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The analysis and simulation for three-dimensional impact with friction

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Abstract This paper presents a full discussion on how to formulate and evaluate the problem of spatial impact with friction in multibody systems. By considering that impact is experiencing a very short but finite time, we have established the differential equations of motion with normal impulse as a 'time-like' independent differential variable to describe the process of impact. A reliable numerical formulation, which can deal with the complex motions appearing in impact such as slip, stick and resumption of slip, has been developed by using difference schemes. The results indicate that the formulation and the numerical method presented in this paper can well solve the problems of 3D impact with friction by a comparison with the theoretical results existing in literature.

Keywords Numerical method · 3D impact · Friction · Impulse · Differential equation

1 Introduction

Principally, there are three kinds of theories in solving the problems of contact/impact [1, 6]: the first, classical stereo-mechanics on the basis of rigid body mechanics developed by Huyghens and Newton; the second, wave theory of impact emerged from the work of St. Venant theory; and the third, finite element method by discretizing the contact area.

The first method can find its application in solving the problems of collision between two hard bodies [1, 2, 5], the multiple simultaneous impacts in rigid multibody systems [3, 4], and the impact occurring in flexible multibody systems when the deformation of colliding

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bodies is small [9]. The second method, which mainly focuses on the vibration of the structure induced by the collision [7, 8], takes the impact as a disturbance propagating away from the impact site, and help us to discover the phenomena of multiple impacts due to the reflection of wave [8]. FE method [10] provides an effective tool to obtain the elaborate information of the contact forces and deformation in contact region.

Both wave theory and FE method often need large computation resources and find difficulties in analyzing the problems of impact in complex multibody systems. Although the method of stereo-mechanics provides a simple structure for the description of impact, some singular phenomena may appear in dynamical equations due to the oversimplified assumption by using the coefficient of restitution, especially in dealing with the problems of frictional impact. Obviously, each method has its own advantages, and with more or less deficiency. Which one is used depends on the accuracy of numerical results and the efficiency of computation we anticipate. The purpose of this paper is to establish an effective formulation and reliable numerical method in the framework of rigid body dynamics to deal with the problem of 3D frictional impact.

Dry friction between colliding bodies, which is often represented by the Amontons–Coulomb law, can make the tangential relative velocity at contact point change its direction continuously or vanish instantaneously [2, 11]. If such changes appearing in the tangential motion is ignored, some singularities will appear in the solution of impulsive dynamics. For example, by adopting the Newton's coefficient, Kane [12] discovered an inconsistency phenomenon of the kinetic energy of system when he analyzed the problem of a double pendulums colliding against a rough plane, and called it 'a puzzle of dynamics'. Such an inconsistent phenomenon is attributed to the action of friction which makes the normal impulse couple with the tangential impulse during impact [2].

For avoiding inconsistent energy appearing, much effort has been made by many authors through using different methods [2, 29]. Some authors try to develop a new definition of coefficient of restitution for describing the process of impact [22, 24]. Chatterjee and Ruina [15] developed a new impact law for three-dimensional rigid-body impact by averaging the total impact impulse. Smith [16] suggested that the average weighted speeds before and after impact should be adopted in order to consider the influences of friction. Brach presented a definition of tangential coefficient of restitution [17] and further provided a generalized coefficient [28] to analyze the three-dimensional frictional collision. Based on the viewpoint of energy, Stronge [2, 27] presented a energetic coefficient for considering the dissipation of energy due to friction. Concerning the problem of a rigid ball colliding against a rough plate with an incident angle [22–26], the Newton's coefficient of restitution is modified by a function relating to the relative velocity at contact point and the material properties of colliding bodies through using experimental and numerical methods. All these methods have found their applications in dealing with some special problems of frictional impact. However, we still need to establish a general formulation to solve 3D impact with friction in multibody systems as these methods mentioned above often aim at the collision of a free body and ignore the influences of the geometrical configuration and initial condition of colliding bodies.

The key point to solve the problems of impact with friction is how to consider the changes of tangential motion due to the action of friction. Routh [21] is the first one to present a semi-graphical method to solve the problem of the impact in plane. In his method, the change of tangential motion at contact point can be determined by the state lines expressed in the plane of normal and tangential impulse. Wang and Mason [11] fully analyzed the phenomenon of inconsistent kinetic energy by applying Routh's graphical technique. However, Routh's method can only deal with the situation of planar collision where the direction of tangential motion is along a straight line.

For 3D frictional impact, friction will result in the direction of tangential velocity continuously changes. In order to explore the variation of tangential motion due to friction, the best way of solving such problems is to discard the instantaneous assumption of impulsive dynamics. Keller [13] and then Stronge [14] developed analytical methods, which takes the normal impulse increased monotonically during impact as a 'time-like' independent differential variable to formulate the dynamical equations of motion, to deal with the problems of 3D frictional impact between two bodies. Both the methods consider that impact is finished in a very short but not infinitesimal time, but the Poisson's coefficient of restitution is used in Keller's method, while the energetic coefficient of restitution is applied in Stronge's method for the calculation of the terminal normal impulse at final separation. Similar methods have been found in the research works of Bhatt [18, 19] for studying the global properties of 3D frictional impact, and the one of Battle [12, 20] for discussing the influence of the value of friction coefficient on the behavior of rough balanced impact where there is no coupling between tangential and normal impulse.

This paper is intended to establish a general theory for dealing with the problem of frictional impact in multibody systems, and then to develop a numerical method for simulating the phenomena of impact existing in many applications of engineering. By abandoning the assumption of impact being instantaneous, we will first formulate the differential equations relating to the motions of slip, stick and the resumption of slip in tangential direction appearing in impact. The formulation of discretization is further established by using difference schemes. Meanwhile, two problems in the implementation of the algorithm will be focused on: the first one is how to determine the duration of integral of the differential equations since normal impulse is taken as the independent differential variable; the second one is how to determine the direction of the tangential velocity when slip resumes. The duration of integral will be determined by using Poisson's coefficient of restitution, while the resuming direction of tangential motion will be determined by a searching algorithm according to the properties of singularities presented by Bhatt [18, 19].

The organization of this paper is as follows: we first establish the governing equations for the spatial impact in multibody systems. Then, a numerical method is developed by using difference schemes and the pseudocodes of the algorithm are presented. Finally, the theoretical analysis of three examples appearing in [2, 20] are briefly introduced, and is used to illustrate the reliability of our numerical method. Summary and conclusions are given in the end of this paper.

