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Collision with friction; Part A: Newton's hypothesis

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Abstract This paper deals with collision with friction. In Part A, equations governing a one-point collision of planar, simple nonholonomic systems are generated. Expressions for the normal and tangential impulses, the normal and tangential velocities of separation of the colliding points, and the change of the system mechanical energy are written for three types of collision (i.e., forward sliding, sticking, etc.). These together with Routh's semigraphical method and Coulomb's coefficient of friction are used to show that the algebraic signs of *four*, newly-defined, configuration-related parameters, not all independent, span five cases of system configuration. For each, the ratio between the tangential and normal components of the velocity of approach, called α, determine the type of collision, which once found, allows the evaluation of the associated normal and tangential impulses and ultimately the changes in the motion variables. The analysis of these cases indicates that the calculated mechanical energy may increase if sticking or reverse sliding occur. In Part B, theories based on Poisson's and Stronge's hypotheses are presented with more encouraging results.

Keywords Collision · Collision with friction · Newton's hypothesis · Routh's graph · Coulombs' coefficient of friction

1 Introduction

The interest in the subject of collision with friction in the context of rigid multibody systems increased dramatically

(a) with the realization that the use of a coefficient of restitution is well suited to cases where numerous collisions with friction occur during motions of interest. In these cases, alternate approaches [1] involving explicit contact, restoring and dissipative forces or finite-element based analysis are complex in implementation and/or increase simulations run-time (i.e., lead to stiff equations); and

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(b) after Kane's observation regarding a possible increase in the calculated mechanical energy of a system undergoing collision with friction [2]. He showed that an energy increase may occur if use is made of Newton's "kinematic" definition of coefficient of restitution (hereinafter referred to as Newton's hypothesis), in conjunction with Coulomb's coefficient of friction. Interestingly, Whittaker [3] comments (Art. 95) "... a collision *generally* results in a decrease of dynamical energy," without discussing exceptions (Mason and Wang [4] point out inconsistencies associated with the Coulomb's coefficient of friction).

Numerous investigators of collision-with-friction problems involving rigid bodies used two-body systems as platforms to convey their ideas. For example, Routh [5] used Poisson's "kinetic" definition of coefficient of restitution (hereinafter referred to as Poisson's hypothesis), in conjunction with the associated compression and restitution phases, to predict seven types of collision, using an ingenious graphical method. Whittaker [3], using Newton's hypothesis, considers (Art. 95) in connection with the 3D collision of two rigid bodies, the loss of kinetic energy in a frictionless collision (in Art. 97, he discusses in general terms what should be done if friction is present). Goldsmith [6], dealing with two-body, 2D, and 3D systems in connection with Newton's hypothesis, also mentions compression and restitution phases. He comments that the coefficient of friction cannot be accurately determined in connection with collision, and is generally defined in a manner corresponding to its noncollision processes counterpart. Brach [7] published an analysis dealing with planar, two-body collisions, suggesting that an impulsive torque is transmitted between the colliding bodies, as well as an impulsive force. He defines a new coefficient of restitution to describe the relative angular motion of the bodies in a manner analogous to that used in connection with the relative linear motion of the contact points.

Keller [8] was the first to react to Kane's observation [2], solving a 3D version of the two-body collision problem with Poisson's hypothesis. Regarding the normal impulse as an independent integration variable, he generated differential equations with components of the relative velocity of the colliding points as dependent variables. He integrated these equations, obtaining the components of the relative velocity of separation, the normal and tangential impulses, and the changes in the motion variables. However, he did not show that his approach leads to the decrease in the system mechanical energy. Battle [9] discussed conditions ensuring energy consistency for both Newton's and Poisson's hypotheses. However, he considered only the energy associated with the normal impulse alone. Stronge, trying to ensure a decrease in the system energy, introduced a new "energetic" definition of coefficient of restitution [10] (hereinafter referred to as Stronge's hypothesis). Expanding Keller's idea, he obtained changes in components of the relative velocity of the colliding points as functions of the normal impulse, forming "slip trajectory" [11, 12], and ultimately solving the collision problem (in [11], Stronge recognized that an "energetic" definition of coefficient of restitution, identical to his definition, was suggested by Boulange [13] in 1939, in connection with problems of collision of bodies that are not perfectly elastic). Keller's idea was also used by Bhatt and Koechling [14], who pointed out ways to overcome singularities that could arise if sticking occurs, and by Battle, who analyzed "balanced collisions" [15] and 3D collisions [16].

It seems that the majority of investigators prefer algebraic solutions of the collision problem rather then solutions involving integrals [8]; for, if viable, these solutions offer a higher computational efficiency. Thus, many continued to investigate different aspects of algebraic solutions. Han and Gilmore [17] tabulated Routh's 7 cases for a two-body planar collision, and generated explicit expressions for the normal and tangential impulses (they also considered multiple-point collisions). Brach [18] conducted a two-body analysis, noting that there are system-dependent limitations of the value of Coulomb's coefficient of friction, which if violated give rise to an increase in the calculated kinetic energy. Smith [19] redefined the tangential impulse in a way leading to the decrease of the kinetic energy in the double pendulum system (used by Kane to show an increase in the kinetic energy). Wang and Mason [20] reviewed Routh's work and generated expressions for normal and tangential impulses for the two-body planar collision using both Newton's and Poisson's hypotheses. They also noted that with Poisson's hypothesis, "tangential" collision (also called "grazing") is possible. Finally, they showed that for the two-body planar collision, an energy gain is impossible with Poissons' hypothesis, whereas this is not necessarily the case with Newton's hypothesis. Ivanov [21] applied the three definitions of coefficient of restitution to the two-body problem. Comparing analysis results with experimental results, he concluded that Stronge's hypothesis is the most realistic. Smith and Pao-Pao [22] found, in connection with a study of collisions of a rigid body with a plane that the three definitions of coefficients of restitution may lead to results differing from one another, and that none of the definitions capture tangential compliance. Moreau [23] suggested the use of two coefficients of restitution, called Normal and Tangential coefficients, defined in a manner similar to Newton's definition. This approach was adopted, e.g., by Pfieffer in his presentation of the idea complementarity in dynamics [24]. Chatterjee and Ruina [25, 1998] discussed limitations of different collision laws, especially in connection with two-body, 2D systems. They pointed out "inaccessible regions" (i.e., regions of parameters with no solutions or multiple solutions, energy inconsistencies, and inapplicability of the common collision laws to 3D problems). Distinguishing between normal and tangential coefficients of restitution, they proposed a new algebraic collision law which overcomes most of the shortcomings of the classical laws. In a different work [26, 1998], they pointed out that Routh's method assumes compliance in the normal direction not in the tangential direction.

A number of works deal with more complex systems. Wittenburg [27] and Wang et al. [28] consider frictionless collisions of multibody systems subject to holonomic constraints. Marghitu and Hurmuzlu [29] considered a tree-topology system with a number of points of end-bodies in contact with different surfaces, and with one point of an end-body hitting a surface. They developed an algorithm which identifies the motion of each of the contacting points after collision termination, using the indicated three definitions of coefficient of restitution. Finally, they commented that Routh's method cannot be applied to 3D systems. Lankarani, in a series of papers culminating in [30], developed a Poisson's hypothesis-based formulation for open and closed chain systems. For each of Routh's seven types of collision, Lankarani generated expressions for the normal and tangential impulses as functions of system parameters, coefficient of friction and coefficient of restitution, and set out a 13-step algorithm for the solution of collision problems, which can be tailored to a multibody simulation. He presented a number of examples showing a decrease in the system kinetic energy. However, he did not prove that the system energy always decreases, nor did he show that his solution is coherent (a coherent solution is one ensuring a positive normal impulse, positive normal velocity of separation, and a positive, zero, or negative tangential velocity of separation for forward sliding, sticking, and reverse sliding, respectively). In fact, coherence is not discussed, e.g., in [20, 26], and [29], all dealing with Poisson's hypothesis and Routh's method.

This survey may lead one to the following conclusions. First, the theories developed to date do not seem to guarantee coherent solutions. Second, the theories do not cover simple, nonholonomic systems. Third, a clear cut definition of the scope of applicability of the different theories does not seem to have been identified. Finally, it has not been proven that Poisson's and Stronge's hypotheses, generally preferred to Newton's hypothesis, lead to the decrease of the system (calculated) mechanical energy. It is the purpose of this paper to fill the indicated gaps. In Sect. 2 of Part A, equations underlying a one-point collision of simple, nonholonomic systems in planar motion are developed, together with a general expression for the associated change of the mechanical energy. Sections 3 and 4 are devoted to the development of a collision theory based on Newton's hypothesis and on Routh's method and to the analysis of the three associated types of collision and their region of application. All possible configurations of one-point collision are mapped, regions of no solution and of multiple solutions are identified, and the system change of mechanical energy is discussed. Kane's double-pendulum example [31] is reviewed in Sect. 5. A short summary in Sect. 6 concludes Part A of this work.

