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A dynamics formulation of general constrained robots

Wen-Hong Zhu · Jean-Claude Piedboeuf · Yves Gonthier

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Abstract The complexity of a standard compact-in-form Lagrangian dynamical expression is proportional to the fourth power of the number of degrees of freedom (DOF) of a robotic system. This fact challenges both simulation and control of robots with hyper degrees of freedom. In this paper, a systematic approach for deriving the dynamical expression of so-called *general constrained robots* is proposed. This proposed approach has two main features. First, it uses the subsystem dynamics such as the dynamics of joints and rigid links to construct the dynamical expression of the entire robotic system in a closed form. The complexity of the resulted dynamic expression is linearly proportional to the number of DOF of a robotic system. Second, it extends the standard dynamical form and properties of the conventional single-arm constrained robots to a class of more general robotic systems including the coordinated multiple-arm robotic systems. Three spaces, namely the *general joint space*, the *general task space*, and the *extended subsystems space*, are connected through corresponding velocity/force mapping matrices.

Keywords Robot dynamics · Robot simulation and control · Subsystem dynamics based formulation · Robots with closed chains · Virtual decomposition control · General constrained robots

1. Introduction

Model-based dynamic control has been extensively developed for both single-arm constrained manipulators and multiple cooperating manipulators. With respect to the single-arm constrained manipulators, hybrid control [1], compliance control [2, 3], nonlinear feedback linearization [4], parallel position/force control [5], and adaptive control [6–10] are all based

W.-H. Zhu (✉) · J.-C. Piedboeuf · Y. Gonthier
Spacecraft Engineering, Space Technologies, Canadian Space Agency, 6767 route de l'Aeroport, Saint-Hubert, QC J3Y 8Y9, Canada
e-mail: wen-hong.zhu@space.gc.ca

on the famous dynamic model

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau - J^T F \tag{1}$$

expressed in the joint space or

$$\underline{M}(q)\ddot{x} + \underline{C}(q,\dot{x})\dot{x} + \underline{G}(q) = u - F \tag{2}$$

expressed in the Cartesian space [11], where

$$\begin{aligned}
\underline{M}(q) &= J^{-T} M(q) J^{-1} \\
\underline{C}(q,\dot{x}) &= J^{-T} C(q,\dot{q}) J^{-1} - J^{-T} M(q) J^{-1} \dot{J} J^{-1} \\
\underline{G}(q) &= J^{-T} G(q) \\
u &= J^{-T} \tau
\end{aligned}$$

when the Jacobian matrix J is invertible. On the other hand, coordinated multiple manipulators are treated as closed kinematic chains. The dynamic model of a group of coordinated manipulators simply consists of the dynamic models of the individual manipulators plus the dynamic model of the held object [12–29], Kreutz and Lokshin [12] pointed out that the loss of motion DOF in a coordinated multiple manipulator system just equals the dimension of the internal force. This argument implies a similarity between the internal forces and the constraint forces such that they can be renamed as "general constraint forces" in the sense that they do not affect the motion of the systems.

Most approaches aforementioned are fundamentally based on the Lagrangian formulation to obtain the dynamical expressions in closed form. It is well known that the complexity of a standard compact-in-form Lagrangian dynamical expression is proportional to the fourth power of the number of degrees of freedom (DOF) of a robotic system. This fact challenges both simulation and control of robots with hyper degrees of freedom. With respect to this difficulty, a novel systematic approach for dynamical modeling by using solely the dynamics of the subsystems (joints and rigid bodies) is proposed in this paper. The separated dynamics of the rigid bodies and joints which comprise a robotic system are much simpler than the standard Lagrangian dynamical expression of the aggregated (coupled) robotic system. Furthermore, the proposed approach is able to express the dynamics of a class of general over-actuated robotic systems including the coordinated multiple-arm robotic systems in a closed form similar to (2). The proposed approach is associated with three spaces connected by corresponding velocity/force mapping matrices. The three spaces, referring to the *general joint space* $\mathcal{A}$, the *general task space* $\mathcal{O}$, and the *extended subsystems space* $\mathcal{S}$, can be formulated in a systematic way. As will be seen below, the velocity/force mapping (kinematics based) matrices connecting the three spaces impose the original coupling (constraints) on the systems and play a crucial role in forming the dynamic expressions. While the proposed modeling approach results in a dynamic expression that possesses a closed form similar to the conventional Lagrangian formulation, the computational complicity is just proportional to the number of degrees of freedom (DOF) of a complex robotic system.

This paper is organized as follows: Section 2 proposes the modeling approach for the *general constrained robots* in which three spaces and two mapping matrices are defined; Section 3 summarizes the modeling procedure followed by two examples; and Section 4 presents an adaptive control design based on the developed dynamic model.

2. General constrained robots

This section presents the procedure for deriving the dynamic expression of a *general constrained robot*.

Without loss of generality, it is assumed that a *general constrained robot* is a base-fixed robot[1] having n_1 single-DOF prismatic or revolute joints and n_3 three-DOF spherical joints. Among the n_1 single-DOF joints, n_{1a} joints are actuated and n_{1u} joints are unactuated. Among the n_3 three-DOF spherical joints, n_{3a} joints are actuated and n_{3u} joints are unactuated. It makes

$$n_1 = n_{1a} + n_{1u} \tag{3}$$

$$n_3 = n_{3a} + n_{3u}. \tag{4}$$

Furthermore, it is assumed that the entire robotic system has m motion DOF.[2] This implies that there exist

$$n_c \stackrel{\text{def}}{=} [n_{1a} + n_{1u} + 3\,(n_{3a} + n_{3u})] - m > 0 \tag{5}$$

overall independent constraints inside the system including the operational constraints[3] and the inherent mechanical constraints.[4]

Two types of coordinate systems are used throughout the paper. The first type is a coordinate system consisting of 3 mutually orthogonal unit axes as basis and the second type is a single-axis coordinate system. The 3-unit-axis orthogonal coordinate system (or called coordinate frame or simply frame) is used to describe the linear/angular velocities and the force/moments of the rigid links, as well as the angular velocities and the moments of the three-DOF spherical joints. The single-axis coordinate system is used to describe the linear/rotational velocities and the force/torques of the single-DOF joints.