2 Governing equations of the impact in multibody systems

Figure 1 shows two multi-rigid-body systems with generalized coordinates $(q_1, q_2, \ldots, q_n)$ colliding with each other at contact point O. It is convenient to set the inertial Cartesian coordinate frame $(\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3)$ with $\mathbf{i}_3$ as normal direction of the tangential plane at contact point O. The coordinates for the contact points O_1 in Body-1 and O_2 in Body-2 can be expressed by generalized coordinates

$$x_i = x_i(q_1, q_2, \ldots, q_n), \qquad x_i{}' = x_i{}'(q_1, q_2, \ldots, q_n), \quad i = 1, 2, 3. \tag{1}$$

In terms of Euler–Lagrange equations, the dynamic equations of the system can be written by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial q_i} - \frac{\partial x_j{}'}{\partial q_i}\right)F_j. \tag{2}$$

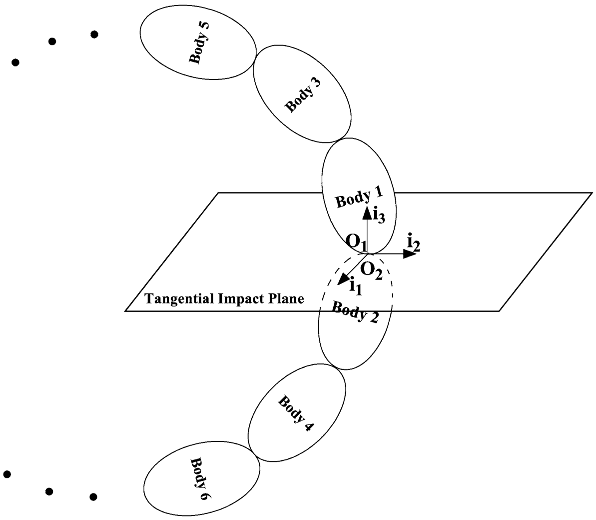


Fig. 1 Multi-rigid-body system impacts

T is kinetic energy of the system, Q_i $(i = 1, 2, \ldots, n)$ are the generalized forces acting on the system, and F_i $(i = 1, 2, 3)$ are the contact forces at contact point.

Supposing that the constraints of the system are time-invariant and holonomic, we can express the kinetic energy of the system as a formulation of quadratic form.

$$T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{\mathbf{q}}^{\mathrm{T}} \mathbf{M} \dot{\mathbf{q}}. \tag{3}$$

Here $\mathbf{M}$ is the $n \times n$ symmetric matrix of mass, $\dot{\mathbf{q}}$ is $n \times 1$ generalized velocity vector, and a_{ij} is the element of $\mathbf{M}$.

There are three assumptions in impulsive dynamics:

the configuration of the system hardly changes during collisions; the generalized impulse contributed by the finite forces is ignored; the impact is finished instantaneously with infinitesimal time;

For considering the changes of tangential motion due to friction, we think impact finishes in a period of $[t_1, t_2]$, but still preserve the first two assumptions. Then integrating equations (2) by taking a more small period $[\xi_1, \xi_2] \in [t_1, t_2]$, we can get the impulsive equations of motion during $[\xi_1, \xi_2]$.

$$\sum_{j=1}^{n} a_{ij} \big(\dot{q}_j(\xi_2) - \dot{q}_j(\xi_1) \big) = \sum_{j=1}^{3} \left(\frac{\partial x_j}{\partial q_i} - \frac{\partial x_j{}'}{\partial q_i} \right) \Delta P_j, \quad i = 1, 2, \ldots, n. \tag{4}$$

Where the contribution of the finite generalized forces are ignored, and $\Delta P_j = \int_{\xi_1}^{\xi_2} F_j \, dt$ $(j = 1, 2, 3)$ are the increment of impulse at contact point during the very small period of $[\xi_1, \xi_2]$.

Let $dP_j \equiv \Delta P_j$ and $d\dot{q}_i \equiv \dot{q}_i(\xi_2) - \dot{q}_i(\xi_1)$. The matrix notation of (4) can be rewritten as

$$\mathbf{M} \, d\dot{\mathbf{q}} = \mathbf{J} \, d\mathbf{P}. \tag{5}$$

$\mathbf{J}_{\mathbf{n} \times \mathbf{3}}$ is the Jacobian matrix of the transformation equations (1) from the coordinates of the contact point to the generalized coordinates with elements $J_{ij} = (\partial x_j / \partial q_i - \partial x_j{}' / \partial q_i)$.

We can decompose the increment of impulse into two parts: $d\mathbf{P} = [d\mathbf{P}_t^T, dP_3]^T$, where $d\mathbf{P_t} = [dP_1, dP_2]^T$ is increment of the tangential impulse, while dP_3 represents the one of normal impulse.

Similarly, Jacobian matrix $\mathbf{J}$ can also be decomposed into $\mathbf{J} = [\mathbf{K}, \mathbf{h}]$, where $\mathbf{K}$ is a $n \times 2$ matrix, and $\mathbf{h}$ is a $n \times 1$ matrix. The elements of matrix $\mathbf{K}$ and $\mathbf{h}$ are expressed as

$$K_{ij} = \frac{\partial x_j}{\partial q_i} - \frac{\partial x_j{}'}{\partial q_i}, \qquad h_i = \frac{\partial x_3}{\partial q_i} - \frac{\partial x_3{}'}{\partial q_i} \quad (i = 1, 2, \dots, n,\ j = 1, 2).$$

Then, we can rewrite (5) as follows

$$\frac{d\dot{\mathbf{q}}}{dP_3} = \mathbf{M}^{-1}\left(\mathbf{K}\frac{d\mathbf{P_t}}{dP_3} + \mathbf{h}\right). \tag{6}$$

Considering the fact that normal force at contact point is always compressive, and the normal impulse increases monotonously during impact, we can take normal impulse as a 'time-like' variable (this idea is first proposed by Keller [13] and then adopted by Stronge [14]). So (6) can be considered as differential equations with respect to the normal impulse P_3.

Since the law of friction is related to the motion of contact point, we should establish the relationship between the increments of generalized velocities and the speed at contact point. By differentiating (1), we have

$$v_i = \sum_{j=1}^{n}\left(\frac{\partial x_i}{\partial q_j} - \frac{\partial x_i{}'}{\partial q_j}\right)\dot{q}_j, \quad \text{where } (i = 1, 2, 3). \tag{7}$$

In terms of the assumption of impulsive dynamics that the configuration of the system are unchanged, the increment of the velocity at contact point in $[\xi_1, \xi_2]$ can be expressed as

$$dv_i = v_i(\xi_2) - v_i(\xi_1) = \sum_{j=1}^{n}\left(\frac{\partial x_i}{\partial q_j} - \frac{\partial x_i{}'}{\partial q_j}\right)\Delta\dot{q}_j \quad (i = 1, 2, 3). \tag{8}$$

The matrix notation for (8) is rewritten by

$$\begin{bmatrix} d\mathbf{v}_t \\ dv_3 \end{bmatrix} = \begin{bmatrix} \mathbf{K}^T \\ \mathbf{h}^T \end{bmatrix} d\dot{\mathbf{q}}. \tag{9}$$

Taking (6) into (9) leads to

$$\begin{cases} \dfrac{d\mathbf{v_t}}{dP_3} = \mathbf{K^T M^{-1} K}\dfrac{d\mathbf{P_t}}{dP_3} + \mathbf{K^T M^{-1} h}, \\[2ex] \dfrac{dv_3}{dP_3} = \mathbf{h^T M^{-1} K}\dfrac{d\mathbf{P_t}}{dP_3} + \mathbf{h^T M^{-1} h}. \end{cases} \tag{10}$$

Where $d\mathbf{v_t} = [dv_1, dv_2]^T$ is the relative velocity in tangential direction at contact point, and dv_3 represents the relative velocity in normal direction.

Since $d\mathbf{P}_t$ will be related with dP_3 based on the friction law and be influenced by the state of the tangential movement at contact point, some complementary equations should be provided in order to solve the differential equations of (10). In the following, we will present the governing differential equations for three cases of motion which may appear during collision: contact point slips on the rough plane; contact point sticks at the rough plane; and slip resumes after the tangential velocity vanishes simultaneously.

