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Discrete Time Transfer Matrix Method for Multibody System Dynamics

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Abstract. A new method for multibody system dynamics is proposed in this paper. This method, named as discrete time transfer matrix method of multibody system (MS-DT-TMM), combines and expands the advantages of the transfer matrix method (TMM), transfer matrix method of vibration of multibody system (MS-TMM), discrete time transfer matrix method (DT-TMM) and the numerical integration procedure. It does not need the global dynamics equations for the study of multibody system dynamics. It has the modeling flexibility and a small size of matrices, and can be applied to a wide range of problems including multi-rigid-body system dynamics and multi-flexible-body system dynamics. This method is simple, straightforward, practical, and provides a powerful tool for the study of multibody system dynamics. Formulations of the method as well as some numerical examples of multi-rigid-body system dynamics and multi-flexible-body system dynamics to validate the method are given.

Keywords: multibody system, dynamics, discrete time, transfer matrix method

1. Introduction

Lots of methods for multibody systems dynamics have been studied by many authors on theory and computational method [1, 2]. In general, almost all methods of the multibody system dynamics have the same characteristics as follows: (1) It is necessary to develop global dynamics equations of the system. And if the system structure is changed, generally, corresponding global dynamics equations must be deduced again. (2) The orders of involved system matrices increase with the increase of the number of the degrees of freedom of the system; hence the orders of matrices involved in global dynamics equations are rather high for complex multibody system.

The transfer matrix method (TMM) has been developed for a long time and has been used widely in structure mechanics and rotor dynamics of linear time invariant system. To linear system, Holzer (1921) initially applied TMM to solve the problems of torsion vibrations of rods [3], Myklestad (1945) applied TMM to determine the bending-torsion modes of beams [4], Thomson (1950) applied TMM to more general vibration problems [5], Pestel (1963) listed transfer matrices for elasto-mechanical elements up to 12th-order [6], Rubin (1964, 1967) provided a general treatment for transfer matrices and their relation to other forms of frequency response matrices [7, 8]. Transfer matrices have been applied to a wide variety of engineering programs by a number of researchers, including Targoff [9], Lin [10], Mercer [11], Lin [12], Mead [13, 14], Henderson [15], McDanie [16, 17] and Murthy [18–20], dealing with beams, beam-type periodic structures, skin-stringer panels, rib-skin structures, curved muti-span structures, cylindrical shells, stiffened rings etc. Dokanish (1972) developed finite element-transfer matrix method to solve the problems of plate structure vibration analysis, by combining finite element method and transfer matrix method [21]. Many researchers, such as, Ohga (1987), Xue (1994) and Loewy (1985, 1999), studied and improved the finite element transfer matrix for structure dynamics [22–25]. Horner (1978) proposed Riccati transfer matrix method in order to circumvent the numerical stability of the boundary value problem [26]. Up to the present, Riccati transform also is an important tool to overcome the ill condition of the transfer matrix method.

Rui and Lu (1989, 2000) developed transfer matrix method of multibody system (MS-TMM) for vibrations analysis of linear multibody system by developing new transfer matrices and orthogonal property of multibody system [27–29]. Kumar and Sankar (1986) developed discrete time transfer matrix method (DT-TMM) for structure dynamics of time variant system by combining the transfer matrix method with the numerical integration procedure [30]. DT-TMM gives an important clue for dynamics of time variant system. MS-TMM provides an important thought that multibody system dynamics may be solved using transfer matrix method.

In this paper, a new analytical method of multibody system dynamics, namely discrete time transfer matrix method of multibody system (MS-DT-TMM) is developed. This method combines and expands the TMM, DT-TMM, MS-TMM and the numerical integration procedure. When using this method, the global dynamics equations of the system are not needed and the orders of involved system matrices are decreased greatly. It can be applied to a wide range of problems of general multibody system dynamics. This method is simple, straightforward, practical, and provides a powerful tool for the multibody system dynamics. Using the new method, dynamics of multi-rigid-body system and dynamics of multi-flexible-body systems, including chain multibody system, branched multibody system, close-looped multibody system, network multibody system, have been discussed in detail. And corresponding numerical examples have been given. The simulating results obtained by MS-DT-TMM and by ordinary dynamic method have a good agreement.

This paper is organized as follows. In Section 2, the sign convention and the steps of the method are shown. In Section 3, the transfer matrices of typical elements are developed, including rigid body moving in space, rigid body moving in plane, elastic hinge, damper hinges, smooth ball-and-socket hinge, and smooth pin hinge. In Section 4, the algorithm for dynamical analysis of multibody system is presented. In Section 5, the numerical results of multibody dynamics got by MS-DT-TMM and by ordinary methods are given, to validate the method. In Section 6, some advantages of the method are shown. The conclusion and future works are presented in Section 7.

2. General Theorems and Steps of the Method

2.1. COORDINATE SYSTEM AND SIGN CONVENTION

We shall use the right-handed inertial Cartesian coordinate system $oxyz$ as the reference system, use x, y and z as the position coordinates of involved points, use space-three-angles 1–2–3, which was defined in the reference [31], as the orientation angles of involved bodies, that is, by rotating about x, y and z axis directions successively in the same coordinate system, then we get θ_1, θ_2 and θ_3 [31].

Use the following sign convention: Positive position coordinates and orientation angles coincide with positive directions of the coordinate system. Inboard forces and outboard torques acting on the elements are positive (negative) if their vectors are in the positive (negative) directions, outboard forces and inboard torques acting on the elements are positive (negative) if their vectors are in the negative (positive) directions. The first subscripts and the second subscripts denote the body indices and the hinge indices of state vectors respectively.

2.2. DISCRETIZATION OF MULTIBODY SYSTEMS

According to the natural attribute of bodies, a complex multibody system may be divided into a certain number of subsystems, which can be represented by various elements including bodies (rigid bodies, elastic bodies, lumped masses etc.) and hinges (joints, ball-and-socket, pins, linear springs, rotary springs, linear dampers and rotary dampers, etc.).

2.3. DYNAMICS EQUATIONS OF ELEMENTS

It should be pointed out that the positions of the bodies and hinges are considered equivalent in transfer equations and transfer matrices in this method. So, the dynamics equations of every body and hinge should be developed relatively to the inertial reference system defined in Section 2.1 respectively.

2.4. LINEARIZATION OF DYNAMICS EQUATIONS OF ELEMENTS

According to numerical integration procedures, the motion parameters of multibody system $\ddot{\boldsymbol{\xi}}$ and $\dot{\boldsymbol{\xi}}$ at the time instant t_i are expressed as the linear function of $\boldsymbol{\xi}$ in form

$$\ddot{\boldsymbol{\xi}}(t_i) = \boldsymbol{\chi}_1 \boldsymbol{\xi}(t_i) + \boldsymbol{\chi}_{2,\xi} \tag{1}$$

$$\dot{\boldsymbol{\xi}}(t_i) = \boldsymbol{\chi}_3 \boldsymbol{\xi}(t_i) + \boldsymbol{\chi}_{4,\xi} \tag{2}$$

where, the variable $\boldsymbol{\xi}$ may represent column matrix of the positions coordinates x, y, z or the orientation angles $\theta_1, \theta_2, \theta_3$ respectively; $\ddot{\boldsymbol{\xi}}$ and $\dot{\boldsymbol{\xi}}$ represent one order and two order derivative of $\boldsymbol{\xi}$ with respect to time, that is, corresponding column matrices of acceleration and velocity, or corresponding column matrices of angular acceleration and angular velocity for planar motion, at the same time instant t_i. The quantities χ_1, $\chi_{2,\xi}$, χ_3 and $\chi_{4,\xi}$, will have different expressions for different numerical integration procedures. There are many accurate and commonly available numerical integration procedures can be chosen, for example, if Newmark- β method [32, 33] is used, then we obtained

$$\chi_1 = \frac{1}{\beta\Delta T^2}\boldsymbol{I}_k,$$

$$\chi_{2,\xi} = -\frac{1}{\beta\Delta T^2}\left[\boldsymbol{\xi}(t_{i-1}) + \dot{\boldsymbol{\xi}}(t_{i-1})\Delta T + \left(\frac{1}{2} - \beta\right)\ddot{\boldsymbol{\xi}}(t_{i-1})\Delta T^2\right] \tag{3}$$

$$\chi_3 = \gamma\chi_1\Delta T, \quad \chi_{4,\xi} = \gamma\chi_{2,\xi}\Delta T + \dot{\boldsymbol{\xi}}(t_{i-1}) + (1-\gamma)\ddot{\boldsymbol{\xi}}(t_{i-1})\Delta T \tag{4}$$

Where, time step $\Delta T = t_i - t_{i-1}$, β and γ are the coefficients of Newmark-β method. Bold capital symbol $\boldsymbol{I}_k$ is the unit matrix, its subscripts k denotes the order of the unit matrix and equals to three for a system moving in space or equals to two for a system moving in plane.

Using the orientation angles θ_1, θ_2 and θ_3 got by rotating about x, y and z successively defined in Section 2.1, the direction cosine matrix can be expressed as follows

$$\boldsymbol{A} = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{bmatrix}$$
$$= \begin{bmatrix} c_2c_3 & s_1s_2c_3 - c_1s_3 & c_1s_2c_3 + s_1s_3 \\ c_2s_3 & s_1s_2s_3 + c_1c_3 & c_1s_2s_3 - s_1c_3 \\ -s_2 & s_1c_2 & c_1c_2 \end{bmatrix} \tag{5}$$

where

$$c_i = \cos\theta_i, \quad s_i = \sin\theta_i, \quad i = 1, 2, 3, \tag{6}$$

Using Taylor expansion theorem, at time instant t_i direction cosine matrix $\boldsymbol{A}(t_i)$ can be approximately expressed with respect to t_{i-1} by the truncated Taylor series of order 3, that is

$$\boldsymbol{A}(t_i) = \boldsymbol{A}(t_{i-1})\tilde{\boldsymbol{T}}_1(t_{i-1})\theta_1(t_i) + \boldsymbol{A}(t_{i-1})\tilde{\boldsymbol{T}}_2(t_{i-1})\theta_2(t_i) + \boldsymbol{A}(t_{i-1})\tilde{\boldsymbol{T}}_3(t_{i-1})\theta_3(t_i)$$
$$+ \boldsymbol{\Phi}(t_{i-1}) + \boldsymbol{B}o(\Delta T^2) \tag{7}$$

where, $\tilde{T}$ is the cross product matrix of T;

$$\begin{aligned}\boldsymbol{\Phi}(t_{i-1}) = \boldsymbol{A}(t_{i-1}) + \boldsymbol{A}(t_{i-1})\Bigg\{&-\tilde{\boldsymbol{T}}_1(t_{i-1})\theta_1(t_{i-1}) - \tilde{\boldsymbol{T}}_2(t_{i-1})\theta_2(t_{i-1})\\ &- \tilde{\boldsymbol{T}}_3(t_{i-1})\theta_3(t_{i-1}) + \frac{\Delta T^2}{2}\tilde{\boldsymbol{T}}_1^2(t_{i-1})\dot{\theta}_1^2(t_{i-1}) + \frac{\Delta T^2}{2}\tilde{\boldsymbol{T}}_2^2(t_{i-1})\dot{\theta}_2^2(t_{i-1})\\ &+\frac{\Delta T^2}{2}\tilde{\boldsymbol{T}}_3^2(t_{i-1})\dot{\theta}_3^2(t_{i-1}) + \tilde{\boldsymbol{T}}_2(t_{i-1})\tilde{\boldsymbol{T}}_1(t_{i-1})\dot{\theta}_2(t_{i-1})\dot{\theta}_1(t_{i-1})\Delta T^2\\ &+\tilde{\boldsymbol{T}}_3(t_{i-1})\tilde{\boldsymbol{T}}_2(t_{i-1})\dot{\theta}_3(t_{i-1})\dot{\theta}_2(t_{i-1})\Delta T^2\\ &+\tilde{\boldsymbol{T}}_3(t_{i-1})\tilde{\boldsymbol{T}}_1(t_{i-1})\dot{\theta}_3(t_{i-1})\dot{\theta}_1(t_{i-1})\Delta T^2\Bigg\}\end{aligned}$$

$$\boldsymbol{B} = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}, \quad \boldsymbol{T}_1 = [1, 0, 0]^T, \quad \boldsymbol{T}_2 = [0, c_1, -s_1]^T,$$
$$\boldsymbol{T}_3 = [-s_2, s_1c_2, c_1c_2]^T \tag{8}$$