Steps similar to those in Sects. 3 and 4 are taken in Part B in connection with Poisson's and Stronge's hypotheses.

2 Preliminaries

Let

$$F_r + F_r^* = 0 \quad (r = 1, \dots, p) \tag{1}$$

be Kane's equations of motion for S, a simple, nonholonomic system of ν particles P_i ($i = 1, \dots, \nu$) of mass m_i, possessing p independent generalized speeds $u_1, \dots, u_p$ and n ($n > p$) generalized coordinates $q_1, \dots, q_n$, where F_r and F_r^* are, respectively, the rth generalized active force and the rth generalized inertia for S (Kane and Levinson [31]). $\mathbf{v}^{P_i}$, the velocity of P_i in N, a Newtonian reference frame, can be expressed in terms of $u_1, \dots, u_p, q_1, \dots, q_n$ and time t as

$$\mathbf{v}^{P_i} = \sum_{r=1}^{p} \mathbf{v}_r^{P_i} u_r + \mathbf{v}_t^{P_i} \quad (i = 1, \dots, \nu) \tag{2}$$

where $\mathbf{v}_r^{P_i}$, called the rth partial velocity of P_i, and $\mathbf{v}_t^{P_i}$, called the remainder partial velocity of P_i, are functions of $q_1, \dots, q_n$ and t. Let B and B' be bodies of S, and let P be a point of body B coming into contact with point P' of body B' during the collision of B with B' occurring between two instants t_1 and t_2. Let $\mathbf{v}^R$, $\mathbf{v}^A$, and $\mathbf{v}^S$ be the relative velocity of points P and P', the velocity of approach, and the velocity of separation, respectively, defined as

$$\mathbf{v}^R \mathrel{\hat{=}} \mathbf{v}^P - \mathbf{v}^{P'}, \tag{3}$$

$$\mathbf{v}^A \mathrel{\hat{=}} \mathbf{v}^R(t_1), \qquad \mathbf{v}^S \mathrel{\hat{=}} \mathbf{v}^R(t_2). \tag{4}$$

Each of the quantities appearing in (3) and (4) can be written similarly to $\mathbf{v}^{P_i}$ in (2). One can thus recognize that

$$\mathbf{v}_r^R = \mathbf{v}_r^P - \mathbf{v}_r^{P'}, \qquad \mathbf{v}_t^R = \mathbf{v}_t^P - \mathbf{v}_t^{P'}, \tag{5}$$

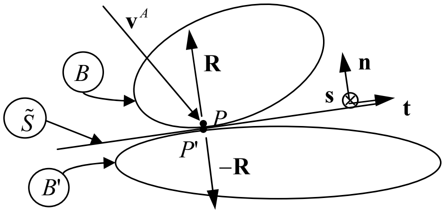
where $\mathbf{v}_r^R$ is the coefficient of u_r in $\mathbf{v}^R$. Suppose that during collision, P' exerts on P a force $\mathbf{R}$, so that P exerts on P' a force $-\mathbf{R}$. Then (1) give way to equations that bring into evidence the contributions of $\mathbf{R}$, i.e.,

$$F_r + F_r^* + \mathbf{R} \cdot \mathbf{v}_r^P - \mathbf{R} \cdot \mathbf{v}_r^{P'} = 0 \quad (r = 1, \dots, p; t_1 \le t \le t_2) \tag{6}$$

or in view of (5a),

$$F_r + F_r^* + \mathbf{R} \cdot \mathbf{v}_r^R = 0 \quad (r = 1, \dots, p). \tag{7}$$

Fig. 1 3D collision



During the collision, P is assumed to maintain contact with P', i.e., to coincide with P'; and a plane $\tilde{S}$ exists which passes through P ($\equiv P'$) and is tangent to B and B' at P if both are locally smooth, or to B' if only B' is locally smooth. Name B and B' such that $\mathbf{n}$, a unit vector perpendicular to $\tilde{S}$, makes $\mathbf{v}^A \cdot \mathbf{n}$ a nonpositive quantity. Aline $\mathbf{t}$, a unit vector lying in $\tilde{S}$, with the projection of $\mathbf{v}^A$ on $\tilde{S}$, making $\mathbf{v}^A \cdot \mathbf{t}$ a non-negative quantity (see Fig. 1). Finally, let $\mathbf{s}$ be a unit vector defined as $\mathbf{s} \mathrel{\hat{=}} \mathbf{n} \times \mathbf{t}$. Then

$$\mathbf{v}^R = \mathbf{v}^R \cdot \mathbf{nn} + \mathbf{v}^R \cdot \mathbf{tt} + \mathbf{v}^R \cdot \mathbf{ss}. \tag{8}$$

For planar collisions

$$\left.\begin{aligned} &\mathbf{v}^R = \mathbf{v}^R \cdot \mathbf{nn} + \mathbf{v}^R \cdot \mathbf{tt}, \\ &\mathbf{v}^A = \mathbf{v}^A \cdot \mathbf{nn} + \mathbf{v}^A \cdot \mathbf{tt} \qquad (\mathbf{v}^A \cdot \mathbf{n} \le 0,\ \mathbf{v}^A \cdot \mathbf{t} \ge 0), \\ &\mathbf{v}^S = \mathbf{v}^S \cdot \mathbf{nn} + \mathbf{v}^S \cdot \mathbf{tt}, \end{aligned}\right\} \tag{9}$$

i.e., both $\mathbf{v}^A$ and $\mathbf{v}^S$ lie in the $\mathbf{n}$–$\mathbf{t}$ plane. Note that $\mathbf{t}$ can be defined

$$\mathbf{t} \mathrel{\hat{=}} \mathbf{n} \times (\mathbf{v}^A \times \mathbf{n}) / |\mathbf{n} \times (\mathbf{v}^A \times \mathbf{n})|. \tag{10}$$

Equation (9a) makes it possible to replace (7) with

$$F_r + F_r^* + \mathbf{R} \cdot \mathbf{n} \mathbf{v}_r^R \cdot \mathbf{n} + \mathbf{R} \cdot \mathbf{t} \mathbf{v}_r^R \cdot \mathbf{t} = 0 \quad (r = 1, \ldots, p;\ t_1 \le t \le t_2). \tag{11}$$

If it is assumed that $t_2 - t_1$ is "small" compared to time constants associated with the motion of S, and that consequently, $q_1, \ldots, q_n$ and t remain constants between t_1 and t_2, then both sides of (11) can be integrated from t_1 to t_2, yielding the equations

$$\sum_{s=1}^{p} m_{rs} \Delta u_s + I_n \mathbf{v}_r^R \cdot \mathbf{n} + I_t \mathbf{v}_r^R \cdot \mathbf{t} = 0 \quad (r = 1, \ldots, p), \tag{12}$$

provided friction-associated impulses arising in the system's joints are disregarded. Here, I_n and I_t are the normal and tangential impulses defined as

$$I_n \mathrel{\hat{=}} \left(\int_{t_1}^{t_2} \mathbf{R}\, \mathrm{d}t \right) \cdot \mathbf{n}; \qquad I_t \mathrel{\hat{=}} \left(\int_{t_1}^{t_2} \mathbf{R}\, \mathrm{d}t \right) \cdot \mathbf{t}, \tag{13}$$

