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Use of Principal Axes as the Floating Reference Frame for a Moving Deformable Body

PARVIZ E. NIKRAVESH and YI-SHIH LIN
Department of Aerospace and Mechanical Engineering, University of Arizona, Tucson, AZ 85721, USA; E-mail: pen@email.arizona.edu

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Abstract. In this paper a method to adopt the principal axes of a deformable body as its body-attached frame is presented. The deformable body in a multibody setting is allowed to deform while it undergoes rigid-body motion. The fundamental concepts of imposing the principal axes as a moving reference frame are that the origin of the frame must remain at the instantaneous mass center and that the three products of inertia must remain zero as the body deforms. These conditions require the construction of several auxiliary matrices that are used in the constraint equations at the position, velocity, and acceleration levels. These auxiliary matrices are constructed only once and remain unchanged through the motion of the deformable body. The presented formulation does not depend on the type of finite element and multibody formulations or any associated assumptions.

Keywords: principal axes, body-attached frame, floating frame, deformable body, multibody dynamics.

1. Introduction

Researchers working on the subject of deformable multibody dynamics must have asked this question at one time or another: Why don't we use the instantaneous principal axes of a deformable body as its floating reference frame? The question is intuitive and the answer seems trivial: We need to define conditions to keep the origin of the reference frame at the instantaneous mass center and also assure that the three products of inertia remain zero as the body deforms. Although the solution seems obvious, defining algebraic equations to enforce these conditions is not easy. This paper derives the necessary algebraic equations for keeping the principal axes of a deformable body as its moving reference frame.

The subject of attaching a reference frame to a moving deformable body has received special attention in the multibody dynamics community. Since a deformable body is normally described as a finite element mesh and nodes, in order to define a moving reference frame, we need to use nodal entities. For this reason, traditionally we select some of the nodes to carry the reference frame with them. This is a simple method that has been borrowed from structural analysis.

In structural finite element analysis, at least six simple conditions are defined to eliminate six of the nodal deflections from the equations of motion. These conditions attach the structure to a reference frame which is assumed stationary and, therefore, the rigid-body degrees of freedom (DoFs) are eliminated. A similar process can be applied to a moving deformable body. A reference frame is attached to one or more nodes by properly setting six of the nodal deflections to zeros. Since the reference frame is allowed to move with respect to an inertial frame, attaching a moving frame to the nodes will not eliminate any of the rigid-body DoFs. We refer to this type of frame as the nodal-fixed axes.

Another less known method for attaching a reference frame to a moving deformable body is to apply the so-called mean axis conditions [1, 2]. The mean axes, based on the minimization of the deformation kinetic energy, impose six conditions on all of the nodes impartially unlike the nodal-fixed conditions. This type of frame is normally referred to as a floating reference frame.

In this paper we present a newly developed formulation that enforces the moving reference frame to coincide with the instantaneous principal axes of a deformable body. This method provides six conditions based on two fundamental concepts: the origin of the reference frame must remain at the instantaneous mass center, and the three products of inertia must remain zeros. The use of such fundamental conditions has been suggested by number of researchers in the past; for example refer to [3]. Although these fundamental conditions are well known, deriving closed-form expressions for the three products of inertia in terms of the nodal masses, coordinates and deflections is not a trivial task. Derivation of these expressions requires clearly defined notation in order to perform the necessary vector-matrix algebra.

The *only* aim of this paper is to show in detail how to enforce the principal axes of a deformable body to be the moving reference frame. It is not the intention of this paper to provide any comparison between the principal axes and other types of reference frames that have been introduced in the past. A review of other types of reference frame is not necessary in order to understand how to implement the formulation for the principal axes. Whether the use of principal axes exhibits any improvements over other types of frames is a question to be answered in the future.

Derivation of the six fundamental conditions for the principal axes is first shown in Section 2 for a deformable body where the nodes only exhibit *translational* DoFs. The process is then repeated in Section 3 for the case where the nodes have both *translational* and *rotational* DoFs. A simple numerical example is provided in Section 4 to help the reader understand the concepts more easily.

Appendix A provides a relatively extensive introduction to the notation, nomenclature and the vector-matrix algebra that are used in the paper. The reader is strongly encouraged to review this appendix first before attempting to follow the derivation of the conditions for principal axes.