Equation (7) will be used in the expressions of position coordinates of corresponding points Equations (21) and (25). In the items of product of the trigonometric functions and the elements of state vectors, the trigonometric functions at time t_i are expanded with respect to t_{i-1} using the truncated Taylor series of order 3, that is

$$\sin\theta(t_i) = \sin[\theta(t_{i-1}) + \Delta\theta] = \bar{s} + o(\Delta T^2)$$
$$\cos\theta(t_i) = \cos[\theta(t_{i-1}) + \Delta\theta] = \bar{c} + o(\Delta T^2)$$

where

$$\begin{aligned}\bar{s} \stackrel{\Delta}{=} \sin\theta(t_{i-1})&\left\{1 - \frac{1}{2}[\dot{\theta}(t_{i-1})\Delta T]^2\right\}\\ &+ \cos\theta(t_{i-1})\left[\dot{\theta}(t_{i-1})\Delta T + \frac{1}{2}\ddot{\theta}(t_{i-1})\Delta T^2\right]\end{aligned} \tag{9}$$

$$\begin{aligned}\bar{c} \stackrel{\Delta}{=} \cos\theta(t_{i-1})&\left\{1 - \frac{1}{2}[\dot{\theta}(t_{i-1})\Delta T]^2\right\}\\ &- \sin\theta(t_{i-1})\left[\dot{\theta}(t_{i-1})\Delta T + \frac{1}{2}\ddot{\theta}(t_{i-1})\Delta T^2\right]\end{aligned} \tag{10}$$

It can be proved that the multinomial in the dynamics equations can be approximated by

$$[a(t_i) - a(t_{i-1})][b(t_i) - b(t_{i-1})] = \dot{a}(t_{i-1})\dot{b}(t_{i-1})\Delta T^2 + o(\Delta T^2) \tag{11}$$

The following equations can be obtained by Equations (2) and (11),

$$\begin{aligned}
\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \\ \dot{\theta}_3^2 \end{bmatrix} &= \boldsymbol{\kappa}_1 \left(\boldsymbol{\chi}_3 \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}_{t_i} + \boldsymbol{\kappa}_3 \right) + \begin{bmatrix} \ddot{\theta}_1^2 \\ \ddot{\theta}_2^2 \\ \ddot{\theta}_3^2 \end{bmatrix}_{t_{i-1}} \Delta T^2 + \begin{bmatrix} o(\Delta T^2) \\ o(\Delta T^2) \\ o(\Delta T^2) \end{bmatrix} \\
\begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_3 \\ \dot{\theta}_1\dot{\theta}_3 \end{bmatrix} &= \boldsymbol{\kappa}_2 \left(\boldsymbol{\chi}_3 \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}_{t_i} + \boldsymbol{\kappa}_3 \right) + \begin{bmatrix} \ddot{\theta}_1\ddot{\theta}_2 \\ \ddot{\theta}_2\ddot{\theta}_3 \\ \ddot{\theta}_1\ddot{\theta}_3 \end{bmatrix}_{t_{i-1}} \Delta T^2 + \begin{bmatrix} o(\Delta T^2) \\ o(\Delta T^2) \\ o(\Delta T^2) \end{bmatrix}
\end{aligned} \tag{12}$$

where

$$\boldsymbol{\kappa}_1 = 2 \begin{bmatrix} \dot{\theta}_1 & 0 & 0 \\ 0 & \dot{\theta}_2 & 0 \\ 0 & 0 & \dot{\theta}_3 \end{bmatrix}_{t_{i-1}}, \quad \boldsymbol{\kappa}_2 = \begin{bmatrix} \dot{\theta}_2 & \dot{\theta}_1 & 0 \\ 0 & \dot{\theta}_3 & \dot{\theta}_2 \\ \dot{\theta}_3 & 0 & \dot{\theta}_1 \end{bmatrix}_{t_{i-1}}, \quad \boldsymbol{\kappa}_3 = \boldsymbol{\chi}_{4,\boldsymbol{\theta}} - \frac{1}{2}\dot{\boldsymbol{\theta}}(t_{i-1})$$

The truncation errors caused in Equations (7) and (9)–(11) are all $o(\Delta T^2)$. If necessary, it is not difficult at all to get higher computational accuracy by multi-steps numerical methods, such as, Houbolt method, predictor-corrector methods [32, 33] etc.

The motion quantities $z(t_{i-1})$, $\dot{z}(t_{i-1})$, $\ddot{z}(t_{i-1})$ at the previous time instant are all known at time instant t_i. Thus, these quantities $\boldsymbol{\chi}_1$, $\boldsymbol{\chi}_{2,\boldsymbol{\xi}}$, $\boldsymbol{\chi}_3$, $\boldsymbol{\chi}_{4,\boldsymbol{\xi}}$, $\bar{c}$ and $\bar{s}$ etc. are all definable for any subsystem for the time interval $(t_i - t_{i-1})$, and hence above formulations are valid.

2.5. STATE VECTORS, TRANSFER EQUATIONS AND TRANSFER MATRICES OF ELEMENTS

In order to describe conveniently, the chain multibody system is taken as an example in the following. According to the dynamics equations of elements and structure of multibody systems (such as closed-loop system, branched system, chain system, network system, etc.), corresponding state vectors, transfer equations and transfer matrices of the bodies and hinges can be developed. The state vectors of the connection point among any rigid bodies and hinges moving in space are defined as

$$\boldsymbol{z} = [x, y, z, \theta_1, \theta_2, \theta_3, m_x, m_y, m_z, q_x, q_y, q_z, 1]^T \tag{13}$$

where x, y, z, θ_1, θ_2 and θ_3 are the position coordinates of the connection point with respect to the inertial reference system and the corresponding orientation angles rotating in the directions of x, y, z, successively defined in Section 2.1; $m_x, m_y, m_z, q_x, q_y, q_z$ are the corresponding internal torques and internal forces in the same reference system respectively.

For a chain system moving in plane, the state vectors of the connection point among any rigid bodies and hinges are defined as

$$z = [x, y, \theta_3, m_z, q_x, q_y, 1]^T \tag{14}$$

where, the meaning of the elements in the state vectors are similar to the meaning described above.

Then the dynamics equations of the jth element that have been linearized using numerical integration procedure can be assembled into a single transfer equation

$$z_{j,j+1}(t_i) = \boldsymbol{U}_j(t_i) z_{j,j-1}(t_i) \tag{15}$$

The meaning of the subscripts of the state vectors z follows the convention in Section 2.1. The transfer equation describes the mutual relationship between the state vectors at two ends of the jth element and has the similar form in contrast with a general transfer equation used in MS-TMM, DT-TMM or TMM. Here; the matrix $\boldsymbol{U}_j(t_i)$ is the transfer matrix of the jth element. It is the functions of the motion quantities ($z(t_k)$, $\dot{z}(t_k)$ and $\ddot{z}(t_k)$, $k = i-1, i-2, \ldots$) which are all known at time instant t_i. Its order always is (13×13) for dynamics of chain multi-rigid-body system moving in space, or (7×7) for dynamics of chain multi-rigid-body system moving in plane.

2.6. TRANSFER EQUATION AND TRANSFER MATRIX OF OVERALL SYSTEM

By using the same method used in MS-TMM, DT-TMM or TMM, the overall system transfer equation and transfer matrix $\boldsymbol{U}$, which relates the state vectors at ends of the system, can be assembled and calculated. That is,

$$z_{n,n+1} = \boldsymbol{U} z_{1,0} \tag{16}$$

$$\boldsymbol{U} = \boldsymbol{U}_n \boldsymbol{U}_{n-1} \ldots \boldsymbol{U}_2 \boldsymbol{U}_1 \tag{17}$$

For a chain system, the order of the overall transfer matrix of the system is equal to the order of the transfer matrix of the element, and it does not increase when the degrees of freedom of system increase. Irrespective of the size of a multibody system, the highest order of the overall transfer matrix $\boldsymbol{U}$ is the same with the order of transfer matrix of single body, that is (13×13) for dynamics of chain multi-rigid-body system moving in space, or (7×7) for dynamics of chain multi-rigid-body system moving in plane. So, the matrices involved in the MS-DT-TMM are always small, which greatly reduces the computational time and the memory storage requirement.

2.7. SOLUTIONS OF SYSTEM MOTION

Once the overall transfer matrix of the multibody system is known, the boundary conditions of the system can then be applied and the unknown quantities in the boundary state vectors can be computed. Now, knowing the boundary state vectors completely, the state vectors and hence the motion quantities at each element at time t_i can be computed by the repeated use of corresponding transfer equations of element similar to Equation (15). The velocity, angular velocity, acceleration and angular acceleration quantities at time t_i are then obtained using Equations (1) and (2) respectively. Then entire procedure can be repeated for time t_{i+1} and so on. It can be seen clearly from Equations (16) and (17) that the global dynamics equations of multibody system are not needed if using MS-DT-TMM to solve the problems of multibody system dynamics. The overall transfer matrix of multibody systems can be assembled easily just using the transfer matrices of elements. So this method simplifies solving procedure of multibody system dynamics.

