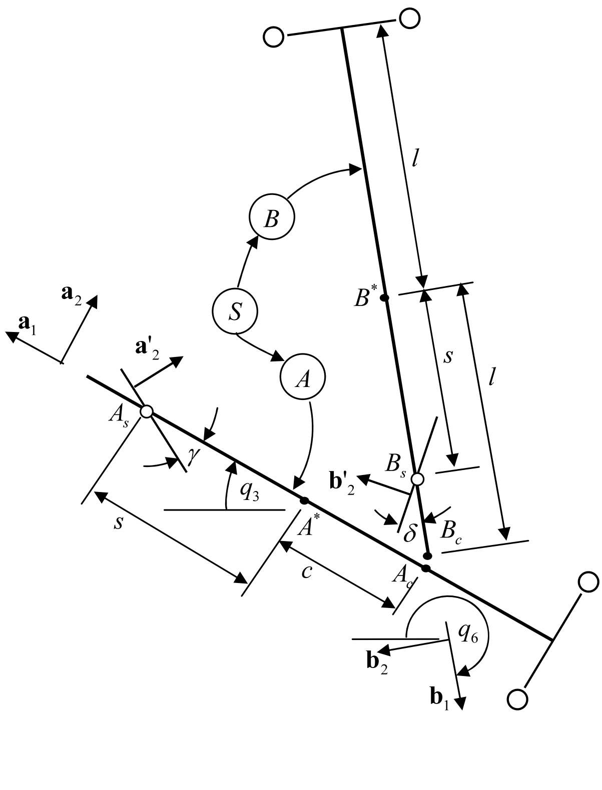


Fig. 1 Impulse diagram

Fig. 2 A two-sled collision



The different lines can be used to describe collision events with the aid of the intersection points G, $H^{(1)}$, $H^{(2)}$ and I. The assumption $\mathrm{v}_t^A > 0$ still applies and all $\tilde{I}_t$–$\tilde{I}_n$ relations start at the origin and vary in accordance with line L_{FS}. Inequalities (A59)–(A61), namely,

$$|m_{nt}|/m_{tt} < \mu => \mu m_{tt} - m_{nt} > 0, \quad \mu m_{tt} + m_{nt} > 0, \qquad \text{(A59) repeated}$$

$$-m_{nt}/m_{tt} > \mu => \mu m_{tt} + m_{nt} < 0 \quad (m_{nt} < 0)\ , \qquad \text{(A60) repeated}$$

$$\left.\begin{array}{l}-m_{nt}/m_{tt} > -\mu => \mu m_{tt} - m_{nt} > 0, \\ -m_{nt}/m_{tt} < -\mu => \mu m_{tt} - m_{nt} < 0 \quad (m_{nt} > 0),\end{array}\right\} \qquad \text{(A61) repeated}$$

remain valid, defining boundaries for sticking, reverse sliding, and forward sliding. Now, suppose that line $L_{MC}^{(1)}$ governs the $\tilde{I}_t$–$\tilde{I}_n$ relation at maximum compression. Then for both sticking (Type 2) and reverse sliding (Type 4), one obtains $(1+e)\tilde{I}_n(H^{(1)}) > \tilde{I}_n(G) > \tilde{I}_n(H^{(1)})$ $(\tilde{I}_n(H^{(1)}) = I_{nc})$, or ((29), (40), and (A58))

$$(1+e)\frac{|\mathrm{v}_n^A|}{m_{nn} - \mu m_{nt}} > \frac{\mathrm{v}_t^A}{\mu m_{tt} - m_{nt}} > \frac{|\mathrm{v}_n^A|}{m_{nn} - \mu m_{nt}} \Rightarrow (1+e) r_m > \alpha > r_m \tag{53}$$

since $\mu m_{tt} - m_{nt} \underset{\text{(A59),(A60)}}{>} 0$ and $m_{nn} - \mu m_{nt} \underset{(29),(40)}{>} 0$ (I_n must be positive).

If the collision ends before values of $\tilde{I}_t$ and $\tilde{I}_n$ at point G are reached, i.e., before sticking or reverse sliding occur, then forward sliding (Type 5) prevails. Accordingly, either $\tilde{I}_n(G) > (1+e)\tilde{I}_n(H^{(1)})$ $(\tilde{I}_n(H^{(1)}) = I_{nc})$ so that in view of (48) and (A58),

$$\frac{\mathrm{v}_t^A}{\mu m_{tt} - m_{nt}} > (1+e)\frac{|\mathrm{v}_n^A|}{m_{nn} - \mu m_{nt}} \Rightarrow \alpha > (1+e) r_m \tag{54}$$

since $\mu m_{tt} - m_{nt} \underset{\text{(A58)}}{>} 0$ (see inequality (A61a)) and $m_{nn} - \mu m_{nt} \underset{(48)}{>} 0$ ($r_m \underset{(36\text{a})}{>} 0$); or $\tilde{I}_n(G) < 0$, in which case $\mu m_{tt} - m_{nt} < 0$ (see inequality (A61b)) and $m_{nn} - \mu m_{nt} \underset{(48)}{>} 0$ ($r_m \underset{\text{(A36a)}}{<} 0$), so that by (51) ($\mathrm{v}_t^S$ must be positive)