2. Nodal Translational DoF's

Assume that a deformable body has n nodes where the nodes are allowed only translational deflections; therefore, the body has $n_{\mathrm{dof}} = 3 \times n$ DoFs. It is not important for our purpose to describe in detail how the body is decritized, what types

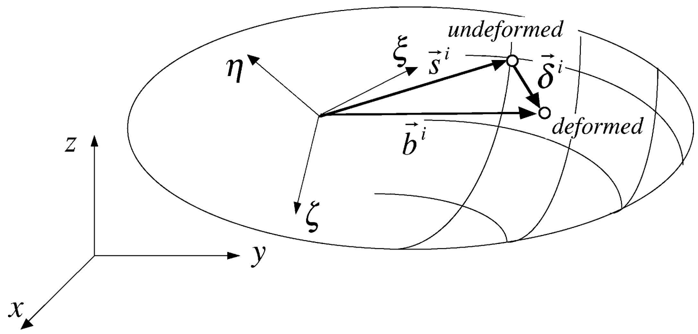

Figure 1. A typical node in a moving deformable body.

of elements are used, or whether we deal with linear or nonlinear material characteristics. Such a deformable body is shown in Figure 1. A nonmoving (inertial) reference frame is defined as x–y–z and a moving reference frame is defined as ξ–η–ζ. The goal is to enforce the moving frame to coincide with the instantaneous principal axes of the body.

For a typical node i, as shown in Figure 1, the translational deflection is denoted by a 3-vector $\boldsymbol{\delta}^i$. This node is positioned from the origin of the body frame in its undeformed state by vector $\boldsymbol{s}^i$, and in the deformed state by vector $\boldsymbol{b}^i = \boldsymbol{s}^i + \boldsymbol{\delta}^i$. For all the nodes, the arrays of nodal deflections, the undeformed positions, and the deformed positions are defined as

$$\boldsymbol{\delta} = \begin{Bmatrix} \boldsymbol{\delta}^1 \\ \vdots \\ \boldsymbol{\delta}^n \end{Bmatrix} \quad \boldsymbol{s} = \begin{Bmatrix} \boldsymbol{s}^1 \\ \vdots \\ \boldsymbol{s}^n \end{Bmatrix} \quad \boldsymbol{b} = \begin{Bmatrix} \boldsymbol{b}^1 \\ \vdots \\ \boldsymbol{b}^n \end{Bmatrix} \tag{1}$$

For this deformable body the mass matrix M is $n_{\text{dof}} \times n_{\text{dof}}$. For the employment of the principal axes conditions, it is not necessary to know how the mass matrix is generated. Also note that in this paper all vectors, arrays, matrices, and the corresponding entities are described in terms of their ξ–η–ζ components unless stated otherwise.

2.1. Rigid-body mode-shapes

A deformable body that is not constrained to the ground, in addition to its deformation DoFs, also exhibits six rigid body DoFs. The translational and rotational *rigid-body mode-shapes* are expressed as

$$\psi^{(t)} = \hat{\mathbf{I}} = \begin{bmatrix} \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix} \quad {}^0\psi^{(r)} = -\hat{\tilde{s}} = \begin{bmatrix} -\tilde{s}^1 \\ \vdots \\ -\tilde{s}^n \end{bmatrix} \quad \psi^{(r)} = -\hat{\tilde{\boldsymbol{b}}} = \begin{bmatrix} -\tilde{\boldsymbol{b}}^1 \\ \vdots \\ -\tilde{\boldsymbol{b}}^n \end{bmatrix} \tag{2}$$

where ${}^0\psi^{(r)}$ and $\psi^{(r)}$ represent the rotational rigid-body mode-shapes at the undeformed and the deformed states, respectively (refer to Appendix A for a description of symbols and notation).

2.2. Mass and rotational inertia matrix

The *total mass* of a deformable body can be found from the mass matrix and the translational rigid-body mode-shapes as

$$\hat{\mathbf{I}}^{\mathrm{T}} M \hat{\mathbf{I}} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} = m\mathbf{I} \tag{3}$$

It is obvious that the total mass is independent of the direction and, therefore, the mass in all three directions is the same.