3. Transfer Matrices of Typical Elements

3.1. RIGID BODY MOVING IN SPACE

As shown in Figure 1, points I, O and C denote the inboard end, outboard end and mass center of the rigid body respectively; the subscript 2 denotes the body-fixed coordinate system whose initial point O_2 is on the inboard end I of the rigid body, $oxyz$ is the inertial coordinate system. So geometrical equations can been obtained

$$\boldsymbol{r}_O = \boldsymbol{r}_I + \boldsymbol{A}\boldsymbol{r}_{2,O} \tag{18}$$

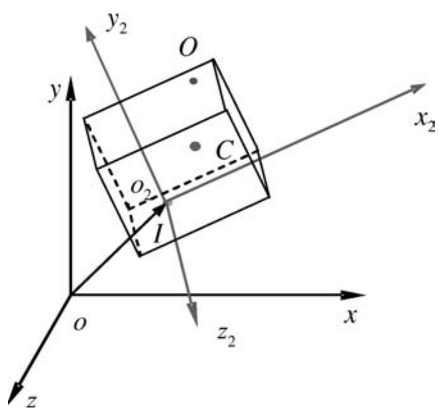


Figure 1. Rigid body moving in space.

where, $\boldsymbol{r}_K$ and $\boldsymbol{r}_{2,K}$ are the column matrices of the position coordinate of point K with respect to the inertial coordinate system and the body-fixed coordinate system respectively, K can be used to represent I, O and C; $\boldsymbol{A}$ is direction cosine matrix that has been given in Equation (5).

Using Newton–Euler method and considering the symbol convention in Section 2.1, the dynamics equations of the rigid body can be gained

$$m\ddot{\boldsymbol{r}}_C = \boldsymbol{q}_I - \boldsymbol{q}_O + \boldsymbol{f}_C \tag{19}$$

$$\dot{\boldsymbol{G}}_I = -\boldsymbol{m}_I + \boldsymbol{m}_O - \widetilde{\boldsymbol{IO}}\boldsymbol{q}_O - m\widetilde{\boldsymbol{IC}}\boldsymbol{a}_I + \boldsymbol{m}_C + \widetilde{\boldsymbol{IC}}\boldsymbol{f}_C \tag{20}$$

where, m and $\ddot{\boldsymbol{r}}_C$ are the mass and the column matrix of mass center acceleration of the rigid body, $\boldsymbol{q}_I$ and $\boldsymbol{q}_O$ are the column matrices of internal forces acted on the point I and O respectively, $\boldsymbol{f}_C$ and $\boldsymbol{m}_C$ are the column matrices of external force and the external torque acted on the mass center of the rigid body; $\boldsymbol{G}_I$ is the column matrix of moment of momentum with respect to point I, $\boldsymbol{m}_I$ and $\boldsymbol{m}_O$ are the column matrices of internal torques acted on the points I and O, $\boldsymbol{IO}$ and $\boldsymbol{IC}$ are the column matrices of position vectors from $\boldsymbol{I}$ to $\boldsymbol{O}$ and to $\boldsymbol{C}$ respectively, $\boldsymbol{a}_I$ is the column matrix of acceleration of point I.

To substitute Equation (7) into (18), we can obtain

$$\boldsymbol{r}_O = [\boldsymbol{I}_3, \boldsymbol{\Psi}_O, \boldsymbol{O}_{3\times3}, \boldsymbol{O}_{3\times3}, \boldsymbol{\Phi}(t_{i-1})\boldsymbol{r}_{2,O}]\boldsymbol{z}_I \tag{21}$$

where

$$\boldsymbol{\Psi}_O = \boldsymbol{A}(t_{i-1})[\tilde{\boldsymbol{T}}_1(t_{i-1})\boldsymbol{r}_{2,O}, \tilde{\boldsymbol{T}}_2(t_{i-1})\boldsymbol{r}_{2,O}, \tilde{\boldsymbol{T}}_3(t_{i-1})\boldsymbol{r}_{2,O}] \tag{22}$$

$\boldsymbol{O}_{3\times3}$ is zero square matrix whose order equal to three; $\boldsymbol{\Phi}(t_{i-1})$ is defined in Equation (8).

Because the orientation angle of any point of the same rigid body is uniform and can be expressed as

$$\boldsymbol{\theta}_O = [\boldsymbol{O}_{3\times3}, \boldsymbol{I}_3, \boldsymbol{O}_{3\times3}, \boldsymbol{O}_{3\times3}, \boldsymbol{O}_{3\times1}]\boldsymbol{z}_I \tag{23}$$

where, $\boldsymbol{\theta}$ is defined as

$$\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^T$$

To substitute Equation (1) into (19), we can obtain

$$m\boldsymbol{\chi}_1\boldsymbol{r}_C(t_i) + m\boldsymbol{\chi}_{2,r_C} = \boldsymbol{q}_I - \boldsymbol{q}_O + \boldsymbol{f}_C \tag{24}$$

To replace subscript O with C, Equation (21) becomes

$$\boldsymbol{r}_C = \boldsymbol{r}_I + \boldsymbol{\Psi}_C \boldsymbol{\theta}_I + \boldsymbol{\Phi}(t_{i-1})\boldsymbol{r}_{2,C} \tag{25}$$

To substitute Equation (25) into (24), we can obtain

$$\boldsymbol{q}_O = [-m\boldsymbol{\chi}_1, -m\boldsymbol{\chi}_1\boldsymbol{\Psi}_C, \boldsymbol{O}_{3\times 3}, \boldsymbol{I}_3, \boldsymbol{U}_{4,5}]\boldsymbol{z}_I \tag{26}$$

where

$$\boldsymbol{U}_{4,5} = \boldsymbol{f}_C - m\boldsymbol{\chi}_1\boldsymbol{\Phi}(t_{i-1})\boldsymbol{r}_{2,C} - m\boldsymbol{\chi}_{2,\boldsymbol{r}_C} \tag{27}$$

The bold capital letters $\boldsymbol{U}_{i,j}$ denote submatrix of the transfer matrix $\boldsymbol{U}$, the first subscript and the second subscript of $\boldsymbol{U}_{i,j}$ denote the sequence numbers of row and column of $\boldsymbol{U}_{i,j}$ in the block matrix of the transfer matrix respectively.

Similar to the process from Equations (19) to (26), substituting Equations (1), (2), (9), (10), (13) and (26) into Equation (20), we can obtain a matrix equation

$$\boldsymbol{m}_O = \left[(\tilde{\boldsymbol{r}}_{IC} - \tilde{\boldsymbol{r}}_{IO})m\boldsymbol{\chi}_1, \boldsymbol{\kappa}_4 - m\tilde{\boldsymbol{r}}_{IO}\boldsymbol{\chi}_1\boldsymbol{\Psi}_C, \boldsymbol{I}_3, \tilde{\boldsymbol{r}}_{IO}, \boldsymbol{U}_{3,5}\right]\boldsymbol{z}_I \tag{28}$$

where

$$\begin{aligned}
\boldsymbol{r}_{IO} &= \bar{\boldsymbol{A}}\boldsymbol{r}_{2,O}, \quad \boldsymbol{r}_{IC} = \bar{\boldsymbol{A}}\boldsymbol{r}_{2,C}, \\
\boldsymbol{U}_{3,5} &= \boldsymbol{\kappa}_5 - \boldsymbol{m}_C + \tilde{\boldsymbol{r}}_{IC}\left(m\boldsymbol{\chi}_{2,\boldsymbol{r}_I} - \boldsymbol{f}_C\right) + \tilde{\boldsymbol{r}}_{IO}\boldsymbol{U}_{4,5} \\
\boldsymbol{\kappa}_4 &= \bar{\boldsymbol{A}}\boldsymbol{J}\boldsymbol{H}\boldsymbol{\chi}_1 + (\boldsymbol{H}_1\boldsymbol{\kappa}_1 + \boldsymbol{H}_2\boldsymbol{\kappa}_2)\boldsymbol{\chi}_3, \quad \boldsymbol{H} = \left[\bar{\boldsymbol{T}}_1, \bar{\boldsymbol{T}}_2, \bar{\boldsymbol{T}}_3\right], \\
\boldsymbol{H}_1 &= \bar{\boldsymbol{A}}\left[\tilde{\bar{\boldsymbol{T}}}_1\boldsymbol{J}\bar{\boldsymbol{T}}_1, \tilde{\bar{\boldsymbol{T}}}_2\boldsymbol{J}\bar{\boldsymbol{T}}_2, \tilde{\bar{\boldsymbol{T}}}_3\boldsymbol{J}\bar{\boldsymbol{T}}_3\right], \\
\boldsymbol{\kappa}_5 &= \bar{\boldsymbol{A}}\boldsymbol{J}\boldsymbol{H}\boldsymbol{\chi}_{2,\boldsymbol{\theta}} + (\boldsymbol{H}_1\boldsymbol{\kappa}_1 + \boldsymbol{H}_2\boldsymbol{\kappa}_2)\boldsymbol{\kappa}_3 + \boldsymbol{H}_1\left[\ddot{\theta}_1^2, \ddot{\theta}_2^2, \ddot{\theta}_3^2\right]_{t_i-1}^T \Delta T^2 \\
&\quad + \boldsymbol{H}_2\left[\ddot{\theta}_1\ddot{\theta}_2, \ddot{\theta}_2\ddot{\theta}_3, \ddot{\theta}_3\ddot{\theta}_1\right]_{t_i-1}^T \Delta T^2 \\
\boldsymbol{H}_2 &= \bar{\boldsymbol{A}}\left[\boldsymbol{J}\bar{\boldsymbol{T}}_{12} + \tilde{\bar{\boldsymbol{T}}}_1\boldsymbol{J}\bar{\boldsymbol{T}}_2 + \tilde{\bar{\boldsymbol{T}}}_2\boldsymbol{J}\bar{\boldsymbol{T}}_1, \boldsymbol{J}\bar{\boldsymbol{T}}_{23} + \tilde{\bar{\boldsymbol{T}}}_2\boldsymbol{J}\bar{\boldsymbol{T}}_3\right. \\
&\quad \left. + \tilde{\bar{\boldsymbol{T}}}_3\boldsymbol{J}\bar{\boldsymbol{T}}_2, \boldsymbol{J}\bar{\boldsymbol{T}}_{31} + \tilde{\bar{\boldsymbol{T}}}_3\boldsymbol{J}\bar{\boldsymbol{T}}_1 + \tilde{\bar{\boldsymbol{T}}}_1\boldsymbol{J}\bar{\boldsymbol{T}}_3\right], \\
\boldsymbol{T}_{12} &= [0, -s_1, -c_1]^T, \quad \boldsymbol{T}_{23} = [-c_2, -s_1 s_2, -c_1 s_2]^T, \\
\boldsymbol{T}_{31} &= [0, c_1 c_2, -s_1 c_2]^T
\end{aligned} \tag{29}$$