Let $\mathcal{T} = [\mathcal{T}_a,\ \mathcal{T}_u]$, with $\mathcal{T}_a \in \Re^{n_c \times (n_{1a}+3n_{3a})}$ and $\mathcal{T}_u \in \Re^{n_c \times (n_{1u}+3n_{3u})}$, be a full row-rank matrix characterizing the n_c constraints as

$$\mathcal{T}_a \dot{q}_a + \mathcal{T}_u \dot{q}_u = 0 \tag{6}$$

where $\dot{q}_a \in \Re^{n_{1a}+3n_{3a}}$ denotes the velocity coordinates of all actuated joints and $\dot{q}_u \in \Re^{n_{1u}+3n_{3u}}$ denotes the velocity coordinates of all unactuated joints. Let n_p denote the rank of matrix

[1] A space robot in which the base is floating can be converted to a base-fixed robot by imposing a virtual zero-mass six-DOF manipulator between the base of the space robot and the absolute base [31].

[2] The number of motion DOF refers to the overall motion degrees of freedom of the entire robotic system, including the motion DOF at the end-effector plus the joint redundant motion DOF targeting at optimization and obstacle avoidance. For instance, with respect to a 8-joint redundant robot manipulator in contact with an environment with 3 motion DOF at the end-effector, it follows that $m = 8 - (6 - 3) = 5$.

[3] This refers to the constraints due to the purpose of operations, such as the constraints imposed on the end-effectors.

[4] This refers to the coupling (constraints) imposed by the mechanical structures of the robots on the joints, such as the mechanical constraints in the parallelogramic four-bar mechanisms or in the Stewart platform parallel robots.

$\mathcal{T}_u$[5] with

$$n_p \leq \min\{n_c, n_{1u} + 3n_{3u}\}. \tag{7}$$

Reorder $\dot{q}_a$ to form $[\dot{q}_{a1}^T, \dot{q}_{a2}^T]^T$ and reorder $\dot{q}_u$ to form $[\dot{q}_{u1}^T, \dot{q}_{u2}^T]^T$ such that (6) can be re-written as

$$\mathcal{T}_{u1}\dot{q}_{u1} + \mathcal{T}_{a1}\dot{q}_{a1} + \mathcal{T}_{a2}\dot{q}_{a2} + \mathcal{T}_{u2}\dot{q}_{u2} = 0 \tag{8}$$

subject to

(1) $\mathcal{T}_{u1} \in \Re^{n_c \times n_p}$ and $\mathcal{T}_{a1} \in \Re^{n_c \times (n_c - n_p)}$ are of full column-rank, and
(2) $[\mathcal{T}_{u1},\ \mathcal{T}_{a1}] \in \Re^{n_c \times n_c}$ is invertible.

The reordering process intends to find the dependent velocity coordinates in the joints imposed by the n_c overall constraints. First, it finds n_p independent columns in $\mathcal{T}_u$ to form a new matrix $\mathcal{T}_{u1}$. Then, it finds $n_c - n_p$ complementary independent columns in $\mathcal{T}_a$ to form a new matrix $\mathcal{T}_{a1}$. Subject to n_c overall constraints, the matrix $\mathcal{T}_{u1}$ defines n_p dependent velocity coordinates in the unactuated joints. The remaining $n_c - n_p$ dependent velocity coordinates are defined by $\mathcal{T}_{a1}$ for the actuated joints. Note that the $n_c - n_p$ constraints imposed on the actuated joints result in the same number of dimensions for the general constraint force.[6]

It follows from (8) that

$$\begin{bmatrix} \dot{q}_{u1} \\ \dot{q}_{a1} \end{bmatrix} = -[\,\mathcal{T}_{u1} \quad \mathcal{T}_{a1}\,]^{-1}[\,\mathcal{T}_{a2} \quad \mathcal{T}_{u2}\,]\begin{bmatrix} \dot{q}_{a2} \\ \dot{q}_{u2} \end{bmatrix} = -\begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{bmatrix}\begin{bmatrix} \dot{q}_{a2} \\ \dot{q}_{u2} \end{bmatrix}. \tag{9}$$

Equation (9) indicates that the joint velocity coordinates $\dot{q}_{a2} \in \Re^{[(n_{1a}+3n_{3a})-(n_c-n_p)]}$ of the actuated joints and $\dot{q}_{u2} \in \Re^{[(n_{1u}+3n_{3u})-n_p]}$ of the unactuated joints form the independent joint velocity coordinates of the system. As mentioned above, n_p dimensional constraints are imposed on the unactuated (passive) joints as

$$\dot{q}_{u1} + \mathcal{T}_{11}\dot{q}_{a2} + \mathcal{T}_{12}\dot{q}_{u2} = 0 \tag{10}$$

and the remaining $n_c - n_p$ dimensional constraints are imposed on the actuated joints as

$$\dot{q}_{a1} + \mathcal{T}_{21}\dot{q}_{a2} + \mathcal{T}_{22}\dot{q}_{u2} = 0. \tag{11}$$

[5] The number n_p specifies the number of unactuated dimensions imposed by the overall constraints. In case of $n_p < n_{1u} + 3n_{3u}$, the difference $n_{1u} + 3n_{3u} - n_p$ will form part of the (unactuated) motion DOF. In typical examples of the parallelogramic four-bar mechanisms or the Stewart platform parallel robots in free motion, it yields $n_p = n_c = n_{1u} + 3n_{3u}$.

[6] The general constraint force in this paper refers to all the constraint forces that can be directly regulated by the force/torque actuators without affecting the motion of the systems. The general constraint forces include the conventional constraint forces for single-arm robots and the internal forces for coordinated multiple-arm robots.

Premultiplying (11) by a $(n_c - n_p) \times (n_c - n_p)$ full-rank matrix, denoted as $\mathcal{T}_c$, yields

$$\mathcal{T}_c \left[I_{(n_c - n_p)} \quad \mathcal{T}_{21} \quad \mathcal{T}_{22} \right] \begin{bmatrix} \dot{q}_{a1} \\ \dot{q}_{a2} \\ \dot{q}_{u2} \end{bmatrix} \stackrel{\text{def}}{=} \mathcal{J}_f \begin{bmatrix} \dot{q}_a \\ \dot{q}_{u2} \end{bmatrix} = 0 \tag{12}$$

where the matrix $\mathcal{T}_c$ maps the $n_c - n_p$ dimensional constraints defined by (11) to the place where the $n_c - n_p$ dimensional general constraint force is defined[7]. Note that $\mathcal{J}_f \in \Re^{(n_c - n_p) \times (n_{1a} + 3n_{3a} + n_{1u} + 3n_{3u} - n_p)}$ is of full row-rank.