The *rotational inertia matrix* at the undeformed state is obtained from the mass matrix and the rotational rigid-body mode-shapes as

$$\hat{\hat{\boldsymbol{s}}}^{\mathrm{T}} \boldsymbol{M} \hat{\hat{\boldsymbol{s}}} = \begin{bmatrix} {}^0 j_{\xi\xi} & {}^0 j_{\xi\eta} & {}^0 j_{\xi\zeta} \\ {}^0 j_{\eta\xi} & {}^0 j_{\eta\eta} & {}^0 j_{\eta\zeta} \\ {}^0 j_{\zeta\xi} & {}^0 j_{\zeta\eta} & {}^0 j_{\zeta\zeta} \end{bmatrix} = {}^0\boldsymbol{J} \tag{4}$$

Similarly, the *instantaneous* rotational inertia matrix at a deformed state is obtained as

$$\hat{\hat{\boldsymbol{b}}}^{\mathrm{T}} M \hat{\hat{\boldsymbol{b}}} = \begin{bmatrix} j_{\xi\xi} & j_{\xi\eta} & j_{\xi\zeta} \\ j_{\eta\xi} & j_{\eta\eta} & j_{\eta\zeta} \\ j_{\zeta\xi} & j_{\zeta\eta} & j_{\zeta\zeta} \end{bmatrix} = \boldsymbol{J} \tag{5}$$

This matrix is expanded as (refer to Equations (A8) and (A9))

$$\begin{aligned} \boldsymbol{J} &= \begin{bmatrix} \boldsymbol{b}^{\mathrm{T}} \bar{\bar{\boldsymbol{u}}}_\xi^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}} \bar{\bar{\boldsymbol{u}}}_\eta^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}} \bar{\bar{\boldsymbol{u}}}_\zeta^{\mathrm{T}} \end{bmatrix} \boldsymbol{M} [\bar{\bar{\boldsymbol{u}}}_\xi \boldsymbol{b} \quad \bar{\bar{\boldsymbol{u}}}_\eta \boldsymbol{b} \quad \bar{\bar{\boldsymbol{u}}}_\zeta \boldsymbol{b}] \\ &= \begin{bmatrix} \boldsymbol{b}^{\mathrm{T}} \bar{\bar{\boldsymbol{u}}}_\xi^{\mathrm{T}} M \bar{\bar{\boldsymbol{u}}}_\xi \boldsymbol{b} & \boldsymbol{b}^{\mathrm{T}} M_{\xi\eta} \boldsymbol{b} & \boldsymbol{b}^{\mathrm{T}} M_{\xi\zeta} \boldsymbol{b} \\ \boldsymbol{b}^{\mathrm{T}} M_{\eta\xi} \boldsymbol{b} & \boldsymbol{b}^{\mathrm{T}} \bar{\bar{\boldsymbol{u}}}_\eta^{\mathrm{T}} M \bar{\bar{\boldsymbol{u}}}_\eta \boldsymbol{b} & \boldsymbol{b}^{\mathrm{T}} M_{\eta\zeta} \boldsymbol{b} \\ \boldsymbol{b}^{\mathrm{T}} M_{\zeta\xi} \boldsymbol{b} & \boldsymbol{b}^{\mathrm{T}} M_{\zeta\eta} \boldsymbol{b} & \boldsymbol{b}^{\mathrm{T}} \bar{\bar{\boldsymbol{u}}}_\zeta^{\mathrm{T}} M \bar{\bar{\boldsymbol{u}}}_\zeta \boldsymbol{b} \end{bmatrix} \end{aligned} \tag{6}$$

where we have defined the following auxiliary matrices:

$$\begin{aligned}\boldsymbol{M}_{\xi\eta} &= \boldsymbol{M}_{\eta\xi}^{\mathrm{T}} \equiv \bar{\bar{\boldsymbol{u}}}_{\xi}^{\mathrm{T}} \boldsymbol{M} \bar{\bar{\boldsymbol{u}}}_{\eta} \\ \boldsymbol{M}_{\eta\zeta} &= \boldsymbol{M}_{\zeta\eta}^{\mathrm{T}} \equiv \bar{\bar{\boldsymbol{u}}}_{\eta}^{\mathrm{T}} \boldsymbol{M} \bar{\bar{\boldsymbol{u}}}_{\zeta} \\ \boldsymbol{M}_{\zeta\xi} &= \boldsymbol{M}_{\xi\zeta}^{\mathrm{T}} \equiv \bar{\bar{\boldsymbol{u}}}_{\zeta}^{\mathrm{T}} \boldsymbol{M} \bar{\bar{\boldsymbol{u}}}_{\xi}\end{aligned} \tag{7}$$

Three additional auxiliary matrices are defined as

$$\begin{aligned}\boldsymbol{M}_{\xi\eta}^{*} &= \boldsymbol{M}_{\xi\eta}^{\mathrm{T}} + \boldsymbol{M}_{\xi\eta} \\ \boldsymbol{M}_{\eta\zeta}^{*} &= \boldsymbol{M}_{\eta\zeta}^{\mathrm{T}} + \boldsymbol{M}_{\eta\zeta} \\ \boldsymbol{M}_{\zeta\xi}^{*} &= \boldsymbol{M}_{\zeta\xi}^{\mathrm{T}} + \boldsymbol{M}_{\zeta\xi}\end{aligned} \tag{8}$$

These matrices will be used in the following sub-section. Note that these auxiliary matrices are generated from the system mass matrix and unit vectors; therefore, they remain constant and need to be computed only once.