$\boldsymbol{J}$ is the inertia matrix, the symbol $\bar{\boldsymbol{T}}$ is the matrix got from matrix $\boldsymbol{T}$ by replaced trigonometric functions s and c using $\bar{s}$ and $\bar{c}$ which defined in Equation (9) and (10) respectively. For example, from $\boldsymbol{A}$ expressed in Equation (7), we can obtain $\bar{\boldsymbol{A}}$, that is,

$$\bar{\boldsymbol{A}} = \begin{bmatrix} \bar{c}_2\bar{c}_3 & \bar{s}_1\bar{s}_2\bar{c}_3 - \bar{c}_1\bar{s}_3 & \bar{c}_1\bar{s}_2\bar{c}_3 + \bar{s}_1\bar{s}_3 \\ \bar{c}_2\bar{s}_3 & \bar{s}_1\bar{s}_2\bar{s}_3 + \bar{c}_1\bar{c}_3 & \bar{c}_1\bar{s}_2\bar{s}_3 - \bar{s}_1\bar{c}_3 \\ -\bar{s}_2 & \bar{s}_1\bar{c}_2 & \bar{c}_1\bar{c}_2 \end{bmatrix}$$

Equations (21), (23), (26) and (28) can be rewritten as the transfer equation

$$z_O = \boldsymbol{U} z_I \tag{30}$$

and the transfer matrix of the rigid body moving in space

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{\Psi}_O & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{\Phi}(t_{i-1})\boldsymbol{r}_{2,O} \\ \boldsymbol{O}_{3\times3} & \boldsymbol{I}_3 & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times1} \\ (\tilde{\boldsymbol{r}}_{IC} - \tilde{\boldsymbol{r}}_{IO})m\boldsymbol{\chi}_1 & \boldsymbol{\kappa}_4 - m\tilde{\boldsymbol{r}}_{IO}\boldsymbol{\chi}_1\boldsymbol{\Psi}_C & \boldsymbol{I}_3 & \tilde{\boldsymbol{r}}_{IO} & \boldsymbol{U}_{3,5} \\ -m\boldsymbol{\chi}_1 & -m\boldsymbol{\chi}_1\boldsymbol{\Psi}_C & \boldsymbol{O}_{3\times3} & \boldsymbol{I}_3 & \boldsymbol{U}_{4,5} \\ \boldsymbol{O}_{1\times3} & \boldsymbol{O}_{1\times3} & \boldsymbol{O}_{1\times3} & \boldsymbol{O}_{1\times3} & 1 \end{bmatrix} \tag{31}$$

where, z_O and z_I are the state vector of the inboard end I and outboard end O on the rigid body, the order of the transfer matrix of the rigid body moving in space is (13×13).

3.2. RIGID BODY MOVING IN PLANE

With similar method used in Section 3.1, we can obtain the transfer matrix of a rigid body moving in plane. And the order of the transfer matrix of a rigid body moving in plane is (7×7). There are the elements of the transfer matrix

$$\begin{aligned}
&u_{h,h} = 1 \quad (h = 1, 2, \ldots, 7), \quad u_{1,3} = -x_{2,O}s - y_{2,O}c, \\
&u_{1,7} = x_{2,O}G_1 - y_{2,O}G_2, \quad u_{2,3} = x_{2,O}c - y_{2,O}s, \\
&u_{2,7} = x_{2,O}G_2 + y_{2,O}G_1, \quad u_{4,1} = m\chi_1(y_{IO} - y_{IC}), \\
&u_{4,2} = m\chi_1(x_{IC} - x_{IO}), \quad u_{4,3} = u_{6,3}x_{IO} - u_{5,3}y_{IO} + J_I\chi_1, \\
&u_{4,5} = -y_{IO}, \quad u_{4,6} = x_{IO}, \\
&u_{4,7} = -m_C + u_{6,7}x_{IO} - u_{5,7}y_{IO} + J_I\chi_{2,\theta} + [m\chi_{2,y_I} - f_{y_C}]x_{IC} \\
&\qquad + [f_{x_C} - m\chi_{2,x_I}]y_{IC}, \\
&u_{5,1} = -m\chi_1, \quad u_{5,3} = m\chi_1(x_{2,C}s + y_{2,C}c), \\
&u_{5,7} = f_{x_C} - m\chi_1(x_{2,C}G_1 - y_{2,C}G_2) - m\chi_{2,x_C}, \\
&u_{6,2} = -m\chi_1, \quad u_{6,3} = -m\chi_1(x_{2,C}c - y_{2,C}s), \\
&u_{6,7} = f_{y_C} - m\chi_1(x_{2,C}G_2 + y_{2,C}G_2) - m\chi_{2,y_C}, \\
&G_1 = c + \theta s - c(\dot{\theta}\Delta T)^2/2, \quad G_2 = s - \theta c - s(\dot{\theta}\Delta T)^2/2,
\end{aligned} \tag{32}$$

Other elements are all zero.
where

$$\begin{aligned}
&x_{IC} = x_{2,C}\bar{c} - y_{2,C}\bar{s}, \;\; y_{IC} = x_{2,C}\bar{s} + y_{2,C}\bar{c}, \\
&x_{IO} = x_{2,O}\bar{c} - y_{2,O}\bar{s}, \;\; y_{IO} = x_{2,O}\bar{s} + y_{2,O}\bar{c}
\end{aligned}$$

the minuscule symbol $u_{i,j}$ denotes the element of the transfer matrix, the first subscript and the second subscript of $u_{i,j}$ denote the sequence numbers of row and column of $u_{i,j}$ in the transfer matrix respectively; angles and angular velocities and trigonometric functions in Equation (32) all have the values of these parameter at time t_{i-1}.

3.3. ELASTIC HINGE AND DAMPER HINGE

Here, the elastic hinge and damper hinge including linear spring hinge, rotary spring hinge, linear damper hinge and rotary damper hinge that are parallel connected each other with elastic hinge. And the mass of the spring and dampers are zero.

According to the equilibrium of forces and torques at point O or I, the following equations can be obtained,

$$\boldsymbol{q}_O + \boldsymbol{k}(\boldsymbol{r}_O - \boldsymbol{r}_I) + \boldsymbol{c}(\dot{\boldsymbol{r}}_O - \dot{\boldsymbol{r}}_I) = 0 \tag{33}$$

$$-\boldsymbol{m}_O + \boldsymbol{k}'(\boldsymbol{\theta}_O - \boldsymbol{\theta}_I) + \boldsymbol{c}'(\boldsymbol{A}_O\boldsymbol{\omega}_O - \boldsymbol{A}_I\boldsymbol{\omega}_I) = 0 \tag{34}$$

where

$$\boldsymbol{k} = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}, \quad \boldsymbol{k}' = \begin{bmatrix} k'_x & 0 & 0 \\ 0 & k'_y & 0 \\ 0 & 0 & k'_z \end{bmatrix},$$

$$\boldsymbol{c} = \begin{bmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & c_z \end{bmatrix}, \quad \boldsymbol{c}' = \begin{bmatrix} c'_x & 0 & 0 \\ 0 & c'_y & 0 \\ 0 & 0 & c'_z \end{bmatrix}$$

k_x, k_y, k_z and k'_x, k'_y, k'_z are the stiffness coefficients of linear springs and rotary springs, c_x, c_y, c_z and c'_x, c'_y, c'_z are damper coefficients of linear dampers and rotary dampers, $\boldsymbol{\omega}_O$ and $\boldsymbol{\omega}_I$ are angular velocity matrices of points O and I expressed in the body fixed coordinate system, $\boldsymbol{A}_O$ and $\boldsymbol{A}_I$ are direction cosine matrix of the body fixed coordinate system of points O and I with respect to the inertia reference system.

Considering that the mass of the hinge is zero, and the forces (torques) acted on the two ends of the hinge are the same. Substituting Equations (2), (5), (9) and (10) into Equation (33), then the transfer matrix of the hinge moving in space can be obtained, that is, there are the submatrices and the elements of this transfer matrix

$$\begin{aligned} &\boldsymbol{U}_{h,h} = \boldsymbol{I}_3 \ (h = 1, 2, 3, 4), \quad \boldsymbol{U}_{1,4} = -(\boldsymbol{k} + \boldsymbol{c}\chi_3)^{-1}, \\ &\boldsymbol{U}_{1,5} = -\boldsymbol{U}_{1,4}\boldsymbol{c}\big(\chi_{4,\boldsymbol{r}_I} - \chi_{4,\boldsymbol{r}_O}\big), \\ &\boldsymbol{U}_{2,3} = (\boldsymbol{k}' + \boldsymbol{c}'\boldsymbol{A}_O\boldsymbol{H}_O\chi_3)^{-1}, \quad \boldsymbol{U}_{2,2} = \boldsymbol{U}_{2,3}(\boldsymbol{k}' + \boldsymbol{c}'\boldsymbol{A}_I\boldsymbol{H}_I\chi_3), \\ &\boldsymbol{U}_{2,5} = \boldsymbol{U}_{2,3}\boldsymbol{c}'\big(\boldsymbol{A}_I\boldsymbol{H}_I\chi_{4,\boldsymbol{\theta}_I} - \boldsymbol{A}_O\boldsymbol{H}_O\chi_{4,\boldsymbol{\theta}_O}\big), \quad u_{13,13} = 1, \end{aligned} \tag{35}$$

Other elements are all zero.

3.4. SMOOTH BALL-AND-SOCKET HINGE

For a smooth ball-and-socket hinge, its mass and size have been neglected, the position coordinates and internal forces are equal and internal torques equal to zero in its two ends.

3.4.1. *Smooth Ball-and-Socket Hinge Whose Outboard Hinge Is Also a Smooth Ball-and-Socket Hinge*

For a smooth ball-and-socket hinge j whose outboard hinge is also a smooth ball-and-socket hinge, according to the internal torques are all zero in two ends of its outboard body, using the transfer equation of the outboard body $j+1$

$$z_{j+1,j+2} = U_{j+1} z_{j+1,j}, \tag{36}$$

the relationship between the orientation angles in two ends of the hinge can be developed

$$\boldsymbol{\theta}_O = [\boldsymbol{U}_{2,1}, \boldsymbol{O}_{3\times3}, \boldsymbol{O}_{3\times3}, \boldsymbol{U}_{2,4}, \boldsymbol{U}_{2,5}]\boldsymbol{z}_I, \tag{37}$$

where

$$\boldsymbol{U}_{2,1} = -\bar{\boldsymbol{U}}_{3,2}^{-1}\bar{\boldsymbol{U}}_{3,1}, \quad \boldsymbol{U}_{2,4} = -\bar{\boldsymbol{U}}_{3,2}^{-1}\bar{\boldsymbol{U}}_{3,4}, \quad \boldsymbol{U}_{2,5} = -\bar{\boldsymbol{U}}_{3,2}^{-1}\bar{\boldsymbol{U}}_{3,5} \tag{38}$$

$\bar{\boldsymbol{U}}_{k,j}$ is the submatrix of the transfer matrix $\bar{\boldsymbol{U}}$ of its outboard body $j+1$.