2.1 The case of slip at contact point

For expressing the relationship between $d\mathbf{P}_t$ and dP_3 in the case that the contact point slips on the rough plane, we adopt the Coulomb's frictional law to represent the action of friction force. Then, the changes of the tangential impulses with the increment of the normal impulse can be expressed as

$$\frac{d\mathbf{P_t}}{dP_3} = -\mu\frac{\mathbf{v_t}}{|\mathbf{v_t}|}, \quad |\mathbf{v_t}| \neq \mathbf{0}. \tag{11}$$

Where $|\mathbf{v_t}|$ represents the value of the relative velocity in tangential direction, and μ is the coefficient of friction. Substituting (11) into (10) leads to

$$\frac{d\mathbf{v_t}}{dP_3} = \mathbf{K^T M^{-1}}\left(-\mu\mathbf{K}\frac{\mathbf{v_t}}{|\mathbf{v_t}|} + \mathbf{h}\right), \tag{12}$$

$$\frac{dv_3}{dP_3} = \mathbf{h^T M^{-1}}\left(-\mu\mathbf{K}\frac{\mathbf{v_t}}{|\mathbf{v_t}|} + \mathbf{h}\right). \tag{13}$$

Equations (12) and (13) represent the ordinary differential equations of the velocity at contact point with normal impulse. We should notice that the direction of tangential velocity appears in the right side of the equations. In generally speaking, such differential equations are not integral unless the direction of tangential velocity can be determined in advance. In the next section, we will present the numerical formulation of such differential equations based on difference schemes.

2.2 The case of stick appearing at contact point

If the value of coefficient of friction is big enough, the tangential movement may disappear during impact, i.e, stick occurs. In such case, the relationship between the increments of tangential and normal impulse will have the following inequality.

$$\mathbf{v_t} = \mathbf{0}, \qquad \frac{|d\mathbf{P_t}|}{dP_3} \leq \mu. \tag{14}$$

Once stick occurs at contact point, the acceleration in tangential direction should be equal to zero, i.e., $d\dot{\mathbf{v}}_t = 0$. So we can get the governing differential equations for stick based on the first equation in (10).

$$\frac{d\mathbf{P_t}}{dP_3} = -\left(\mathbf{K^T M^{-1} K}\right)^{-1}\mathbf{K^T M^{-1} h}. \tag{15}$$

Taking (15) into the second equation of (10), then

$$\frac{dv_3}{dP_3} = -\mathbf{h^T M^{-1} K}\left(\mathbf{K^T M^{-1} K}\right)^{-1}\mathbf{K^T M^{-1} h} + \mathbf{h^T M^{-1} h}. \tag{16}$$

Equation (16) is the differential equation of normal movement once stick appears in tangential direction. Reminding of the assumption of impulsive dynamics that the configuration of the system is not changed during impact, the matrices of **h**, **M** and **K** are constant matrix. Therefore, the ratio between the increments of tangential and normal impulse is also a constant according to (15). Such ratio can not exceed the value of coefficient of friction based on the condition of stick expressed in (14). Therefore, slip can not resume once stick appears, i.e., the separating velocity in tangential direction after impact will equal zero. Meanwhile, the normal velocity of contact point during the process of stick will change linearly with the increase of normal impulse until impact finishes.

2.3 The case of resumption of slip

The tangential velocity may disappear but immediately resume along a new direction during impact. Although this case also has the instant that tangential velocity equals zero during impact, it is not related to a stick as slip can not resume once stick occurs. Therefore, the governing equations for such case should be the differential equations for slip (see (12) and (13)) but not the one for stick. For distinguishing it from the case of stick, we would provide the following condition for identifying the occurrence of slip resuming.

Denoting P_3^* is the normal impulse corresponding to the instant that tangential velocity equals zero, we can first suppose stick occurs, and by using (15) to calculate the increment of tangential impulse $d\mathbf{P}_t$. If the ratio of the increments of tangential and normal impulse satisfy the following inequality

$$(\mathbf{v_t})_{P_3=P_3^*} = \mathbf{0}, \qquad \left(\frac{|d\mathbf{P_t}|}{dP_3}\right)_{P_3=P_3^*} > \mu \tag{17}$$

slip will resume after tangential velocity disappears instantaneously. Since friction force is related to the direction of tangential motion, the difficulty for such case in simulation is how to determine the direction of subsequent motion after tangential velocity disappears. By analyzing the properties of differential equations for slip at the point of tangential velocity equaling zero, Bhatt [18, 19] proved the new direction of slip resuming will be along a stable invariant direction.

In fact, the uniqueness of the direction of resumption can be explained as follows: although the tangential velocity at contact point equals zero, which makes slip vanish simultaneously, the acceleration in tangential direction is not equal to zero in this case, i.e., $d\dot{\mathbf{v}}_t \neq 0$. Therefore, the direction of tangential velocity starting from the moment of slip vanishing should be just align with the direction of tangential acceleration at this instant. Since such direction of tangential acceleration at contact point is unique but not known in advance, we have to provide a searching arithmetic to find such new direction of slip resuming. This will be illustrated in the following section.

3 Numerical method based on normal impulse discretization

3.1 Introduction of the numerical method

Developing a reliable numerical integration algorithm is significant for solving 3D-frictional impact in complex multibody systems as its analytical solutions can not be easily obtained due to the changes of tangential motion. In recent years, the revival of interest in the development of the so-called time-stepping methods for non-smooth dynamics has been triggered in part by a concern with important development of LCP theories of contact/impact problem in multibody systems. Time-stepping method uses the formulation of difference schemes including fully the complementarity conditions and impact laws to deal with the problem of vary-structure systems, which may contain impulsive and non-impulsive forces together with all inequalities involved. A good review for such methods and its application in a woodpecker toy can be found in [4] by Glocker.

Similar to the time-stepping method, we adopt the Euler's difference scheme to discretize the differential equations of motion. But it is different from the time-stepping methods which often use the optimal theories, such as convex optimization, to solve the multiple contact/impact problems in multibody systems, our numerical method presented here will directly solve the equations relating to different states of motion according to the condition of inequalities.

Since the normal impulse is the independent variable for the differential equations of motion, we will discretize the differential equations and inequalities involved in impact with respect to normal impulse but not to time. So we can call the process of discretization as 'Normal impulse-stepping' method. Assuming ΔP_3^l represents the increment of normal impulse at the step of number l. If the tangential velocity at contact point $|\mathbf{v_t}^l| \neq \mathbf{0}$, this means that contact point slides on the contact plane. By using (12) and (13), we can get the formulations of dicretization for slip.

$$\mathbf{v_t}^{l+1} = \mathbf{v_t}^l + \mathbf{K^T M^{-1}}\left(-\mu\mathbf{K}\frac{\mathbf{v_t}^l}{|\mathbf{v_t}^l|} + \mathbf{h}\right)\cdot \Delta P_3^l, \tag{18}$$

$$v_3^{l+1} = v_3^l + \mathbf{h^T M^{-1}}\left(-\mu\mathbf{K}\frac{\mathbf{v_t}^l}{|\mathbf{v_t}^l|} + \mathbf{h}\right)\cdot \Delta P_3^l. \tag{19}$$

When slip exists at contact point, the increments of tangential and normal impulse at this numerical step will comply with the law of friction defined by (11). Then, the tangential impulses at step $l+1$ can be calculated by

$$\mathbf{P_t}^{l+1} = \mathbf{P_t}^l - \mu\frac{\mathbf{v_t}^l}{|\mathbf{v_t}^l|}\Delta P_3{}^l. \tag{20}$$

If the tangential speed $|\mathbf{v_t}| = 0$ at the numerical step l, two possibilities of the movement in tangential direction may occur at the next step: one that stick appear in tangential direction; the other that slip will resume immediately along a new direction. Which one appearing depends on the ratio of the tangential impulse to the normal impulse at such instant.

