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On the use of moment-matching to build reduced order models in flexible multibody dynamics

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Abstract An important issue in the field of flexible multibody dynamics is the reduction of the flexible body's degrees of freedom. For this purpose, often modal reduction through projection onto a subspace spanned by some dominant eigenvectors is used. However, as in this method the dynamical boundary conditions are not taken into account, a large number of eigenmodes is required to obtain a good approximation and also the selection of the dominant modes can be quite difficult. Therefore, the authors propose an approach based on accounting for the flexible body as an input-output system in the frequency domain. The reduced order model is generated by imposing a set of interpolation conditions concerning the values and derivatives of the system's transfer function in a predefined frequency range. This procedure is known as moment-matching and can be realised through projection onto so-called Krylov-subspaces. As this technique allows the incorporation of the frequency content and the spatial distribution of the loads, in the chosen frequency range more accurate reduced order models can be obtained compared to other model reduction techniques available in structural mechanics. The calculation of the Krylov-subspaces can be implemented very efficiently, using the Arnoldi or Lanczos procedure in connection with sparse matrix techniques. The capability of the proposed technique is demonstrated by means of a numerical example.

Keywords Flexible multibody systems · Model reduction · Krylov-subspaces · Moment-matching

1. Introduction

The method of flexible multibody systems (FMBS) is an important tool for the simulation and analysis of complex mechanisms including flexible components. To describe the behaviour of a flexible body within the mechanism, often the well-established floating frame of reference formulation is used. In this methodology, the flexible body's motion is subdivided into the

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nonlinear reference motion and the linearised elastic deformation. The flexible part of the displacement field is approximated using a Ritz-approach by the product of known shape functions and time-dependent elastic coordinates. Using the finite element method (FEM) to describe the flexible body leads to a large number of elastic coordinates and dynamic analysis of the flexible multibody system becomes computationally very demanding. Therefore, the number of the flexible body's degrees of freedom has to be diminished considerably. According to [32], the use of suitable reduction methods can be justified as the large number of elastic coordinates is often not caused by the complexity of the flexible body's dynamic response, but rather by its complex topology or rapid changes in the system properties. In this regard, a classical method is the use of modal reduction. However, the special character of the load acting on the flexible body is not considered in the basic method and as a consequence, convergence is often very slow, see [29]. To improve convergence, different approaches can be found in the literature, for example the supplementation of the basis of eigenvectors with assumed modes [9] or the selection of the eigenmodes according to some influence coefficients accounting for the nature of the load acting on the flexible body, see [15, 47].

Here, a different approach is presented by formulating the elastic body as an input-output system and using Krylov-subspace based projection methods to reduce the system. Applying this method, the reduced system interpolates the frequency response of the full model and a given number of its derivatives at a specified number of points. As a consequence, in a predefined frequency range more accurate reduced order models can be obtained than, e.g., by an application of modal reduction. The calculation of the Krylov-subspace can be implemented very efficiently using iterative methods like the Arnoldi or Lanczos procedure, and also a combination with modal reduction is possible in order to improve its approximation capabilities in a predefined frequency range. Starting from the constructed reduction base, the so-called standard input data file (SID file) [41] representing the flexible substructure can be generated and subsequently imported by FMBS programs like SIMPACK [26] or NEWEUL [25].

2. The floating-frame of reference formulation

In this section, the basic idea behind the floating-frame of reference formulation is presented in brief. More detailed explanations on this topic can be found in the literature, e.g. [41, 44].

2.1. Reference kinematics

One possibility to describe the configuration of an elastic body is the simultaneous use of two sets of coordinates [44]. First, the reference coordinates, describing the global motion of the body's reference system $\boldsymbol{K}_i$ with respect to the inertial frame of reference $\boldsymbol{K}_I$ and second, the elastic coordinates, describing the body's deformation with respect to the reference frame.

According to Figure 1, the position $\boldsymbol{r}_k(t) = \boldsymbol{r}(\boldsymbol{R}_{ik}, t)$ of an arbitrary frame $\boldsymbol{K}_k$ on the body with the material coordinate $\boldsymbol{R}_{ik}$ can be written as

$$\boldsymbol{r}_k(t) = \boldsymbol{r}_i(t) + \boldsymbol{d}_{ik}(t) \tag{1}$$

using the global position of the body's reference system $\boldsymbol{r}_i(t)$ and the local position vector $\boldsymbol{d}_{ik}(t)$. Similarly, for the transformation matrix of frame $\boldsymbol{K}_k$ with respect to frame $\boldsymbol{K}_I$, the relation

$$\boldsymbol{A}_{Ik}(t) = \boldsymbol{A}_{Ii}(t) \cdot \boldsymbol{A}_{ik}(t) \tag{2}$$

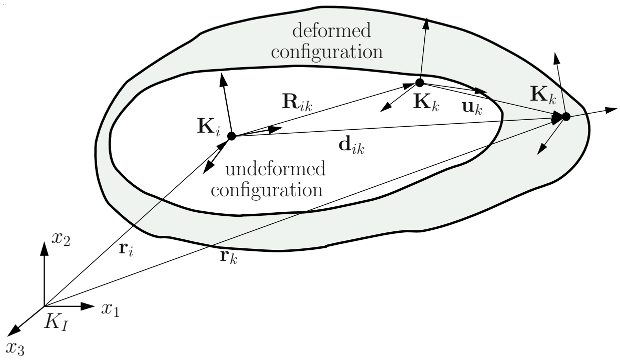

Fig. 1 Kinematic description of a flexible body

holds and a coordinate transformation from frame $\boldsymbol{K}_i$ to frame $\boldsymbol{K}_I$ can be written as

$$ {}_I\boldsymbol{r}_k(t) = \boldsymbol{A}_{Ii}(t) \cdot {}_i\boldsymbol{r}_k(t). \tag{3}$$

In the case of rigid bodies, the position vector ${}_i\boldsymbol{d}_{ik}(t)$ and the transformation matrix $\boldsymbol{A}_{ik}(t)$ remain constant. Below, if no subscript is given, all vectors are assumed to be represented by coordinates with respect to the basis defined by frame $\boldsymbol{K}_i$.

For elastic bodies, the vector $\boldsymbol{d}_{ik}(t)$ can be split up in two parts

$$\boldsymbol{d}_{ik}(t) = \boldsymbol{R}_{ik} + \boldsymbol{u}_k(t) \tag{4}$$

with $\boldsymbol{R}_{ik}$ representing the constant position of $\boldsymbol{K}_k$ in the undeformed state and $\boldsymbol{u}_k(t) = \boldsymbol{u}(\boldsymbol{R}_{ik})$ denoting the elastic displacement of this frame.

Equivalently, the time dependent transformation matrix $\boldsymbol{A}_{ik}(t)$ is separated in a constant part $\boldsymbol{\Gamma}_{ik}$ and a part $\boldsymbol{\Theta}_k(t)$ depending on the angular elastic deformation

$$\boldsymbol{A}_{ik}(t) = \boldsymbol{\Gamma}_{ik} \cdot \boldsymbol{\Theta}_k(t). \tag{5}$$

The elements of the matrix $\boldsymbol{\Theta}_k(t)$ can, e.g., be determined using three angles, collected in the vector $\boldsymbol{\vartheta}_k(t) = \boldsymbol{\vartheta}(\boldsymbol{R}_{ik}, t)$.