So the transfer equation and transfer matrix of the smooth ball-and-socket hinge can be obtained as follows

$$\boldsymbol{z}_O = \boldsymbol{U}\boldsymbol{z}_I \tag{39}$$

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times1} \\ -\bar{\boldsymbol{U}}_{3,2}^{-1}\bar{\boldsymbol{U}}_{3,1} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & -\bar{\boldsymbol{U}}_{3,2}^{-1}\bar{\boldsymbol{U}}_{3,4} & -\bar{\boldsymbol{U}}_{3,2}^{-1}\bar{\boldsymbol{U}}_{3,5} \\ \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{I}_3 & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times1} \\ \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{O}_{3\times3} & \boldsymbol{I}_3 & \boldsymbol{O}_{3\times1} \\ \boldsymbol{O}_{1\times3} & \boldsymbol{O}_{1\times3} & \boldsymbol{O}_{1\times3} & \boldsymbol{O}_{1\times3} & 1 \end{bmatrix} \tag{40}$$

3.4.2. *Smooth Ball-and-Socket Hinge Whose Outboard Hinge Is a Fictitious Hinge*

For a smooth ball-and-socket hinge whose outboard hinge is a fictitious hinge (that is, this end is free), the internal torques all equal to zero in two ends of its outboard body too. So the transfer equation and transfer matrix of this kind of smooth ball-and-socket hinge are the same as Equations (39) and (40).

3.4.3. *Smooth Ball-and-Socket Hinge Whose Outboard Hinge Is Neither a Smooth Ball-and-Socket Hinge Nor a Fictitious Hinge*

For a smooth ball-and-socket hinge j whose outboard hinges $j+1, j+2, \ldots, j+r-1$ are neither smooth ball-and-socket hinge nor fictitious hinge until smooth ball-and-socket hinge or fictitious hinge $j+r$, the transfer matrix can be obtained from Equation (40) by replacing $\bar{\boldsymbol{U}}_{j+1} \rightarrow \bar{\boldsymbol{U}}_{j,j+r}$, where $\bar{\boldsymbol{U}}_{j,j+r}$ denote the submatrices of the transfer matrix of the subsystem described the transfer relationship between the state vectors $z_{j,j}$ and $z_{jj+r-1,j+r}$, that is $\bar{\boldsymbol{U}}_{j,j+r} = \boldsymbol{U}_{j+r-1}\boldsymbol{U}_{j+r-2}\cdots\boldsymbol{U}_{j+2}\boldsymbol{U}_{j+1}$.

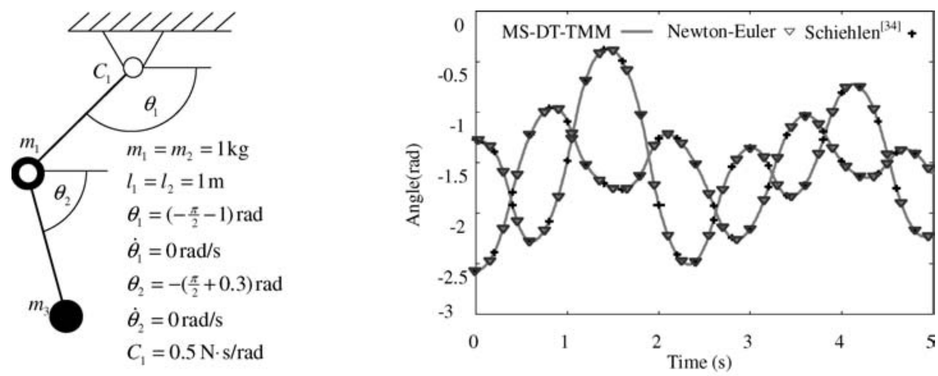
3.5. SMOOTH PIN HINGE

For a multibody system moving in plane, the transfer matrix of the smooth pin hinge can be obtained using similar method to Section 3.4. In fact, the transfer matrix of the smooth pin hinge can be obtained directly from the transfer matrix of smooth ball-and-socket hinge as the special case of space motion.

4. Algorithm for Dynamics Analysis

Following the formulations given above, the motion quantities of a multibody system at different time instants for different subsystems can now be obtained as follows:

1. Decide the initial conditions and system properties of the multibody system.
2. Set $i = 1$.
3. Knowing the initial conditions $\boldsymbol{r}(t_{i-1})$, $\dot{\boldsymbol{r}}(t_{i-1})$, $\ddot{\boldsymbol{r}}(t_{i-1})$, $\boldsymbol{\theta}(t_{i-1})$, $\dot{\boldsymbol{\theta}}(t_{i-1})$, $\ddot{\boldsymbol{\theta}}(t_{i-1})$ etc. and the system properties at time t_i, calculate the quantities $\boldsymbol{\chi}_1$, $\boldsymbol{\chi}_2$, $\boldsymbol{\chi}_3$, $\boldsymbol{\chi}_4$, $\bar{c}$ and $\bar{s}$ etc. for each subsystem.
4. Formulate the transfer matrix for each subsystem and the overall transfer matrix respectively.
5. Apply boundary conditions to the end state vectors of the system and calculate the unknown quantities in the boundary state vectors as a function of the elements of the overall transfer matrix.
6. Now, knowing all the elements in the boundary state vector, the motion quantities at each subsystem at time instant t_i can be computed by successive multiplication of the transfer matrices, using Equation (15).
7. By using the computed values of the position coordinates $\boldsymbol{r}(t_i)$ and orientation angles $\boldsymbol{\theta}(t_i)$ at t_i, compute the values of $\dot{\boldsymbol{r}}(t_i)$, $\ddot{\boldsymbol{r}}(t_i)$, $\dot{\boldsymbol{\theta}}(t_i)$, $\ddot{\boldsymbol{\theta}}(t_i)$ etc. using Equations (1) and (2).
8. Let $i = i + 1$, use the computed values of the last step as the initial conditions, and return to step 3, until the time required for complete analysis.



(a) Model of a rigid double pendulum (b) Time history of the angles of the system

Figure 2. Motion of rigid double pendulum.

5. Applications and Numerical Examples

5.1. DYNAMICS OF MULTI-RIGID-BODY SYSTEM MOVING IN PLANE

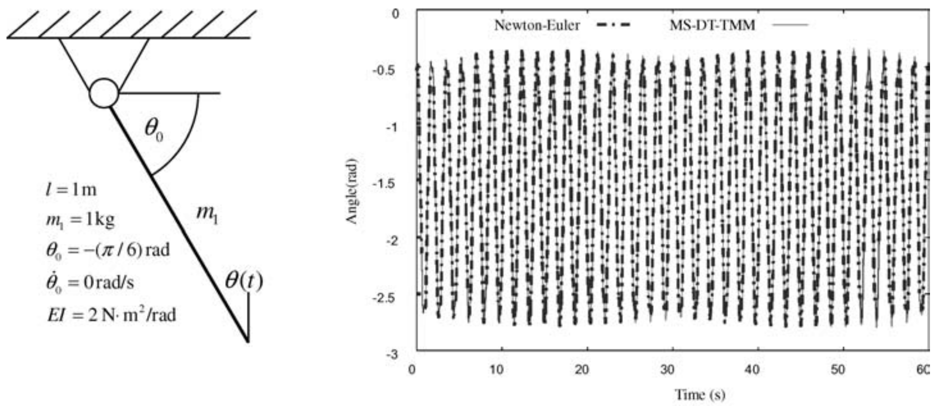
Numerical example 1: Consider a rigid double pendulum moving in plane under the effect of gravity. The initial conditions and salient parameters of the system are listed in Figure 2(a).

The motion of the system is first computed by developing the global dynamics equations of the system using Newton–Euler method and integrating the equations using Runge–Kutta (R–K) method and Schiehlen method [34] respectively. Then the same system is simulated by the proposed MS-DT-TMM procedure. The simulation results are shown in Figure 2(b). For these cases, the same integration time step is taken. As can be seen from Figure 2(b), the motion of the system obtained by MS-DT-TMM is almost identical to the motion got by other methods of multibody dynamics.

5.2. DYNAMICS OF MULTI-FLEXIBLE-BODY SYSTEM MOVING IN PLANE

Numerical example 2: Compute the large planar motion of a flexible pendulum under the effect of the gravity. The initial parameters of the system are listed in Figure 3(a).

The dynamical equations of the beam system can be developed with an ordinary method multi-flexible-body system. However, for making comparison between the ordinary dynamics method and MS-DT-TMM, the flexible beam is divided into twenty same parts represented by rigid bodies connected with rotary spring hinges successively. The stiffness of the rotary spring hinges and the mass of the rigid bodies can be decided by the method described in reference [35, 36]. The large motion of the beam system is first computed by developing the global dynamics equations of the system with Newton-Euler method and solved with R–K method. Then the large motion is computed with MS-DT-TMM. The large angle motion



(a) Model of a flexible pendulum (b) Time history of the angle at pendulum end

Figure 3. Motion of the flexible pendulum.

of the beam end got by the two methods, as shown in Figure 2(b), have a good agreement. These confirm the suitability of MS-DT-TMM for the dynamics of multi-flexible-body system.

It can be seen from this example and Figure 3(b) that:

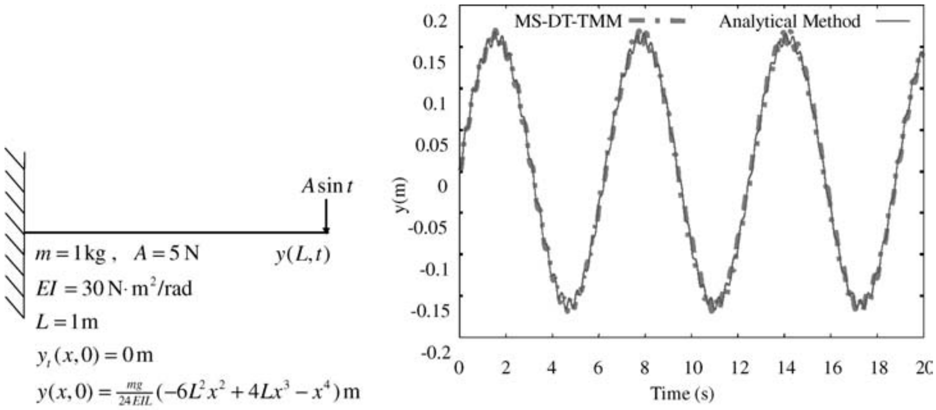
1. The order of the system matrix of corresponding global dynamics equations is forty. But the order of corresponding overall transfer matrix of the system is only seven. And the computational time required decreases greatly if using MS-DT-TMM, see Figure 10.
2. With time step properly chosen, we have not found problems in the computational stability using MS-DT-TMM for the large system in the long time history.
3. The MS-DT-TMM is valid for the large system including twenty bodies.

Numerical example 3: Compute the response of the cantilever beam moving in plane under the effect of external force $A \sin t$ in the free end. The initial conditions and salient parameters of the system are listed in Figure 4(a).