We can first assume stick will occur at $l+1$ step. By discretizing (15) and (16) with respect to normal impulse, we can get the difference equations for stick.

$$\mathbf{P_t}^{l+1} = \mathbf{P_t}^l - \left(\mathbf{K^T M^{-1} K}\right)^{-1}\mathbf{K^T M^{-1} h}\cdot \Delta P_3{}^l, \tag{21}$$

$$v_3^{l+1} = v_3^l + \left(-\mathbf{h^T M^{-1} K}\left(\mathbf{K^T M^{-1} K}\right)^{-1}\mathbf{K^T M^{-1} h} + \mathbf{h^T M^{-1} h}\right)\cdot \Delta P_3{}^l. \tag{22}$$

If the increment of tangential impulse satisfies with the following inequality,

$$\left|\mathbf{P_t}^{l+1} - \mathbf{P_t}^l\right| \le \mu \cdot \Delta P_3^l.$$

This means that stick is truly appearing when the tangential velocity equals zero. Based on the analysis in above section, the final tangential velocity at separation will also equal zero since the tangential motion can not get rid of the constraint of the cone of friction. However, if

$$\left|\mathbf{P_t}^{l+1} - \mathbf{P_t}^l\right| > \mu \cdot \Delta P_3^l.$$

This illustrates the assumption of stick appearing is wrong. In this case, the tangential motion will not stick at the contact point but be along a new direction in the subsequent movement.

Thus, we should adopt (21) and (22) to obtain the solutions of motion. Considering the new direction of slip is unique, we can establish an arithmetic to search for such new direction.

Assuming σ is the unit vector of such direction, and by (18) together with $|\mathbf{v_t}^l| = 0$, σ should satisfy the following difference equations.

$$\left|\mathbf{v_t}^{l+1}\right|\sigma = \left(-\mu\mathbf{K^T M^{-1} K}\sigma + \mathbf{K^T M^{-1} h}\right)\Delta P_3{}^l. \tag{23}$$

Selecting an arbitrary unit vector on contact plane and scanning it from 0 to 2π, we can find such unique direction σ which makes (23) come into existence. This direction is just the one of slip resuming.

3.2 The integrating duration of the differential equations

The other issue relating to the numerical algorithm is how to determine the integrating duration expressed in the value of normal impulse. This can be provided by Poisson's law of impact which divided the process of impact into compressional and restitution phases in normal direction. First, the value of normal impulse during compressional period can be calculated by summing the value of normal impulse from start of impact to the instant that the normal velocity at contact point vanishes. Supposing N is the number of step for $v_3^N = 0$, we can get the normal impulse during compression by

$$P_3{}^c = \sum_{i=1}^{N} \Delta P_3{}^i. \tag{24}$$

Then, the value of normal impulse during impact is

$$P_3 = (1 + e_P) P_3{}^c. \tag{25}$$

e_P is the Poisson's coefficient of restitution. Once P_3 is calculated, the total impulse $P = [P_1\ \ P_2\ \ P_3]^T$ at contact point can be obtained by differential equations of motion. Furthermore, the jump of generalized velocities and the relative velocity when impact finishes can be calculated by

$$\Delta\dot{\mathbf{q}} = \mathbf{M}^{-1}\begin{bmatrix}\mathbf{K}\\ \mathbf{h}\end{bmatrix}\mathbf{P^T}, \tag{26}$$

$$\Delta\mathbf{v} = \begin{bmatrix}\mathbf{K^T} & \mathbf{h^T}\end{bmatrix}\mathbf{M}^{-1}\begin{bmatrix}\mathbf{K}\\ \mathbf{h}\end{bmatrix}\mathbf{P}. \tag{27}$$

3.3 Pseudocode of the algorithm

The main steps of the proposed numerical method are summarized in the follows.

Initial Input: initial configuration, μ, e_P.
Calculated **K**, **M**, **h** via the initial configuration.
Data: relative speed $\mathbf{v_t}^l$, $v_3{}^l$, and the impulse $\mathbf{P_t}^l$, $P_t{}^l$ at current step l

MainProgramme
(1)**if** $v_3{}^l < 0$ (in the compression phase)
Execute **SubProgramme**
else

(2)**if** $v_3{}^l = 0$ (the normal compression phase is finished)
Calculated $P_3{}^c$, via. (24)
Calculated P_3, via. (25) (the integrating duration)
else
(3)**if** $\sum_{i=1}^{l} \Delta P_3^i \le P_3$
Execute **SubProgramme**
else
Impact comes to end.
endif(3)
endif(2)
endif(1)

SubProgramme
(1)**if** $|\mathbf{v_t}{}^l| \neq \mathbf{0}$ (sliding at the contact point)
Calculate $\mathbf{v_t}{}^{l+1}$, $v_3{}^{l+1}$, $P_t{}^{l+1}$, $P_3{}^{l+1}$ at step $(l+1)$, via. (18), (19), (20).
else $|\mathbf{v_t}{}^l| = \mathbf{0}$, assume stick holds
Calculate $\mathbf{v_t}{}^{l+1}$, $v_3{}^{l+1}$, $P_t{}^{l+1}$, $P_3{}^{l+1}$, via. (21), (22).
(2)**if** $|\mathbf{P_t}{}^{l+1} - \mathbf{P_t}{}^l| > \mu(P_3{}^{l+1} - P_3{}^l)$(the slip resumes)
Search for the new slip direction σ via. (23)
Recalculate $\mathbf{v_t}{}^{l+1}$, $v_3{}^{l+1}$, ,$P_t{}^{l+1}$ and $P_3{}^{l+1}$.
endif(2)
endif(1)
Calculated the post-state after impact via. (26), (27).
stop.

The bisection method is used in searching for the critical points such as $|\mathbf{v_t}{}^l| = 0$ and $v_3^N = 0$ in order to guarantee the accuracy of the numerical scheme.

4 Application examples

4.1 A homogeneous ellipsoid collision with a fixed surface

The example of a homogeneous ellipsoid collides with a fixed surface is shown in Fig. 2, which is used by Battle [16] to analyze the case of collinear collision with friction. Since there is no coupling between tangential and normal impulse, the analytical solution for such example can be obtained.