$$\alpha > 0. \tag{55}$$

Next, suppose $L_{MC}^{(2)}$ governs the $\tilde{I}_t$–$\tilde{I}_n$ relation at maximum compression. Then forward sliding is ruled out (point G must be reached). If sticking (Type 1) occurs, then L_{ST} governs the $\tilde{I}_t$–$\tilde{I}_n$ relation, and $\tilde{I}_n(H^{(2)}) > \tilde{I}_n(G)$ (see Fig. 1). Using (52) and (A56) to find $\tilde{I}_n(H^{(2)})$, one has by substitution, $(|\mathrm{v}_n^A| m_{tt} + \mathrm{v}_t^A m_{nt})/\Delta > \mathrm{v}_t^A/(\mu m_{tt} - m_{nt})$ noting that $\tilde{I}_n(G)$ is given by (A58). Rearrangements with $\Delta \underset{(25\text{a}),(27)}{>} 0$ (which together with $|\mathrm{v}_n^A| m_{tt} + \mathrm{v}_t^A m_{nt} > 0$ ensure $\mathrm{v}_n^S \underset{(27)}{>} 0$ and $I_n \underset{(25\text{a})}{>} 0$) lead to

$$\frac{|\mathrm{v}_n^A|}{m_{nn} - \mu m_{nt}} > \frac{\mathrm{v}_t^A}{\mu m_{tt} - m_{nt}} \Rightarrow 0 < \alpha < r_m \tag{56}$$

if $m_{nn} - \mu m_{nt} > 0$, and to

$$\frac{|\mathrm{v}_n^A|}{m_{nn} - \mu m_{nt}} < \frac{\mathrm{v}_t^A}{\mu m_{tt} - m_{nt}} \Rightarrow \alpha > 0. \tag{57}$$

if $m_{nn} - \mu m_{nt} < 0$. When reverse sliding (Type 3) follows, L_{RS} governs the $\tilde{I}_t$–$\tilde{I}_n$ relation and $\tilde{I}_n(I) > \tilde{I}_n(G)$ (Fig. 1). Using (52) and (A57) to find $\tilde{I}_n(I)$, one has by substitution, $|\mathrm{v}_n^A|/(m_{nn} + \mu m_{nt}) > \mathrm{v}_t^A(m_{nn} - \mu m_{nt})/[(m_{nn} + \mu m_{nt})(\mu m_{tt} - m_{nt})]$. This relation reduces to inequality (56) since for Type 3 $m_{nn} + \mu m_{nt} \underset{(37)}{>} 0$ (v_n^S must be positive), $m_{nn} - \mu m_{nt} > 0$ and $\mu m_{tt} - m_{nt} \underset{\text{(A60)}}{>} 0$ ($m_{nt} \underset{\text{(A60)}}{<} 0$).

3.2 The determination of the type of collision

Equations (26)–(27), (29)–(30), (36)–(38), (44)–(46), and (49)–(51), can be used together with inequalities (A59)–(A61) (μ-bounds) and (53)–(57) (α-regions) to determine the type of collision. To this end, one can invoke a table similar to Table 1 in Part A, describing cases with consistent permutations of the algebraic signs of g, h, p and q, where

$$g \hat{=} \mu m_{tt} - m_{nt}, \quad h \hat{=} m_{nn} - \mu m_{nt}, \quad p \hat{=} \mu m_{tt} + m_{nt}, \quad q \hat{=} m_{nn} + \mu m_{nt}. \quad \text{(A35) repeated}$$

Regarding collision Types 1 and 2, Cases 1, 2, and 4 comply with inequalities (A59). The argumentation preceding inequality (56) ensures $I_n \underset{(25a)}{>} 0$ and $\mathrm{v}_n^S \underset{(27)}{>} 0$ for Type 1, and inequalities (56) and (57) indicate the range of α for Cases 1 and 2 ($h > 0$) and for Case 4 ($h < 0$), respectively. Turning to Type 2, one must have $h > 0$ to ensure $I_n > 0$ (29a), and can verify that $\mathrm{v}_n^S > 0$ only in Cases 1 and 2, provided the α-region in inequality (53) is satisfied. As for Types 3 and 4, Case 3 complies with inequality (A60), ensuring $I_n > 0$, $\mathrm{v}_n^S > 0$ and $\mathrm{v}_t^S < 0$ for both types, as can be shown straightforwardly with the aid of inequalities (56) (Type 3) and (53) (Type 4). Finally, all cases satisfy either inequality (61a) or (61b) in Type 5. However, $h > 0$ and $\alpha > (1+e)r_m$, ensuring $I_n \underset{(49a)}{>} 0$, $\mathrm{v}_n^S \underset{(50)}{>} 0$, and $\mathrm{v}_t^S \underset{(51)}{>} 0$, in all but Case 4 (in Case 5 $\alpha > 0$ since $r_m < 0$).