Below, only small deformations are considered. This allows a linearisation of the kinematic equations with respect to the elastic deformation and the transformation matrix $\boldsymbol{\Theta}_k(t)$ can be written as

$$\boldsymbol{\Theta}_k(t) = \boldsymbol{E} + \tilde{\boldsymbol{\vartheta}}_k(t). \tag{6}$$

Introducing a Ritz-Ansatz, the small deformation variables $\boldsymbol{u}(\boldsymbol{R}, t)$ and $\boldsymbol{\vartheta}(\boldsymbol{R}, t)$ are approximated by known shape functions $\boldsymbol{\Phi}(\boldsymbol{R})$ and $\boldsymbol{\Psi}(\boldsymbol{R})$ and time dependent weighting coefficients, collected in a vector $\boldsymbol{q}(t)$

$$\boldsymbol{u}(\boldsymbol{R}, t) = \boldsymbol{\Phi}(\boldsymbol{R}) \cdot \boldsymbol{q}(t), \qquad \boldsymbol{\vartheta}(\boldsymbol{R}, t) = \boldsymbol{\Psi}(\boldsymbol{R}) \cdot \boldsymbol{q}(t). \tag{7}$$

If a finite element model is used to describe the flexible body, $\boldsymbol{\Phi}(\boldsymbol{R})$ and $\boldsymbol{\Psi}(\boldsymbol{R})$ comprise the shape functions of the finite elements and $\boldsymbol{q}(t)$ are the nodal coordinates [41].

2.2. Equations of motion of an elastic body

Using the kinematic equations of Section 2.1, three sets of variables are defined. The position is represented by

$$z^I = [\boldsymbol{r}_i, \quad \boldsymbol{\beta}_i, \quad \boldsymbol{q}], \tag{8}$$

with $\boldsymbol{r}_i$ as the position of the body's reference system, $\boldsymbol{\beta}_i$ as a set of variables describing the rotation of the reference system and the coordinates $\boldsymbol{q}$ representing the body's deformation. To describe the velocity, the set

$$z^{II} = [\boldsymbol{v}_i, \quad \boldsymbol{\omega}_i, \quad \dot{\boldsymbol{q}}] \tag{9}$$

is used. Here, $\boldsymbol{v}_i$ denotes the translational velocity and $\boldsymbol{\omega}_i$ the rotational velocity of the reference system. For the flexible part, the time derivative of the deformation coordinates $\dot{\boldsymbol{q}}$ is taken into account. To represent the acceleration,

$$z^{III} = [\boldsymbol{a}_i, \quad \boldsymbol{\alpha}_i, \quad \ddot{\boldsymbol{q}}] \tag{10}$$

is defined, with the translational acceleration of the reference system $\boldsymbol{a}_i$, its angular acceleration $\boldsymbol{\alpha}_i$ and the second derivative of the deformation coordinates $\ddot{\boldsymbol{q}}$. Expressing Jourdain's principle of virtual power [41] through the velocity variations δz^{II} results in

$$\delta z^{II} \cdot \left(\boldsymbol{M} \cdot z^{III} - \boldsymbol{h}^{ac} - \boldsymbol{h}^{e}\right) = 0. \tag{11}$$

Since the variations δz^{II} are independent, Equation (11) is only valid if

$$\boldsymbol{M} \cdot z^{III} = \boldsymbol{h}^{ac} + \boldsymbol{h}^{e} \tag{12}$$

holds. In Equation (12), $\boldsymbol{M}$ is the generalized mass matrix, $\boldsymbol{h}^{ac}$ collects generalized inertia forces, gravitational forces and forces acting on the body's surface due to force elements or kinematic constraints. The vector $\boldsymbol{h}^{e}$ comprises internal forces due to deformations. Partitioning of the matrices in Equation (12) according to their translational, rotational and elastic shares yields

$$\boldsymbol{M} = \begin{bmatrix} m\boldsymbol{I} & m\tilde{\boldsymbol{c}}^T(\boldsymbol{q}) & \boldsymbol{C}_t^T(\boldsymbol{q}) \\ m\tilde{\boldsymbol{c}}(\boldsymbol{q}) & \boldsymbol{J}(\boldsymbol{q}) & \boldsymbol{C}_r^T(\boldsymbol{q}) \\ \boldsymbol{C}_t(\boldsymbol{q}) & \boldsymbol{C}_r(\boldsymbol{q}) & \boldsymbol{M}_e \end{bmatrix}, \quad \boldsymbol{h}^{ac} = \begin{bmatrix} \boldsymbol{h}_t^{ac}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ \boldsymbol{h}_r^{ac}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ \boldsymbol{h}_e^{ac}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{bmatrix} \quad \text{and}$$

$$\boldsymbol{h}^{e} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ -\boldsymbol{K}_e \cdot \boldsymbol{q} - \boldsymbol{D}_e \cdot \dot{\boldsymbol{q}} \end{bmatrix}. \tag{13}$$

2.3. Definition of the body reference frame

Location $\boldsymbol{r}_k$ and orientation $\boldsymbol{A}_k$ of an arbitrary frame are determined by six generalized coordinates. Using the floating frame of reference approach, twelve variables are used in order to describe the location and orientation of the frame. This redundancy has to be eliminated, using at least six reference conditions. These conditions are not unique and can be chosen according to kinematic or dynamic criteria. The minimum requirement the shape functions $\boldsymbol{\Phi}(\boldsymbol{R})$ and $\boldsymbol{\Psi}(\boldsymbol{R})$ have to meet is their ability to fulfil the geometric boundary conditions implied by choosing a certain reference system. Such shape functions are called admissible functions [29].

In [42], several methods to meet these conditions are proposed. One possibility is to use a so-called chord-frame that is defined by the three points P_1, P_2 and P_3 with the material coordinates $\boldsymbol{R}_{i1}$, $\boldsymbol{R}_{i2}$ and $\boldsymbol{R}_{i3}$. The origin of the reference system $\boldsymbol{K}_i$ is at P_1, two of its basis vectors are required to belong to a plane defined by the three points and the third basis vector is chosen to be perpendicular to this plane. For the deformation variables, constraints

$$\boldsymbol{u}(\boldsymbol{R}_{i1}, t) = \boldsymbol{0}, \quad u_2(\boldsymbol{R}_{i2}, t) = 0, \quad u_3(\boldsymbol{R}_{i2}, t) = 0, \quad u_3(\boldsymbol{R}_{i3}, t) = 0 \tag{14}$$

equivalent to shape functions satisfying the conditions

$$\boldsymbol{\Phi}(\boldsymbol{R}_{i1}) = \boldsymbol{0}, \quad \boldsymbol{\Phi}_{2*}(\boldsymbol{R}_{i2}) = \boldsymbol{0}, \quad \boldsymbol{\Phi}_{3*}(\boldsymbol{R}_{i2}) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{\Phi}_{3*}(\boldsymbol{R}_{i3}) = \boldsymbol{0} \tag{15}$$

apply. Similarly, the tangent-frame is defined by identifying the reference system $\boldsymbol{K}_i$ with the frame of the body having the material coordinate $\boldsymbol{R} = \boldsymbol{0}$.

Besides the kinematical relations used so far, also dynamical constraints to define the reference system can be used. For example the Buckens-frame satisfying the condition

$$\int_V \boldsymbol{u}(\boldsymbol{R}, t) \cdot \boldsymbol{u}(\boldsymbol{R}, t)\, dm \overset{!}{=} \text{minimum} \tag{16}$$

can be defined. Although the applied load effects the physical deformation, the choice of the reference frame determines the displacements with respect to this frame. As Equation (16) reveals, the choice of the Buckens-frame leads to the smallest elastic deformation possible. For arbitrary shape functions, condition Equation (16) must be enforced by imposing an algebraic constraint while solving the system's equations of motion.

However, it can be shown, that by using eigenfunctions of unsupported structures, so called free-free modes, and deleting the six rigid body modes, constraint Equation (16) is automatically met.

3. Model reduction for flexible multibody dynamics

In this section, a general framework for the reduction of the dynamical equations (12) with the partitioning (13) is described. The basic idea of this methodology known as reduction by projection is to generate a reduced order model by approximating the solution in a suitable low-dimensional subspace and imposing constraints related with another low-dimensional subspace. Also classical methods like modal reduction can be interpreted in the scope of this framework. Below only some fundamental relations will be explained, a detailed study on projection methods can be found in [2, 8, 18].