The collision point of the ellipsoid lies to the equator and coincides with Pole axes 3. the normal impulse P_3 at contact point just passes through the center of the mass for such ellipsoid. Supposing the generalized velocity vector is

$$\dot{\mathbf{q}}^T = [\,\dot{x} \quad \dot{y} \quad \dot{z} \quad \Omega_1 \quad \Omega_2 \quad \Omega_3\,].$$

Here $(\dot{x}, \dot{y}, \dot{z})$ and $(\Omega_1, \Omega_2, \Omega_3)$ are the velocity of mass center and the angular velocity in the fixed reference coordinates, respectively. We can express the configuration matrices for such the system as

$$\mathbf{K}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & -r & 0 \\ 0 & 1 & 0 & r & 0 & 0 \end{bmatrix}, \qquad \mathbf{h}^T = [\,0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0\,].$$

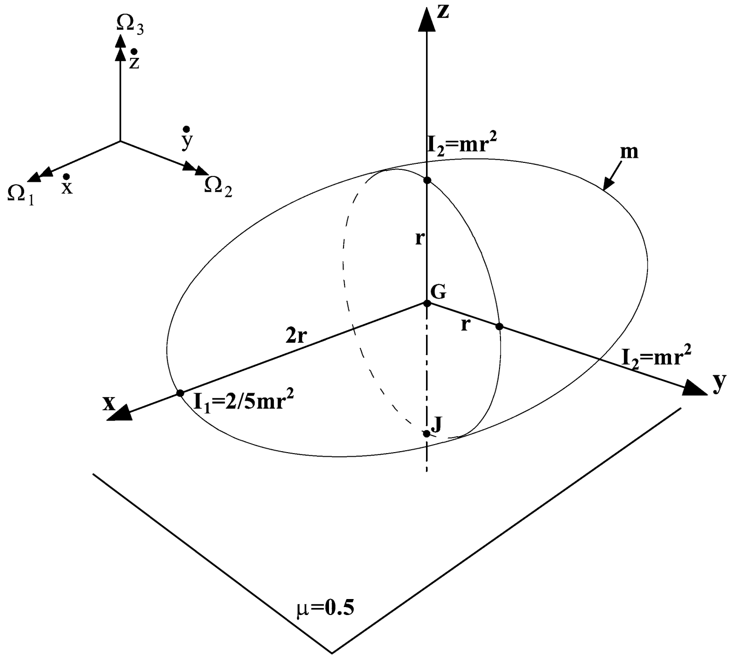


Fig. 2 Homogeneous ellipsoid collision with a fixed surface

The mass matrix is

$$\mathbf{M}=\frac{1}{m}\begin{bmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&\frac{2}{5}r^2&&\\&&&&r^2&\\&&&&&r^2\end{bmatrix}.$$

Then,

$$\mathbf{K}^T\mathbf{M}^{-1}\mathbf{K}=\frac{1}{m}\begin{bmatrix}2&0\\0&\frac{7}{2}\end{bmatrix},\qquad \mathbf{h}^T\mathbf{M}^{-1}\mathbf{h}=\frac{1}{m}\quad\text{and}\quad \mathbf{K}^T\mathbf{M}^{-1}\mathbf{h}=\begin{bmatrix}0\\0\end{bmatrix}. \tag{28}$$

Substituting (28) into (10), we have

$$\begin{bmatrix}dv_1\\dv_2\end{bmatrix}=\frac{1}{m}\begin{bmatrix}2&0\\0&\frac{7}{2}\end{bmatrix}\begin{bmatrix}dP_1\\dP_2\end{bmatrix}, \tag{29}$$

$$dv_3=\frac{1}{m}dP_3, \tag{30}$$

v_1, v_2 and v_3 are the components of the velocity at contact point. Obviously, there is no coupling between tangential and normal impulse in the equations of the motion for such system. In terms of law of friction, the relationship between the increments of tangential

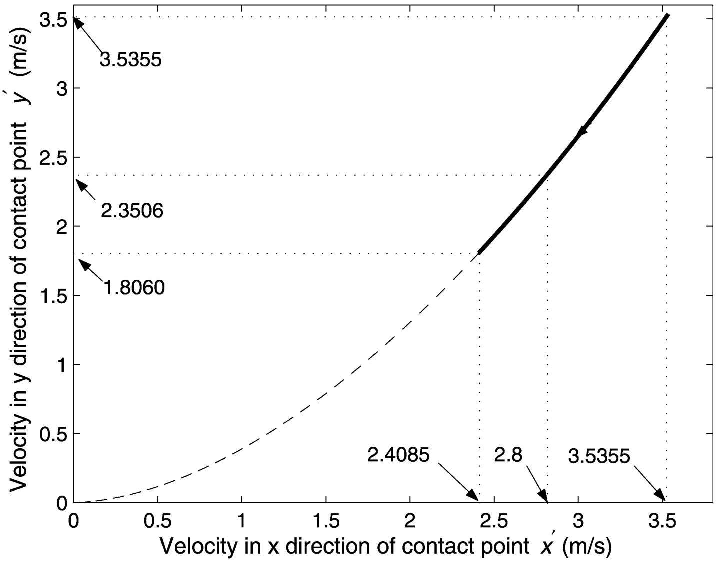


Fig. 3 Hodogragh of the sliding velocity in tangential plane during impact

and normal impulse during slip is

$$dP_1 = -\mu \frac{v_1}{\sqrt{v_1^2 + v_2^2}} dP_3, \qquad dP_2 = -\mu \frac{v_2}{\sqrt{v_1^2 + v_2^2}} dP_3. \tag{31}$$

Denoting the initial relative velocity of impact is (v_1^o, v_2^o, v_3^o), and substituting (31) into (29), the analytical solution for the ellipsoid can be expressed as

$$v_2 = v_2{}^o \left(\frac{v_1}{v_1{}^o} \right)^{\frac{7}{4}} \tag{32}$$

and

$$v_3{}^e = -(1 + e_P) v_3{}^o, \tag{33}$$

where $v_3{}^e$ is the post-speed of the normal relative velocity at contact point after impact finishes. For verifying our numerical method by comparison with the analytical solution expressed in (32) and (33), we set the parameters of the system and initial conditions of impact as follows: $m = 1$, $r = 1$, $e_P = 0.5$, and $\mathbf{d\dot{q}}^T = [5/\sqrt{2}, 5/\sqrt{2}, -1, 0, 0, 0]$. By observing Fig. 3, we can find the numerical results for the tangential velocities represented by the solid-line in Fig. 3 coincide very well with the theoretical solution of (35).

Figure 4 shows the state of tangential movement can be influenced by the initial condition of impact even if the collision is collinear. Supposing $d\dot{\mathbf{q}} = [0.5, 0.5, -1, 0, 0, 0]$, we can observe stick will appear during collision. In this case, the final velocity at separation in tangential direction will be equal to zero as stick will be preserved until impact finishes.