An algorithm for the solution of a collision problem can now be established, provided μ and e are known. α, m_{nn}, m_{tt}, m_{nt}, g, h, p, q, r_m and Δ are calculated for t_1, the collision time. The collision type is determined with the aid of Table 1 and used to evaluate I_n and I_t with (26), (29), (36), (44), or (49), and then $\Delta u_1, \ldots, \Delta u_p$ with (A16).

3.3 Energy considerations

Expressions for ΔE in the five types of collision can be obtained by substitutions of I_n, I_t, v_n^S, and v_t^S in (A29). It is expedient to express ΔE as polynomial of degree 2 in α and I_n',

Table 1 Admissible sign-variations of g, h, p and q; T1, ..., T5 refer to Type 1, ..., Type 5, respectively

No	g	h	p	q	Sticking (T1&T2)	R. Sliding (T3&T4)	F. Sliding
1	>0	>0	>0	>0	T1 $0<\alpha<r_m$	–	$\alpha>(1+e)r_m$
					T2 $r_m<\alpha<(1+e)r_m$		
2	>0	>0	>0	<0	T1 $0<\alpha<r_m$	–	$\alpha>(1+e)r_m$
					T2 $r_m<\alpha<(1+e)r_m$		
3	>0	>0	<0	>0	–	T3 $0<\alpha<r_m$	$\alpha>(1+e)r_m$
						T4 $r_m<\alpha<(1+e)r_m$	
4	>0	<0	>0	>0	T1 $\alpha>0$	–	–
5	<0	>0	>0	>0	–	–	$\alpha>0$

positive quantities, as follows

$$
\begin{aligned}
2\Delta E\big|_1/\big|\mathrm{v}_n^A\big|^2 &\underset{(25)-(27)}{=} -1/m_{tt}\alpha^2-\rho\Delta/m_{tt}I_n'^2\\
2\Delta E\big|_2/\big|\mathrm{v}_n^A\big|^2 &\underset{(29),(30)}{=} -1/m_{tt}\alpha^2-2(m_{nt}/m_{tt})I_n'\alpha\\
&\quad+[(1+e)\Delta-2m_{tt}(m_{nn}-\mu m_{nt})]/[m_{tt}(1+e)]I_n'^2\\
2\Delta E\big|_3/\big|\mathrm{v}_n^A\big|^2 &\underset{(36)-(38)}{=} 4\mu m_{nt}/(\mu m_{tt}-m_{nt})^2\alpha^2\\
&\quad-[2\mu(\mu m_{tt}+m_{nt})/(\mu m_{tt}-m_{nt})]I_n'\alpha\\
&\quad+[-\rho(m_{nn}+\mu m_{nt})+\mu(\mu m_{tt}+m_{nt})]I_n'^2\\
2\Delta E\big|_4/\big|\mathrm{v}_n^A\big|^2 &\underset{(44)-(46)}{=} 4\mu m_{nt}/(\mu m_{tt}-m_{nt})^2\alpha^2\\
&\quad-[2\mu(\mu m_{tt}+3m_{nt})/(\mu m_{tt}-m_{nt})]I_n'\alpha\\
&\quad+[-\rho(m_{nn}-\mu m_{nt})+2\mu m_{nt}+\mu(\mu m_{tt}+m_{nt})]I_n'^2\\
2\Delta E\big|_5/\big|\mathrm{v}_n^A\big|^2 &\underset{(49)-(51)}{=} -\{(1-e)+\mu\alpha+\mu[\alpha-(1+e)r_m]\}I_n'
\end{aligned}
\tag{58}
$$

where

$$\rho=(1-e)/(1+e),\qquad I_n'(\alpha)\hat{=}I_n/\big|\mathrm{v}_n^A\big|. \tag{59}$$

Now, $\Delta E|_1<0$ and $\Delta E|_5 \underset{(54),(55)}{<} 0$, as can be verified by inspection (each of the right-hand side terms is negative). Moreover, $\Delta E|_2$, $\Delta E|_3$ and $\Delta E|_4$ become maximal when $\alpha=-m_{nt}I_n'$, $\alpha=gpI_n'/(4m_{nt})$ and $\alpha=g(p+2m_{nt})I_n'/(4m_{nt})$, respectively, as their second derivatives with respect to α are negative. These maxima have all positive derivatives with respect to e, hence become maximal for $e=1$, reading $\Delta E|_{2max@e=1} \underset{(58b)}{=} \mu m_{nt}I_n^2$, $\Delta E|_{3max@e=1} \underset{(58c)}{=} -\mu p(g-2m_{nt})I_n^2/(8m_{nt})$ and $\Delta E|_{4max@e=1} \underset{(58d)}{=} -\mu p(p+2m_{nt})I_n^2/(8m_{nt})$. By Table 1 $\Delta E|_{3max@e=1}<0$ and $\Delta E|_{4max@e=1}<0$, whereas $\Delta E|_{2max@e=1}<0$ only if $m_{nt}<0$ (Case 2, Table 1). In Case 1 m_{nt} can be positive, however, $\Delta E|_2$ becomes maximal at $\alpha=-m_{nt}I_n'$, which now is negative, i.e., outside the range of α; and, since $\Delta E|_2$ is negative on the borders of α, i.e., $\Delta E|_2=\Delta E|_1(\alpha=r_m)<0$ and $\Delta E|_2=\Delta E|_5$ $(\alpha=(1+e)r_m)<0$ (see comment ***a*** below), then $\Delta E|_2$ is negative everywhere, as are $\Delta E|_3$ and $\Delta E|_4$.