$\mathbf{v}_r^R \mathrel{\hat{=}} \mathbf{v}_r^R(t_2) = \mathbf{v}_r^R(t_1)$ $(r = 1, \ldots, p)$ (see (5a)), Δu_s $(s = 1, \ldots, p)$ are defined as

$$\Delta u_s \mathrel{\hat{=}} u_s(t_2) - u_s(t_1) \quad (s = 1, \ldots, p), \tag{14}$$

and m_{rs} is the entry in row r, column s of the mass matrix $\mathbf{M}$ associated with (1) (see (68), Appendix A). Note that $\mathbf{n}$ and $\mathbf{t}$, defined only for $t_1 \le t \le t_2$, remain fixed in N during the collision. Also note that $\mathbf{R}\cdot\mathbf{n}(t_1 \le t \le t_2) > 0$ (P' cannot "pull" P), hence

$$I_n > 0. \tag{15}$$

Equations (12) comprise p equations with $p+2$ unknowns $\Delta u_1, \dots, \Delta u_p$, I_n and I_t. The quantities of interest in the context of simulations of motion of multibody systems undergoing collisions are $\Delta u_1, \dots, \Delta u_p$, with which the simulations can be kept running; and these can be obtained after I_n and I_t have been identified. To this end, consider the matrix form of (12), solved for $\Delta u_s (s = 1, \dots, p)$, namely

$$|\Delta u_1 \quad \dots \quad \Delta u_p|^T = -I_n \mathbf{M}^{-1} |\mathbf{v}_1^R \cdot \mathbf{n} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{n}|^T - I_t \mathbf{M}^{-1} |\mathbf{v}_1^R \cdot \mathbf{t} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{t}|^T, \tag{16}$$

where $\mathbf{M}$ and $\mathbf{M}^{-1}$ are negative definite matrices. Now, $\mathbf{v}^S - \mathbf{v}^A$ can be written as

$$\mathbf{v}^S - \mathbf{v}^A = |\mathbf{v}_1^R \quad \dots \quad \mathbf{v}_p^R||\Delta u_1 \quad \dots \quad \Delta u_p|^T \tag{17}$$

when use is made of (4), (3), (2), (5), and (14). If both sides of (17) are dot-multiplied by $\mathbf{n}$, and if $|\Delta u_1 \ \dots \ \Delta u_p|^T$ is eliminated with the aid of (16), one has

$$\begin{aligned} \mathbf{v}^S \cdot \mathbf{n} - \mathbf{v}^A \cdot \mathbf{n} = &-|\mathbf{v}_1^R \cdot \mathbf{n} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{n}|\mathbf{M}^{-1}|\mathbf{v}_1^R \cdot \mathbf{n} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{n}|^T I_n \\ &-|\mathbf{v}_1^R \cdot \mathbf{n} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{n}|\mathbf{M}^{-1}|\mathbf{v}_1^R \cdot \mathbf{t} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{t}|^T I_t, \end{aligned} \tag{18}$$

noting that $\mathbf{v}_t^S = \mathbf{v}_t^A$ (5b). Similarly, if both sides of (17) are dot-multiplied by $\mathbf{t}$, and if $|\Delta u_1 \quad \dots \quad \Delta u_p|^T$ is eliminated with the aid of (16), one obtains

$$\begin{aligned} \mathbf{v}^S \cdot \mathbf{t} - \mathbf{v}^A \cdot \mathbf{t} = &-|\mathbf{v}_1^R \cdot \mathbf{t} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{t}|\mathbf{M}^{-1}|\mathbf{v}_1^R \cdot \mathbf{n} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{n}|^T I_n \\ &-|\mathbf{v}_1^R \cdot \mathbf{t} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{t}|\mathbf{M}^{-1}|\mathbf{v}_1^R \cdot \mathbf{t} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{t}|^T I_t. \end{aligned} \tag{19}$$

The coefficients of I_n and I_t in (18) and (19) are constants between t_1 and t_2. Defining m_{nn}, m_{nt} and m_{tt} as

$$\left. \begin{aligned} m_{nn} &\hat{=} -|\mathbf{v}_1^R \cdot \mathbf{n} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{n}|\mathbf{M}^{-1}|\mathbf{v}_1^R \cdot \mathbf{n} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{n}|^T > 0, \\ m_{nt} &\hat{=} -|\mathbf{v}_1^R \cdot \mathbf{n} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{n}|\mathbf{M}^{-1}|\mathbf{v}_1^R \cdot \mathbf{t} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{t}|^T, \\ m_{tt} &\hat{=} -|\mathbf{v}_1^R \cdot \mathbf{t} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{t}|\mathbf{M}^{-1}|\mathbf{v}_1^R \cdot \mathbf{t} \quad \dots \quad \mathbf{v}_p^R \cdot \mathbf{t}|^T > 0, \end{aligned} \right\} \tag{20}$$

in agreement with Lankarani's definition in [30]; and $\mathrm{v}_n^A, \mathrm{v}_t^A, \mathrm{v}_n^S$ and v_t^S as

$$\mathrm{v}_n^A \hat{=} \mathbf{v}^A \cdot \mathbf{n}, \quad \mathrm{v}_t^A \hat{=} \mathbf{v}^A \cdot \mathbf{t}, \quad \mathrm{v}_n^S \hat{=} \mathbf{v}^S \cdot \mathbf{n}, \quad \mathrm{v}_t^S \hat{=} \mathbf{v}^S \cdot \mathbf{t} \tag{21}$$

one can write (18) and (19) as

$$\mathrm{v}_n^S - \mathrm{v}_n^A = m_{nn} I_n + m_{nt} I_t, \tag{22}$$

$$\mathrm{v}_t^S - \mathrm{v}_t^A = m_{nt} I_n + m_{tt} I_t, \tag{23}$$

where m_{nn} and m_{tt} are positive numbers. The coefficient matrix of (22)–(23) can be written $\mathbf{C}(-\mathbf{M}^{-1})\mathbf{C}^T$ where $\mathbf{C}$ is the $2 \times p$ matrix $\mathbf{C} \hat{=} |\mathrm{v}_1^R \cdot \mathbf{n} \ \dots \ \mathrm{v}_p^R \cdot \mathbf{n} \ \mathrm{v}_1^R \cdot \mathbf{t} \ \dots \ \mathrm{v}_p^R \cdot \mathbf{t}|$, hence is positive definite, so that

$$\Delta \hat{=} m_{nn} m_{tt} - m_{nt}^2 > 0. \tag{24}$$

Equations (22)–(23) are two equations with four unknowns I_n, I_t, v_n^S and v_t^S. Two additional equations can be obtained if use is made of the concept of coefficient of restitution and of Coulomb's coefficient of friction, and contribute to the evaluation of I_n and I_t required in (16).

Before this objective is pursued, however, an expression for ΔE, the loss in the mechanical energy for the system, in terms of I_n, I_t, $\mathbf{v}^A \cdot \mathbf{n}$ and $\mathbf{v}^A \cdot \mathbf{t}$, is generated. To this end, both sides of the rth of (12) are multiplied by Δu_r for $r = 1, \dots, p$, and the respective sides of all the resulting equations are added, giving rise to

$$\sum_{r=1}^{p}\left(\sum_{s=1}^{p} m_{rs}\Delta u_s\right)\Delta u_r + I_n \sum_{r=1}^{p} \mathbf{v}_r^R \Delta u_r \cdot \mathbf{n} + I_t \sum_{r=1}^{p} \mathbf{v}_r^R \Delta u_r \cdot \mathbf{t} = 0$$

or, in view of (14), (5), and (4),

$$\sum_{r=1}^{p}\left(\sum_{s=1}^{p} m_{rs}\Delta u_s\right)\Delta u_r + I_n(\mathbf{v}^S - \mathbf{v}^A)\cdot \mathbf{n} + I_t(\mathbf{v}^S - \mathbf{v}^A)\cdot \mathbf{t} = 0. \tag{25}$$

Similarly, if both sides of the rth of (12) are multiplied by $u_r(t_1)$, and the respective sides of all the resulting equations are added, then

$$\sum_{r=1}^{p}\left(\sum_{s=1}^{n} m_{rs}\Delta u_s\right)u_r(t_1) + I_n\mathbf{v}^A \cdot \mathbf{n} + I_t\mathbf{v}^A \cdot \mathbf{t} - I_n\mathbf{v}_t^A \cdot \mathbf{n} - I_t\mathbf{v}_t^A \cdot \mathbf{t} = 0. \tag{26}$$

The energy loss during collision is given by (72) in Appendix A, which can be replaced with

$$\Delta E = -1/2\sum_{r=1}^{p}\sum_{s=1}^{p} m_{rs}\Delta u_r \Delta u_s - \sum_{r=1}^{p}\sum_{s=1}^{p} m_{rs}\Delta u_r u_s(t_1) + \sum_{i=1}^{\nu} m_i \mathbf{v}_t^{P_i} \cdot [\mathbf{v}^{P_i}(t_2) - \mathbf{v}^{P_i}(t_1)]. \tag{27}$$