2.3. Principal axis conditions

In order for the ξ–η–ζ axes to be the principal axes (PA) of the deformable body, in the *undeformed* state, the origin of the reference frame must be positioned at the mass center; i.e.,

$$\hat{\mathbf{I}}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{s} = \mathbf{0} \tag{9}$$

This equation represents three algebraic equations known as the first moment conditions. Additionally, for the three axes to be along the body PA, we must have

$${}^{0}j_{\xi\eta} = {}^{0}j_{\eta\zeta} = {}^{0}j_{\zeta\xi} = 0 \tag{10}$$

Equation (10) can also be expressed, as it will be seen shortly, as

$$\boldsymbol{s}^{\mathrm{T}} \boldsymbol{M}_{\xi\eta} \boldsymbol{s} = \boldsymbol{s}^{\mathrm{T}} \boldsymbol{M}_{\eta\zeta} \boldsymbol{s} = \boldsymbol{s}^{\mathrm{T}} \boldsymbol{M}_{\zeta\xi} \boldsymbol{s} = 0 \tag{11}$$

In a *deformed* state, the condition to keep the origin at the instantaneous mass center is

$$\hat{\mathbf{I}}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{b} = \mathbf{0} \tag{12}$$

Subtracting Equation (9) from Equation (12) yields the following condition (three algebraic equations):

$$\hat{\mathbf{I}}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{\delta} = \mathbf{0} \tag{13}$$

For the reference frame to remain along the PA in a deformed state, we must enforce that the three products of inertia remain zeros. Referring to Equation (6), we can write

$$\begin{aligned} j_{\xi\eta} &= \boldsymbol{b}^{\mathrm{T}} \boldsymbol{M}_{\xi\eta} \boldsymbol{b} = 0 \\ j_{\eta\zeta} &= \boldsymbol{b}^{\mathrm{T}} \boldsymbol{M}_{\eta\zeta} \boldsymbol{b} = 0 \\ j_{\zeta\xi} &= \boldsymbol{b}^{\mathrm{T}} \boldsymbol{M}_{\zeta\xi} \boldsymbol{b} = 0 \end{aligned} \tag{14}$$

Therefore, at the position level, we must impose Equations (13) and (14) (in total six algebraic equations) in order to keep the ξ–η–ζ frame as the PA of the body. Obviously this requires that the axes to be defined at the undeformed state as the PA; i.e., Equations (9) and (11) must also be satisfied. Note that Equation (11) is the special case of Equation (14) when the array of deflections is set to zero.

In multibody dynamics, in addition to the position constraints, we may also need the corresponding constraints at the velocity and acceleration levels. For the velocity constraints, taking time derivative of Equations (13) and (14) yields

$$\hat{\mathbf{I}}^{\mathrm{T}} \boldsymbol{M} \dot{\boldsymbol{\delta}} = \mathbf{0} \tag{15}$$

$$\begin{aligned} \boldsymbol{b}^{\mathrm{T}} \boldsymbol{M}^{*}_{\xi\eta} \dot{\boldsymbol{\delta}} &= 0 \\ \boldsymbol{b}^{\mathrm{T}} \boldsymbol{M}^{*}_{\eta\zeta} \dot{\boldsymbol{\delta}} &= 0 \\ \boldsymbol{b}^{\mathrm{T}} \boldsymbol{M}^{*}_{\zeta\xi} \dot{\boldsymbol{\delta}} &= 0 \end{aligned} \tag{16}$$

where the auxiliary matrices of Equation (8) have been used. Similarly, taking time derivative of Equations (15) and (16) yields the acceleration constraints as

$$\hat{\mathbf{I}}^{\mathrm{T}} \boldsymbol{M} \ddot{\boldsymbol{\delta}} = \mathbf{0} \tag{17}$$