Define the system dimensions as

$$n \stackrel{\text{def}}{=} (n_{1a} + 3n_{3a} + n_{1u} + 3n_{3u}) - n_p = m + (n_c - n_p). \tag{13}$$

The system dimensions include m motion degrees of freedom and $n_c - n_p$ dimensions for the general constraint forces.

Remark 2.1. The n_c constraints of a robotic system represent the pure mechanical constraints inside the robotic system (such as the constraints among the linkages) and the constraints between the robot and the environment. Among the n_c overall constraints, n_p constraints are satisfied by releasing the corresponding motion coordinates associated with the unactuated joints (as performed by (10)). The remaining $n_c - n_p$ constraints will be reflected to the ultimate dynamic equation.

2.1. General joint space $\mathcal{A}$

The *general joint space* $\mathcal{A}$ consists of $n_{1a} + 3n_{3a}$ actuated dimensions and $n_{1u} + 3n_{3u} - n_p$ unactuated dimensions and, therefore, possesses n dimensions.

Let

$$\dot{q} = \begin{bmatrix} \dot{q}_a \\ \dot{q}_{u2} \end{bmatrix} \in \Re^n \tag{14}$$

be the velocity coordinates and let

$$\tau = \begin{bmatrix} \tau_a \\ 0 \end{bmatrix} \in \Re^n \tag{15}$$

be the corresponding torque coordinates, in the *general joint space* $\mathcal{A}$.

2.2. General task space $\mathcal{O}$

The *general task space* $\mathcal{O}$ consists of a m-dimensional motion space and a $(n_c - n_p)$-dimensional constraint force space. Thus, it possesses n dimensions, and is analogous to the Cartesian space. The space $\mathcal{O}$ is used to express the velocity and force control specifications and, therefore, is application oriented and control objective driven.

[7] The matrix $\mathcal{T}_c$ can be identity if the constraint Equation (11) is directly defined in terms of operations.

In view of (12) and (14), the constraint force space is governed by $\mathcal{J}_f \dot{q} = 0$. Accordingly, the constraint force transferred to the *general joint space* can be expressed as $\mathcal{J}_f^T \eta_f$, where $\eta_f \in \Re^{n_c - n_p}$ denotes the general constraint force coordinates that do not transfer power[8]. Besides the general constraint forces, the dynamic contact force coordinates in the motion space [30] can be denoted as $\eta_m \in \Re^m$ which represents the forces that are state-dependent such as the frictional forces and the contact forces with compliant environments. As a result, the overall general constraint force coordinates and the dynamic contact force coordinates converted to the *general joint space* can be written as

$$\tau^* = \begin{bmatrix} \mathcal{J}_f^T & \mathcal{D}_m^T \end{bmatrix} \begin{bmatrix} \eta_f \\ \eta_m \end{bmatrix} \tag{16}$$

where $\mathcal{D}_m \in \Re^{m \times n}$ is a matrix.

In the motion space, let $\mathcal{V}_m \in \Re^m$ be the independent velocity coordinates subject to

$$\mathcal{V}_m = \mathcal{J}_m \dot{q} \tag{17}$$

where $\mathcal{J}_m \in \Re^{m \times n}$ is of full row-rank.

Since the rows of $\mathcal{J}_f$ span the space for the general constraint forces and the rows of $\mathcal{J}_m$ span the configuration space for the motion DOF, the orthogonality of $\mathcal{J}_f$ and $\mathcal{J}_m$, i.e. $\mathcal{J}_f^T \mathcal{J}_m = 0$, can be ensured [30]. Furthermore, consider the fact that both $\mathcal{J}_m$ and $\mathcal{J}_f$ are of full row-rank, the composed matrix

$$\begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix} \in \Re^{n \times n}$$

is of full-rank and, therefore, is invertible such that

$$\begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-1} = \begin{bmatrix} \Omega & \Psi \end{bmatrix} \tag{18}$$

where $\Omega \in \Re^{n \times m}$ and $\Psi \in \Re^{n \times (n-m)}$ are two matrices of full column-rank.

In view of (12) (14), and (17), it follows that

$$\mathcal{V} \stackrel{\text{def}}{=} \begin{bmatrix} \mathcal{V}_m \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix} \dot{q}. \tag{19}$$

Consequently, it yields

$$\dot{q} = \Omega \mathcal{V}_m. \tag{20}$$

[8] η_f is equivalent to the Lagrangian multiplier.

2.3. Extended subsystems space $\mathcal{S}$

The *extended subsystems space* $\mathcal{S}$ in which the dynamics of rigid links and joints are expressed casts the kernel of the proposed approach for dynamic modeling. This is due to the fact that the separated dynamics of the rigid bodies and joints which comprise a robotic system are much simpler than the standard Lagrangian dynamical expression of the aggregated (coupled) robotic system, particularly for robots with high DOF [31].

Three types of subsystems, namely the rigid links (rigid bodies), the single-DOF prismatic or revolute joints, and the three-DOF spherical joints, are studied. Let Φ be a set containing all frames each is fixed to a corresponding rigid link, Θ_1 be a set containing the sequential numbers of all single-DOF joints, and Θ_3 be a set containing the sequential numbers of all three-DOF spherical joints.

The dynamics of a rigid body expressed in the body-fixed frame $\alpha \in \Phi$ is [31]

$$M_\alpha \frac{d}{dt}({}^\alpha X) + C_\alpha {}^\alpha X + G_\alpha =^\alpha \underline{F}, \tag{21}$$

where the explicit expressions of $M_\alpha \in \Re^{6\times 6}$, $C_\alpha \in \Re^{6\times 6}$, and $G_\alpha \in \Re^6$ are given by [31] (p. 418), M_α is constant and symmetric and C_α is skew-symmetric; ${}^\alpha X \in \Re^6$ denotes the generalized linear/angular velocity of frame α and expressed in frame α, and ${}^\alpha\underline{F} \in \Re^6$ denotes the net force/moment of the rigid body expressed in frame α.