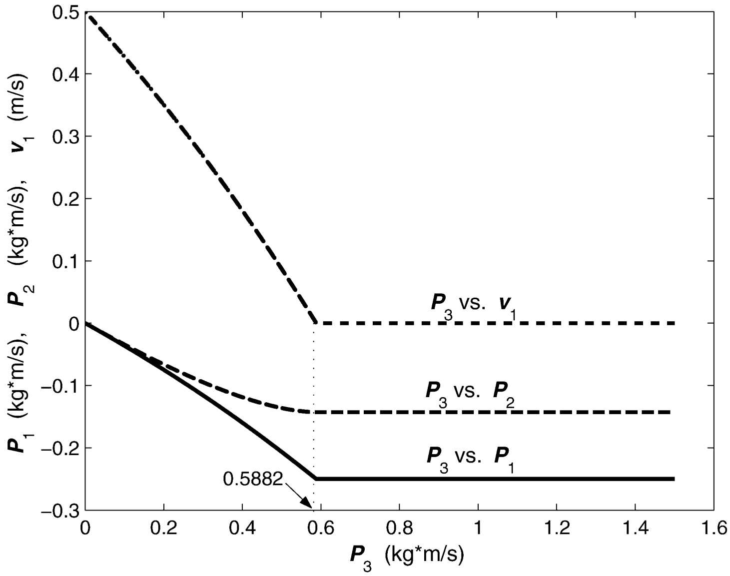
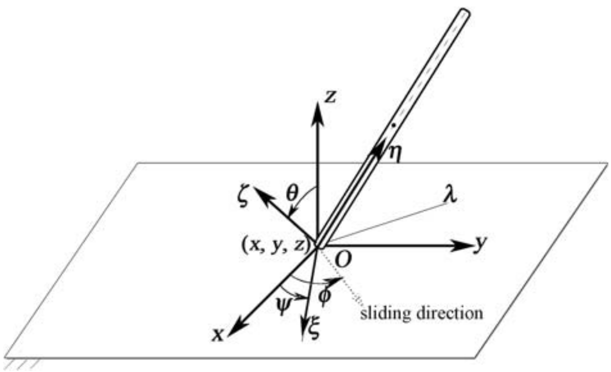


Fig. 4 Tangential impulse and velocity change with normal impulse during impact

Fig. 5 A 3D uniform slender rod colliding against a fixed surface



4.2 A 3-D uniform slender rod impact with a rough fixed surface

For the eccentric collision, the coupling between tangential and normal impulse will complicate the process of impact. Figure 5 shows a rod with length $2l$ and mass m collides against a rough fixed surface. This example firstly introduced by Stronge [2].

Let (x_0, y_0, z_0) and (x, y, z) be the positions of mass center and the contact point in the inertial reference frame, and (ψ, θ) represents the Euler's angles of the slender rod. We define the generalized velocity of such system as $\dot{\mathbf{q}}^T = [\dot{x}_0, \dot{y}_0, \dot{z}_0, \dot{\psi}, \dot{\theta}]$. In this subsection, we will first introduce some theoretical analysis by Stronge for such example, then compare it with the numerical results based on our algorithm.

4.2.1 Theoretical analysis for the 3D slender rod impact

The matrices of configuration for such spatial rod can be expressed as

$$\mathbf{K}^T = \begin{bmatrix} 1 & 0 & 0 & l\cos\psi\cos\theta & -l\sin\psi\sin\theta \\ 0 & 1 & 0 & l\sin\psi\cos\theta & l\cos\psi\sin\theta \end{bmatrix},$$
$$\mathbf{h}^T = [\,0 \quad 0 \quad 1 \quad 0 \quad -l\cos\theta\,].$$

and the mass matrix

$$\mathbf{M} = m\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \frac{1}{3}l^2\cos^2\theta & \\ & & & & \frac{1}{3}l^2 \end{bmatrix}.$$

Without lose generality, we can set $\psi = 0$. The differential equations of motion at contact point during impact can be written as follows by (10).

$$\begin{cases} d\dot{x} = \dfrac{4}{m}dP_x, \\ d\dot{y} = \dfrac{1+3\sin^2\theta}{m}dP_y - \dfrac{3\sin\theta\cos\theta}{m}dP_z, \\ d\dot{z} = -\dfrac{3\sin\theta\cos\theta}{m}dP_y + \dfrac{1+3\cos^2\theta}{m}dP_z. \end{cases} \tag{34}$$

For getting the analytical results for the 3D-frictional impact of spatial rod, Stronge defined a critical coefficient of friction which causes the rod slides along the initial direction of slip. Supposing ϕ_0 is the initial direction of tangential motion at contact point, the condition for the direction of tangential motion unchanged during impact is

$$\frac{d\dot{x}}{d\dot{y}} = \frac{\dot{x}}{\dot{y}} = \frac{\dot{x}_0}{\dot{y}_0} = \cot\phi_0. \tag{35}$$

Denoting μ_c is such critical coefficient of friction making the direction of tangential motion unchanged, and in terms of the law of friction, we have

$$\begin{cases} dP_x = -\mu_c\dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}dP_z = -\mu_c\cos\phi dP_z, \\ dP_y = -\mu_c\dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}dP_z = -\mu_c\sin\phi dP_z. \end{cases} \tag{36}$$

By (34), (35) and (36), the critical coefficient of friction μ_c can be expressed as follows.

$$\mu_c = \frac{\tan\theta}{\sin\phi_0}. \tag{37}$$

For identifying whether stick appears or not during impact, another critical value of the coefficient of friction μ_{st} is introduced. In terms of the condition for the occurrence of stick

($d\dot{x} = d\dot{y} = 0$), the coefficient of friction μ_{st} should satisfy with the following relationship by (34).

$$\mu_{st} \equiv \left| \frac{\sqrt{dP_x^2 + dP_y^2}}{dP_z} \right| = \frac{3\sin\theta\cos\theta}{1 + 3\sin^2\theta}, \qquad \text{if } 0 < \theta < \pi/2. \tag{38}$$

If $\mu = \mu_c$, the direction of slip will remain constant during impact, which is termed an isoclinic or a separatrix in [2]; If $\mu \neq \mu_c$, the direction of slip will asymptotically approach different isoclinic with the increase of the normal impulse. If $\mu > \mu_{st}$, stick will appear once the velocity in tangential direction vanishes before separation; If $\mu < \mu_{st}$, stick will not occur even though the situation of tangential velocity equals zero appears, and slip will resume and be along a fixed direction.

4.2.2 Numerical results

We can use our numerical method to illustrate the properties of the impact for 3D rod. The physical and geometrical parameters for the rod are $l = 1$, $m = 1$, $e_P = 0.8$, $\theta = 0.46$ and $x = y = z = \psi = 0$. We provide two sets of initial impacting velocity: $\dot{x} = \pm 0.2$, $\dot{y} = 0.2\sqrt{3}$, $\dot{z} = -2$ and $\dot{\psi} = \dot{\theta} = 0$. The values of two critical coefficient of friction are $\mu_c = 0.5721$ and $\mu_{st} = 0.75$.

The evolution of tangential velocity for different values of coefficient of friction is shown in Fig. 6. For the case of $\mu/\mu_c = 0$ which corresponds to a impact without friction, the component of velocity in x-direction will keep constant, while the one in y-direction will be changed with the increase of the normal impulse.

When $\mu/\mu_c = 1$, the direction of slip during impact coincides with the initial direction of slip. In other cases, the direction of slip varies continuously with the increase of normal impulse. If $\mu/\mu_c = 0.5$, the direction of slip asymptotically approaches an isoclinic which is the y-direction for such rod, and stick will not appear during impact.