3.4 Comments

***a*.** I_n, I_t, v_n^S and v_t^S are continuous functions of α. This continuity is maintained through the passages from one type of collision to another, occurring at $\alpha=(1+e)r_m$ and $\alpha=r_m$; i.e., $I_n|_2[\alpha=(1+e)r_m]=I_n|_5[\alpha=(1+e)r_m]$, etc. Consequently, equal signs can be added to the inequalities defining the regions of α in Table 1.

b*.** If $\mu=0$, then $r_m \underset{(36a)}{=} -m_{nt}/m_{nn}$. By inequalities (A59) ($m_{nt}=0$, $r_m=0$), (A60) ($m_{nt}<0$, $r_m>0$), and (A61) ($m_{nt}<0$ or $m_{nt}>0$) one has Types 1 and 2 for $\alpha \underset{(56),(53)}{=} 0$, Type 3 for $\alpha \underset{(56)}{<} r_m$, Type 4 for $r_m<\alpha \underset{(53)}{<} (1+e)r_m$, and Type 5 for $\alpha \underset{(54)}{>} (1+e)r_m$ (if $m_{nt}>0$, then $r_m<0$, hence $\alpha \underset{(55)}{>} 0$). In all events, I_n, I_t, v_n^S, v_t^S and ΔE are identical with their counterparts in Newton's hypothesis (see comment ***b in Part A).

***c*.** $v_n^S|_{e=0} = 0$ for all the types of collisions (see (27), (30), (37), (45), and (50)). In connection with Types 2 and 4, note that if $e = 0$, then $\alpha = r_m$, and the region of α reduces to zero. Moreover, v_n^S is a linear function of e, and it can be verified that $\partial v_n^S/\partial e > 0$ for all types of collision.

d*.** Direct impact ($v_n^A < 0$, $v_t^A = 0$, $\alpha \to 0$) and grazing ($v_n^A = 0$, $v_t^A > 0$, $\alpha \to \infty$) can be followed by collision types 1, 3, and 5 (not 2 and 4, for which $0 < (1 + ce)r_m < \alpha < (1 + e)r_m < \infty$ $(0 < c < 1, r_m > 0)$ by Table 1). Expressions for I_n, I_t, v_n^S and v_t^S are obtained by the substitution of $v_t^A = 0$ and $v_n^A = 0$ in (26)–(27) (Type 1), (36)–(38) (Type 3) and in (49)–(51) (Type 5), respectively, and are identical with their counterparts in Part A except $v_n^S = e(m_{nt}/m_{tt})v_t^A$ in Type 1 and $v_n^S = e[2\mu m_{nt}/(\mu m_{tt} - m_{nt})]v_t^A$ in Type 3 rather than $v_n^S = 0$ (see comment ***d Part A).

e*.** When $m_{nt} = 0$ ('balanced collision'), no reverse sliding is possible since by inequality (A60) $m_{tt} < 0$, in contradiction with (A20b), which permits only $m_{tt} > 0$. This is also true in connection with Newton's hypothesis (see comment ***e, Part A). Moreover, by (26) and (27) (Type 1) and (29) and (30) (Type 2), $I_n = (1 + e)|v_n^A|/m_{nn}$, $I_t = -v_t^A/m_{tt}$ and $v_n^S = e|v_n^A|$, expressions identical to those obtained from (A47) and (A30) with Newton's hypothesis.

***f*.** Both Newton's and Poisson's hypotheses lead to identical results for forward sliding ((A52), (A53), (A30), and (A67) are identical with (49) (and (48)), (51), (50), and (58e), respectively, subject to inequalities (A61)). Keller [7] arrived at a similar result in connection with a two-body collision.

***g*.** h, g and/or q, g and/or h, and h, in collision Types 2–5, respectively, are not allowed to vanish, or else I_n and/or I_t go to infinity (see (26), (29), (36), (44), and (49)). Equal signs can thus be added to the remaining inequalities in Table 1, Columns 2–5, leaving the solutions unique. This can be shown to always be the case; for example, if $p = 0$, then Case 1, Type 1 and Case 3, Type 3, and similarly, Case 1, Type 2, and Case 3, Type 4 lead to the same I_n, I_t, v_n^S and v_t^S $(= 0)$, hence to the same solution.

***h*.** Inequalities (A59)–(A61) do not depend on collision hypotheses, and hence are used in connection with both Poisson's and Stronge's hypotheses, discussed presently.