The dynamics of the jth single-DOF joint is

$$\underline{\tau}_{1j} \triangleq I^*_{1j}\ddot{q}_{1j} + \xi_{1j}(t) = \tau_{1j} - \text{proj}(F)_{1j}, \;\; j \in \Theta_1 \tag{22}$$

where $I^*_{1j} \in \Re$ is the equivalent mass or rotational inertia, $q_{1j} \in \Re$ is the joint displacement, $\xi_{1j}(t) \in \Re$ is the frictional force/torque, $\tau_{1j} \in \Re$ is the joint control force/torque,[9] $\underline{\tau}_{1j} \in \Re$ represents the net force/torque devoted to the joint dynamics, while$-\text{proj}(F)_j \in \Re$ is the projected force/moment from the links onto the joint axis.

The dynamics of the ith three-DOF spherical joint is

$$\underline{\tau}_{3i} \triangleq \xi_{3i}(t) = \tau_{3i} - \text{proj}(F)_{3i}, \quad i \in \Theta_3 \tag{23}$$

where $\xi_{3i}(t) \in \Re^3$ denotes the frictional moment, $\tau_{3i} \in \Re^3$ is the joint control torque[10], $\underline{\tau}_{3i} \in \Re^3$ represents the net moment devoted to the spherical joint dynamics, while $-\text{proj}(F)_{3i} \in \Re^3$ is the projected moment from the links onto the three joint axes. Note that in the spherical joint dynamics, the inertial term is omitted, since the mass properties of the spherical joints are fully incorporated into the corresponding links. Therefore, only the three-dimensional friction moments are handled in (23).

Remark 2.2. Note that the Equations (21), (22), and (23) cannot be treated as independent equations, since all the velocity coordinates in space $\mathcal{S}$ are subject to the inherent mechanical constraints inside the robotic system.

[9] $\tau_{1j} = 0$ holds for a unactuated joint.

[10] $\tau_{3i} = 0$ holds for a unactuated spherical joint.

The velocity and net force/moment coordinates in $\mathcal{S}$ are

$$\underline{\mathcal{X}} \triangleq \begin{bmatrix} \vdots \\ \dot{q}_{1j} \\ \vdots \\ \dot{q}_{3i} \\ \vdots \\ {}^{\alpha}X \\ \vdots \end{bmatrix}; \qquad \underline{\mathcal{F}} \triangleq \begin{bmatrix} \vdots \\ \underline{\tau}_{1j} \\ \vdots \\ \underline{\tau}_{3i} \\ \vdots \\ {}^{\alpha}\underline{F} \\ \vdots \end{bmatrix}.$$

Note that $\dot{q}_{1j}$ in $\underline{\mathcal{X}}$ and $\underline{\tau}_{1j}$ in $\underline{\mathcal{F}}$ appear in a corresponding row. So do $\dot{q}_{3i}$ and $\underline{\tau}_{3i}$, and ${}^{\alpha}X$ and ${}^{\alpha}\underline{F}$. In view of (21), (22), and (23), it follows from the definitions of $\underline{\mathcal{X}}$ and $\underline{\mathcal{F}}$ that

$$\underline{\mathcal{F}} = \underline{\mathcal{M}}\,\dot{\underline{\mathcal{X}}} + \underline{\mathcal{C}}\,\underline{\mathcal{X}} + \underline{\mathcal{G}} \tag{24}$$

where

$$\begin{aligned} \underline{\mathcal{M}} &= \operatorname{diag}\{\ldots, I_{1j}^{*}, \ldots, 0, \ldots, M_{\alpha}, \ldots\} \\ \underline{\mathcal{C}} &= \operatorname{diag}\{\ldots, 0, \ldots, 0, \ldots, C_{\alpha}, \ldots\} \\ \underline{\mathcal{G}} &= [\ldots, \xi_{1j}(t), \ldots, \xi_{3i}^{T}(t), \ldots, G_{\alpha}^{T}, \ldots]^{T}. \end{aligned}$$

Based on the definition of $\dot{q} \in \Re^{n}$ in $\mathcal{A}$, the extended velocity $\underline{\mathcal{X}}$ in $\mathcal{S}$ can be expressed as

$$\underline{\mathcal{X}} = T_{S}\dot{q}. \tag{25}$$

The physical meaning of (25) is that the velocities of all subsystems (rigid links, single-DOF joints, and three-DOF joints) are completely dependent on the velocities in $\mathcal{A}$[11]. In general, T_S has more rows than columns.

With $\eta_f = 0$ (by virtually breaking up the constraints $\mathcal{J}_f\dot{q} = 0$ defined in (12)) and $\eta_m = 0$, the power received by all rigid links equals the power generated by all joints, i.e.

$$\sum_{\alpha\in\Phi} {}^{\alpha}X^{T}\,{}^{\alpha}\underline{F} = \sum_{j\in\Theta_1}\{\dot{q}_{1j}[\operatorname{proj}(F)_{1j}]\} + \sum_{i\in\Theta_3}\left\{\dot{q}_{3i}^{T}[\operatorname{proj}(F)_{3i}]\right\}. \tag{26}$$

Substituting $\operatorname{proj}(F)_{1j} = \tau_{1j} - \underline{\tau}_{1j}$ from (22) and $\operatorname{proj}(F)_{3i} = \tau_{3i} - \underline{\tau}_{3i}$ from (23) into (26) yields

$$\underline{\mathcal{X}}^{T}\underline{\mathcal{F}} = \dot{q}^{T}\tau \tag{27}$$

in which $\sum_{j\in\Theta_1}(\dot{q}_{1j}\tau_{1j}) + \sum_{i\in\Theta_3}(\dot{q}_{3i}^{T}\tau_{3i}) = \dot{q}^{T}\tau$ is used.

[11] Equation (25) imposes kinematic constraints on (21), (22), and (23).