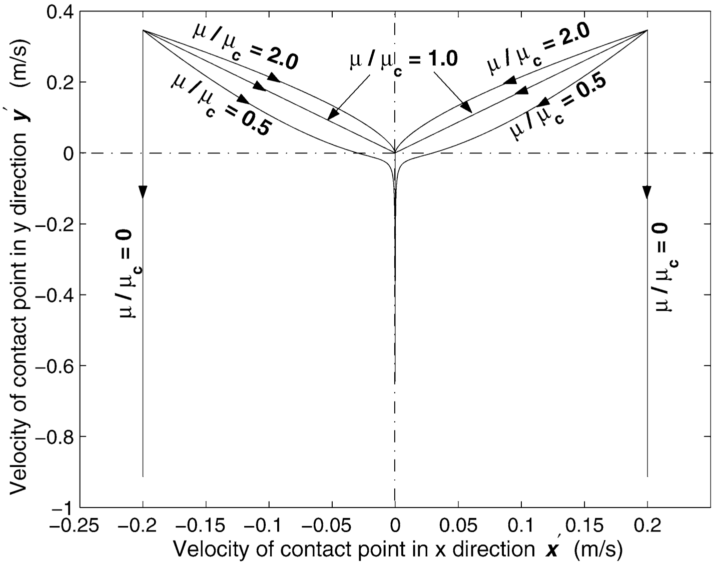


Fig. 6 Different modes of tangential movements in different values of friction coefficient

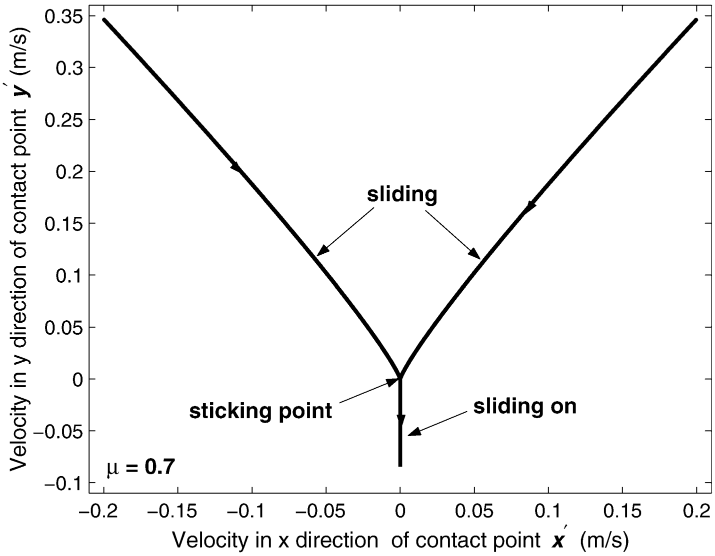


Fig. 7 Slip resumes after tangential speed vanishes

For the larger value of coefficient of friction, for example $\mu/\mu_c = 2$, its value is greater than μ_{st}, stick will appear and hold on until impact finishes. However, if $\mu_c < \mu \leq \mu_{st}$, there is an instant when the tangential velocity equals zero but immediately resumes along a stable invariant direction. Such property are illustrated by Fig. 7 for the value of coefficient of friction $\mu = 0.7$. All the results based on our numerical method well reflect the global properties by using analytical method.

4.3 Spherical pendulum colliding against rough ground

Figure 8 shows a spherical pendulum colliding against a rough ground, which represents a situation of an eccentric impact with velocity constraint [14]. The pendulum is composed of a massless rod with length l and a particle bob with mass m, and pivoted at a stationary point O. (ξ, η, z) and (x, y, z) are the inertial coordinate frame and the coordinates frames fixed in the pendulum, respectively. We select ψ and θ as the generalized coordinates of the system.

4.3.1 Theoretical analysis for the spherical pendulum

We will first summary some analytical results in [14]. The configuration matrices **M**, **K**, and **h** for the pendulum are

$$\mathbf{M} = \begin{bmatrix} \frac{1}{3}ml^2\cos^2\theta & 0 \\ 0 & \frac{1}{3}ml^2 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} -l\sin\psi\cos\theta & l\cos\psi\cos\theta \\ -l\cos\psi\sin\theta & -l\sin\psi\sin\theta \end{bmatrix},$$
$$\mathbf{h}^T = [0, -l\cos\theta].$$

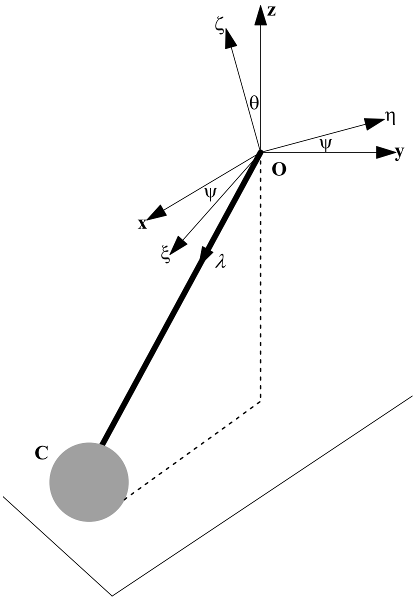


Fig. 8 Spherical pendulum

By letting the initial Euler angle $\psi = 0$ and in terms of (10), we can obtain the differential equations of motion as follows

$$\begin{cases} d\dot{x} = \dfrac{3\sin^2\theta}{m} dP_x + \dfrac{3\sin\theta\cos\theta}{m} dP_z, \\ d\dot{y} = \dfrac{3}{m} dP_y, \\ d\dot{z} = \dfrac{2\sin\theta\cos\theta}{m} dP_x + \dfrac{3\cos^2\theta}{m} dP_z. \end{cases} \tag{39}$$

The velocity at contact point C is expressed by generalized coordinates.

$$\begin{cases} \dot{x} = -l\dot{\theta}\sin\theta, \\ \dot{y} = l\dot{\psi}\cos\theta, \\ \dot{z} = -l\dot{\theta}\cos\theta. \end{cases} \tag{40}$$

Similar to the process for analyzing a spatial rod colliding against a rough plane, the critical coefficient of friction μ_c, which makes the direction of slip not vary during impact and be along the initial direction of tangential velocity, is

$$\mu_c = \frac{\tan\theta}{\cos\phi_0}. \tag{41}$$

The coefficient of friction μ_{st} for stick appearing is

$$\mu_{st} = \cot\theta. \tag{42}$$

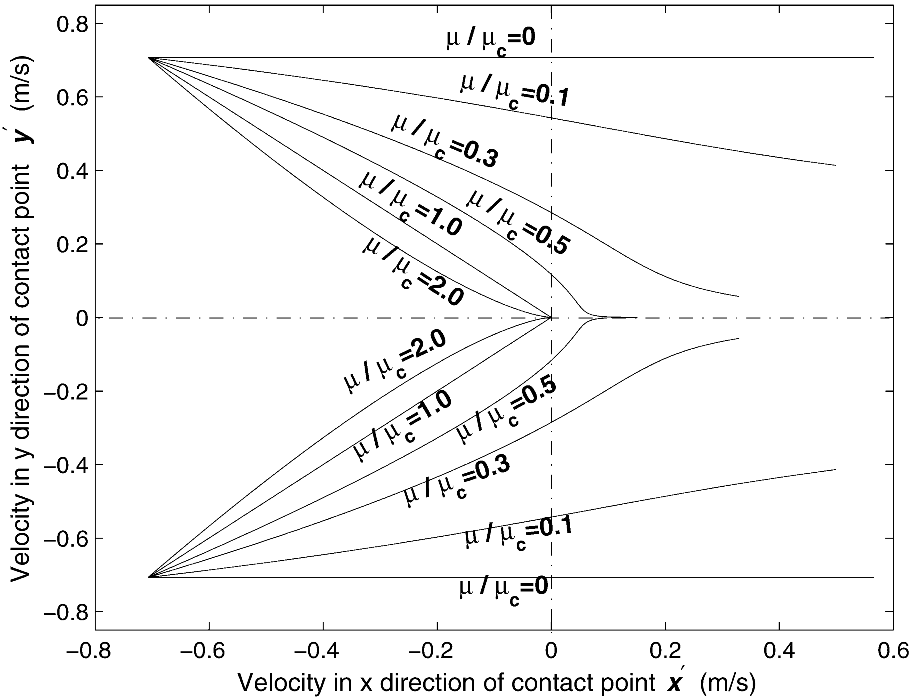


Fig. 9 Tangential motion in different values of coefficient of friction

Basing on the critical coefficients of friction μ_c and μ_{st}, we can understand the properties of tangential motion during impact, especially for identifying whether stick occurs when the tangential velocity equals zero.