The motion of the beam is studied using the similar model to that used in Numerical example 2. Then the motion of the beam system is computed with MS-DT-TMM and with analytical method respectively. The time history of the deflection response of the cantilever beam end got by MS-DT-TMM is almost identical to that got by analytical method, as shown in Figure 4(b).

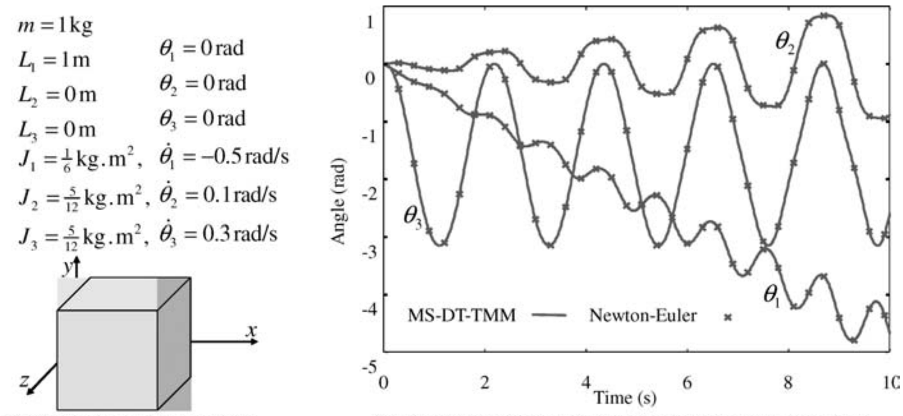
5.3. DYNAMICS OF MULTI-RIGID-BODY SYSTEM MOVING IN SPACE

Numerical example 4: Compute the motion of a space pendulum moving in space under the effect of the gravity. The initial parameters of the system are listed in Figure 5(a).



(a) Model of cantilever beam (b) Deflection response of the cantilever beam end

Figure 4. Response of the cantilever beam.



(a) Model of a spatial pendulum (b) Time history of the orientation angle of the space pendulum

Figure 5. Motion of the space pendulum.

Time history of the orientation angles of the space pendulum is computed by MS-DT-TMM procedure and by using R–K method and integrating the dynamics equations of the system respectively. The results of the system motion got by MS-DT-TMM are almost identical to that got by Newton-Euler method, as shown in Figure 5(b).

Numerical example 5: Compute the system motion of rigid three pendulums moving in space under the effect of the gravity, as shown in Figure 6(a). In which, three same rigid bodies connected by three smooth ball-and-socket hinges successively, and the parameters of every rigid body is the same with the parameters of the rigid body of the example 5, the initial orientated angles of the three bodies and the initial angular velocities of body 2 and body 4 are all zero, the initial angular velocities of body 6 is $[0, 0.1, 0]^T$ rad/s.

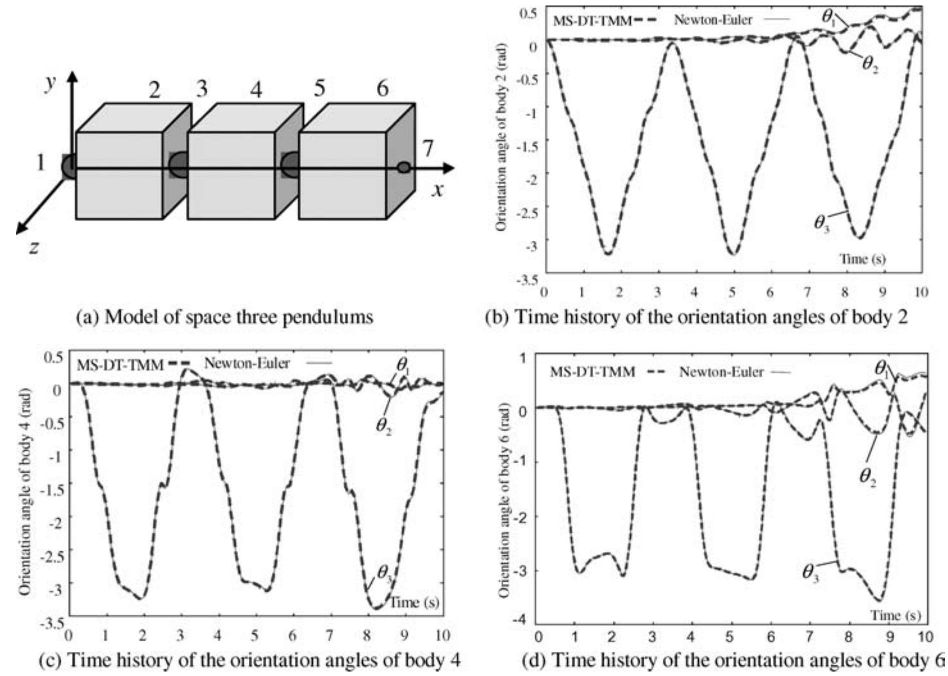


Figure 6. Motion of the space three pendulum.

For making comparison between the ordinary dynamics method and MS-DT-TMM, the procedure to study the multibody system dynamics using two methods is given in detail as follows.

If using ordinary methods of multibody dynamics, for example, Newton-Euler method, the global dynamics equations of the system can be obtained as follows

$$\begin{bmatrix} \boldsymbol{E}_{22} & \boldsymbol{E}_{24} & \boldsymbol{E}_{26} \\ \boldsymbol{E}_{42} & \boldsymbol{E}_{44} & \boldsymbol{E}_{46} \\ \boldsymbol{E}_{62} & \boldsymbol{E}_{64} & \boldsymbol{E}_{66} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_2 \\ \dot{\boldsymbol{\omega}}_4 \\ \dot{\boldsymbol{\omega}}_6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_2 \\ \boldsymbol{F}_4 \\ \boldsymbol{F}_6 \end{bmatrix} \tag{41}$$

where

$$\begin{aligned}
&\boldsymbol{E}_{66} = \boldsymbol{J}, \quad \boldsymbol{E}_{44} = \boldsymbol{J} - m\tilde{\boldsymbol{r}}_{2,O}\tilde{\boldsymbol{r}}_{2,O}, \quad \boldsymbol{E}_{22} = \boldsymbol{J} - 2m\tilde{\boldsymbol{r}}_{2,O}\tilde{\boldsymbol{r}}_{2,O}, \\
&\boldsymbol{E}_{24} = -\tilde{\boldsymbol{r}}_{2,O}\boldsymbol{A}_2^T\boldsymbol{A}_4 m(\tilde{\boldsymbol{r}}_{2,C} + \tilde{\boldsymbol{r}}_{2,O}), \quad \boldsymbol{E}_{26} = -m\tilde{\boldsymbol{r}}_{2,O}\boldsymbol{A}_2^T\boldsymbol{A}_6\tilde{\boldsymbol{r}}_{2,C}, \\
&\boldsymbol{E}_{42} = -m(\tilde{\boldsymbol{r}}_{2,O} + \tilde{\boldsymbol{r}}_{2,C})\boldsymbol{A}_4\boldsymbol{A}_2^T\tilde{\boldsymbol{r}}_{2,O}, \quad \boldsymbol{E}_{46} = -m\tilde{\boldsymbol{r}}_{2,O}\boldsymbol{A}_4^T\boldsymbol{A}_6\tilde{\boldsymbol{r}}_{2,C}, \\
&\boldsymbol{E}_{62} = -m\tilde{\boldsymbol{r}}_{2,C}\boldsymbol{A}_6^T\boldsymbol{A}_2\tilde{\boldsymbol{r}}_{2,O}, \quad \boldsymbol{E}_{64} = -m\tilde{\boldsymbol{r}}_{2C}\boldsymbol{A}_6^T\boldsymbol{A}_4\tilde{\boldsymbol{r}}_{2,O}, \\
&\boldsymbol{F}_2 = -\tilde{\boldsymbol{\omega}}_2\boldsymbol{J}\boldsymbol{\omega}_2 - mg[\tilde{\boldsymbol{r}}_{2,C} + 2\tilde{\boldsymbol{r}}_{2,O}]\boldsymbol{A}_2^T[0\,1\,0]^T - \tilde{\boldsymbol{r}}_{2,O}\boldsymbol{A}_2^T[2m\boldsymbol{A}_2\tilde{\boldsymbol{\omega}}_2\tilde{\boldsymbol{\omega}}_2\boldsymbol{r}_{2,O} \\
&\qquad + \boldsymbol{A}_4\tilde{\boldsymbol{\omega}}_4\tilde{\boldsymbol{\omega}}_4 m(\boldsymbol{r}_{2,O} + \boldsymbol{r}_{2,C}) + \boldsymbol{A}_6\tilde{\boldsymbol{\omega}}_6\tilde{\boldsymbol{\omega}}_6 m\boldsymbol{r}_{2,C}],
\end{aligned}$$

$$\begin{aligned}\boldsymbol{F}_4 &= -\tilde{\boldsymbol{\omega}}_4\boldsymbol{J}\tilde{\boldsymbol{\omega}}_4 - mg[\tilde{\boldsymbol{r}}_{2,C} + \tilde{\boldsymbol{r}}_{2,O}]\boldsymbol{A}_4^T[0\,1\,0]^T - \tilde{\boldsymbol{r}}_{2,O}\boldsymbol{A}_4^T m[\boldsymbol{A}_2\tilde{\boldsymbol{\omega}}_2\tilde{\boldsymbol{\omega}}_2\boldsymbol{r}_{2,O} \\ &\quad + \boldsymbol{A}_4\tilde{\boldsymbol{\omega}}_4\tilde{\boldsymbol{\omega}}_4\boldsymbol{r}_{2,O} + \boldsymbol{A}_6\tilde{\boldsymbol{\omega}}_6\tilde{\boldsymbol{\omega}}_6\boldsymbol{r}_{2,C}] - m\tilde{\boldsymbol{r}}_{2,C}\boldsymbol{A}_4^T\boldsymbol{A}_2\tilde{\boldsymbol{\omega}}_2\tilde{\boldsymbol{\omega}}_2\boldsymbol{r}_{2,O}, \\ \boldsymbol{F}_6 &= -\tilde{\boldsymbol{\omega}}_6\boldsymbol{J}\boldsymbol{\omega}_6 - mg\tilde{\boldsymbol{r}}_{2,C}\boldsymbol{A}_6^T[0\,1\,0]^T \\ &\quad - m\tilde{\boldsymbol{r}}_{2,C}\boldsymbol{A}_6^T[\boldsymbol{A}_2\tilde{\boldsymbol{\omega}}_2\tilde{\boldsymbol{\omega}}_2\boldsymbol{r}_{2,O} + \boldsymbol{A}_4\tilde{\boldsymbol{\omega}}_4\tilde{\boldsymbol{\omega}}_4\boldsymbol{r}_{2,O}] \end{aligned} \tag{42}$$

g is the gravity acceleration. Because the structure parameters of three bodies are the same, here we do not select the different denoted symbol. The order of the system matrix of the system is eighteen in ordinary method.