This can be shown if Δu_r and $\Delta u_s (r, s = 1, \dots, p)$ appearing on the right-hand side of (27) are replaced with $u_r(t_2) - u_r(t_1)$ and $u_s(t_2) - u_s(t_1)$ $(r, s = 1, \dots, p)$, in accordance with (14). Then after expansions and cancellations, one arrives at (72) in Appendix A. The use of (25) and (26) in (27) yields the expression

$$\Delta E = 1/2 I_n(\mathbf{v}^S - \mathbf{v}^A)\cdot \mathbf{n} + 1/2 I_t(\mathbf{v}^S - \mathbf{v}^A)\cdot \mathbf{t} + I_n\mathbf{v}^A \cdot \mathbf{n} + I_t\mathbf{v}^A \cdot \mathbf{t} - I_n\mathbf{v}_t^A \cdot \mathbf{n} - I_t\mathbf{v}_t^A \cdot \mathbf{t} + \sum_{i=1}^{\nu} m_i \mathbf{v}_t^{P_i} \cdot [\mathbf{v}^{P_i}(t_2) - \mathbf{v}^{P_i}(t_1)].$$

If and only if

$$-I_n\mathbf{v}_t^A \cdot \mathbf{n} - I_t\mathbf{v}_t^A \cdot \mathbf{t} + \sum_{i=1}^{\nu} m_i \mathbf{v}_t^{P_i} \cdot [\mathbf{v}^{P_i}(t_2) - \mathbf{v}^{P_i}(t_1)] = 0, \tag{28}$$

then (27) reduces to

$$\Delta E = 1/2 I_n(\mathbf{v}^S + \mathbf{v}^A)\cdot \mathbf{n} + 1/2 I_t(\mathbf{v}^S + \mathbf{v}^A)\cdot \mathbf{t}, \tag{29}$$

an equation reminiscent of Routh's [5] and Smith's [22] results produced for a planar, two-body collision. If (28) is violated, then ΔE may become positive as in the example of Appendix B. Note that if $\mathbf{v}_t^{P_i} = 0$ ($i = 1, \ldots, \nu$), then (28) is satisfied.

The remainder of Part A and Part B of this work are dedicated to the discussion of collision theories, based on Newton's, Poisson's, and Stronge's hypotheses for the determination of I_n and I_t.

3 A collision theory with Newton's hypothesis

In accordance with Newton's hypothesis [32], a quantity e called coefficient of restitution is defined as

$$e \hat{=} -\mathbf{v}^S \cdot \mathbf{n}/\mathbf{v}^A \cdot \mathbf{n} \quad (0 \le e \le 1). \tag{30}$$

The associated theory of collision with friction (given, e.g., in [31]) stipulates that if

$$|I_t| < \mu I_n \tag{31}$$

where μ is Coulomb's static coefficient of friction, then sticking occurs, i.e.,

$$\mathbf{v}^S \cdot \mathbf{t} = 0. \tag{32}$$

If inequality (31) is violated, then forward sliding or reverse sliding take place, hence

$$I_t \hat{=} -\mu' I_n \mathbf{v}^S \cdot \mathbf{t}/|\mathbf{v}^S \cdot \mathbf{t}|, \tag{33}$$

where μ' is Coulomb's dynamic coefficient of friction. In this work, no distinction is made between μ' and μ (see, e.g., Whittaker's [6] and Keller's [8] analyses) as it may lead to inconclusive solutions discussed in Sect. 4.

Equations (22)–(23) make it possible to reformulate the theory in terms of m_{nn}, m_{tt} and m_{nt}, in conjunction with α, g, h, p, q, r_m and r_p, quantities defined as

$$\alpha \hat{=} \mathrm{v}_t^A/|\mathrm{v}_n^A| \quad (0 < \alpha < \infty) \tag{34}$$

$$g \hat{=} \mu m_{tt} - m_{nt}, \quad h \hat{=} m_{nn} - \mu m_{nt},$$

$$p \hat{=} \mu m_{tt} + m_{nt}, \quad q \hat{=} m_{nn} + \mu m_{nt}, \tag{35}$$

$$r_m \hat{=} \frac{g}{h} = \frac{(\mu m_{tt} - m_{nt})}{(m_{nn} - \mu m_{nt})}, \qquad r_p \hat{=} \frac{p}{q} = \frac{(\mu m_{tt} + m_{nt})}{(m_{nn} + \mu m_{nt})} \tag{36}$$

(Stronge [12] defines a quantity similar to r_m); and with the aid of the relations

$$\Delta = [(m_{nn} - \mu m_{nt})(\mu m_{tt} + m_{nt}) + (m_{nn} + \mu m_{nt})(\mu m_{tt} - m_{nt})]/2\mu \tag{37}$$

which reduce to (24) after expansion and with

$$r_m + r_p \underset{(36),(37)}{=} \frac{2\mu\Delta}{(m_{nn} - \mu m_{nt})(m_{nn} + \mu m_{nt})}.^{[1]} \tag{38}$$

[1] Numbers appearing under equal signs refer to equations numbered correspondingly.

To this end, expressions for I_n, I_t and v_t^S must be generated, e.g., with the aid of following observation, namely, that there might be an instant $\bar{t}(t_1 < \bar{t} < t_2)$ where

$$\bar{\mathrm{v}}_t \mathrel{\hat{=}} \mathbf{v}^R(\bar{t}) \cdot \mathbf{t} = 0. \tag{39}$$

In this case, the initial forward sliding is completed during collision, and is followed by either sticking or reverse sliding. Defining I_n', I_n'', I_t' and I_t'' as

$$I_n' \mathrel{\hat{=}} \left(\int_{t_1}^{\bar{t}} \mathbf{R}\,\mathrm{d}t\right) \cdot \mathbf{n}, \qquad I_n'' \mathrel{\hat{=}} \left(\int_{\bar{t}}^{t_2} \mathbf{R}\,\mathrm{d}t\right) \cdot \mathbf{n}, \qquad I_n = I_n' + I_n'', \tag{40}$$

$$I_t' \mathrel{\hat{=}} \left(\int_{t_1}^{\bar{t}} \mathbf{R}\,\mathrm{d}t\right) \cdot \mathbf{t}, \qquad I_t'' \mathrel{\hat{=}} \left(\int_{\bar{t}}^{t_2} \mathbf{R}\,\mathrm{d}t\right) \cdot \mathbf{t}, \qquad I_t = I_t' + I_t'', \tag{41}$$

and dividing (22)–(23) accordingly, that is,

$$\bar{\mathrm{v}}_n - \mathrm{v}_n^A = m_{nn} I_n' + m_{nt} I_t', \tag{42}$$

$$\mathrm{v}_n^S - \bar{\mathrm{v}}_n = m_{nn} I_n'' + m_{nt} I_t'', \tag{43}$$

$$\bar{\mathrm{v}}_t - \mathrm{v}_t^A = m_{nt} I_n' + m_{tt} I_t', \tag{44}$$

$$\mathrm{v}_t^S - \bar{\mathrm{v}}_t = m_{nt} I_n'' + m_{tt} I_t'', \tag{45}$$

one can generate I_n, I_t and v_t^S for the following three types of collision.

Type 1 *Sticking*, comprising forward sliding ($t_1 \div \bar{t}$) and sticking ($\bar{t} \div t_2$). Type 1 is characterized by

$$I_t' = -\mu I_n', \quad \bar{\mathbf{v}} \cdot \mathbf{t} = 0, \quad \mathbf{v}^S \cdot \mathbf{t} = 0. \tag{46}$$

Substitutions from (46), (24), (40c), and (41c) in (44), (45) and (22) lead to

$$I_n = [(1+e)m_{tt} + \alpha m_{nt}]|\mathrm{v}_n^A|/\Delta, \qquad I_t = -[\alpha m_{nn} + (1+e)m_{nt}]|\mathrm{v}_n^A|/\Delta \tag{47}$$

(note the absence of μ from these equations).