In view of (27) and (25), it follows that

$$\dot{q}^T T_S^T \underline{\mathcal{F}} = \dot{q}^T \tau. \tag{28}$$

In a particular case that only the kth element of $\dot{q}$ is non-zero, i.e. mathematically, $\dot{q}_k \neq 0$ and $\dot{q}_j = 0$ for $j = 1, 2, \ldots, k-1, k+1, \ldots, n$, it follows from (28) that

$$\left(T_S^T \underline{\mathcal{F}}\right)_k = \tau_k \tag{29}$$

where $(T_S^T \underline{\mathcal{F}})_k$ and τ_k denote the kth elements of $T_S^T \underline{\mathcal{F}}$ and τ, respectively. Equation (29) gives a force transformation from $\mathcal{S}$ toward $\mathcal{A}$. Since the force transformation depends on the configuration only, (29) will be valid for $k \in \{1, n\}$. Thus, it yields

$$T_S^T \underline{\mathcal{F}} = \tau. \tag{30}$$

Note that equation (30) demonstrates how the net force/moments of a robotic system are mapped into the *general joint space*. The actual joint torques, however, will include the torques that are transfered from the general constraint forces η_f (by reinstalling the constraints $\mathcal{J}_f \dot{q} = 0$) and the dynamic contact forces η_m as formulated by (16). Therefore, the force transformation equation should be re-written as

$$T_S^T \underline{\mathcal{F}} = \tau - \tau^*. \tag{31}$$

2.4. The Dynamic Model

Based on (31), (24), (25), (16), and (19), the dynamic model of a *general constrained robot* expressed in $\mathcal{O}$ is written as

$$\mathcal{M} \begin{bmatrix} \dot{\mathcal{V}}_m \\ 0 \end{bmatrix} + \mathcal{C} \begin{bmatrix} \mathcal{V}_m \\ 0 \end{bmatrix} + \mathcal{G} = \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-T} \tau - \begin{bmatrix} 0 \\ \eta_f \end{bmatrix} \tag{32}$$

where

$$\begin{aligned}
\mathcal{M} &= \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-T} T_S^T \underline{\mathcal{M}} T_S \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-1} \\
\mathcal{C} &= \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-T} T_S^T \underline{\mathcal{M}} \frac{d}{dt} \left(T_S \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-1} \right) + \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-T} T_S^T \underline{\mathcal{C}} T_S \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-1} \\
\mathcal{G} &= \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-T} T_S^T \underline{\mathcal{G}} + \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-T} \mathcal{D}_m^T \eta_m.
\end{aligned}$$

Remark 2.3. In Equation (32), $\mathcal{V}_m \in \Re^m$ denotes the independent velocity coordinates in the *general task space*. Among the m dimensions of $\mathcal{V}_m \in \Re^m$, there exist $(n_{1u} + 3n_{3u} - n_p)$

unactuated dimensions. Therefore,

$$m \geq n_{1u} + 3n_{3u} - n_p \tag{33}$$

holds. Meanwhile, the dimensions for the general constraint force are $n_c - n_p$.

Remark 2.4. In simulations, the independent variables of integration are $\dot{\mathcal{V}}_m \in \Re^m$. The effective velocity coordinates of the entire system are denoted as $[\mathcal{V}_m^T, 0^T]^T \in \Re^n$, where $\mathcal{V}_m \in \Re^m$ is the integral of $\dot{\mathcal{V}}_m \in \Re^m$.

Remark 2.5. In view of the dynamic equation of the *general constrained robots* represented by (32), two groups of components are needed. The first group consists of two kinematics mapping matrices $[{\mathcal{J}_m \atop \mathcal{J}_f}]$ and T_S; and the second group consists of $\underline{\mathcal{M}}$, $\underline{\mathcal{C}}$, and $\underline{\mathcal{G}}$, which can be formed in terms of the separated dynamics of the rigid links and joints [31]. Since the dynamic formulation in $\mathcal{S}$ is standard, the task which needs to be done for a particular application is to find the two kinematics mapping matrices.

Remark 2.6. Using subsystem dynamics to represent the complete dynamics of a robot can also be found in [32]. In general, the approach in [32] applies to both rigid and flexible robot arms, and the approach proposed in this paper particularly addresses the high dimensional rigid-link robots with complex kinematic constraints.

Remark 2.7. Note that $\underline{\mathcal{M}}$ is time invariant and $\underline{\mathcal{C}}$ is skew-symmetric. Therefore, it follows that

$$\frac{1}{2}\dot{\mathcal{M}} - \mathcal{C}$$

is skew-symmetric.[12]

Remark 2.8. Newton-Euler formulation has been well known for its recursive forms in either forward dynamics [33, 34] or inverse dynamics [35, 31]. The dynamic formulation presented in this paper can be considered as a closed form of Newton-Euler formulation that may have exactly the same application forum as the Lagrangian formulation, but its simplicity. In other words, the closed form of Newton-Euler formulation is applicable to both forward dynamics based simulation and inverse dynamics based control [36, 37].

[12] In view of (24), $\underline{\mathcal{M}}$ is constant and symmetric due to the constant $I_{1j}^* \in \Re$ and the constant and symmetric $M_\alpha \in \Re^{6\times 6}$, and $\underline{\mathcal{C}}$ is skew-symmetric due to the skew-symmetric $C_\alpha \in \Re^{6\times 6}$[31]. It follows from (32) that $\frac{1}{2}\dot{\mathcal{M}} - \mathcal{C} = \frac{1}{2}\frac{d}{dt}\left(\begin{bmatrix}\mathcal{J}_m \\ \mathcal{J}_f\end{bmatrix}^{-T} T_S^T\right)\underline{\mathcal{M}} T_S \begin{bmatrix}\mathcal{J}_m \\ \mathcal{J}_f\end{bmatrix}^{-1} - \frac{1}{2}\begin{bmatrix}\mathcal{J}_m \\ \mathcal{J}_f\end{bmatrix}^{-T} T_S^T \underline{\mathcal{M}} \frac{d}{dt}\left(T_S \begin{bmatrix}\mathcal{J}_m \\ \mathcal{J}_f\end{bmatrix}^{-1}\right) - \begin{bmatrix}\mathcal{J}_m \\ \mathcal{J}_f\end{bmatrix}^{-T} T_S^T \underline{\mathcal{C}} T_S \begin{bmatrix}\mathcal{J}_m \\ \mathcal{J}_f\end{bmatrix}^{-1}$ is skew-symmetric.