With reference to the example of Part A ($g > 0$, $h > 0$, $p < 0$, $q > 0$), one can show with the aid of Table 1, row 3, that the five cases are of Type 4, and that the changes in the system mechanical energies are −0.1063, −0.1265, −0.1337, −0.1192, and −0.1115 J, respectively.

4 A collision theory with Stronge's hypothesis

Let ΔE_n be the change in the system kinetic energy associated with I_n (first term on the right-hand side of (A29)), and let ΔE_{nc} and ΔE_{nr} be the parts of ΔE_n associated, respectively, with the compression and the restitution phases. Then according to Storge's hypothesis [8], the coefficient of restitution e_e is defined

$$e_e^2 \hat{=} -\Delta E_{nr}/\Delta E_{nc} = -(\Delta E_n - \Delta E_{nc})/\Delta E_{nc}, \quad 0 \le e_e \le 1. \tag{60}$$

In view of (A29),

$$\Delta E_n = 1/2 I_n \left(\mathbf{v}^S + \mathbf{v}^A\right) \cdot \mathbf{n}, \qquad \Delta E_{nc} \underset{(2)}{=} 1/2 I_{nc} \left(\bar{\mathbf{v}} + \mathbf{v}^A\right) \cdot \mathbf{n},$$

where $\bar{\mathbf{v}} \cdot \mathbf{n} \underset{(1)}{=} 0$; and, substitutions in (60) yields

$$e_e^2 = 1 - (1 + e_i)(1 - e_v) \tag{61}$$

where e_v and e_i refer to Newton's and Poisson's definitions of coefficient of restitution given by (A30) and (6), respectively (without the subscripts appearing in (61)). Now, by reference to (27), (30), (37), (45), and (50), one can express for each type of collision, e_v in terms of e_i, eliminate e_v from (61) and obtain a relation between e_i and e_e. For Type 1

$$\mathrm{v}_n^S / \left|\mathrm{v}_n^A\right| \underset{(A30)}{=} e_v \underset{(27)}{=} e_i(1+r), \quad r \hat{=} \alpha m_{nt}/m_{tt}, \tag{62}$$

and eliminating e_v from (61) one has after rearrangements,

$$(1+r)e_i^2 + re_i - e_e^2 = 0. \tag{63}$$

This equation can be solved for e_i, yielding

$$e_i = \frac{-r + \sqrt{r^2 + 4(1+r)e_e^2}}{2(1+r)}. \tag{64}$$

Now, $1 + r \underset{(62)}{>} 0$ (then $\mathrm{v}_n^S \underset{(27)}{>} 0$), therefore, the root in (64) is chosen positive to ensure $e_i > 0$ and $\partial e_i / \partial e_e \geq 0$.

For Type 2 one has

$$\mathrm{v}_n^S / \left|\mathrm{v}_n^A\right| \underset{(A30)}{=} e_v \underset{(30),(62b)}{=} (1+r+y)e_i + y, \quad y \hat{=} r(r_m/\alpha - 1). \tag{65}$$

Substitution in (61) leads to

$$(1+r+y)e_i^2 + (r+2y)e_i + y - e_e^2 = 0. \tag{66}$$

For Cases 1 and 2 $1 + r + y \underset{(62b),(65b),(A36)}{=} \Delta/[m_{tt}(m_{nn} - \mu m_{nt})] > 0$ (Table 1), and

$$e_i = \frac{-(r+2y) + \sqrt{(r+2y)^2 + 4(1+r+y)(e_e^2 - y)}}{2(1+r+y)}. \tag{67}$$

With a positive-sign root, (67) reduces to (64) when $\alpha = r_m$ (then $y = 0$), ensuring $e_i > 0$ and $\partial e_i / \partial e_e \geq 0$.

With regard to Type 3,

$$\mathrm{v}_n^S / \left|\mathrm{v}_n^A\right| \underset{(A30)}{=} e_v \underset{(37)}{=} e_i(1+s), \qquad s \hat{=} 2\mu m_{nt}\alpha/(\mu m_{tt} - m_{nt}) \tag{68}$$

and after the elimination of e_v from (61),

$$(1+s)e_i^2 + se_i - e_e^2 = 0, \tag{69}$$

$$e_i = \frac{-s + \sqrt{s^2 + 4(1+s)e_e^2}}{2(1+s)}. \tag{70}$$

Here, $1+s \underset{(68)}{>} 0$ (then $v_n^S \underset{(37)}{>} 0$), and the positive root was chosen to ensure $e_i > 0$ and $\partial e_i/\partial e_e \geq 0$. Similarly for Type 4

$$v_n^S/\left|v_n^A\right| \underset{(A30)}{=} e_v \underset{(45)}{=} (1+s+z)e_i + z, \quad z \hat{=} s(r_m/\alpha - 1), \tag{71}$$

$$(1+s+z)e_i^2 + (s+2z)e_i + z - e_e^2 = 0. \tag{72}$$

For Case 3 $1+s+z \underset{(68b),(71b),(A36)}{=} (m_{nn}+\mu m_{nt})/(m_{nn}-\mu m_{nt}) > 0$ (Table 1), and

$$e_i = \frac{-(s+2z) + \sqrt{(s+2z)^2 + 4(1+s+z)(e_e^2 - z)}}{2(1+s+z)}. \tag{73}$$

With a positive-sign root (73) reduces to (70) when $\alpha = r_m$ (then $z = 0$), ensuring $e_i > 0$ and $\partial e_i/\partial e_e \geq 0$. Lastly, for Type 5,

$$v_n^S/\left|v_n^A\right| \underset{(A30)}{=} e_v \underset{(50)}{=} e_i \underset{(61)}{=} e_e, \tag{74}$$

which means that (49)–(51) remain valid with e_e replacing e_i.