3.1. Model reduction using projection methods

The method will be illustrated by means of a linear time-invariant MIMO-system connecting input $\boldsymbol{u}(t)$ and output $\boldsymbol{y}(t)$

$$\begin{aligned} \boldsymbol{E} \cdot \dot{\boldsymbol{x}}(t) - \boldsymbol{A} \cdot \boldsymbol{x}(t) &= \boldsymbol{B} \cdot \boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C} \cdot \boldsymbol{x}(t) \end{aligned} \tag{17}$$

with the descriptor matrix $\boldsymbol{E} \in \mathbb{R}^{n \times n}$, the system matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, the input matrix $\boldsymbol{B} \in \mathbb{R}^{n \times p}$ and the output matrix $\boldsymbol{C} \in \mathbb{R}^{r \times n}$. In a first step, the solution $\boldsymbol{x}$ is approximated in a subspace $\mathcal{V}$ of dimension $m < n$. As this subspace can be represented by a rectangular matrix $\boldsymbol{V} \in \mathbb{R}^{n \times m}$ with $\operatorname{colsp}\{\boldsymbol{V}\} = \mathcal{V}$, the approximation can be written as

$$\boldsymbol{x}(t) \approx \boldsymbol{V} \cdot \bar{\boldsymbol{x}}(t). \tag{18}$$

Inserting Equation (18) into Equation (17) leads to an overdetermined system of differential equations

$$\begin{aligned} \boldsymbol{E} \cdot \boldsymbol{V} \cdot \dot{\bar{\boldsymbol{x}}}(t) - \boldsymbol{A} \cdot \boldsymbol{V} \cdot \bar{\boldsymbol{x}}(t) &= \boldsymbol{B} \cdot \boldsymbol{u}(t) + \boldsymbol{r}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C} \cdot \boldsymbol{V} \cdot \bar{\boldsymbol{x}}(t). \end{aligned} \tag{19}$$

As the real solution $\boldsymbol{x}(t)$ of Equation (17) can not be expected to be an element of subspace $\mathcal{V}$ in general, a residual $\boldsymbol{r}(t) \neq \boldsymbol{0}$ is taken into account in Equation (19). In order to obtain a unique solution, m constraints have to be imposed on the residual of Equation (19). These conditions are associated with a second subspace $\mathcal{W}$ spanned by the columns of a rectangular matrix $\boldsymbol{W} \in \mathbb{R}^{n \times m}$, that means $\operatorname{colsp}\{\boldsymbol{W}\} = \mathcal{W}$. Typically, constraints in the form of orthogonality conditions, so-called Petrov-Galerkin conditions, are applied by premultiplying Equation (19) with $\boldsymbol{W}^T$. By this premultiplication we can take advantage of the orthogonality $\boldsymbol{W}^T \cdot \boldsymbol{r} = \boldsymbol{0}$ and make sure to get a feasible solution. The result is a reduced system of dimension $m \times m$

$$\begin{aligned} \boldsymbol{W}^T \cdot \boldsymbol{E} \cdot \boldsymbol{V} \cdot \dot{\bar{\boldsymbol{x}}}(t) - \boldsymbol{W}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V} \cdot \bar{\boldsymbol{x}}(t) &= \boldsymbol{W}^T \cdot \boldsymbol{B} \cdot \boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C} \cdot \boldsymbol{V} \cdot \bar{\boldsymbol{x}}(t). \end{aligned} \tag{20}$$

A projection method is called orthogonal, if the two subspaces $\mathcal{V}$ and $\mathcal{W}$ are identical, otherwise it is referred to as an oblique projection.

The same approach, adopted to the flexible part of Equation (12) by setting

$$\boldsymbol{q}(t) \approx \boldsymbol{V} \cdot \bar{\boldsymbol{q}}(t) \tag{21}$$

and imposing the orthogonality condition related to $\boldsymbol{W}$, leads to the following reduced set of equations

$$\begin{bmatrix} m\boldsymbol{I} & m\tilde{\boldsymbol{c}}^T(\bar{\boldsymbol{q}}) & \boldsymbol{C}_t^T(\bar{\boldsymbol{q}}) \cdot \boldsymbol{V} \\ m\tilde{\boldsymbol{c}}(\bar{\boldsymbol{q}}) & \boldsymbol{J}(\bar{\boldsymbol{q}}) & \boldsymbol{C}_r^T(\bar{\boldsymbol{q}}) \cdot \boldsymbol{V} \\ \boldsymbol{W}^T \cdot \boldsymbol{C}_t(\bar{\boldsymbol{q}}) & \boldsymbol{W}^T \cdot \boldsymbol{C}_r(\bar{\boldsymbol{q}}) & \boldsymbol{W}^T \cdot \boldsymbol{M}_e \cdot \boldsymbol{V} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{a}_i \\ \boldsymbol{\alpha}_i \\ \ddot{\bar{\boldsymbol{q}}} \end{bmatrix}$$
$$= \begin{bmatrix} \boldsymbol{h}_t^{ac}(\bar{\boldsymbol{q}}, \dot{\bar{\boldsymbol{q}}}) \\ \boldsymbol{h}_r^{ac}(\bar{\boldsymbol{q}}, \dot{\bar{\boldsymbol{q}}}) \\ \boldsymbol{W}^T \cdot \boldsymbol{h}_e^{ac}(\bar{\boldsymbol{q}}, \dot{\bar{\boldsymbol{q}}}) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ -\boldsymbol{W}^T \cdot \boldsymbol{K}_e \cdot \boldsymbol{V} \cdot \bar{\boldsymbol{q}} - \boldsymbol{W}^T \cdot \boldsymbol{D}_e \cdot \boldsymbol{V} \cdot \dot{\bar{\boldsymbol{q}}} \end{bmatrix}. \tag{22}$$

3.2. Methods for model reduction

So far, the projection matrices $\boldsymbol{V}$ and $\boldsymbol{W}$ introduced in Section 3.1 are not specified. This task is associated with the choice of a specific model reduction method. The need for accurate and efficient model reduction methodologies arises in various fields of application and there exist a large amount of techniques, often closely related to one of these applications. Amongst others, model reduction methods are used for micro-electro-mechanical systems (MEMS) [36], the simulation of electronic circuits [10], the simplification of dynamic controllers [8], weather prediction [46] and, of course, structural mechanics [32].

In the following, some of the developed approaches are inspected and assessed especially with regard to flexible multibody dynamics. In the literature, see [1, 3, 8], mainly two categories of reduction techniques are distinguished.

The first group comprises techniques based on the optimisation of the reduced-order model with respect to a given criteria. In [1] they are referred to as SVD-based methods as they have their roots in the singular value decomposition (SVD). Two important approaches within this context are the balanced truncation [31] and the Hankel norm approximation [17]. When applied to stable models, both approaches guarantee to preserve stability and further on, a global error bound is available by computation of the so-called Hankel singular values. Despite of these favourable properties, the application of the SVD-based methods within the context of flexible multibody dynamics is arguable. These approaches require at least the solution of two linear matrix equations, the Lyapunov equations, as well as the computation of an SVD. Consequently, the computational effort grows with the cubic power in the number of degrees of freedom, making the application to large-scale systems at least difficult. Another drawback is, that originally they were developed for first order systems and are not directly applicable to second order systems like this was done for the reduced equation of motion of the flexible body Equation (22). Meanwhile, there exist some approaches to adapt the balanced truncation method to second order systems, see [30, 40, 46]. However, these methods lack some of the aforementioned properties of SVD-methods. In this respect, the methods presented in [30, 46] do not provide a global error bound and do not guarantee stability of the reduced system. An alternative way to benefit from the properties offered by the SVD-methods is given in [15, 23]. Here, it is shown that for proportionally damped mechanical structures, the modal model is already almost balanced and approximate Hankel singular values can be obtained with little computational effort. This procedure can be applied to second order systems, too.