$$\begin{aligned} \boldsymbol{b}^{\mathrm{T}} \boldsymbol{M}^{*}_{\xi\eta} \ddot{\boldsymbol{\delta}} &= -\dot{\boldsymbol{\delta}}^{\mathrm{T}} \boldsymbol{M}^{*}_{\xi\eta} \dot{\boldsymbol{\delta}} \\ \boldsymbol{b}^{\mathrm{T}} \boldsymbol{M}^{*}_{\eta\zeta} \ddot{\boldsymbol{\delta}} &= -\dot{\boldsymbol{\delta}}^{\mathrm{T}} \boldsymbol{M}^{*}_{\eta\zeta} \dot{\boldsymbol{\delta}} \\ \boldsymbol{b}^{\mathrm{T}} \boldsymbol{M}^{*}_{\zeta\xi} \ddot{\boldsymbol{\delta}} &= -\dot{\boldsymbol{\delta}}^{\mathrm{T}} \boldsymbol{M}^{*}_{\zeta\xi} \dot{\boldsymbol{\delta}} \end{aligned} \tag{18}$$

3. Nodal Translational and Rotational DoF's

When the nodes of the deformable body have both translational and rotational deflections, the body contains $n_{\mathrm{dof}} = 6 \times n$ DoFs. The definitions of the arrays for the translational entities remain the same as in the previous section. The rotational deflection of a typical node i is denoted by a 3-vector $\boldsymbol{\theta}^i$. The arrays of nodal translational and rotational deflections for the body are defined as

$$\boldsymbol{\delta} = \begin{Bmatrix} \boldsymbol{\delta}^1 \\ \vdots \\ \boldsymbol{\delta}^n \end{Bmatrix} \quad \boldsymbol{\theta} = \begin{Bmatrix} \boldsymbol{\theta}^1 \\ \vdots \\ \boldsymbol{\theta}^n \end{Bmatrix} \tag{19}$$

For this deformable body, the symmetric mass matrix $\boldsymbol{M}$ is $n_{\text{dof}} \times n_{\text{dof}}$ and it is partitioned into four sub-matrices as

$$\boldsymbol{M} = \begin{bmatrix} {}^{\delta\delta}\boldsymbol{M} & {}^{\delta\theta}\boldsymbol{M} \\ {}^{\theta\delta}\boldsymbol{M} & {}^{\theta\theta}\boldsymbol{M} \end{bmatrix} \tag{20}$$

3.1. Rigid-body mode-shapes

The translational and rotational rigid-body mode-shapes are expressed as

$$\boldsymbol{\psi}^{(t)} = \begin{bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{O}} \end{bmatrix} \qquad {}^0\boldsymbol{\psi}^{(r)} = \begin{bmatrix} -\hat{\tilde{\boldsymbol{s}}} \\ \hat{\mathbf{I}} \end{bmatrix} \qquad \boldsymbol{\psi}^{(r)} = \begin{bmatrix} -\hat{\tilde{\boldsymbol{b}}} \\ \hat{\mathbf{I}} \end{bmatrix} \tag{21}$$

3.2. Mass and rotational inertia matrix

The total mass of the deformable body is found as (refer to Equation (A6))

$$\left[\hat{\mathbf{I}}^{\mathrm{T}} \quad \hat{\mathbf{O}}^{\mathrm{T}}\right] \boldsymbol{M} \begin{bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{O}} \end{bmatrix} = \hat{\mathbf{I}}^{\mathrm{T}\,\delta\delta}\boldsymbol{M}\,\hat{\mathbf{I}} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} = m\mathbf{I} \tag{22}$$

The rotational inertia matrix at the undeformed state is obtained from the mass matrix and the rotational rigid body mode-shapes as

$$[-\hat{\tilde{\boldsymbol{s}}}^{\mathrm{T}} \quad \hat{\mathbf{I}}^{\mathrm{T}}] \begin{bmatrix} {}^{\delta\delta}\boldsymbol{M} & {}^{\delta\theta}\boldsymbol{M} \\ {}^{\theta\delta}\boldsymbol{M} & {}^{\theta\theta}\boldsymbol{M} \end{bmatrix} \begin{bmatrix} -\hat{\tilde{\boldsymbol{s}}} \\ \hat{\mathbf{I}} \end{bmatrix} = \begin{bmatrix} {}^0j_{\xi\xi} & {}^0j_{\xi\eta} & {}^0j_{\xi\zeta} \\ {}^0j_{\eta\xi} & {}^0j_{\eta\eta} & {}^0j_{\eta\zeta} \\ {}^0j_{\zeta\xi} & {}^0j_{\zeta\eta} & {}^0j_{\zeta\zeta} \end{bmatrix} = {}^0\boldsymbol{J} \tag{23}$$