When $\alpha = (1+e_i)r_m$, then (66) and (72) reduce to $e_i = e_e$. Thus, the type of collision can be determined uniquely from Table 1: $\alpha < r_m$ indicates Types 1 or 3, $\alpha > (1+e_e)r_m$ indicates Type 5, and $r_m < \alpha < (1+e_e)r_m$ indicates Types 2 or 4. Having identified the collision type, one can use (64), (67), (70), (73), or (74) to evaluate e_i, and then calculate I_n and I_t (Sect. 3) and ultimately $\Delta u_1, \ldots, \Delta u_p$ (A16).

Now, (64), (67), (70), and (73) can lead to e_i which is greater than 1, even if $0 \leq e_e \leq 1$. Subintervals of e_e ensuring $0 \leq e_i \leq 1$ can be identified with the aid of (64), (67), (70), and (73), as shown in Table 2. Starting with Type 1, one can show with (64), that if $0 \leq e_e^2 \leq 1+2r$, then $0 \leq e_i \leq 1$ in Cases 1, 2, and 5. It can be shown that with regards to Type 2, $0 \leq e_i \leq 1$ in Cases 1 and 2 if $0 \leq e_e^2 \leq l$, where $l \hat{=} 1 + 2(r+2y) = m_{nt}/m_{tt}(2r_m - \alpha) > 0$, provided $m_{nt} > 0$. If $m_{nt} < 0$, then $l < 0$, hence either $y \leq e_e^2 \leq l$ or $0 \leq e_e^2 \leq$ (the smaller of l and y).

The procedure of Sect. 3 does not require $e_i < 1$, however, questionable results may arise if $e_i > 1$. This can be illustrated with the example of Part A. Suppose that the coefficients given in Column 2 of Table 2 are e_e, and note that Cases 1–5 result in collisions of Type 4, as in Sect. 3. Evaluating e_i first by substitutions in (73) for Cases 1–5, one obtains $e_i =$ 0.8454, 4.374, 4.097, 4.734 and 5.115 with energy losses of −0.08118, 0.0650, 0.04796, 0.08802, and 0.1154J, respectively (58d). Cases 2–5 involve $e_i > 1$, and lead to positive energy changes. Indeed, by Table 2, the limit of e_e^2 ensuring $e_i < 1$ is 0.353 for Case 1, and −0.228 for Cases 2–5, implying that Cases 2–5 have no energy consistent solution.

Table 2 Regions of e_e ensuring $0 \leq e_i \leq 1$;* $l \hat{=} 1 + 2(r+2y)$

Type 1	Equation (64)	Cases 1, 2, 4 $\Rightarrow 0 \leq e_e^2 \leq 1+2r$
Type 2	Equation (67)	Cases 1, 2 $\Rightarrow m_{nt} > 0 : 0 \leq e_e^2 \leq l^*$
		$m_{nt} < 0 : \begin{cases} e_e^2 - y \geq 0 \Rightarrow y \leq e_e^2 \leq l \\ e_e^2 - y \leq 0 \Rightarrow 0 \leq e_e^2 \leq l, y \end{cases}$
Type 3	Equation (70)	Case 3 $\Rightarrow 0 \leq e_e^2 \leq 1+2s$
Type 4	Equation (73)	Case 3 as in Type 2, Cases 1, 2 with $s\&z$ replacing $r\&y$
Type 5	Equation (74)	Cases 1–3, 5 $\Rightarrow 0 \leq e_e^2 \leq 1$

5 The sled example ([9], p. 9)

Figure 2 shows two identical sleds A and B comprising rods of length $2l$ and mass m, supported by massless knife-edges with steering angles γ and δ, touching ground at points A_s and B_s a distance s from their mass centers A^* and B^*, respectively; and supported by two back sliders. Let $u_1, \ldots, u_6$ be generalized speeds defined such that the velocities of A^* and B^*, and the angular velocities of A and B in N, are given by

$$\mathbf{v}^{A*} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2, \quad \omega^A = u_3\mathbf{a}_3, \quad \mathbf{v}^{B*} = u_4\mathbf{b}_1 + u_5\mathbf{b}_2, \quad \omega^B = u_6\mathbf{b}_3,$$