Type 2 *Reverse sliding*, comprising forward sliding ($t_1 \div \bar{t}$) and reverse sliding ($\bar{t} \div t_2$). Type 2 is characterized by

$$I_t' = -\mu I_n', \quad \bar{\mathbf{v}} \cdot \mathbf{t} = 0, \quad I_t'' = \mu I_n''. \tag{48}$$

Expressing I_n' and I_n'' from (44)–(45) in conjunction with (48) in terms of v_t^A and v_t^S, and substituting in (22), one can find v_t^S and show that

$$\left.\begin{aligned} I_n &= [(1+e) + 2\alpha\mu m_{nt}/(\mu m_{tt} - m_{nt})]|\mathrm{v}_n^A|/(m_{nn} + \mu m_{nt}), \\ I_t &= \mu[(1+e) - 2\alpha m_{nn}/(\mu m_{tt} - m_{nt})]|\mathrm{v}_n^A|/(m_{nn} + \mu m_{nt}), \end{aligned}\right\} \tag{49}$$

$$\mathrm{v}_t^S = -r_p/r_m[\alpha - (1+e)r_m]|\mathrm{v}_n^A|. \tag{50}$$

Type 3 *Forward sliding* ($t_1 \div t_2$). Type 3 is characterized by

$$I_t = -\mu I_n. \tag{51}$$

Substitutions from (51) in (22)–(23) lead to

$$I_n = (1+e)|\mathrm{v}_n^A|/(m_{nn} - \mu m_{nt}), \quad I_t = -\mu I_n, \tag{52}$$

$$\mathrm{v}_t^S = [\alpha - (1+e) r_m]|\mathrm{v}_n^A|. \tag{53}$$

I_n and I_t are thus available from (47), (49), or (52), depending on the type of collision, which must now be uncovered. This can be accomplished with the aid of additional information derived from Routh's graph [7], as follows.

4 Routh-based semi-graphical method

4.1 Routh's graph

Consider two variables $\tilde{I}_n$ and $\tilde{I}_t$ defined as

$$\tilde{I}_n \mathrel{\hat{=}} \left(\int_{t_1}^{t} \mathbf{R}\, \mathrm{d}t\right) \cdot \mathbf{n}, \qquad \tilde{I}_t \mathrel{\hat{=}} \left(\int_{t_1}^{t} \mathbf{R}\, \mathrm{d}t\right) \cdot \mathbf{t} \tag{54}$$

so that $\tilde{I}_n$ is a monotonously growing quantity, and $I_n \underset{(13),(54)}{\hat{=}} \tilde{I}_n(t_2)$, $I_t \underset{(13),(54)}{\hat{=}} \tilde{I}_t(t_2)$. Replacing I_n and I_t in (33) and (23) with $\tilde{I}_n$ and $\tilde{I}_t$, respectively, and setting $\mathrm{v}_t^S = 0$ in (23), one obtains equations of lines in the $\tilde{I}_t - \tilde{I}_n$ plane, namely,

$$\text{forward sliding line} \quad L_{FS} \Rightarrow \tilde{I}_t \underset{(33)}{=} -\mu \tilde{I}_n, \tag{55}$$

$$\text{sticking line} \quad L_{ST} \Rightarrow \tilde{I}_t \underset{(23)}{=} -\left(\tilde{I}_n m_{nt} + \mathrm{v}_t^A\right)/m_{tt}, \tag{56}$$

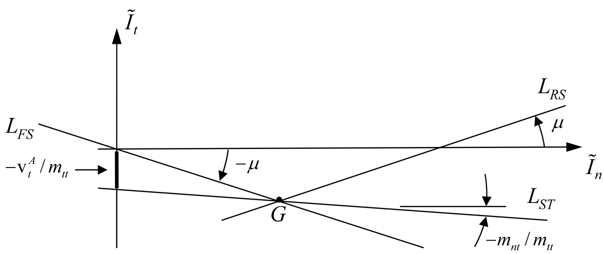
$$\text{reverse sliding line} \quad L_{RS} \underset{(33),(55),(56)}{\Longrightarrow} \tilde{I}_t = \mu \tilde{I}_n - 2\mu \mathrm{v}_t^A/(\mu m_{tt} - m_{nt}), \tag{57}$$

with $\tilde{I}_n$ regarded as an independent variable. Note that the reverse sliding line crosses point G, the intersection point of lines L_{FS} and L_{ST}; and that, by (55) and (56),

$$\tilde{I}_n(G) = \mathrm{v}_t^A/(\mu m_{tt} - m_{nt}). \tag{58}$$

Lines L_{FS}, L_{ST} and L_{RS}, drawn in Fig. 2, and, in particular, their respective slopes $-\mu, -m_{nt}/m_{tt}$ and μ, can be used to look at values of $\tilde{I}_n$ and $\tilde{I}_t$ as the collision proceeds, and to describe events occurring during collisions. With $\mathrm{v}_t^A \underset{(9\mathrm{b}),(21\mathrm{b})}{>} 0$, all $\tilde{I}_t - \tilde{I}_n$ relations

Fig. 2 Impulse plane



start at the origin and vary in accordance with L_{FS}, until forward sliding is completed either before point G is reached—then the collision ends with forward sliding characterized by $-m_{nt}/m_{tt} > -\mu$ (if $\tilde{I}_n(G) > 0$) or by $-m_{nt}/m_{tt} < -\mu$ (if $\tilde{I}_n(G) < 0$); or at point G. In that event, either sticking—governed by L_{ST}, or reverse sliding—governed by L_{RS}, follow depending on whether $|m_{nt}|/m_{tt} < \mu$ (then the inertial forces are not large enough to overcome friction and produce reverse sliding) or $-m_{nt}/m_{tt} > \mu$, respectively. The conditions for sticking, reverse sliding, and forward sliding can thus be written, respectively, as

$$|m_{nt}|/m_{tt} < \mu \Rightarrow \mu m_{tt} - m_{nt} > 0, \qquad \mu m_{tt} + m_{nt} > 0, \tag{59}$$

$$-m_{nt}/m_{tt} > \mu \Rightarrow \mu m_{tt} + m_{nt} < 0 \quad (m_{nt} < 0), \tag{60}$$

$$\left.\begin{aligned} -m_{nt}/m_{tt} > -\mu &\Rightarrow \mu m_{tt} - m_{nt} > 0, \\ -m_{nt}/m_{tt} < -\mu &\Rightarrow \mu m_{tt} - m_{nt} < 0 \quad (m_{nt} > 0). \end{aligned}\right\} \tag{61}$$

Now, if sticking follows forward sliding (Type 1), then $I_n > \tilde{I}_n(G) > 0$ (Fig. 2). Substitutions from (47a) and (58) lead in view of inequality (59a) to

$$[\alpha - (1+e)r_m]/(\Delta r_m) < 0. \tag{62}$$

If reverse sliding follows forward sliding (Type 2), then again $I_n > \tilde{I}_n(G) > 0$. Substitutions from (49a) and (58) yield $\alpha/(m_{nn} + \mu m_{nt}) < (1+e)r_m/(m_{nn} + \mu m_{nt})$ (since $m_{nt} \underset{(60)}{<} 0$, and hence $r_m \underset{(36\text{a})}{>} 0$) indicating that

$$\left.\begin{aligned} m_{nn} + \mu m_{nt} > 0 &\Rightarrow \alpha < (1+e)r_m, \\ m_{nn} + \mu m_{nt} < 0 &\Rightarrow \alpha > (1+e)r_m. \end{aligned}\right\} \tag{63}$$

Lastly, if a collision terminates with forward sliding (Type 3), then either $\tilde{I}_n(G) > I_n > 0$ or $\tilde{I}_n(G) < 0$. Substitutions from (52a) and (58) yield, respectively,

$$\alpha \underset{(61\text{a})}{>} (1+e)r_m, \qquad \alpha \underset{(61\text{b})}{>} 0. \tag{64}$$

4.2 The determination of the type of collision

Coherent solution requires $I_n > 0$, $\mathrm{v}_n^S > 0$ and v_t^S greater, equal, or smaller than zero, as the case may be; or $[(1+e)m_{tt} + \alpha m_{nt}]|\mathrm{v}_n^A|/\Delta(\underset{(47)}{=} I_n) > 0$, etc., if use is made of (47), (49), (50), (52), and (53) to obtain explicit expressions. These, together with inequalities (59)–(61) (μ-bounds) and (62)–(64) (α-regions), can be expressed in terms of g, h, p, and q, therefore, the algebraic signs of these parameters determine the satisfaction—or violation—of the inequalities. Now, each of the types of collision is coherent if and only if a certain subset of the indicated inequalities is satisfied, an event dependent solely on the algebraic signs of g, h, p, and q and on the range of α. The different possibilities can be explored in an orderly fashion if it is noted that a one point collision of a simple, nonholonomic system gives rise to five possible variations of the algebraic signs of g, h, p, and q (see (35)) arranged in columns 2–5 of Table 1. In fact, 16 sign-variations exist, however, the interdependence of g, h, p, and q ((35) and (37)) rejects the consistency of 11. That is, variations with either $g < 0$ or $h < 0$ (implying $m_{nt} > 0$) and with either $p < 0$ or $q < 0$ (implying $m_{nt} < 0$), or variations with $h \cdot p + g \cdot q$ $(= 2\mu\Delta) < 0$ (contradicting $\Delta \underset{(24)}{>} 0$) will never occur, hence are ignored.