The second category of model reduction techniques includes methods which preserve exactly a predefined selection of parameters of the full-order model. These parameters can be the dominant eigenfrequencies of the original system as in modal reduction or the values of the transfer function and a selected number of its derivatives as in the moment-matching methods. These moment-matching conditions can be imposed explicitly through so-called Padé approximation or implicitly by projection on certain Krylov-subspaces [3, 18]. In our context, only the projection methods are applicable and, furthermore, they are known to be the numerically more stable approach [3]. Using these methods, very accurate models within a selected frequency range can be obtained and, different from the SVD-based methods, also large-scale systems can be handled by exploiting the sparsity of the involved matrices. Similar to SVD-based methods, the Krylov-based methods originally were developed for state-space systems such as Equation (17) but meanwhile there exist powerful adaptations that can be applied directly to second-order systems, like the methods presented in [4, 46]. Nevertheless, there are also some drawbacks one has to face when working with Krylov-subspace methods. Because of the local nature of the reduction procedure, it is very difficult to develop global error bounds [1], as it is done in [19] for one special implementation, the Lanczos procedure. In practise, often heuristic error estimates are used as proposed in [6, 18, 38]. Further on, stability of the reduced model is not guaranteed in general even if the full order model is stable. However, for systems of special structure and the restriction to orthogonal projection, stable reduced order models can be obtained, see [11].

Additionally, in [1] a third group of reduction techniques is introduced, that aims to combine the favourable properties of SVD- and Krylov-based methods to so-called SVD-Krylov methods. So far, only approaches based on the state space formulation Equation (17) exist and consequently these methods are not considered here.

As most model reduction methods are derived for state-space systems, one can wonder if it is advisable to use this formulation also for the description of the flexible body. Suitable methods for the linearisation are available in the literature, see [45]. Here, this is not done for several reasons. From a practical point of view, it is much easier to integrate the flexible body's dynamics in a general multibody formalism if it is formulated in a second order form. In addition, as stated in [4, 46], structural properties like symmetry or positive definiteness as well as the physical meaning of the original system are lost.

4. Model reduction using moment-matching

In Section 3.2 it was exposed, that moment-matching methods constitute a promising approach also in the context of flexible multibody dynamics. Actually, variants of these methods have been applied successfully to structural mechanics problems, see [20, 22, 27, 33, 34, 48]. Additionally, in this work recent advances based amongst others on the work presented in [4, 18, 46] are considered and adapted to structural mechanics and flexible multibody dynamics, respectively.

4.1. Moment-matching and Krylov-subspaces

The moment-matching model reduction technique, as it is established here, is based on regarding the transfer function of a linear system. If zero initial conditions are assumed, application of the Laplace transformation

$$\boldsymbol{X}(s) = \int_0^\infty \boldsymbol{x}(t) e^{-st} dt \tag{23}$$

to Equation (17) leads to an algebraic system of equations

$$\begin{aligned} s\boldsymbol{E} \cdot \boldsymbol{X}(s) - \boldsymbol{A} \cdot \boldsymbol{X}(s) &= \boldsymbol{B} \cdot \boldsymbol{U}(s), \\ \boldsymbol{Y}(s) &= \boldsymbol{C} \cdot \boldsymbol{X}(s) \end{aligned} \tag{24}$$

from which the transfer function

$$\boldsymbol{H}(s) = \frac{\boldsymbol{Y}(s)}{\boldsymbol{U}(s)} = \boldsymbol{C} \cdot (s\boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B} \tag{25}$$

is directly deduced. Making use of the Neumann expansion [18]

$$(\boldsymbol{I} - \eta \boldsymbol{G})^{-1} = \sum_{j=0}^{\infty} (\eta \boldsymbol{G})^j, \tag{26}$$

the transfer function in Equation (25) can be expanded into a power series about a point σ that is not a pole of this transfer function. At first, Equation (25) is written, see [13], as

$$\begin{aligned} \boldsymbol{H}(s) &= \boldsymbol{C} \cdot (\sigma \boldsymbol{E} - \boldsymbol{A} - (\sigma - s)\boldsymbol{E})^{-1} \cdot \boldsymbol{B} \\ &= \boldsymbol{C} \cdot (\boldsymbol{I} - (\sigma \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{E}(\sigma - s))^{-1} \cdot (\sigma \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B}. \end{aligned} \tag{27}$$

Now, application of the Neumann expansion (26) leads to

$$\begin{aligned} \boldsymbol{H}(s) &= \sum_{j=0}^{\infty} \boldsymbol{C} \cdot ((\sigma \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{E})^j \cdot (\sigma \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B}(\sigma - s)^j \\ &= \sum_{j=0}^{\infty} \boldsymbol{T}_j^{\sigma} (\sigma - s)^j \end{aligned} \tag{28}$$

with the j-th moment

$$\boldsymbol{T}_j^{\sigma} = \boldsymbol{C} \cdot ((\sigma \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{E})^j \cdot (\sigma \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B} = \frac{1}{j!} \frac{\partial^j \boldsymbol{H}(s)}{\partial s^j} \bigg|_{s=\sigma} \tag{29}$$

of a transfer function around an expansion point σ. It is the approximation idea of the Krylov-based or moment-matching methods to generate a reduced order model $\bar{\boldsymbol{H}}(s)$ such that the conditions

$$\boldsymbol{T}_{j(k)}^{\sigma_k} = \bar{\boldsymbol{T}}_{j(k)}^{\sigma_k}, \quad k = 1, \ldots, K; \quad j(k) = 0, \ldots, J_k \tag{30}$$

hold. If the transfer function is expanded about infinity, the problem is also known as partial realisation and the moments are denoted as Markov parameter. Otherwise, the method belongs to the group of Padé approximations, see [1]. A numerically reliable approach to generate reduced order models satisfying condition (30) is the use of Krylov-subspace methods that allow moment-matching without the numerically unstable necessity to calculate the moments explicitly. In general, a (block-) r-th Krylov-subspace is defined by

$$\mathcal{K}_r(\boldsymbol{M}, \boldsymbol{R}) = \text{span}\{\boldsymbol{R}, \boldsymbol{M} \cdot \boldsymbol{R}, \ldots, \boldsymbol{M}^{r-1} \cdot \boldsymbol{R}\} \tag{31}$$

with matrix $\boldsymbol{M}$ and starting matrix $\boldsymbol{R}$, see [10]. The particular moment-matching property is defined by the specific choice of $\boldsymbol{M}$ and $\boldsymbol{R}$. A very general choice is the rational Krylov method in [18] given by

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_b(k)}((\sigma_k \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{E}, (\sigma_k \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B}) \quad \subseteq \mathcal{V} = \mathrm{colsp}\{\boldsymbol{V}\}, \tag{32}$$

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_c(k)}((\sigma_k \boldsymbol{E} - \boldsymbol{A})^{-T} \cdot \boldsymbol{E}^T, (\sigma_k \boldsymbol{E} - \boldsymbol{A})^{-T} \cdot \boldsymbol{C}^T) \subseteq \mathcal{W} = \mathrm{colsp}\{\boldsymbol{W}\}. \tag{33}$$

Using Equations (32) and (33), the moments in Equation (30) satisfy the condition

$$\boldsymbol{T}^{\sigma_k}_{j(k)} = \bar{\boldsymbol{T}}^{\sigma_k}_{j(k)} \tag{34}$$

for $k = 1, \ldots, K$ and $j(k) = 0, \ldots, J_b(k) - 1 + J_c(k) - 1$. The proof of this relation is given in [13] or, in a slightly generalized manner, also in [18]. To demonstrate the basic idea in the argumentation used thereby, the matching of the zeroth moment about σ_k similar to the representation in [39] is shown.