3. Modeling procedure and examples

3.1. Modeling procedure

In this subsection, the modeling approach presented in the last section is summarized into steps as follows:

Step 1: For a given base-fixed (or equivalent) robotic system, count the numbers of the single-DOF prismatic or revolute joints and the three-DOF spherical joints. It yields n_1 and n_3. Among the n_1 single-DOF prismatic or revolute joints, count the numbers of the actuated joints and the unactuated joints. It yields n_{1a} and n_{1u}. Among the n_3 three-DOF spherical joints, count the numbers of the actuated joints and the unactuated joints. It yields n_{3a} and n_{3u}.

Step 2: Determine the number of degrees of freedom of the system in motion and obtain m. Calculate n_c (the dimensions of the overall constraints) in terms of (5).

Step 3: Specify the m-dimensional velocity configuration space and specify the n_c overall constraints. Assign appropriate coordinate frames for motion/force descriptions. Furthermore, assign coordinate frames to each rigid link or joint [31].

Step 4: Specify $\dot{q}_a \in \Re^{n_{1a}+3n_{3a}}$ and $\dot{q}_u \in \Re^{n_{1u}+3n_{3u}}$, and form $\mathcal{T}_a$, $\mathcal{T}_u$ subject to (6) accordingly. It yields the number n_p – the rank of $\mathcal{T}_u$.

Step 5: Partition $\mathcal{T}_a$ and $\mathcal{T}_u$ to form $\mathcal{T}_{a1}$, $\mathcal{T}_{a2}$, $\mathcal{T}_{u1}$, and $\mathcal{T}_{u2}$ such that $[\mathcal{T}_{u1},\ \mathcal{T}_{a1}] \in \Re^{n_c \times n_c}$ is invertible.

Step 6: Calculate $\mathcal{T}_{21}$ and $\mathcal{T}_{22}$ in terms of (9).

Step 7: Specify the constraint force coordinates $\eta_f \in \Re^{n_c - n_p}$, determine the matrix $\mathcal{T}_c$, and then form $\mathcal{J}_f$ in terms of (12).

Step 8: Determine n in terms of (13) and form $\dot{q}$ and τ in terms of (14) and (15) in space $\mathcal{A}$.

Step 9: Specify the dynamic contact force coordinates η_m; and form the transfer matrix $\mathcal{D}_m$ accordingly.

Step 10: Specify the m-dimensional independent velocity coordinates $\mathcal{V}_m$ in space $\mathcal{O}$ and form the full row-rank mapping matrix $\mathcal{J}_m$. Calculate $\begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-1}$.

Step 11: Calculate the velocity mapping matrix T_s in terms of (25).

Step 12: Write the block diagonal matrices and vector $\underline{\mathcal{M}}$, $\underline{\mathcal{C}}$, and $\underline{\mathcal{G}}$ in space $\mathcal{S}$. Finally, form the dynamic model (32).

Two examples are presented below to demonstrate the modeling steps in details.

3.2. Constrained single-arm manipulator

The first example is a six single-DOF joint manipulator grasping an object in contact with a plane. The system has $n_1 = n_{1a} = 6$ actuated single-DOF joints, $n_{1u} = 0$ unactuated single-DOF joint, and $n_3 = 0$ three-DOF spherical joint in Step 1.

Since the robot end-effector is in contact with a plane, it yields $m = 3$ (the linear motion frem the links onto the two tangential directions of the plane and the rotational motion along the normal vector of the plane). Thus, it follows $n_c = 6 + 0 - 3 = 3$ from (5) in Step 2.

In Step 3, the linear motion along the two tangential directions of the contact plane and the rotational motion along the normal vector of the contact plane specify the motion configuration space. The linear motion along the normal vector of the contact plane and the rotational motion along the two tangential directions of the contact plane specify the constraints. A frame O is fixed to the robot end-effector in a way that its x and y axes lie on the contact plane and its z axis coincides with the normal vector of the contact plane. Meanwhile, the six joints are numbered sequentially from the base towards the end-effector with the jth joint connecting the jth link with the $j-1$th link, $j=1,\ldots,6$. There are six auxiliary frames L_j, $j=1,2,\ldots,6$, each is fixed to link j with its z axis coincident with the jth joint.

In Step 4, $\dot{q}_a=[\dot{q}_1,\ldots,\dot{q}_6]^T\in\Re^6$ and $\dot{q}_u=0$ are specified. Matrix $\mathcal{T}_a$ is formed as

$$\mathcal{T}_a = T_f J$$

with

$$T_f = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} {}^{L_1}U_O^T Z_1 & {}^{L_2}U_O^T Z_2 & \ldots & {}^{L_6}U_O^T Z_6 \end{bmatrix}$$

where ${}^{\beta}U_{\alpha}\in\Re^{6\times6}$ denotes a force/moment transformation matrix which transforms a force/moment measured and expressed in frame α to that measured and expressed in frame β; and $Z_j=[0\ 0\ 1\ 0\ 0\ 0]^T$ for a prismatic joint and $Z_j=[0\ 0\ 0\ 0\ 0\ 1]^T$ for a revolute joint, $j=1,\ldots,6$. Since there is no unactuated joint in the system. It yields $\mathcal{T}_u=0$ and, therefore, $n_p=0$.

Steps 5 and 6 are skipped, since no unactuated joint is presented.

In Step 7, the 3-dimensional constraint forces are specified as the force along the z and the two moments along the x and y axes of frame O. It follows that

$$\eta_f = \begin{bmatrix} f_z \\ m_x \\ m_y \end{bmatrix}$$

where $f_z\in\Re$ denotes the force along the z axis of frame O and $m_x\in\Re$ and $m_y\in\Re$ denote the moments along the x and y axes of frame O. Consequently, $\mathcal{J}_f$ is formed as

$$\mathcal{J}_f = \mathcal{T}_a.$$

In Step 8, $\dot{q}=\dot{q}_a\in\Re^6$ and $\tau=\tau_a=[\tau_1,\ldots,\tau_6]^T\in\Re^6$ represent the actual joint velocities and control torques.