Table 1 Addmissible sign-variations of g, h, p and q

No	g	h	p	q	Sticking	R. Sliding	F. Sliding
1	>0	>0	>0	>0	$\alpha<(1+e)r_m$	–	$\alpha>(1+e)r_m$
2	>0	>0	>0	<0	$\alpha<(1+e)r_m$	–	$\alpha>(1+e)r_m$
3	>0	>0	<0	>0	–	$\alpha<(1+e)r_m$	$\alpha>(1+e)r_m$
4	>0	<0	>0	>0	$\alpha>0$	–	–
5	<0	>0	>0	>0	–	–	$\alpha>0$

Considering sticking, one may conclude that only Cases 1, 2 and 4 comply with inequalities (59). Moreover, inequality (62) indicates that the sticking-associated α-regions are those appearing in the sticking column in Table 1 (in Case 4 $r_m<0$ hence $\alpha>0$) for which $I_n \underset{(47a)}{>} 0$ and $|I_t| \underset{(31)}{<} \mu I_n$. Regarding reverse sliding, inequality (60) is satisfied only in Case 3. The range α is determined by inequality (63a) and is reported in the reverse sliding column of Table 1. It can be verified that in Case 3 $I_n \underset{(49a)}{>} 0$ and $\mathrm{v}_t^S \underset{(50)}{<} 0$. Finally, all cases satisfy either inequality (61a) or (61b) in forward sliding. However, $h>0$ and $\alpha>(1+e)r_m$, ensuring $I_n \underset{(52a)}{>} 0$ and $\mathrm{v}_t^S \underset{(53)}{>} 0$, in all but Cases 4 (in Case 5 $\alpha>0$ since $r_m<0$).

An algorithm for the solution of a collision problem can now be established provided μ and e are known. $\alpha, m_{nn}, m_{tt}, m_{nt}, g, h, p, q, r_m$ and Δ are calculated for t_1, the collision time. The collision type is determined with the aid of Table 1, and used to evaluate I_n and I_t with (47), (49), or (52), and then $\Delta u_1, \ldots, \Delta u_p$ with (16).

It should finally be noted that the distinction between μ' and $\mu(\mu'<\mu)$ gives rise to additional regions of ambiguity. For example, consider the first case in Table 1 with $\mu'<\mu$. The regions of sticking and forward sliding become $0<\alpha<(1+e)r_m$ and $(1+e)r_m'<\alpha<\infty$, where r_m' is defined as r_m in (36a) with μ' replacing μ. However, $\partial r_m/\partial\mu>0$ since $\Delta>0$, hence $r_m'<r_m$. Consequently, conditions for both forward sliding and sticking are satisfied when $(1+e)r_m'<\alpha<(1+e)r_m$. This state of affairs is typical of transitions from sticking to sliding regions, also appearing in Part B; and leads to the following conclusion, namely, that the distinction between μ' and μ gives rise to regions where solutions are not unique. Hence, no such distinction is made in this work.

4.3 Energy considerations

The change in the system mechanical energy associated with sticking can be obtained by substitutions from (30) and (32) in (29), which lead to $\Delta E|_1 = 1/2 I_n(1-e)\mathrm{v}_n^A + 1/2 I_t \mathrm{v}_t^A$, or, in view of (34) and (47),

$$2\Delta E|_1/|\mathrm{v}_n^A| = -\{1-e+\alpha[\alpha m_{nn}+(1+e)m_{nt}]/[(1+e)m_{tt}+\alpha m_{nt}]\}I_n. \tag{65}$$

With regard to reverse and forward sliding, substitutions from (30) and (33) in (29) yield $\Delta E = -1/2 I_n(1-e)|\mathrm{v}_n^A| + 1/2 I_n(I_t/I_n)(\mathrm{v}_t^S+\mathrm{v}_t^A)|\mathrm{v}_n^A|/|\mathrm{v}_n^A|$, or

$$2\Delta E|_2/|\mathrm{v}_n^A| \underset{(49),(50)}{=} -\{(1-e)-\mu(l-2\alpha m_{nn})/(l+2\mu\alpha m_{nt}) \times [\alpha(1-r_p/r_m)+(1+e)r_p]\}I_n \tag{66}$$

for reverse sliding, where $l=(1+e)(\mu m_{tt}-m_{nt})$; and

$$2\Delta E|_3/|\mathrm{v}_n^A| \underset{(53)}{=} -\{(1-e)+\mu[2\alpha-(1+e)r_m]\}I_n \tag{67}$$

for forward sliding. Equations (65)–(67) show that whereas. $\Delta E|_3 < 0$ for forward sliding as can be shown straightforwardly for all cases of Table 1 (see inequality (64)), this is not necessarily the case with sticking and reverse sliding. For instance, if $e = 1$, ΔE reduces to $\Delta E|_{e=1} \underset{(29),(30)}{=} 1/2 I_t \Sigma \mathrm{v}_t$ where $\Sigma \mathrm{v}_t \hat{=} \mathrm{v}_t^A + \mathrm{v}_t^S$. Equation (65) becomes $\Delta E|_1 = -1/2|\mathrm{v}_n^A|^2[2\alpha m_{nt} + \alpha^2 m_{nn}]/\Delta$, and $\Delta E|_1$ has for the first case of Table 1, a maximum at $\alpha = -m_{nt}/m_{nn}$ (within the sticking region), which equals $\Delta E|_{1\,\max\,@e=1} = 1/2|\mathrm{v}_n^A|^2 m_{nt}^2/(m_{nn}\Delta) > 0$. For reverse sliding, the values of I_t and $\Sigma \mathrm{v}_t$ are at $\alpha = 0$ $I_t|_{\alpha=0} \underset{(49)}{=} 2\mu|\mathrm{v}_n^A|/(m_{nn} + \mu m_{nt}) > 0$ and $\Sigma \mathrm{v}_t|_{\alpha=0} \underset{(50)}{=} 2r_p|\mathrm{v}_n^A| < 0$, and at $\alpha = 2r_m$ $I_t|_{\alpha=2r_m} \underset{(49)}{=} -2\mu|\mathrm{v}_n^A|/(m_{nn} - \mu m_{nt}) < 0$ and $\Sigma \mathrm{v}_t|_{\alpha=2r_m} \underset{(50)}{=} 2r_m|\mathrm{v}_n^A| > 0$. Moreover, $\alpha(I_t = 0) = (\mu m_{tt} - m_{nt})/m_{nn} \neq \alpha(\Sigma \mathrm{v}_t = 0) = 2r_p r_m/(r_p - r_m)$, hence there is a region of α where both I_t and $\Sigma \mathrm{v}_t$ have identical signs, and $\Delta E|_2 > 0$.

4.4 Comments

a. I_n, I_t, v_n^S and v_t^S are continuous functions of α. This continuity is maintained through the passages from one type of collision to another, e.g., at $\alpha = (1+e)r_m$ in Cases 1 and 2 of Table 1, where $I_n|_1[\alpha = (1+e)r_m] \underset{(47a),(52a)}{=} I_n|_3[\alpha = (1+e)r_m]$, and in Case 3, where $I_t|_2 I_t[\alpha = (1+e)r_m] \underset{(49b),(52b)}{=} I_t|_3[\alpha = (1+e)r_m]$. Consequently, equal signs can be added to the inequalities defining the regions of α in Table 1.

b. If $\mu = 0$, then $r_m \underset{(36)}{=} -r_p = -m_{nt}/m_{nn}$, hence $r_m + r_p = 0$. Inequalities (59) ($m_{nt} = 0$), (60) ($m_{nt} < 0$) and (61) ($m_{nt} < 0$ or $m_{nt} > 0$) indicate sticking, reverse sliding or forward sliding when $\alpha \underset{(62)}{=} 0, \alpha \underset{(63),(50)}{<} (1+e)r_m$ or $\alpha \underset{(64a)}{>} (1+e)r_m$, respectively (if $m_{nt} > 0$, then $r_m < 0$, hence $\alpha \underset{(64b)}{>} 0$). In all cases $I_n = (1+e)|\mathrm{v}_n^A|/m_{nn}$ ((47), (49) and (52)) and $2\Delta E/|\mathrm{v}_n^A| = -(1-e)I_n < 0$ ((65)–(67)).