Using MS-DT-TMM and considering the symbol convention in Section 2.1, we can obtain the overall transfer equation of the system

$$\boldsymbol{Z}_{6,7} = \boldsymbol{U}\boldsymbol{Z}_{2,1} \tag{43}$$

transfer matrix of the system

$$\boldsymbol{U} = \boldsymbol{U}_6\boldsymbol{U}_5\boldsymbol{U}_4\boldsymbol{U}_3\boldsymbol{U}_2 \tag{44}$$

transfer equations of the elements

$$\begin{aligned}&\boldsymbol{Z}_{2,3} = \boldsymbol{U}_2\boldsymbol{Z}_{2,1}, \quad \boldsymbol{Z}_{4,3} = \boldsymbol{U}_3\boldsymbol{Z}_{2,3} \quad \boldsymbol{Z}_{4,5} = \boldsymbol{U}_4\boldsymbol{Z}_{4,3} \quad \boldsymbol{Z}_{6,5} = \boldsymbol{U}_5\boldsymbol{Z}_{4,5} \\ &\boldsymbol{Z}_{6,7} = \boldsymbol{U}_6\boldsymbol{Z}_{6,5}\end{aligned} \tag{45}$$

and boundary conditions of the system

$$\begin{aligned}z_{2,1} &= [0, 0, 0, \theta_1, \theta_2, \theta_3, 0, 0, 0, q_x, q_y, q_z, 1]^T \\ z_{6,7} &= [x, y, z, \theta_1, \theta_2, \theta_3, 0, 0, 0, 0, 0, 0, 1]^T\end{aligned} \tag{46}$$

where, $\boldsymbol{U}_2$, $\boldsymbol{U}_4$, $\boldsymbol{U}_6$ and $\boldsymbol{U}_3$, $\boldsymbol{U}_5$ are the transfer matrices of corresponding rigid bodies 2, 4, 6 and smooth ball-and-socket hinges 3, 5 described in Section 3 respectively. The order of the transfer matrix of the system is thirteen in the MS-DT-TMM.

The system motion (orientated angles of three space pendulums) can be got by direct integrating the dynamics equations of the system using R–K method and by solving the transfer equations of the system using MS-DT-TMM procedure respectively. The results obtained by two methods are almost identical, as shown in Figure 6(b–d).

5.4. BRANCHED SYSTEMS, NETWORK SYSTEMS, CLOSED-LOOP SYSTEMS

TMM and MS-TMM are applicable not only for chain systems, but also for branched systems, network systems, closed-loop systems etc. [6, 29]. In the MS-DT-TMM, various complex multibody systems can be modeled in the same way as in the

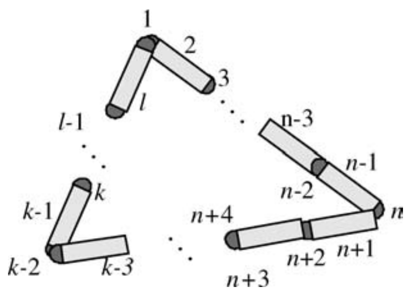


Figure 7. Closed-loop system.

TMM and MS-TMM. Showing the applicability of the proposed method for general multibody systems, some explanatory examples about branched system, network system, closed-loop system, have been discussed in this section. A detailed study of these is yet to be undertaken.

5.4.1. *Dynamics of Closed-Loop System*

For a closed-loop system moving in space or in plane, as shown in Figure 7, the boundary conditions are not so obvious. In the same way as in the TMM, we "cut" and start at any arbitrary point of the closed system, where the state vector is $z_{l,1}$. Then the closed-loop system can be considered as a chain system, proceeding around the system until we return to the original point from which we started. Then the overall transfer equations

$$\boldsymbol{Z}_{l,1} = \boldsymbol{U}\boldsymbol{Z}_{l,1}$$

or

$$\boldsymbol{U}_{\text{all}}\boldsymbol{Z}_{l,1} = \boldsymbol{0} \tag{47}$$

and the transfer matrix of the system

$$\boldsymbol{U}_{\text{all}} = \boldsymbol{U} - \boldsymbol{I} \tag{48}$$

are obtained, where

$$\boldsymbol{U} = \boldsymbol{U}_l\boldsymbol{U}_{l-1}\ldots\boldsymbol{U}_{n-1}\boldsymbol{U}_{n-2}\ldots\boldsymbol{U}_2\boldsymbol{U}_1 \tag{49}$$

The motion of the closed-loop system can be simulated by solving the transfer equations of the system as same as chain system.

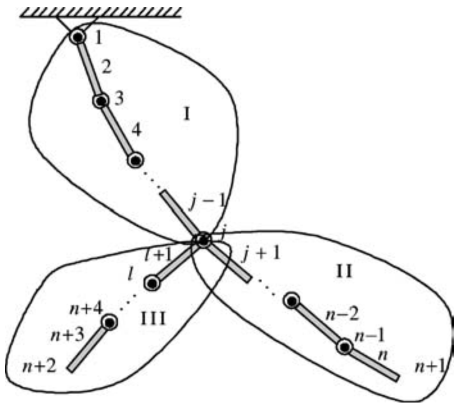


Figure 8. Branched system.

5.4.2. *Dynamics of Branched System*

A branched system moving in space or in plane, can be divided into three branches I, II and III respectively, each branch can be considered as a chain subsystem and is connected at branched point j by hinge, as shown in Figure 8.

There are transfer equations for these branches

$$z_{j-1,j} = \boldsymbol{U}_{\mathrm{I}} z_{2,1}, \quad z_{n,n+1} = \boldsymbol{U}_{\mathrm{II}} z_{j+1,j}, \quad z_{l+1,j} = \boldsymbol{U}_{\mathrm{III}} z_{n+3,n+2} \tag{50}$$

where $\boldsymbol{U}_{\mathrm{I}}$, $\boldsymbol{U}_{\mathrm{II}}$ and $\boldsymbol{U}_{\mathrm{III}}$ are the transfer matrix of the chain subsystem I, II and III.

If the system moves in a plane and the hinge on the branched point j is smooth pin hinges, because the displacement is continuing and the forces are equilibrium as well as the torques are zero in the branched point j, we can obtain

$$\begin{aligned}
&z_{j-1,j}(1) = z_{j+1,j}(1) = z_{l+1,l}(1),\\
&z_{j-1,j}(2) = z_{j+1,j}(2) = z_{l+1,l}(2),\\
&z_{j-1,j}(5) - z_{j+1,j}(5) + z_{l+1,l}(5) = 0,\\
&z_{j-1,j}(6) - z_{j+1,j}(6) + z_{l+1,l}(6) = 0,\\
&z_{j-1,j}(4) = z_{j+1,j}(4) = z_{l+1,l}(4) = 0
\end{aligned} \tag{51}$$

where, symbol $z(i)$ means the element of corresponding state vector, the number i in brackets is the sequence number of the element in corresponding state vector.

Substitute Equation (50) into (51), we can obtain

$$\begin{aligned}
&\boldsymbol{U}_{b1}\boldsymbol{U}_{\mathrm{I}}\boldsymbol{z}_{2,1} = \boldsymbol{U}_{b1}\boldsymbol{U}_{\mathrm{II}}^{-1}\boldsymbol{z}_{n,n+1} - \boldsymbol{U}_{b1}\boldsymbol{U}_{\mathrm{III}}\boldsymbol{z}_{n+3,n+2},\\
&\boldsymbol{U}_{b2}\boldsymbol{U}_{\mathrm{I}}\boldsymbol{z}_{2,1} = \boldsymbol{U}_{b2}\boldsymbol{U}_{\mathrm{II}}^{-1}\boldsymbol{z}_{n,n+1} = \boldsymbol{U}_{b2}\boldsymbol{U}_{\mathrm{III}}\boldsymbol{z}_{n+3,n+2}, \boldsymbol{U}_{b3}\boldsymbol{U}_{\mathrm{I}}\boldsymbol{z}_{2,1} = 0,\\
&\boldsymbol{U}_{b3}\boldsymbol{U}_{\mathrm{II}}^{-1}\boldsymbol{z}_{n,n+1} = 0, \quad \boldsymbol{U}_{b3}\boldsymbol{U}_{\mathrm{III}}\boldsymbol{z}_{n+3,n+2} = 0
\end{aligned} \tag{52}$$

$$\begin{aligned}
&\boldsymbol{U}_{b1} = [\boldsymbol{O}_{2\times 4}, \boldsymbol{I}_2, \boldsymbol{O}_{2\times 1}], \quad \boldsymbol{U}_{b2} = [\boldsymbol{I}_2, \boldsymbol{O}_{2\times 5}],\\
&\boldsymbol{U}_{b3} = [\boldsymbol{O}_{1\times 3}, 1, \boldsymbol{O}_{1\times 3}]
\end{aligned} \tag{53}$$

So the overall transfer equation can be obtained

$$\boldsymbol{U}_{\mathrm{all}}\left[\boldsymbol{z}_{2,1}^T \ \ \boldsymbol{z}_{n,n+1}^T \ \ \boldsymbol{z}_{n+3,n+2}^T\right]^T = \boldsymbol{0} \tag{54}$$

where

$$\boldsymbol{U}_{\mathrm{all}} = \begin{bmatrix}
\boldsymbol{U}_{b1}\boldsymbol{U}_{\mathrm{I}} & -\boldsymbol{U}_{b1}\boldsymbol{U}_{\mathrm{II}}^{-1} & \boldsymbol{U}_{b1}\boldsymbol{U}_{\mathrm{III}}\\
\boldsymbol{U}_{b2}\boldsymbol{U}_{\mathrm{I}} & -\boldsymbol{U}_{b2}\boldsymbol{U}_{\mathrm{II}}^{-1} & \boldsymbol{O}_{2\times 7}\\
\boldsymbol{U}_{b2}\boldsymbol{U}_{\mathrm{I}} & \boldsymbol{O}_{2\times 7} & -\boldsymbol{U}_{b2}\boldsymbol{U}_{\mathrm{III}}\\
\boldsymbol{U}_{b3}\boldsymbol{U}_{\mathrm{I}} & \boldsymbol{O}_{1\times 7} & \boldsymbol{O}_{1\times 7}\\
\boldsymbol{O}_{1\times 7} & \boldsymbol{U}_{b3}\boldsymbol{U}_{\mathrm{II}}^{-1} & \boldsymbol{O}_{1\times 7}\\
\boldsymbol{O}_{1\times 7} & \boldsymbol{O}_{1\times 7} & \boldsymbol{U}_{b3}\boldsymbol{U}_{\mathrm{III}}
\end{bmatrix} \tag{55}$$

The motion of the branched system can be simulated by using boundary conditions and solving the transfer equations of the system as same as chain system.

5.4.3. *Dynamics of the Network System*

A network system as shown in Figure 9, can be divided into two subsystems I, and II respectively, each subsystem can be considered as a chain system and is connected at network point j by hinge.