According to Equations (29) and (20), the zeroth moment of the reduced system can be written as

$$\bar{\boldsymbol{T}}^{\sigma_k}_0 = \boldsymbol{C} \cdot \boldsymbol{V} \cdot (\sigma_k \boldsymbol{W}^T \cdot \boldsymbol{E} \cdot \boldsymbol{V} - \boldsymbol{W}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V})^{-1} \cdot \boldsymbol{W}^T \cdot \boldsymbol{B}. \tag{35}$$

Keeping the definition of the input Krylov-subspace (32) in mind, the columns of the matrix $(\sigma_k \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B}$ can be written as a linear combination of the columns $\boldsymbol{V}$, that is

$$\exists \boldsymbol{H}_0 \in \mathbb{C}^{m \times n}: \quad (\sigma_k \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B} = \boldsymbol{V} \cdot \boldsymbol{H}_0. \tag{36}$$

Combing Eqs. (35) and (36) leads to the desired result

$$\begin{aligned}
\bar{\boldsymbol{T}}^{\sigma}_0 &= \boldsymbol{C} \cdot \boldsymbol{V} \cdot (\sigma_k \boldsymbol{W}^T \cdot \boldsymbol{E} \cdot \boldsymbol{V} - \boldsymbol{W}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V})^{-1} \cdot \boldsymbol{W}^T \cdot (\sigma_k \boldsymbol{E} - \boldsymbol{A}) \cdot \boldsymbol{V} \cdot \boldsymbol{H}_0 \\
&= \boldsymbol{C} \cdot \boldsymbol{V} \cdot (\sigma_k \boldsymbol{W}^T \cdot \boldsymbol{E} \cdot \boldsymbol{V} - \boldsymbol{W}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V})^{-1} \\
&\quad \cdot (\sigma_k \boldsymbol{W}^T \cdot \boldsymbol{E} \cdot \boldsymbol{V} - \boldsymbol{W}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V}) \cdot \boldsymbol{H}_0 \\
&= \boldsymbol{C} \cdot \boldsymbol{V} \cdot \boldsymbol{H}_0 \\
&= \boldsymbol{C} \cdot (\sigma_k \boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B} \\
&= \boldsymbol{T}^{\sigma_k}_0.
\end{aligned} \tag{37}$$

Methods in which the input Krylov-subspace $\boldsymbol{V}$ and output Krylov-subspace $\boldsymbol{W}$ are used to build the reduced model are denoted as two-sided Krylov-subspace methods. In a one-sided Krylov-subspace method, only one of the subspaces (32) or (33) is used and the other subspace is chosen in such a way, that $\boldsymbol{W}^T \cdot \boldsymbol{A} \cdot \boldsymbol{V}$ is nonsingular. Yet, only half of the moments can be matched.

In practise, it is not advisable to use the definitions of the Krylov-subspaces (32) and (33) directly to build the bases $\boldsymbol{V}$ and $\boldsymbol{W}$. As stated in [10], the vectors of the Krylov-sequence (31) will quickly converge to dominant eigenspaces of the matrix $\boldsymbol{M}$ and, in finite-precision arithmetic, even for a small number of iterations k they contain only information about these eigenspaces. A remedy is to construct orthogonal bases for $\mathcal{V}$ and $\mathcal{W}$ using the Lanczos process or the Arnoldi algorithm. In this work, the Arnoldi algorithm is used as it is numerically more stable and easier to implement. A comparison of the two methods is given in [18, 39].

4.2. Adaptation to second order systems

The Krylov-subspace method introduced in Section 4.1 was established using the generalised state-space description Equation (17). However, as in Section 3.1 a projection method was applied directly to the model of a flexible body described in second order form, structure-preserving adaptations of this technique have to be used. For this purpose, three approaches are available. One method is, to transform the second order system into state-space form and to apply a suitable model reduction technique for such systems. If the model reduction technique preserves certain conditions defined in [28, 30], the reduced order model can be rewritten as second order system using an appropriate state-space coordinate transformation. With this approach, a reduced order model of size $2n$ in state-space description can match up to $4n - 1$ interpolation conditions in the SISO case. However, this method also has remarkable drawbacks that make it hardly applicable in the application area considered here. In this respect, properties of the original system such as stability and symmetry are not maintained and application to MIMO-systems is also limited. Furthermore, the matrices $\boldsymbol{W}$ and $\boldsymbol{V}$ are not directly accessible but this is necessary to build some of the elements in Equation (22).

Another technique is based on the assumption that the damping matrix can be neglected when building the reduced order model. Thereby, the methods developed for state-space systems can be used with little modifications.

The third possibility to apply moment-matching methods to second order systems, is to introduce second order Krylov-subspaces first used in [43] and recently revisited in [4]. The last two methods are considered in the following.

4.2.1. Adaptation to proportionally damped systems

This approach focuses on problems where it can be assumed that the damping is proportional, see [5], i.e.

$$\boldsymbol{\phi}_i \cdot \boldsymbol{D}_e \cdot \boldsymbol{\phi}_j = 2\ \omega_i\ \xi_i\ \delta_{ij} \tag{38}$$

with the i-th eigenvector $\boldsymbol{\phi}_i$, the corresponding frequency of vibration ω_i of the undamped system, the modal damping parameter ξ_i and δ_{ij} as the Kronecker delta. Such a damping matrix is obtained using the Caughey series

$$\boldsymbol{D}_e = \boldsymbol{M}_e \cdot \sum_{k=0}^{r-2} a_k \left[\boldsymbol{M}_e^{-1} \cdot \boldsymbol{K}_e\right]^k \tag{39}$$

that for $r = 2$ reduces to Rayleigh damping

$$\boldsymbol{D}_e = \alpha \boldsymbol{M}_e + \beta \boldsymbol{K}_e. \tag{40}$$

As stated in [5, 14], this assumption is adequate for many mechanical systems with known exceptions like structures with widely varying material properties or with concentrated dampers at the support parts. Also, structures resulting from a component modes synthesis exhibit non proportional damping even when the sub-systems used are proportionally damped [34]. In [16], the consistency of this approach with the assumption that the structure is lightly damped is proven and, consequently, the neglect of damping can be justified when the reduction basis is built. This is also in the spirit of other well-known reduction methods like Guyan reduction or modal reduction.

Neglecting the damping matrix $\boldsymbol{D}_e$, the transfer function of the linear elastic part in Equation (22) can be written as

$$\boldsymbol{H}(s) = \boldsymbol{C} \cdot (\lambda \boldsymbol{M}_e - (-\boldsymbol{K}_e))^{-1} \cdot \boldsymbol{B} \tag{41}$$

with $\lambda = s^2$. Taking advantage of the technical similarity with Equation (25), the rational Krylov method introduced in Section 4.1 can be applied directly when substituting $\boldsymbol{E}$ by $\boldsymbol{M}_e$ and $\boldsymbol{A}$ by $(-\boldsymbol{K}_e)$ in Equations (32) and (33). As a result, the derivatives of the original model's transfer function $\boldsymbol{H}(\lambda(s))$ and the transfer function of the reduced model $\bar{\boldsymbol{H}}(\lambda(s))$ with respect to λ match up to a certain order J, i.e.

$$\left.\frac{\partial^i}{\partial \lambda^i} \bar{\boldsymbol{H}}(\lambda(s))\right|_{\lambda=\lambda(\sigma)} = \left.\frac{\partial^i}{\partial \lambda^i} \boldsymbol{H}(\lambda(s))\right|_{\lambda=\lambda(\sigma)}, \quad i = 1, \ldots, J. \tag{42}$$

Now, by applying mathematical induction it can be shown that relation (42) implies also matching of the derivatives with respect to s

$$\left.\frac{\partial^i}{\partial s^i} \bar{\boldsymbol{H}}(s)\right|_{s=\sigma} = \left.\frac{\partial^i}{\partial s^i} \boldsymbol{H}(s)\right|_{s=\sigma}, \quad i = 0, \ldots, J. \tag{43}$$

According to Equation (29), this is equivalent to the moment-matching property for the second order transfer function (41).

Using the chain rule, for $i = 1$ the relation

$$\frac{\partial}{\partial s} \boldsymbol{H}(\lambda(s)) = \frac{\partial}{\partial \lambda} \boldsymbol{H}(\lambda(s)) \, \frac{d\lambda}{ds} = \boldsymbol{M}_1 \; a_1^1 \tag{44}$$

holds and as the coefficient a_1^1 does not depend on the reduction procedure, relation (42) comprises relation (43).