4.3.2 Numerical results

The parameters of the pendulum used in numerical simulation are set as follows: $m = 1$, $l = 1$, $e_P = 0.8$, $\psi = 0$, $\theta = \pi/4$; the generalized velocities $\dot{\psi}_0 = \pm 1$ and $\dot{\theta}_0 = 1$. In terms of (40), we can calculate the initial speed of contact point: $\dot{x}_0 = -\sqrt{2}/2$, $\dot{y}_0 = \pm\sqrt{2}/2$ and $\dot{z}_0 = -\sqrt{2}/2$. The values of two critical coefficients of friction are $\mu_c = \sqrt{2}$ and $\mu_{st} = 1$.

Figure 9 shows the numerical results of the trajectories of tangential speed by setting different values of the coefficient of friction. Since $\mu = \mu_c > \mu_{st}$, tangential motion will be along the direction of the initial tangential velocity and approach the point of stick. If $\mu \neq \mu_c$, the tangential motion will continuously change its direction during impact. Slip will not disappear during impact if $\mu < \mu_{st}$, but stick will occur if $\mu > \mu_c$.

5 Summary and conclusions

The problems of 3D frictional impact in multibody systems can be represented by the differential equations of motion with normal impulse as independent differential variable. We have extended such method, firstly proposed by Keller and Stronge for dealing with the impact of single body, into the problem of 3D frictional impact in multi-body systems. By using difference scheme to discretize the differential equations of motion, we have developed a reliable numerical method, called Normal impulse-stepping method, to deal with the complex motions appearing in impact such as slip, stick and slip resumption. Based on the global properties of frictional impact discovered in existing literature, the reliability of the numerical method is verified by the comparison with the theoretical analysis in three examples. This study provides a systematic theory for analyzing the problem of frictional impact in multibody systems and an effective method for simulating the phenomena of impact appearing in mechanical engineering.

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References

1. Ivanov, A.P.: The problem of constrained impact. J. Appl. Math. Mech. **61**(3), 341–353 (1997)
2. Stronge, W.J.: Impact Mechanics. Cambridge University Press, Cambridge (2000)
3. Pheiffer, F., Glocker, C.: Multibody Dynamics with Unilateral Contacts. Wiley, New York (1996)
4. Glocker, C., Studer, C.: Formulation and preparation for numerical evaluation of linear complementarity systems in dynamics. Multibody Syst. Dyn. **13**, 447–463 (2005)
5. Yao, W., Chen, B., Liu, C.: Energetic coefficient of restitution for planar impact in multi-rigid-body systems with friction. Int. J. Impact Eng. **31**, 255–265 (2005)
6. Schiehlen, W., Seiferied, R.: Three approaches for elastodynamic contact in multibody systems. Multibody Syst. Dyn. **12**, 1–16 (2004)
7. Hu, B., Eberhard, P., Schiehlen, W.: Comparison of analytical and experimental results for longitudinal impacts on elastic rods. J. Vib. Control **9**, 157–174 (2003)
8. Seiferied, R., Schiehlen, W., Eberhard, P.: Numerical and experimental evaluation of coefficient of restitution for repeated impacts. Int. J. Impact Eng. **32**, 508–524 (2005)
9. Escalona, J.L., Sany, J.R., Shabana, A.A.: On the use of the restitution condition in flexible body dynamics. Nonlinear Dyn. **30**, 71–86 (2002)
10. Seifried, R., Hu, B., Eberhard, P.: Numerical and experimental investigation of radial impacts on a half-circular plate. Multibody Syst. Dyn. **9**, 265–281 (2003)
11. Yu, W., Matthew, T.M.: Two-dimensional rigid-body collisions with friction. Trans. ASME J. Appl. Mech. **59**, 635–642 (1992)
12. Kane, T.R., Levinson, D.A.: Dynamics: Theory and Applications. McGraw–Hill, New York (1985)
13. Keller, J.B.: Impact with friction. Trans. ASME J. Appl. Mech. **53**, 1–4 (1986)
14. Stronge, W.J.: Swerve during three-dimensional impact of rough rigid bodies. Trans. ASME J. Appl. Mech. **61**, 605–611 (1994)
15. Chatterjee, A., Ruina, A.: Two interpretations of rigidity in rigid body collisions. Trans. ASME J. Appl. Mech. **65**, 894–900 (1997)
16. Smith, C.E.: Predicting rebounds using rigid-body dynamics. Trans. ASME J. Appl. Mech. **58**, 754–758 (1991)
17. Brach, R.M.: Impact coefficient and tangential impacts. Trans. ASME J. Appl. Mech. **64**, 1014–1016 (1997)
18. Bhatt, V., Koechling, J.: Partitioning the parameter space according to different behaviors during three-dimensional. Trans. ASME J. Appl. Mech. **62**, 740–746 (1995)
19. Bhatt, V.: Three-dimensional frictional rigid-body impact. Trans. ASME J. Appl. Mech. **62**, 893–898 (1995)
20. Batlle, J.A.: Rough balanced collisions. Trans. ASME J. Appl. Mech. **63**, 168–172 (1996)
21. Routh, E.J.: Dynamics of a System of Rigid Bodies. Macmillan, London (1897)
22. Kuninaka, H., Hayakawa, H.: Anomalous behavior of the coefficient of normal restitution in oblique impact. Phys. Rev. Lett. **93**(15), 154301 (2004)
23. Weir, W., Tallon, S.: The coefficient of restitution for normal incident, low velocity particle impacts. Chem. Eng. Sci. **60**, 3637–3647 (2005)
24. Louge, M.Y., Adams, M.E.: Anomalous behavior of normal kinematic restitution in the oblique impacts of a hard sphere on an elastoplastic plate. Phys. Rev. E **65**, 021303 (2002)
25. Wu, C.Y., Li, L.Y., Thornton, C.: Rebound behaviour of spheres for plastic impacts. Int. J. Impact Eng. **28**, 929–946 (2003)

26. Sondergaard, R., Chaney, K., Brennen, C.: Measurements of sloid sphere bouncing off flat plates. J. Appl. Mech. **57**, 694–699 (1990)
27. Stronge, W.J.: Rigid body collisions with friction. Proc. Roy. Soc. Lond. A **431**(1881), 169–181 (1990)
28. Brach, R.M.: Formulation of rigid body impact problems using generalized coefficient. Int. J. Eng. Sci. **36**, 61–71 (1998)
29. Brogliato, B.: Nonsmooth Impact Mechanics: Models, Dynamics and Control, 2nd edn., pp. 157–159. Springer, London (1996)