c. $e = 0$ indicates $\mathrm{v}_n^S = 0$ but not $\mathrm{v}_t^S = 0$ ((50) and (53)).

d. The cases $\mathrm{v}_n^A < 0$, $\mathrm{v}_t^A = 0$ ($\alpha \to 0$), called direct impact [10, 30], and $\mathrm{v}_n^A = 0$, $\mathrm{v}_t^A > 0$ ($\alpha \to \infty$), called grazing [4, 12, 20], can be followed by all three types of collision; and I_n, I_t, v_n^S and v_t^S can be found from (47) (sticking), (49) and (50) (reverse sliding) and (52) and (53) (forward sliding), by direct substitution. For direct impact ($\mathrm{v}_t^A = 0$) $I_n \underset{(47a)}{=} (1+e)m_{tt}/\Delta|\mathrm{v}_n^A|$ and $I_t \underset{(47b)}{=} -m_{nt}/m_{tt} I_n$ (sticking), $I_n \underset{(49a)}{=} (1+e)/(m_{nn} + \mu m_{nt})|\mathrm{v}_n^A|$ and $\mathrm{v}_t^S \underset{(50)}{=} (1+e)|\mathrm{v}_n^A| r_p$ (reverse sliding); and for grazing ($\mathrm{v}_n^A = 0$) $I_n \underset{(47a)}{=} m_{nt}/\Delta \mathrm{v}_t^A$ and $I_t \underset{(47b)}{=} -m_{nn}/\Delta \mathrm{v}_t^A$ (sticking), and $\mathrm{v}_t^S \underset{(50)}{=} -r_p/r_m \mathrm{v}_t^A$, $I_n \underset{(49a)}{=} 2\mu m_{nt}/[(\mu m_{tt} - m_{nt})(m_{nn} + \mu m_{nt})]|\mathrm{v}_n^A|$ and $I_t \underset{(49b)}{=} -m_{nn}/m_{nt} I_n$ (reverse sliding).

e. When $m_{nt} = 0$ ('balanced collision', [11, 15, 22]), then g, h, p, and q (35) are all positive, hence only the first case of Table 1 can occur (no reverse sliding is possible). Also, $2\Delta E/|\mathrm{v}_n^A| \underset{(65)}{=} -\{1 - e + \alpha^2 m_{nn}/[(1+e)m_{tt}]\} I_n < 0$ for sticking, hence in both sticking and forward sliding $\Delta E < 0$.

f. The first term in (29) is always negative (see (30)). However, it approaches zero as $e \to 1$, enhancing the possibility for $\Delta E > 0$.

g. g and/or q, and h, in collision types 2 and 3, respectively, are not allowed to vanish, or else I_n and/or I_t go to infinity (see (47), (49), and (52)). Equal signs can thus be added to the remaining inequalities in Table 1, columns 2–5, leaving the solutions unique. This can be shown to always be the case; for example, if $p=0$, then Case 1, Type 2 and Case 3, Type 4 lead to the same I_n, I_t, v_n^S and $v_t^S(=0)$, hence to the same solution.

h. A collision theory can be formulated without the bisection of the collision time, i.e., without the introduction of $\bar{t}$ (39), as in Appendix C. This theory underlies that used by Kane and Levinson in [31]. It can be shown, with the aid of a table similar to Table 1 that it offers unique and coherent, but energy inconsistent solutions.

5 Kane and Levinson's example [31, p. 348]

Figure 3 shows a double pendulum S consisting of uniform rods A and B, each of length l and mass m. Let q_1 and q_2 be the orientation angles of the rods, and let $u_i=\dot{q}_i$ $(i=1,2)$. Suppose that at time t_1 the endpoint $\bar{B}$ of B strikes H, a flat surface, and that at t_1 $q_1=20$, $q_2=30$ deg and $u_1=-0.1$, $u_2=-0.2$ rad/sec. It is required to evaluate the change in the kinetic energy of S following the collision, for $m=3$ kg and $l=2$ m. To this end, **n** and **t** are identified as $\mathbf{n}=-\mathbf{n}_1$ and $\mathbf{t}=-\mathbf{n}_2$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit vectors shown in Fig. 3. Next, the velocity of $\bar{B}$ at t_1, which is the velocity of approach, and the equation of motion of S are generated and cast as follows

$$\mathbf{v}^{\bar{B}}(t_1)=\mathbf{v}^A=-0.2684\mathbf{n}+0.5343\mathbf{t},$$
$$-1/3ml^2\left[4\dot{u}_1+3/2\cos(q_1-q_2)\dot{u}_2+3/2\sin(q_1-q_2)u_2^2\right]=0,$$
$$-1/3ml^2\left[3/2\cos(q_1-q_2)\dot{u}_1+\dot{u}_2-3/2\sin(q_1-q_2)u_1^2\right]=0.$$

Substitutions in (20), (24), and (34) yield $m_{nn}=0.3365$, $m_{tt}=0.8134$, $m_{nt}=-0.5071$, $\Delta=0.0166$, and $\alpha=1.9908$. The top rows 1–4 of Table 2 show cases with different values of e and μ corresponding to the four cases of [31], for which $\mu m_{tt}-m_{nt}>0$, $\mu m_{tt}+m_{nt}<0$ and $\Delta>0$; and if solved as in Appendix C (Kane and Levinson's solution), then inequalities (74) reduce to $\alpha<(1+e)r_m$ and $\alpha>-(1+e)r_p$. These together with inequalities (77) and (80), lead to the types of collision reported in column 7 of Table 2 and to the energy changes in column 11 in agreement with [31]. Referring next to the Routh-based method and to Table 1, and noting that in all cases $g>0$, $h>0$, $p<0$, $q>0$ (Table 1, case 3), one has reverse sliding for all cases, as noted in column 7, bottom rows 1–4, with a decrease of the kinetic energy. However, row 5 shows that a larger coefficient of restitution can lead

Fig. 3 Double pendulum

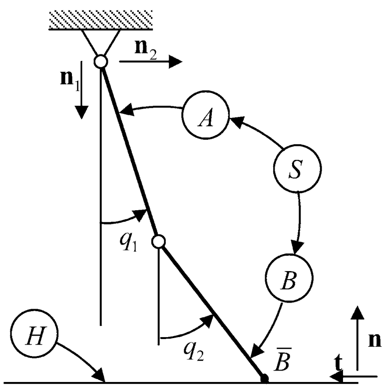


Table 2 The double pendulum collision problem: two solutions

	e	μ	r_m	$(1+e)r_m$	$-(1+e)r_p$	Type	Δu [r/s,r/s]	v_n^S [m/s]	v_t^S [m/s]	ΔE [J]
Newton's hypothesis; no bisection of the collision time, Appendix C										
1	0.5	0.2	1.529	2.294	2.198	2	[−0.15, 0.51]	0.13	−0.06	−0.033
2	0.5	0.5	1.549	2.323	1.815	1	[−0.23, 0.56]	0.13	0.00	0.163
3	0.3	0.5	1.549	2.013	1.573	1	[−0.10, 0.41]	0.08	0.00	−0.119
4	0.7	0.5	1.549	2.633	2.057	2	[−0.34, 0.69]	0.19	−0.018	0.489
5	0.9	0.5	1.549	2.943	2.299	–	[–, –]	–	–	–
Newton's hypothesis; with bisection of the collision time, Sect. 3										
1	0.5	0.2	1.529	2.294	2.198	2	[−0.12, 0.47]	0.13	−0.078	−0.095
2	0.5	0.5	1.549	2.323	1.815	2	[−0.13, 0.49]	0.13	−0.070	−0.073
3	0.3	0.5	1.549	2.013	1.573	2	[−0.09, 0.41]	0.08	−0.005	−0.131
4	0.7	0.5	1.549	2.633	2.057	2	[−0.17, 0.58]	0.19	−0.130	−0.002
5	0.9	0.5	1.549	2.943	2.299	2	[−0.22, 0.66]	0.24	−0.200	0.083

to $\Delta E > 0$ (see Comment f). It may be concluded that the bisection of the collision time (compare Sect. 3 with Appendix C) does not rectify energy-related inconsistencies associated with Newton's hypothesis.

6 Summary

A thorough investigation of a one-point planar collision with friction theory based on Newton's hypothesis, in conjunction with the Routh's semigraphical method and Coulomb's coefficient of friction, was conducted with the aid of explicit expressions for I_n, I_t, v_n^S and v_t^S written for three types of collision. It was shown that the algebraic signs of four parameters g, h, p, and q, and the range of α, determine the type of collision, as reported in Table 1, spanning five possible cases of simple, nonholonomic systems undergoing planar collisions. Table 1 also shows that the incorporation of Routh's method, does not prevent Newton's hypothesis from leading to a possible increase in the calculated mechanical energy of systems undergoing collisions if sticking or reverse sliding occur. This, in general, is an unacceptable proposition for simulations of motion of multibody systems undergoing collisions.