Now it is assumed, that for a given i the statement

$$\frac{\partial^i}{\partial s^i} \boldsymbol{H}(\lambda(s)) = \sum_{k=1}^{i} \boldsymbol{M}_k \; a_k^i(\lambda) \tag{45}$$

with $\boldsymbol{M}_k = \partial^k \boldsymbol{H}(\lambda(s))/\partial \lambda^k$, is true. Technically, also the relations $a_0^i \equiv 0$ and $a_{i+1}^i \equiv 0$ hold.

Proceeding to step $i + 1$ yields

$$\begin{aligned}
\frac{\partial^{i+1}}{\partial s^i}\boldsymbol{H}(\lambda(s)) &= \frac{\partial}{\partial\lambda}\left(\sum_{k=1}^{i}\boldsymbol{M}_k\ a_k^i(\lambda)\right)\frac{d\lambda}{ds}\\
&= \sum_{k=1}^{i}\boldsymbol{M}_{k+1}\ a_k^i(\lambda)\ \frac{d\lambda}{ds} + \boldsymbol{M}_k\left(\frac{d}{d\lambda}\ a_k^i(\lambda)\right)\frac{d\lambda}{ds}\\
&= \sum_{k=1}^{i+1}\boldsymbol{M}_k\left(a_{k-1}^i(\lambda)\ \frac{d\lambda}{ds} + \frac{\partial}{\partial\lambda}a_k^i(\lambda)\ \frac{d\lambda}{ds}\right)\\
&= \sum_{k=1}^{i+1}\boldsymbol{M}_k\ a_k^{i+1}(\lambda).
\end{aligned}\tag{46}$$

In other words, the i-th derivative of the transfer function with respect to s, compare Equation (43), can be written as a weighted sum of the derivatives with respect to λ, see Equation (42), and the associated coefficients a_k^i only depend on the mapping $f : s \mapsto \lambda(s)$.

Finally, it is also worth to mention that, if an orthogonal projection is performed and the corresponding projection matrix $\boldsymbol{V}$ is generated considering only one expansion point $\sigma = 0$, compare Equation (32), then the proposed method is closely related to the techniques presented in [20, 33, 48]. Nevertheless, the reasoning used in this work is quite different.

4.2.2. Second order Krylov-subspaces

For some mechanical systems the assumption made in Section 4.2.1 are not valid and the effect of damping has to be taken into account in order to build an appropriate reduced order model. For these purposes, many authors, e.g. in [22, 24, 34], introduce a state-space formulation similar to Equation (17). However, in [43], the structure of a Krylov-sequence for such a state-space system originating from a second order system is analysed and an iterative scheme to generate reduction bases directly applicable to second order systems is proposed. In [4] this scheme is denoted as a second order Krylov-subspace. Here, already a block version

$$\mathcal{G}_r(\boldsymbol{M}, \boldsymbol{N}; \boldsymbol{R}) = \operatorname{span}\{\boldsymbol{R}_0, \boldsymbol{R}_1, \boldsymbol{R}_1, \ldots, \boldsymbol{R}_{r-1}\}\tag{47}$$

with the matrices

$$\begin{aligned}
\boldsymbol{R}_0 &= \boldsymbol{R},\\
\boldsymbol{R}_1 &= \boldsymbol{M}\cdot\boldsymbol{R}_0,\\
\boldsymbol{R}_l &= \boldsymbol{M}\cdot\boldsymbol{R}_{l-1} + \boldsymbol{N}\cdot\boldsymbol{R}_{l-2} \qquad \text{for } l \geq 2
\end{aligned}\tag{48}$$

is considered. Further on, in [4] the method is extended for moment-matching about selected expansion points $\sigma_k \neq 0$. To build the input Krylov-subspace, the matrices in Equation (47) have to be chosen as

$$\begin{aligned} \boldsymbol{M} &= \left(\sigma_k^2 \boldsymbol{M}_e + \sigma_k \boldsymbol{D}_e + \boldsymbol{K}_e\right)^{-1} \cdot (2\sigma_k \boldsymbol{M}_e + \boldsymbol{D}_e), \\ \boldsymbol{N} &= -\left(\sigma_k^2 \boldsymbol{M}_e + \sigma_k \boldsymbol{D}_e + \boldsymbol{K}_e\right)^{-1} \cdot \boldsymbol{M}_e, \\ \boldsymbol{R} &= \left(\sigma_k^2 \boldsymbol{M}_e + \sigma_k \boldsymbol{D}_e + \boldsymbol{K}_e\right)^{-1} \cdot \boldsymbol{B}. \end{aligned} \tag{49}$$

Accordingly, also an output Krylov-subspace can be defined. A prove of the moment-matching property as well as an efficient algorithm to generate the second order Krylov-subspace is given in [4]. In this work a straight-forward extension to the rational Krylov method [18] is considered.

The shift points σ_k used in Equation (49) are complex numbers in general. Thus, while the original problem is real, to build the second order Krylov-subspace (47) according to the definitions in Equation (49), complex matrices occur and also the resulting projection matrices $\boldsymbol{V}$ and $\boldsymbol{W}$ turn out to be complex. The change to complex linear algebra not only increases the computational effort in generating the matrices, but also the FMBS programs mentioned in Section 2 would be forced to handle complex numbers. For real systems, a remedy directly adaptable to the second order case is given in [37]. The basic idea is, that with σ_k implicitly an expansion about its conjugate $\bar{\sigma}_k$ is executed by using the relation

$$\begin{aligned} Re\{\boldsymbol{H}(\sigma_k) \cdot \boldsymbol{x}\} &= \frac{1}{2}[(\boldsymbol{H}(\sigma_k) + \boldsymbol{H}(\bar{\sigma}_k)) \cdot \boldsymbol{x}], \\ Im\{\boldsymbol{H}(\sigma_k) \cdot \boldsymbol{x}\} &= \frac{1}{2i}[(\boldsymbol{H}(\sigma_k) - \boldsymbol{H}(\bar{\sigma}_k)) \cdot \boldsymbol{x}] \end{aligned} \tag{50}$$

that holds for every real vector $\boldsymbol{x}$ with matrix $\boldsymbol{H}(\sigma_k) = (\sigma_k^2 \boldsymbol{M}_e + \sigma_k \boldsymbol{D}_e + \boldsymbol{K}_e)^{-1} \cdot (2\sigma_k \boldsymbol{M}_e + \boldsymbol{D}_e)$ or $\boldsymbol{H}(\sigma_k) = (\sigma_k^2 \boldsymbol{M}_e + \sigma_k \boldsymbol{D}_e + \boldsymbol{K}_e)^{-1}$, respectively. Thus, two complex vectors in the basis of the subspace defined by Equations (47), (48) and (49) can be substituted by two real vectors without changing the spanned subspace.

4.2.3. Stability of the reduced order model

As annotated above, the Krylov-subspace method in its general form does not guarantee the preservation of the original model's stability properties in the reduced order model. For some applications this might not be of interest, but if the model is used for time integration, this property is essential. However, if the system matrices $\boldsymbol{M}_e$, $\boldsymbol{K}_e$ and, if already available, $\boldsymbol{D}_e$ are assumed to be symmetric and in addition the important special case of $\boldsymbol{C} = \boldsymbol{B}^T$ is considered, the input and output Krylov-subspaces correspond to each other and the oblique projection turns into an orthogonal projection.

Accordingly, the reduced system matrices show the same properties with respect to symmetry and positive (semi)definiteness, see [49], as the original matrices and the stability properties are preserved in the reduced order model. If the condition of symmetry of in- and output does not hold and thereby an oblique projection method is used, the stability of the reduced model has to be examined and restored when necessary. Appropriate methods are described in [18]. In the following, the symmetry of in- and output is assumed.