Similarly, the instantaneous rotational inertia matrix at a deformed state is obtained as

$$[-\hat{\tilde{\boldsymbol{b}}}^{\mathrm{T}} \quad \hat{\mathbf{I}}^{\mathrm{T}}] \begin{bmatrix} {}^{\delta\delta}\boldsymbol{M} & {}^{\delta\theta}\boldsymbol{M} \\ {}^{\theta\delta}\boldsymbol{M} & {}^{\theta\theta}\boldsymbol{M} \end{bmatrix} \begin{bmatrix} -\hat{\tilde{\boldsymbol{b}}} \\ \hat{\mathbf{I}} \end{bmatrix} = \begin{bmatrix} j_{\xi\xi} & j_{\xi\eta} & j_{\xi\zeta} \\ j_{\eta\xi} & j_{\eta\eta} & j_{\eta\zeta} \\ j_{\zeta\xi} & j_{\zeta\eta} & j_{\zeta\zeta} \end{bmatrix} = \boldsymbol{J} \tag{24}$$

This matrix can be expanded as (refer to Equations (A8) and (A9))

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{b}^{\mathrm{T}}\bar{\tilde{\boldsymbol{u}}}_{\xi}^{\mathrm{T}} & \hat{\boldsymbol{u}}_{\xi}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}}\bar{\tilde{\boldsymbol{u}}}_{\eta}^{\mathrm{T}} & \hat{\boldsymbol{u}}_{\eta}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}}\bar{\tilde{\boldsymbol{u}}}_{\zeta}^{\mathrm{T}} & \hat{\boldsymbol{u}}_{\zeta}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} {}^{\delta\delta}\boldsymbol{M} & {}^{\delta\theta}\boldsymbol{M} \\ {}^{\theta\delta}\boldsymbol{M} & {}^{\theta\theta}\boldsymbol{M} \end{bmatrix} \begin{bmatrix} \bar{\tilde{\boldsymbol{u}}}_{\xi}\boldsymbol{b} & \bar{\tilde{\boldsymbol{u}}}_{\eta}\boldsymbol{b} & \bar{\tilde{\boldsymbol{u}}}_{\zeta}\boldsymbol{b} \\ \hat{\boldsymbol{u}}_{\xi} & \hat{\boldsymbol{u}}_{\eta} & \hat{\boldsymbol{u}}_{\zeta} \end{bmatrix}$$

Further expansion of this matrix leads to the following expressions for the products of inertia:

$$\begin{aligned} \boldsymbol{j}_{\xi\eta} &= \begin{bmatrix} \boldsymbol{b}^{\mathrm{T}}\bar{\tilde{\boldsymbol{u}}}_{\xi}^{\mathrm{T}} & \hat{\boldsymbol{u}}_{\xi}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} {}^{\delta\delta}\boldsymbol{M} & {}^{\delta\theta}\boldsymbol{M} \\ {}^{\theta\delta}\boldsymbol{M} & {}^{\theta\theta}\boldsymbol{M} \end{bmatrix} \begin{bmatrix} \bar{\tilde{\boldsymbol{u}}}_{\eta}\boldsymbol{b} \\ \hat{\boldsymbol{u}}_{\eta} \end{bmatrix} \\ &= \boldsymbol{b}^{\mathrm{T}\delta\delta}\boldsymbol{M}_{\xi\eta}\boldsymbol{b} + \boldsymbol{b}^{\mathrm{T}\delta\theta}\boldsymbol{m}_{\xi\eta} + {}^{\theta\delta}\boldsymbol{m}'_{\xi\eta}\boldsymbol{b} + {}^{\theta\theta}\boldsymbol{m}_{\xi\eta} \\ j_{\eta\zeta} &= \begin{bmatrix} \boldsymbol{b}^{\mathrm{T}}\bar{\tilde{\boldsymbol{u}}}_{\eta}^{\mathrm{T}} & \hat{\boldsymbol{u}}_{\eta}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} {}^{\delta\delta}\boldsymbol{M} & {}^{\delta\theta}\boldsymbol{M} \\ {}^{\theta\delta}\boldsymbol{M} & {}^{\theta\theta}\boldsymbol{M} \end{bmatrix} \begin{bmatrix} \bar{\tilde{\boldsymbol{u}}}_{\zeta}\boldsymbol{b} \\ \hat{\boldsymbol{u}}_{\zeta} \end{bmatrix} \\ &= \boldsymbol{b}^{\mathrm{T}\delta\delta}\boldsymbol{M}_{\eta\zeta}\boldsymbol{b} + \boldsymbol{b}^{\mathrm{T}\delta\theta}\boldsymbol{m}_{\eta\zeta} + {}^{\theta\delta}\boldsymbol{m}'_{\eta\zeta}\boldsymbol{b} + {}^{\theta\theta}\boldsymbol{m}_{\eta\zeta} \\ j_{\zeta\xi} &= \begin{bmatrix} \boldsymbol{b}^{\mathrm{T}}\bar{\tilde{\boldsymbol{u}}}_{\zeta}^{\mathrm{T}} & \hat{\boldsymbol{u}}_{\zeta}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} {}^{\delta\delta}\boldsymbol{M} & {}^{\delta\theta}\boldsymbol{M} \\ {}^{\theta\delta}\boldsymbol{M} & {}^{\theta\theta}\boldsymbol{M} \end{bmatrix} \begin{bmatrix} \bar{\tilde{\boldsymbol{u}}}_{\xi}\boldsymbol{b} \\ \hat{\boldsymbol{u}}_{\xi} \end{bmatrix} \\ &= \boldsymbol{b}^{\mathrm{T}\delta\delta}\boldsymbol{M}_{\zeta\xi}\boldsymbol{b} + \boldsymbol{b}^{\mathrm{T}\delta\theta}\boldsymbol{m}_{\zeta\xi} + {}^{\theta\delta}\boldsymbol{m}'_{\zeta\xi}\boldsymbol{b} + {}^{\theta\theta}\boldsymbol{m}_{\zeta\xi} \end{aligned} \tag{25}$$

where the following auxiliary matrices have been defined:

$$\begin{aligned} &{}^{\delta\delta}\boldsymbol{M}_{\xi\eta} \equiv \bar{\tilde{\boldsymbol{u}}}_{\xi}^{\mathrm{T}\,\delta\delta}\boldsymbol{M}\bar{\tilde{\boldsymbol{u}}}_{\eta}, \quad {}^{\delta\theta}\boldsymbol{m}_{\xi\eta} \equiv \bar{\tilde{\boldsymbol{u}}}_{\xi}^{\mathrm{T}\,\delta\theta}\boldsymbol{M}\hat{\boldsymbol{u}}_{\eta}, \quad {}^{\theta\delta}\boldsymbol{m}'_{\xi\eta} \equiv \hat{\boldsymbol{u}}_{\xi}^{\mathrm{T}\,\theta\delta}\boldsymbol{M}\bar{\tilde{\boldsymbol{u}}}_{\eta}, \\ &{}^{\theta\theta}m_{\xi\eta} \equiv \hat{\boldsymbol{u}}_{\xi}^{\mathrm{T}\,\theta\theta}\boldsymbol{M}\hat{\boldsymbol{u}}_{\eta}, \quad {}^{\delta\delta}\boldsymbol{M}_{\eta\zeta} \equiv \bar{\tilde{\boldsymbol{u}}}_{\eta}^{\mathrm{T}\,\delta\delta}\boldsymbol{M}\bar{\tilde{\boldsymbol{u}}}_{\zeta}, \quad {}^{\delta\theta}\boldsymbol{m}_{\eta\zeta} \equiv \bar{\tilde{\boldsymbol{u}}}_{\eta}^{\mathrm{T}\delta\theta}\boldsymbol{M}\hat{\boldsymbol{u}}_{\zeta}, \\ &{}^{\theta\delta}\boldsymbol{m}'_{\eta\zeta} \equiv \hat{\boldsymbol{u}}_{\eta}^{\mathrm{T}\theta\delta}\boldsymbol{M}\bar{\tilde{\boldsymbol{u}}}_{\zeta}, \quad {}^{\theta\theta}m_{\eta\zeta} \equiv \hat{\boldsymbol{u}}_{\eta}^{\mathrm{T}\,\theta\theta}\boldsymbol{M}\hat{\boldsymbol{u}}_{\zeta}, \quad {}^{\delta\delta}\boldsymbol{M}_{\zeta\xi} \equiv \bar{\tilde{\boldsymbol{u}}}_{\zeta}^{\mathrm{T}\delta\delta}\boldsymbol{M}\bar{\tilde{\boldsymbol{u}}}_{\xi}, \\ &{}^{\delta\theta}\boldsymbol{m}_{\zeta\xi} \equiv \bar{\tilde{\boldsymbol{u}}}_{\zeta}^{\mathrm{T}\delta\theta}\boldsymbol{M}\hat{\boldsymbol{u}}_{\xi}, \quad {}^{\theta\delta}\boldsymbol{m}'_{\zeta\xi} \equiv \hat{\boldsymbol{u}}_{\zeta}^{\mathrm{T}\theta\delta}\boldsymbol{M}\bar{\tilde{\boldsymbol{u}}}_{\xi}, \quad {}^{\theta\theta}m_{\zeta\xi} \equiv \hat{\boldsymbol{u}}_{\zeta}^{\mathrm{T}\theta\theta}\boldsymbol{M}\hat{\boldsymbol{u}}_{\xi} \end{aligned} \tag{26}$$