In Step 9, the dynamic contact forces are identified as the frictional forces and torque in the motion space. It yields

$$\mathcal{D}_m = T_m J$$

with

$$T_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In Step 10, it follows that

$$\mathcal{V}_m = \begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix}$$

where $v_x \in \Re$ and $v_y \in \Re$ denote the linear velocities of the end-effector along the x and y axes of frame O and $\omega_z \in \Re$ denotes the angular velocity of the end-effector along the z axis of frame O. Accordingly, it yields

$$\mathcal{J}_m = T_m J.$$

In Step 11, the velocity mapping matrix T_s is

$$T_S = \begin{bmatrix} I_6 \\ T_S^* \\ J \end{bmatrix}$$

$$T_S^* = \begin{bmatrix} Z_1 & & & \\ {}^{L_1}U_{L_2}^T Z_1 & Z_2 & & \\ \vdots & \ddots & \ddots & \\ {}^{L_1}U_{L_6}^T Z_1 & {}^{L_2}U_{L_6}^T Z_2 & \dots & Z_6 \end{bmatrix}.$$

In Step 12, $\underline{\mathcal{M}}$, $\underline{\mathcal{C}}$, and $\underline{\mathcal{G}}$ in space $\mathcal{S}$ are obtained as

$$\underline{\mathcal{M}} = \mathrm{diag}\{I_1^*, \dots, I_6^*, M_{L_1}, \dots, M_{L_6}, M_O\}$$

$$\underline{\mathcal{C}} = \mathrm{diag}\{0, \dots, 0, C_{L_1}, \dots, C_{L_6}, C_O\}$$

$$\underline{\mathcal{G}} = [\xi_1(t), \dots, \xi_6(t), G_{L_1}^T, \dots, C_{L_6}^T, G_O^T]^T.$$

3.3. Coordinated multiple manipulators

Consider a system composed of h manipulators grasping a rigid object moving in free space without kinematic singularity. Each manipulator has six actuated single-DOF joints. It follows that $n_1 = n_{1a} = 6h$, $n_{1u} = 0$, and $n_3 = 0$ in Step 1. Meanwhile, it yields $m = 6$ and $n_c = 6(h-1)$ in Step 2.

In Step 3, the 6-dimensional motion belongs to the held object and the $6(h-1)$-dimensional constraints are imposed on the end-effectors of the h manipulators. Frame O is fixed to the held object. Frame L_{ij}, which has the same definition as frame L_j in the last subsection, is assigned to the jth link of the ith manipulator, $i = 1, 2, \dots, h$, $j = 1, 2, \dots, 6$.

In Step 4, $\dot{q}_a = [\dot{q}_1^T, \dot{q}_2^T, \cdots, \dot{q}_h^T]^T \in \Re^{6h}$ with $\dot{q}_i = [\dot{q}_{i1}, \dot{q}_{i2}, \cdots, \dot{q}_{i6}]^T \in \Re^6$, $i = 1, 2, \cdots, h$, and $\dot{q}_u = 0$ are

$$\mathcal{T}_a = \mathcal{T}_f \mathcal{J}$$

with

$$\mathcal{T}_f = \begin{bmatrix} I_6 & -I_6 & 0 & \dots & 0 \\ 0 & I_6 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -I_6 & 0 \\ 0 & \dots & 0 & I_6 & -I_6 \end{bmatrix} \in \Re^{6(h-1)\times 6h}$$

$$\mathcal{J} = \mathrm{diag}\{J_1, J_2, \dots, J_h\} \in \Re^{6h\times 6h}$$

where $J_i, i = 1, 2, \dots, h$, is exactly the same as J defined in the last subsection. Meanwhile, it follows that $\mathcal{T}_u = 0$ and $n_p = 0$.

Steps 5 and 6 are skipped, since no unactuated joint is presented.

In Step 7, the constraint forces are located among the end-effectors of the h manipulators. Therefore, it follows that

$$\eta_f = \begin{bmatrix} \eta_{12} \\ \eta_{23} \\ \vdots \\ \eta_{(h-1)h} \end{bmatrix} \in \Re^{6(h-1)}$$

where $\eta_{(i-1)i} \in \Re^6$, $i = 2, \dots, 6$, denotes the internal force between the $i - 1$ manipulator and the ith manipulator, expressed in frame O. Consequently, it yields

$$\mathcal{J}_f = \mathcal{T}_a.$$

In Step 8, $n = 6h$ is obtained from (13). Furthermore, it yields

$$\dot{q} = \dot{q}_a \in \Re^6$$

and

$$\tau = \tau_a = \left[\tau_1^T, \tau_2^T, \dots, \tau_h^T\right]^T \in \Re^{6h}$$

with $\tau_i = [\tau_{i1}, \tau_{i2}, \dots, \tau_{i6}] \in \Re^6$.

Since all the end-effectors are rigidly holding a rigid object, there is no dynamic contact force. It gives $\eta_m = 0$ in Step 9.

In Step 10, the independent velocity coordinates $\mathcal{V}_m$ in space $\mathcal{O}$ are specified as $\mathcal{V}_m = {}^O X$, where ${}^O X \in \Re^6$ denotes the linear/angular velocities of frame O and expressed in frame O [31]. Accordingly, $\mathcal{J}_m$ is obtained as

$$\mathcal{J}_m = \mathcal{H}\mathcal{J}$$

where

$$\mathcal{H} = [\gamma_1 I_6, \gamma_2 I_6, \dots, \gamma_h I_6]$$

with $\gamma_i \geq 0$ subject to $\sum_{i=1}^h \gamma_i = 1$.

In Step 11, the velocity mapping matrix T_s defined by (25) is obtained as

$$T_S = \begin{bmatrix} I_{6h} \\ T_S^* \\ \mathcal{H}\mathcal{J} \end{bmatrix}$$

$$T_S^* = \mathrm{diag}\left\{T_{S1}^*, T_{S2}^*, \ldots, T_{Sh}^*\right\}$$

where T_{Si}^*, $i = 1, 2, \ldots, h$, has exactly the same format as T_S^* defined in the last subsection.