In Part B of this work, investigations of Poisson's and Stronge's collision hypotheses are conducted and their generality examined with the aid of a table similar to Table 1 with more encouraging results.

Appendix A

The entry m_{rs} in row r, column s of the $p \times p$ mass matrix associated with (1) is the coefficient of $\dot{u}_s$ in the rth equation, and is given by

$$m_{rs} \hat{=} -\sum_{i=1}^{\nu} m_i \mathbf{v}_r^{P_i} \cdot \mathbf{v}_s^{P_i} \quad (r,s=1,\dots,p). \tag{68}$$

This can be shown formally if $\mathbf{a}^{P_i}$, the acceleration of P_i in N, is expressed as

$$\mathbf{a}^{P_i} = \sum_{r=1}^{p} \mathbf{v}_r^{P_i} \dot{u}_r + \dot{\mathbf{v}}_r^{P_i} u_r + \dot{\mathbf{v}}_t^{P_i} \quad (i = 1, \ldots, \nu) \tag{69}$$

(see (2)). The inertia force associated with P_i equals $-m_i \mathbf{a}^{P_i}$, hence its contribution to the rth of (1) is given by $-m_i \mathbf{a}^{P_i} \cdot \mathbf{v}_r^{P_i}$. Summation of such contributions from all the particles comprise the rth generalized inertia force, namely, $F_r^* = \sum_{i=1}^{\nu} -m_i \mathbf{a}^{P_i} \cdot \mathbf{v}_r^{P_i}$. Substitutions from (69) for $i = 1, \ldots, \nu$ make it possible to verify that coefficient of $\dot{u}_s$ is that given by (68). Note that $m_{rs} = m_{sr}$, hence that the mass matrix is symmetric. Next, the kinetic energy of a simple nonholonomic system defined in Sect. 1 is given by

$$E = 1/2 \sum_{i=1}^{\nu} m_i \left[\sum_{r=1}^{p} \mathbf{v}_r^{P_i} u_r + \mathbf{v}_t^{P_i} \right] \left[\sum_{s=1}^{p} \mathbf{v}_s^{P_i} u_s + \mathbf{v}_t^{P_i} \right] \tag{70}$$

or when use is made of m_{rs} defined in (68),

$$E = -1/2 \sum_{r=1}^{p} \sum_{s=1}^{p} m_{rs} u_r u_s + \sum_{i=1}^{\nu} m_i \mathbf{v}_t^{P_i} \mathbf{v}^{P_i} - 1/2 \sum_{i=1}^{\nu} m_i (\mathbf{v}_t^{P_i})^2. \tag{71}$$

The loss of mechanical energy during collision is $\Delta E \mathrel{\hat{=}} E(t_2) - E(t_1)$, and since $\mathbf{v}_t^{P_i}(t_2) \mathrel{\hat{=}} \mathbf{v}_t^{P_i}(t_1)$ $(i = 1, \ldots, \nu)$, then

$$\begin{aligned} \Delta E = {} & -1/2 \sum_{r=1}^{p} \sum_{s=1}^{p} m_{rs} u_r(t_2) u_s(t_2) + 1/2 \sum_{r=1}^{p} \sum_{s=1}^{p} m_{rs} u_r(t_1) u_s(t_1) \\ & + \sum_{i=1}^{\nu} m_i \mathbf{v}_t^{P_i} \cdot [\mathbf{v}^{P_i}(t_2) - \mathbf{v}^{P_i}(t_1)]. \end{aligned} \tag{72}$$

Appendix B

Consider a particle P dropped on a disk which is made to rotate with a constant angular speed Ω about its vertically-posed axis, hitting a point P' of the disk a distance r from the indicated axis. Then in accordance with the definitions of $\mathbf{n}$ and $\mathbf{t}$, the former is vertical and points upward, and the latter is tangent to the path of P' at t_1. Thus, if $u_1 \mathrel{\hat{=}} \mathbf{v}^P \cdot \mathbf{n}$, then $\mathbf{v}^P(t_1) = u_1(t_1)\mathbf{n}$. Also $\mathbf{v}^{P'}(t_1) = \mathbf{v}_t^{P'} = \Omega r \mathbf{t}$. Next, let $e = 1$, so that in accordance with (30), $\mathbf{v}^P(t_2) \cdot \mathbf{n} = -\mathbf{v}^P(t_1) \cdot \mathbf{n}$ (or $u_1(t_2) = -u_1(t_1)$). Finally, assume that due to the rough surface of the disk $\mathbf{v}^P(t_2) = u_1(t_2)\mathbf{n} + \mathrm{v_T}\mathbf{t}$, i.e., the velocity of P is imparted with a tangential component $\mathrm{v_T}\mathbf{t}$. Clearly, the mechanical energy of the system increases although the mechanical energy of the disk remains constant since its angular speed is prescribed. This example shows that if $\mathbf{v}_t^{P_i} \neq 0$, then an increase in the mechanical energy of the system following a collision does not necessarily indicate a flaw in the collision theory.

Appendix C

Expressions for I_n, I_t, v_n^S and v_t^S can be obtained for the three types of collision without the bisection of the collision time as follows.

Type 1 *Sticking*. Type 1 is characterized by $\mathrm{v}_t^S = 0$. By (22), (23), (30), and (32)

$$I_n = |\mathrm{v}_n^A|[(1+e)m_{tt} + \alpha m_{nt}]/\Delta, \qquad I_t = -|\mathrm{v}_n^A|[\alpha m_{nn} + (1+e)m_{nt}]/\Delta. \tag{73}$$

Sticking prevails if inequality (31) is satisfied or with the substitution of I_n and I_t from (73) in inequality (31)

$$-\mu[(1+e)m_{tt} + \alpha m_{nt}]/\Delta < -[\alpha m_{nn} + (1+e)m_{nt}]/\Delta < \mu[(1+e)m_{tt} + \alpha m_{nt}]/\Delta,$$

which becomes if use is made of r_m and r_p defined in (36),

$$\frac{m_{nn} - \mu m_{nt}}{\Delta}[\alpha - (1+e)r_m] < 0, \qquad \frac{m_{nn} + \mu m_{nt}}{\Delta}[\alpha + (1+e)r_p] > 0. \tag{74}$$

Type 2 *Reverse sliding*. Type 2 is characterized by $\mathrm{v}_t^S < 0$ and $I_t \underset{(33)}{=} \mu I_n$. Equations (22), (23), (30), and (33) yield

$$I_n = (1+e)|\mathrm{v}_n^A|/(m_{nn} + \mu m_{nt}), \qquad I_t = \mu I_n, \tag{75}$$

$$\mathrm{v}_t^S = [\alpha + (1+e)r_p]|\mathrm{v}_n^A|. \tag{76}$$

Reverse sliding prevails if

$$\alpha < -(1+e)r_p. \tag{77}$$

Type 3 *Forward sliding*. Type 3 is characterized by $\mathrm{v}_t^S > 0$ and $I_t \underset{(33)}{=} -\mu I_n$. Equations (22), (23), (30), and (33) yield

$$I_n = (1+e)|\mathrm{v}_n^A|/(m_{nn} - \mu m_{nt}), \qquad I_t = -\mu I_n, \tag{78}$$

$$\mathrm{v}_t^S = [\alpha - (1+e)r_m]|\mathrm{v}_n^A|. \tag{79}$$

Forward sliding prevails if

$$\alpha > (1+e)r_m. \tag{80}$$

Equations (73), (78), and (79) are identical with (47), (52), and (53), respectively. However, (75) and (76) differ from (49) and (50). Here, the collision time is not bisected and, therefore, inequalities (59)–(64) cannot be used. A table similar to Table 1 can be built in conjunction with inequalities (74), (77) and (80), however. One then obtains for Cases 1–2 and 4–5 α—regions identical with those of Table 1. In Case 3, one has, by inequalities (74), (77) and (80), $(1+e)r_m > \alpha > -(1+e)r_p$, $\alpha < -(1+e)r_p$ and $\alpha > (1+e)r_m$ for sticking, reverse sliding and forward sliding, respectively $(-r_p < r_m)$. Hence solutions are always coherent and unique.

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