and are subject to constraint $\mathbf{v}^{A_s} \cdot \mathbf{a}'_2 = 0$, $\mathbf{v}^{B_s} \cdot \mathbf{b}'_2 = 0$ imposed by the knife-edges, where $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{a}'_i$ and $\mathbf{b}'_i$ $(i = 1, 2, 3)$ are sets of three dextral, mutually perpendicular unit vectors fixed in A and B, as shown in Fig. 2. These constraint equations, when written explicitly and solved for u_2 and u_5 read

$$u_2 = \tan\gamma u_1 - su_3, \qquad u_5 = \tan\delta u_4 - su_6,$$

and lead with u_1, u_3, u_4, and u_6 regarded as independent generalized speeds to the following equations, governing motions of A and B:

$$\begin{aligned} &-\dot{u}_1/\cos^2\gamma + s\tan\gamma\dot{u}_3 - su_3^2 = 0, \qquad && s\tan\gamma\dot{u}_1 - \left(l^2/3 + s^2\right)\dot{u}_3 + su_1u_3 = 0, \\ &-\dot{u}_4/\cos^2\delta + s\tan\delta\dot{u}_6 - su_6^2 = 0, \qquad && s\tan\delta\dot{u}_4 - \left(l^2/3 + s^2\right)\dot{u}_6 + su_4u_6 = 0. \end{aligned}$$

Defining generalized coordinates $q_1, \ldots, q_6$ as $q_1 \hat{=} \mathbf{p}^{A^*} \cdot \mathbf{n}_1$, $q_2 \hat{=} \mathbf{p}^{A^*} \cdot \mathbf{n}_2$, $\dot{q}_3 \hat{=} u_3$ and $q_4 \hat{=} \mathbf{p}^{B^*} \cdot \mathbf{n}_1$, $q_5 \hat{=} \mathbf{p}^{B^*} \cdot \mathbf{n}_2$, $\dot{q}_6 \hat{=} u_6$, one can replace the constraint equations with

$$-\sin(\gamma + q_3)\dot{q}_1 + \cos(\gamma + q_3)\dot{q}_2 + \cos\gamma s\dot{q}_3 = 0,$$
$$-\sin(\delta + q_6)\dot{q}_4 + \cos(\delta + q_6)\dot{q}_5 + \cos\delta s\dot{q}_6 = 0,$$

nonintegrable differential equations, which make the system nonholonomic. Next, suppose that at time t_1 the endpoint B_c of B collides with point A_c of A located a distance c from A^*; and that it is required to evaluate the associated change in the generalized speeds and in the system kinetic energy. To this end, the velocities of A_c and B_c are expressed as

$$\mathbf{v}^{A_c} = u_1\mathbf{a}_1 + (u_2 + cu_3)\mathbf{a}_2, \qquad \mathbf{v}^{B_c} = u_4\mathbf{b}_1 + (u_5 + cu_6)\mathbf{b}_2.$$

With $\mathbf{n}$ and $\mathbf{t}$ identified as $\mathbf{n} = \mathbf{a}_2$ and $\mathbf{t} = \pm\mathbf{a}_1$ (making $\mathrm{v}_t^A > 0$), the components of the relative velocity $\mathbf{v}^R = \mathbf{v}^{B_c} - \mathbf{v}^{A_c}$ of the colliding points are written, with $\lambda \hat{=} q_3 - q_6$ as

$$\mathrm{v}_n^R = -\tan\gamma u_1 - (c - s)u_3 - (\sin\lambda - \tan\delta\cos\lambda)u_4 - (s - l)\cos\lambda u_6,$$
$$\mathrm{v}_t^R = \pm[-u_1 + (\cos\lambda + \tan\delta\sin\lambda)u_4 + (l - s)\sin\lambda u_6].$$

The mass matrix and the relative velocity components can now be used in (A20) to evaluate m_{nn}, m_{nt} and m_{tt} and then, with (A35), g, h, p and q needed to uncover the type of collision and the associated quantities I_n, I_t, Δu_r $(r = 1, 3, 4, 6)$, v_n^A, v_n^S, v_t^A, v_t^S and ΔE. With $m = 3$ kg, $l = 1$, $s = 0.75$, $c = -0.5$ m, $q_3(t_1) = \pi/4$, $q_6(t_1) = 7\pi/4$ rad, $u_1(t_1) = u_4(t_1) = 1$ m/sec, $u_3(t_1) = u_6(t_1) = 0.1$ rad/sec, and for e, μ, γ and δ given in columns 1–4 of Table 3, one obtains results recorded in columns 5–9 for the three hypotheses. Regarding Stronge's hypothesis, Poisson's coefficient of restitution equivalent to $e_e = 0.8$ is found for collision Types 5, 3, and 1 by substitutions in (74), (70), and (64), respectively.