Finally in Step 12, $\underline{\mathcal{M}}$, $\underline{\mathcal{C}}$, and $\underline{\mathcal{G}}$ in space $\mathcal{S}$ are obtained as

$$\begin{aligned}
\underline{\mathcal{M}} &= \mathrm{diag}\left\{I_{11}^*, \ldots, I_{16}^*, I_{21}^*, \ldots, I_{26}^*, \ldots, I_{h1}^*, \ldots, I_{h6}^*,\right. \\
&\quad \left. M_{L_{11}}, \ldots, M_{L_{16}}, M_{L_{21}}, \ldots, M_{L_{26}}, \ldots, M_{L_{h1}}, \ldots, M_{L_{h6}}, M_O\right\} \\
\underline{\mathcal{C}} &= \mathrm{diag}\{0, \ldots, 0, C_{L_{11}}, \ldots, C_{L_{16}}, C_{L_{21}}, \ldots, C_{L_{26}}, \ldots, C_{L_{h1}}, \ldots, C_{L_{h6}}, C_O\} \\
\underline{\mathcal{G}} &= [\xi_{11}(t), \ldots, \xi_{16}(t), \xi_{21}(t), \ldots, \xi_{26}(t), \ldots, \xi_{h1}(t), \ldots, \xi_{h6}(t), \\
&\quad G_{L_{11}}^T, \ldots, C_{L_{16}}^T, G_{L_{21}}^T, \ldots, C_{L_{26}}^T, \ldots, G_{L_{h1}}^T, \ldots, C_{L_{h6}}^T, G_O^T]^T .
\end{aligned}$$

4. Control

With the dynamic model of a *general constrained robot* described by (32), the *virtual decomposition control* algorithm originally presented in [31] for controlling fully actuated robot manipulators can be written in a compact form as

$$\tau = T_S^T \mathcal{Y}_r \hat{\mathcal{P}}_r + \mathcal{Y}_m \hat{\mathcal{P}}_m + \mathcal{J}_f^T \eta_{fr} + \mathcal{K}_A \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_{mr} - \mathcal{V}_m \\ 0 \end{bmatrix} \tag{34}$$

where $\mathcal{V}_{mr} \in \Re^m$ denotes the required vector of $\mathcal{V}_m$ and $\eta_{fr} \in \Re^{n-m}$ denotes the required vector of η_f; $\mathcal{K}_A \in \Re^{n\times n}$ denotes a positive-definite feedback gain matrix in space $\mathcal{A}$; $\hat{\mathcal{P}}_r$ denotes the estimate of $\mathcal{P}_r$ which contains all the constant parameters of rigid links and joints; $\mathcal{Y}_r$ is the regressor matrix defined by

$$\begin{aligned}
\mathcal{Y}_r\mathcal{P}_r &= \underline{\mathcal{M}}\, \underline{\dot{\mathcal{X}}_r} + \underline{\mathcal{C}}\, \underline{\mathcal{X}_r} + \underline{\mathcal{G}} \\
&= \begin{bmatrix} \vdots \\ Y_{1j}P_{1j} \\ \vdots \\ Y_{3i}P_{3i} \\ \vdots \\ Y_\alpha P_\alpha \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ I_{1j}^*\ddot{q}_{1jr} + \xi_{1j}(t) \\ \vdots \\ \xi_{3i}(t) \\ \vdots \\ M_\alpha \frac{d}{dt}({}^\alpha X_r) + C_\alpha {}^\alpha X_r + G_\alpha \\ \vdots \end{bmatrix}
\end{aligned} \tag{35}$$

with

$$\underline{\mathcal{X}}_r = \begin{bmatrix} \vdots \\ \dot{q}_{1jr} \\ \vdots \\ \dot{q}_{3ir} \\ \vdots \\ {}^{\alpha}X_r \\ \vdots \end{bmatrix} = T_S \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_{mr} \\ 0 \end{bmatrix}. \tag{36}$$

Accordingly, $\hat{\mathcal{P}}_m$ denotes the estimate of $\mathcal{P}_m$ which contains all the constant parameters of the dynamic contact forces and $\mathcal{Y}_m$ is the regressor matrix defined by

$$\mathcal{Y}_m \mathcal{P}_m = \mathcal{D}_m^T \eta_m. \tag{37}$$

By defining

$$\mathcal{Y} = \begin{bmatrix} T_S^T \mathcal{Y}_r & \mathcal{Y}_m \end{bmatrix}$$
$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_r \\ \mathcal{P}_m \end{bmatrix}$$

the parameter adaptation law can be written as

$$\begin{aligned} \dot{\hat{\mathcal{P}}}_k &= \rho_k \kappa_k s_k \\ s_k &= \mathcal{Y}_k^T \begin{bmatrix} \mathcal{J}_m \\ \mathcal{J}_f \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_{mr} - \mathcal{V}_m \\ 0 \end{bmatrix} \\ \kappa_k &= \begin{cases} 0 & \hat{\mathcal{P}}_k \le \mathcal{P}_k^- \quad \text{and} \quad s_k \le 0 \\ 0 & \hat{\mathcal{P}}_k \ge \mathcal{P}_k^+ \quad \text{and} \quad s_k \ge 0 \\ 1 & \text{otherwise} \end{cases} \end{aligned} \tag{38}$$

where $\mathcal{P}_k$ denotes the kth parameter (kth element) of $\mathcal{P}$; $\rho_k > 0$ denotes the update gain for the kth parameter; $\mathcal{Y}_k$ denotes the kth column of $\mathcal{Y}$; and $\mathcal{P}_k^-$ and $\mathcal{P}_k^+$ denote the lower and upper bounds of the kth parameter $\mathcal{P}_k$.

5. Conclusion

In this paper, a novel modeling approach addressing a class of *general constrained robots* has been proposed. Under this modeling approach, both single-arm constrained robots and coordinated multiple robots can be united and expressed in a form very much similar to the conventional Lagrangian model. The information required by this modeling approach includes two velocity mapping matrices and the separated dynamics of individual rigid links and joints. Since the dynamics of the subsystems are simple and standard, the unique information required for a particular application is going to be the two velocity mapping matrices only. Thus, the dynamics issue of a complex robotic system is virtually converted into the kinematics issue plus the use of the standard subsystem dynamics. Two examples have been presented to demonstrate the applications of this approach to single-arm constrained robots and coordinated multiple robots. An application to the *virtual decomposition* based adaptive control has also been presented.

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