4.2.4. Adaptation to multibody dynamics

In flexible multibody dynamics, some additional requirements have to be met by a model reduction formalism. First, as stated in Section 2.3, the reduction base $\boldsymbol{V}$ is not allowed to contain any rigid body modes. If a statically determined constraint set as the tangent- or chord-frame is used, this is automatically fulfilled. If a Buckens-frame is chosen, the original substructure as well as the reduction base generated with the Krylov-subspace method described above contain rigid body modes, although only a linear combination of the elastic modes is allowed. This problem can be avoided if the orthogonality of the eigenmodes $\boldsymbol{\phi}_i$ with respect to the mass matrix $\boldsymbol{M}_e$

$$\boldsymbol{\phi}_i \cdot \boldsymbol{M}_e \cdot \boldsymbol{\phi}_j = \mu_i \delta_{ij} \tag{51}$$

is taken into account. Hence, if the columns of the basis $\boldsymbol{V}$ are constructed mass orthogonal, i.e. $\boldsymbol{V}^T \cdot \boldsymbol{M}_e \cdot \boldsymbol{V} = \boldsymbol{I}$, and during construction additional orthogonalisations with respect to the rigid body modes are made, the requirements for a Buckens-frame are met, too.

Sometimes it may be reasonable to start with a basis of eigenmodes, e.g. all eigenmodes with corresponding eigenvalues in the frequency range of interest, collected as columns in a matrix $\boldsymbol{\Phi}_F$ and to use the Krylov-subspace method to enhance this basis in order to improve the accuracy in the predefined spectrum. In this case, it is likewise appropriate to execute additional (mass)orthogonalisations with respect to the initial basis $\boldsymbol{\Phi}_F$ to avoid linear dependence of the new basis vectors. This procedure is also in the spirit of the residual component modes used in component mode synthesis, see [7].

In practise, often the damping matrix $\boldsymbol{D}_e$ is not available and the model reduction procedure is based only on the mass matrix $\boldsymbol{M}_e$ and the stiffness matrix $\boldsymbol{K}_e$. Subsequently, a damping matrix based on available modal damping parameters ξ_i can be constructed easily, if a transformed basis matrix $\hat{\boldsymbol{V}}$ spanning the same subspace as $\boldsymbol{V}$ is used, but additionally exhibiting the property, that the reduced model is diagonal

$$\boldsymbol{I} \cdot \ddot{\bar{\boldsymbol{q}}} + \mathbf{diag}\{\omega_i^2\} \cdot \bar{\boldsymbol{q}} = \bar{\boldsymbol{B}} \tag{52}$$

with $\bar{\boldsymbol{B}} = \hat{\boldsymbol{V}}^T \cdot \boldsymbol{B}$. In Equation (52), the diagonal entries ω_i^2 of the reduced stiffness matrix are real eigenvalues, corresponding to the preselected modes in $\boldsymbol{\Phi}_F$, or pseudo eigenvalues, originating from the Krylov-subspace method. According to [12], the transformed basis matrix $\hat{\boldsymbol{V}}$ can be calculated from $\boldsymbol{V}$ in two steps:

1. Solve the small eigenproblem

$$\left(\omega_i^2 \boldsymbol{I} - \bar{\boldsymbol{K}}_e\right) \cdot \boldsymbol{X} = \boldsymbol{0} \tag{53}$$

   with $\bar{\boldsymbol{K}}_e = \boldsymbol{V}^T \cdot \boldsymbol{K}_e \cdot \boldsymbol{V}$.
2. Transform the reduction basis

$$\hat{\boldsymbol{V}} = \boldsymbol{V} \cdot \boldsymbol{X}. \tag{54}$$

When necessary, the pseudo eigenvalues ω_i^2 can be used as a criterion to further reduce the model.

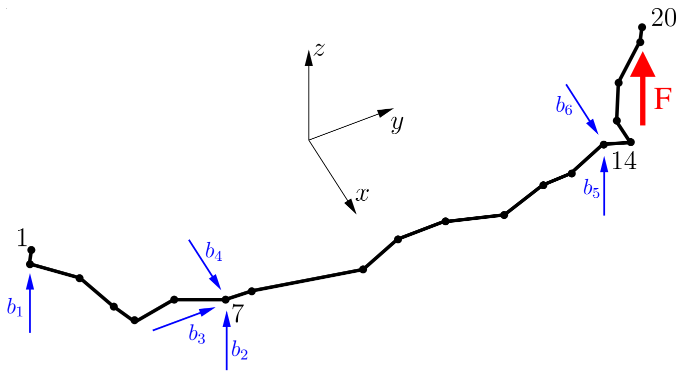

Fig. 2 FEM model of the stabilisation linkage from [47]

5. Numerical example: Stabilisation linkage

The methods described above will be evaluated using the stabilisation linkage analysed in [47]. The corresponding FEM model, generated using the commercial FEM software ANSYS, consists of 19 beam elements with a total of 114 elastic and 6 rigid degrees of freedom. At the nodes 1, 7 and 14, the stabilisation linkage is loaded with the associated inputs $\boldsymbol{b}_1$ to $\boldsymbol{b}_6$, see Figure 2. At node 20 an additional force is assumed, acting in z-direction of the global coordinate system. The damping matrix $\boldsymbol{D_e}$ is not directly available from the ANSYS model and is reconstructed from mass matrix $\boldsymbol{M}_e$ and stiffness matrix $\boldsymbol{K}_e$ with the assumption of Rayleigh damping given by Equation (40). Based upon the FEM model and the selected inputs, a SISO- and a MIMO-system, respectively, are derived by introducing different reference frames. Reduced order models based on these systems are built using the Krylov-subspace techniques presented in Sections 4.2.1 and 4.2.2. These models are compared to models obtained from application of two well-established reduction techniques, the modal reduction and an extension thereof, the frequency response mode (frm) technique proposed in [9]. To asses the accuracy of the reduced order models, the distribution of the relative reduction error

$$\epsilon(f) = \frac{\|\boldsymbol{H}(i\ 2\pi f) - \bar{\boldsymbol{H}}(i\ 2\pi f)\|_F}{\|\boldsymbol{H}(i\ 2\pi f)\|_F} \tag{55}$$

in the desired frequency interval $[f_{\min}, f_{\max}]$ is used. To deal with the MIMO-system a matrix-norm such as the Frobenius-norm

$$\|\boldsymbol{A}\|_F = \sqrt{\sum_{i,k=1}^{n} |A_{ik}|^2} \tag{56}$$

has to be introduced. Based on the error (55), also an integral quality factor is defined by

$$Q = \int_{f_{\min}}^{f_{\max}} \epsilon(f)\, df. \tag{57}$$

Table 1 The examined reduced order models in the SISO-case

Nr.	Method	Modes	σ_k (order)	Error Q
1	Krylov (Section 4.2.1)	–	$i\ 10 \cdot 2\pi$ (1) $i\ 150 \cdot 2\pi$ (5) $i\ 240 \cdot 2\pi$ (3) $i\ 300 \cdot 2\pi$ (1)	$2.69 \cdot 10^{-9}$
2	Second order Krylov (Section 4.2.2)	–	$i\ 10 \cdot 2\pi$ (0) $i\ 150 \cdot 2\pi$ (2) $i\ 240 \cdot 2\pi$ (1) $i\ 300 \cdot 2\pi$ (0)	$4.67 \cdot 10^{-9}$
3	Modal + frm	1–13	0(0)	$1.44 \cdot 10^{-2}$
4	Modal	1–14	–	11.95

5.1. The SISO-case

First, the stabilisation linkage is assumed to be kinematically constraint at the suspension nodes 1, 7 and 14 according to the directions given in Figure 2. This corresponds to a reference frame chosen as the chord-frame defined in Section 2.3. Thus, only the force acting on node 20 in z-direction is considered as in- and output, respectively.

The boundaries of the frequency interval are set to $f_{\min} = 0$ Hz and $f_{\max} = 300$ Hz. The size of the reduced order models is 14, which is about the sum of the number of the eigenfrequencies in the range between 0 Hz and 2 $f_{\max} = 600$ Hz and the number of inputs. The design parameters of the considered models are given in Table 1.