Table 3 Two-sled collision problem: three solutions

e	μ	γ	δ	Type	$[\Delta u_1, \Delta u_3, \Delta u_4, \Delta u_6]$ [m/s, r/s, m/s, r/s]	v_n^A, v_n^S [m/s]	v_t^A, v_t^S [m/s]	ΔE [J]
Newton's hypothesis; type 1 => sticking, type 2 => reverse sliding, type 3 => forward sliding								
0.8	0.3	0.2	0.20	3	[−0.209, 0.984, −0.667, −0.052]	−1.08, 0.86	0.77, 0.71	−0.724
0.8	0.3	0.2	0.85	2	[−0.193, 1.197, −0.405, −0.329]	−1.08, 0.86	0.16, −0.186	−0.292
0.8	0.7	0.2	0.85	1	[−0.286, 1.232, −0.342, −0.242]	−1.08, 0.86	0.16, 0.0	−0.224
Poisson's hypothesis; equivalent with Newton's hypothesis for forward sliding (comment f)								
0.8	0.3	0.2	0.20	5	[−0.209, 0.984, −0.667, −0.052]	−1.08, 0.86	0.77, 0.71	−0.724
0.8	0.3	0.2	0.85	3	[−0.187, 1.169, −0.396, −0.322]	−1.08, 0.82	0.16, −0.035	−0.298
0.8	0.7	0.2	0.85	1	[−0.276, 1.198, −0.334, −0.238]	−1.08, 0.81	0.16, 0.0	−0.288
Stronge's hypothesis; the coefficients of restitutions are Poisson's, equivalent to Stronge's $e_e = 0.8$								
0.800	0.3	0.2	0.20	5	[−0.209, 0.984, −0.667, −0.052]	−1.08, 0.86	0.77, 0.71	−0.724
0.874	0.3	0.2	0.85	3	[−0.387, 1.206, −0.305, −0.708]	−1.08, 0.89	0.16, −0.033	−0.207
0.887	0.7	0.2	0.85	1	[−0.494, 1.188, −0.296, −1.049]	−1.08, 0.90	0.16, 0.0	−0.178

A comment regarding multiple collisions is in order. Collisions of the type in question are events of a very short duration, typically on the order of a few milliseconds, hence are rarely simultaneous, and can be dealt with one at a time. If, nevertheless, m collisions are simultaneous, then (A12) must be replaced by

$$\sum_{s=1}^{p} m_{rs}\Delta u_s + \sum_{i=1}^{m}\left(I_n^{(i)}\mathbf{v}_r^{R(i)}\cdot\mathbf{n}^{(i)} + I_t^{(i)}\mathbf{v}_r^{R(i)}\cdot\mathbf{t}^{(i)}\right) = 0 \quad (r = 1,\dots,p). \tag{75}$$

With Poisson's and Stronge's hypotheses, the number of possible solutions rises dramatically from 5, if $m = 1$, to 5^m (and from 3 to 3^m with Newton's hypothesis), the determination of the solution in a particular case becomes complex; and it may occur that there is no coherent solution or there are multiple solutions. Authors dealing with this state of affaires make simplifying assumptions which allow admissible solutions. For example, Ivanov [10] uses Newton's hypothesis for the investigation of multiple collisions of unconstrained bodies, assuming that sliding does not reverse direction during collision. Wolfsteiner and Pfieffer [11] use Poisson's hypothesis to generate normal impulses and (friction dependent) tangential impulses, neglecting transitions from sliding to sticking and from sticking to sliding.

Regarding (75), one assumption that comes to mind is that (26), (29), (36), (44), and (49) associated with the five types of collision apply to $i = 1,\dots,m$; and can be used one at a time, for 5^m evaluations of $\Delta u_1,\dots,\Delta u_p$, provided $I_n^{(i)} > 0$ $(i = 1,\dots,m)$. Coherence must then be tested, requiring $v_n^{S(i)} > 0$ $(i = 1,\dots,m)$ and $v_t^{S(i)}$ $(i = 1,\dots,m)$ which accommodates the respective type of collision. The investigation of such assumptions is, however, lengthy and is left for future work.

6 Summary

An analysis similar to that in Part A was conducted in Part B with reference to Poisson's hypothesis. It was shown that as in Part A, the signs of g, h, p and q—spanning five configuration-related cases of the systems in question—and the range of α determine the type of collision, with which I_n and I_t and the changes in the motion variables can be calculated. The different cases were arranged as in Table 1, and used not only as a base for collision-type identification algorithm, but also as a tool for the comparison of the three theories. It was shown that Poisson-based theory leads always to unique, coherent, and energy-consistent solutions. With regards to Stronge's hypothesis, it was shown that the type of collision can be determined with the aid of Table 1 with e_e replacing e_i, and that the one-to-one correspondence existing for the five types of collision between Stronge's and Poisson's definitions of coefficient of restitution, enable the evaluation of the latter, given the former. One can then proceed with the evaluation of the normal and tangential impulses and the changes in the motion variables. It is shown, however, that Poisson's coefficient of restitution, now regarded as a parameter, may exceed unity, giving rise to questionable results. Thus, limitations defined in Table 2 are imposed on the permissible values of Stronge's coefficient. In that regard, Poisson's hypothesis is advantageous. Poisson's hypothesis is also advantageous when compared with Newton's hypothesis (Table 1, Part A), as no energy discrepancies occur. In conclusion, Poisson's hypothesis is superior to both Newton's and Stronge's hypotheses for the solution of planar collision problems.

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