There are transfer equations for these subsystem I and II

$$\boldsymbol{z}_{j-1,j} = \boldsymbol{U}_{\mathrm{I}}\boldsymbol{z}_{2,1}, \quad \boldsymbol{z}_{n,j} = \boldsymbol{U}_{\mathrm{II}}\boldsymbol{z}_{j+1,j} \tag{56}$$

where, $\boldsymbol{U}_{\mathrm{I}}$ and $\boldsymbol{U}_{\mathrm{II}}$ are the transfer matrix of the chain subsystem I and II.

$$\boldsymbol{U}_{\mathrm{I}} = \boldsymbol{U}_{j-1}\boldsymbol{U}_{j-2}\ldots\boldsymbol{U}_3\boldsymbol{U}_2, \quad \boldsymbol{U}_{\mathrm{II}} = \boldsymbol{U}_n\boldsymbol{U}_{n-1}\ldots\boldsymbol{U}_{j+1} \tag{57}$$

If the system moves in a plane and the hinge on the network point j is smooth pin hinges, because the displacement is continuing and the forces are equilibrium as well as the torques are zero in the network point j, we can obtain

$$\begin{aligned}
&\boldsymbol{U}_{b1}\boldsymbol{U}_{\mathrm{I}}\boldsymbol{z}_{2,1} = \boldsymbol{U}_{b1}\boldsymbol{z}_{j+1,j} - \boldsymbol{U}_{b1}\boldsymbol{U}_{\mathrm{II}}\boldsymbol{z}_{j+1,j}, \quad \boldsymbol{U}_{b3}\boldsymbol{U}_{\mathrm{I}}\boldsymbol{z}_{2,1} = 0,\\
&\boldsymbol{U}_{b2}\boldsymbol{U}_{\mathrm{I}}\boldsymbol{z}_{2,1} = \boldsymbol{U}_{b2}\boldsymbol{z}_{j+1,j} = \boldsymbol{U}_{b2}\boldsymbol{U}_{\mathrm{II}}\boldsymbol{z}_{j+1,j}, \quad \boldsymbol{U}_{b3}\boldsymbol{U}_{\mathrm{II}}\boldsymbol{z}_{j+1,j} = 0
\end{aligned} \tag{58}$$

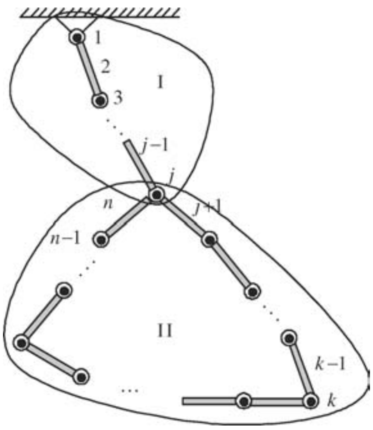


Figure 9. Network system.

where, $\boldsymbol{U}_{b1}$, $\boldsymbol{U}_{b2}$, and $\boldsymbol{U}_{b3}$ are defined by Equation (53).

So the overall transfer equation can be obtained

$$\boldsymbol{U}_{\text{all}}\left[\boldsymbol{z}_{2,1}^T, \boldsymbol{z}_{j+1,j}^T\right]^T = \boldsymbol{0} \tag{59}$$

where

$$\boldsymbol{U}_{\text{all}} = \begin{bmatrix} \boldsymbol{U}_{b1}\boldsymbol{U}_{\text{I}} & \boldsymbol{U}_{b1}(\boldsymbol{I}_7 - \boldsymbol{U}_{\text{II}}) \\ \boldsymbol{U}_{b2}\boldsymbol{U}_{\text{I}} & -\boldsymbol{U}_{b2} \\ \boldsymbol{O}_{2\times 7} & \boldsymbol{U}_{b2}(\boldsymbol{I}_7 - \boldsymbol{U}_{\text{II}}) \\ \boldsymbol{U}_{b3}\boldsymbol{U}_{\text{I}} & \boldsymbol{O}_{1\times 7} \\ \boldsymbol{O}_{1\times 7} & \boldsymbol{U}_{b3}\boldsymbol{U}_{\text{II}} \end{bmatrix} \tag{60}$$

The motion of network system can be simulated by using boundary conditions of the system and solving the transfer equations of the system as same as chain system.

5.5. COMPUTATIONAL TIME, MEMORY STORAGE AND ACCURACY

Consider the motion of a chain multibody system including rigid bodies connected with rotary spring hinges, the initial parameters of the system are as follows: the number of the rigid bodies is n, the length and mass of each rigid body are $(1/n)$ m and $(1/n)$ kg respectively, the stiffness of each rotary spring hinge is $20N\cdot$ m/rad, the initial angles and initial angular velocities of the bodies are all $(-\pi/3)$ and zero.

The computational time required of the system dynamics using MS-DT-TMM and ordinary method is shown in Figure 10. And the CPU of the used microcomputer is an AMD-1.2 GHz. It can be seen clearly from Figure 10 that the computational time increase very slowly when the number of the degrees of freedom increases

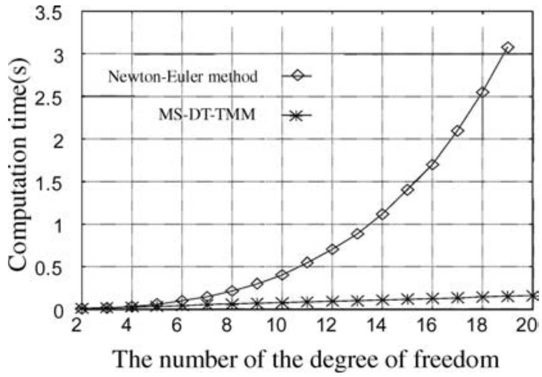


Figure 10. Computational time vary with number of the degrees of freedom.

if using MS-DT-TMM, and the computational time increases very quickly when the number of the degrees of freedom increases if using ordinary methods, the MS-DT-TMM has a higher computational speed.

Of course, now, the building of the global dynamics equations of multibody systems can also be relatively simple by using some effective methods; and by using the sparse matrix techniques, the computer time is direct proportion to the number of the bodies for chain system so that computational speed can also be increased greatly [37].

There is no general method of exactly estimating the accumulated errors involved in the computed solution, so far in the authors' experience with MS-DT-TMM in solving problems of varied multibody system sizes, with time step properly chosen, the truncation error accumulation was not significant. On the other hand, the round off error accumulation is a function of the word length of computing system as well as the number of elements to be modeled and is very significant for MS-DT-TMM. It can be seen from numerical examples shown in Section 5.2 that MS-DT-TMM is also valid for large system. The simulation results of the motion of the large system got by MS-DT-TMM and by ordinary dynamics methods always have a good agreement in over thirty periods. In fact, MS-DT-TMM is also valid for very large system if the Riccati transform is introduced. In authors' experience in computing the motion of the system including one hundred thousand rigid bodies connected by elastic hinges, it is difficult to obtain correct results if only MS-DT-TMM is used. But by using MS-DT-TMM and Riccati transform, we have not found difficulty in the computational stability and accuracy for this very large system. The order of the overall transfer matrix of the very large system is only seven, and the order of the system matrix will be two hundred thousand if using ordinary method, which greatly reduces the computational time and the memory storage requirement. A detailed study of these errors on the accuracy of MS-DT-TMM solutions is yet to be undertaken and will be discussed in detail in another paper.

6. Highlights of the Method

1. The method avoids global dynamic equations of multibody system and simplifies solving procedure of multibody system dynamics.
2. Irrespective of the size of a multibody system, the matrices involved in the MS-DT-TMM are always small, which greatly increases the computational speed.
3. The method avoids the computing difficulties caused by too high matrix orders of complex multibody system.
4. In contrast with the conventional TMM and MS-TMM, the matrices involved in the MS-DT-TMM are always real, even when the damping is included. This simplifies numerical computation algorithms of multibody system dynamics.
5. Unlike the TMM and MS-TMM, which are restricted to small vibration of system alone, MS-DT-TMM is capable of analyzing linear time invariant, linear time variant, nonlinear, multi-rigid-body and multi-flexible-body systems.
6. The method provides flexibility in modeling multibody systems with varying configuration. That is, by creating a library of transfer matrices for commonly occurring elements, such as rigid bodies moving in plane, rigid bodies moving in space, elastic beams moving in plane, elastic beams moving in space, rotary spring hinges, linear spring hinges, smooth ball-and-socket hinges, smooth pin hinges, linear damper hinges and rotary damper hinges, etc., and by assembling these at the required locations, various configurations can be modeled easily.
7. Any suitable numerical integration scheme [32, 33] can be included in this method, thus providing researchers with flexibility in the computation of multibody systems dynamics.

7. Conclusions

A new method, named as discrete time transfer matrix method of multibody system, is introduced for the study of multibody systems dynamics. The formulation of the method as well as some numerical examples of multi-rigid-body system dynamics and multi-flexible-body system dynamics to validate the method are given. Several possible areas of applications are identified. The proposed method is based on the TMM, DT-TMM, MS-TMM and the numerical solution procedures of differential equations. It combines the advantages of these methods. It has the modeling flexibility, smaller core size requirement of TMM, DT-TMM, MS-TMM and the extended applicability of numerical integration procedures. It avoids global dynamics equations of multibody system so that simplifies solution procedure of multibody system dynamics. This method is simple, straightforward and practical for the simulation of multibody system dynamics. By developing transfer matrices of flexible body elements, such as, elastic beam and shell elements etc. this method would be a more power tool for dynamics of multi-flexible-body system with flexible body elements; this is one subject that we are studying.

It should be noted that the propagation of round off error in TMM, and the influence of time step selection on computational accuracy in numerical integration procedures will also be important for the proposed method. The numerical difficulties will arise when analyzing very large systems on small computing systems in using transfer matrices. The accumulation of round off error, which is a function of the word length of computing system as well as the number of elements to be modeled, is very significant. Sometimes the multiplication of the transfer matrices of the different elements of the system lead to very large numbers for the larger system, when several very large numbers are multiplied and several very small numbers are multiplied successively in nature sequence. Because the word length of computing system is limited, when computing the multiplication of the transfer matrices, the solution process probably eventually collapses, if the absolute values of these numbers are too large or too small. To avoid the problem of numerical difficulties arising in the multiplication of the transfer matrices, there are some methods can be chosen, such as the Delta-matrix method [6], Modified transfer-matrix method [6, 38] etc. One of effective methods to avoid the problem of numerical difficulties is to adjust the computing sequence of these numbers, so that large absolute value is multiplied by small absolute value successively when computing the multiplication. Riccati transfer matrix method [26] provides an effective approach for numerical stability of the boundary value problem. Riccati transform can be used not only to decrease the orders of matrices involved in the MS-DT-TMM, but also to overcome the problem of numerical difficulties, see Section 5.5.

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