Figure 3 shows the distribution of the relative error $\epsilon(f)$ for the several reduced order models. Obviously, the least accurate model is obtained by the use of modal reduction. If in the reduction basis of this model, the eigenmode associated with the highest eigenfrequency is substituted with the 0-th Krylov vector, the relative error diminishes considerably, particularly in the lower frequency range. This technique corresponds to the aforementioned frequency

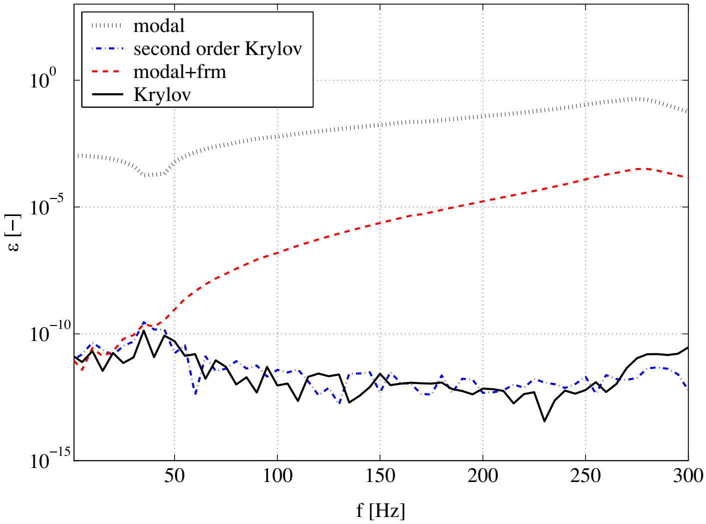

Fig. 3 Relative error of the reduced order models in the SISO-case

Table 2 The examined reduced order models in the MIMO-case

Nr.	Method	Modes	σ_k (order)	Error Q
1	Krylov (Section 4.2.1)	–	$i\ 80 \cdot 2\pi$ (1) $i\ 260 \cdot 2\pi$ (0)	$4.01 \cdot 10^{-4}$
2	Modal+Second order Krylov (Section 4.2.2)	7–13	$i\ 150 \cdot 2\pi$ (0) $i\ 150 \cdot 2\pi$ (2)	$1.20 \cdot 10^{-3}$
3	Modal+frm	7–20	$i\ 10 \cdot 2\pi$ (0)	$1.08 \cdot 10^{-3}$
4	Modal	7–27	–	1.57

response mode approach. Even more accurate models in the examined frequency range can be obtained, when the reduction procedure is based exclusively on one of the Krylov-subspace methods proposed above. For both approaches, the relative error ϵ is for a comparable effort much lower in the whole regarded frequency range, compared to the methods based on modal reduction.

5.2. The MIMO-case

Now, the reference frame is chosen to be the Buckens-frame defined in Section 2.3. Hence, the unconstrained structure has to be used in the model reduction process. The MIMO-system considered holds seven inputs and outputs each, given by the inputs $\boldsymbol{b}_1$ to $\boldsymbol{b}_6$ and the force $\boldsymbol{F}$ acting in z-direction. With the same justification as above, reduced order models of dimension 21 are considered and also the frequency range considered is the same as in the SISO case. The description of the used reduction techniques is given in Table 2. According to Figure 4, the modal reduction shows the worst behaviour in the regarded spectrum. However, the difference to the other methods has diminished compared to the SISO-case. Furthermore, the other methods roughly show a similar magnitude of accuracy. In the logarithmic scaling of Figure 4, the combination of the second order Krylov-subspace and a few eigenmodes seems

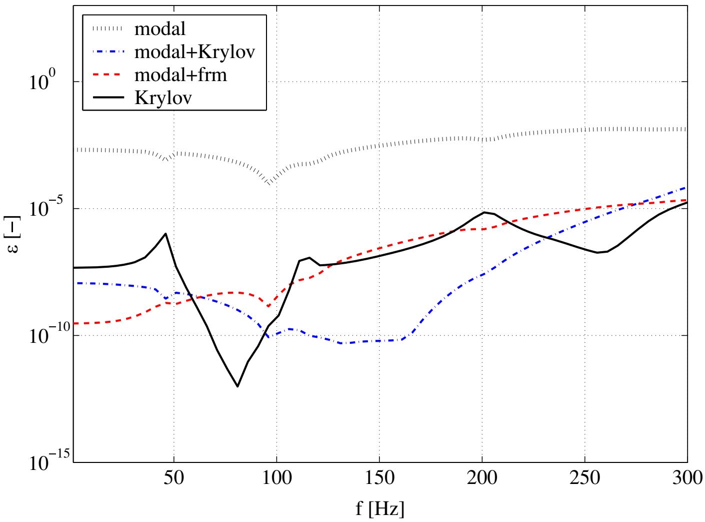

Fig. 4 Relative error of the reduced order models in the MIMO-case

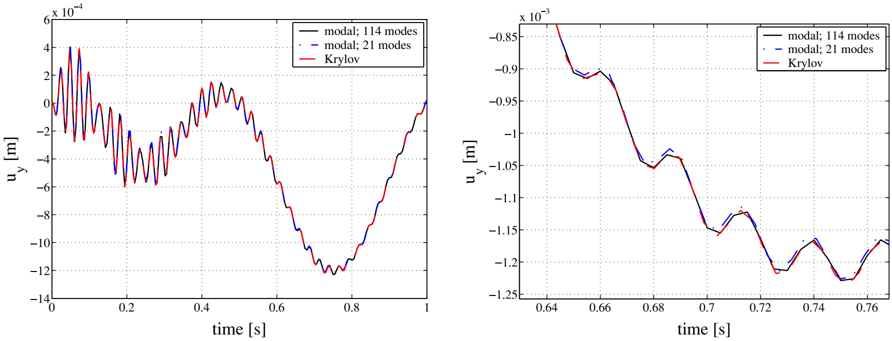

Fig. 5 Dynamic simulation (a) deflection u_y at node 20 (b) cutaway view

to be the most accurate method. Yet, if the integral measure Q is taken into account, the pure Krylov-subspace methods gives the best results. So far, the frequency domain was used to compare the different reduced order models. Yet, as the time domain is of great importance in flexible multibody dynamics, also a dynamic simulation is considered. Therefore, a time-dependent force

$$\boldsymbol{F} = \begin{bmatrix} 0 \\ 0 \\ -1000\ \sin(2\pi t) \end{bmatrix} N$$

at node 20 is assumed. The resulting DAE-system is treated numerically with the well-established solver RADAU5 [21] using an index-2 approach and the simulation time is one second. As a reference, a modal model consisting of all 114 flexible modes is used. In Figure 5, the deflection of node 20 in y-direction u_y of the reference model is compared with the results obtained using two of the reduced order models described in Table 2. As the dimension of the reduction base is rather high for the given problem, both models show quite good agreement with the reference solution. However, taking a look at the enlarged cutaway view, the results of the reduced order model obtained by the Krylov subspace method show a slightly better agreement with the reference solution.

6. Conclusion

In this work, moment-matching was used to build reduced order models for application in flexible multibody dynamics. Therefore, two Krylov-subspace based methods were proposed that give very accurate reduced order models in a predefined range of the spectrum. The reason for the superior accuracy compared to modal reduction is the consideration of the spatial distribution of the loads as well as the flexibility concerning the choice of the interpolation conditions. By defining these conditions in an appropriate way, an arbitrary frequency range can be emphasised easily. In the authors' opinion, this is more convenient than the use of modal reduction where in addition to the dominant eigenmodes in the chosen range of the spectrum, also a set of correction modes, that account for the influence of the eigenmodes not included in the reduction base, has to be determined to ensure good convergence. Referring to this, it is also worth to mention, that in combination with modal reduction, the moment-matching method contains the frequency response modes approach developed in [9] and can be used to build a set of correction modes for an initial modal reduction basis. Further enhancements could comprise the development of error estimators, the application of oblique projection and the usage of sparse matrix techniques to handle large-scale systems.

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