Three more auxiliary matrices are defined as

$$\begin{aligned} {}^{\delta\delta}\boldsymbol{M}^{*}_{\xi\eta} &= {}^{\delta\delta}\boldsymbol{M}^{\mathrm{T}}_{\xi\eta} + {}^{\delta\delta}\boldsymbol{M}_{\xi\eta} \\ {}^{\delta\delta}\boldsymbol{M}^{*}_{\eta\zeta} &= {}^{\delta\delta}\boldsymbol{M}^{\mathrm{T}}_{\eta\zeta} + {}^{\delta\delta}\boldsymbol{M}_{\eta\zeta} \\ {}^{\delta\delta}\boldsymbol{M}^{*}_{\zeta\xi} &= {}^{\delta\delta}\boldsymbol{M}^{\mathrm{T}}_{\zeta\xi} + {}^{\delta\delta}\boldsymbol{M}_{\zeta\xi} \end{aligned} \tag{27}$$

These matrices will be used in the following sub-section. Like the auxiliary matrices in Equations (7) and (8), the auxiliary matrices of Equations (26) and (27) are constants and need to be evaluated only once.

3.3. Principal axis conditions

At the *undeformed* state, the origin of the reference frame must be positioned at the mass center of the body. This requires the following condition (three algebraic equations) to be satisfied:

$$\hat{\mathbf{I}}^{\mathrm{T}\delta\delta}\boldsymbol{M}\,\boldsymbol{s} = \mathbf{0} \tag{28}$$

For the three axes to be aligned along the body principal axes, we must have

$${}^{0}j_{\xi\eta} = {}^{0}j_{\eta\zeta} = {}^{0}j_{\zeta\xi} = 0 \tag{29}$$

In a *deformed* state, the condition to keep the origin at the instantaneous mass center is

$$\hat{\mathbf{I}}^{\mathrm{T}\delta\delta}\boldsymbol{M}\,\boldsymbol{b} = \mathbf{0} \tag{30}$$

Subtracting Equation (28) from Equation (29) yields

$$\hat{\mathbf{I}}^{\mathrm{T}\delta\delta}\boldsymbol{M}\,\boldsymbol{\delta} = \mathbf{0} \tag{31}$$

In order to enforce the three products of inertia to be zeros, we refer to Equation (25) to get

$$\begin{aligned}
&\boldsymbol{b}^{\mathrm{T}\;\delta\delta}\boldsymbol{M}_{\xi\eta}\,\boldsymbol{b} + \left({}^{\delta\theta}\boldsymbol{m}_{\xi\eta}^{\mathrm{T}} + {}^{\theta\delta}\boldsymbol{m}'_{\xi\eta}\right)\boldsymbol{b} + {}^{\theta\theta}m_{\xi\eta} = 0\\
&\boldsymbol{b}^{\mathrm{T}\;\delta\delta}\boldsymbol{M}_{\eta\zeta}\,\boldsymbol{b} + \left({}^{\delta\theta}\boldsymbol{m}_{\eta\zeta}^{\mathrm{T}} + {}^{\theta\delta}\boldsymbol{m}'_{\eta\zeta}\right)\boldsymbol{b} + {}^{\theta\theta}m_{\eta\zeta} = 0\\
&\boldsymbol{b}^{\mathrm{T}\;\delta\delta}\boldsymbol{M}_{\zeta\xi}\,\boldsymbol{b} + \left({}^{\delta\theta}\boldsymbol{m}_{\zeta\xi}^{\mathrm{T}} + {}^{\theta\delta}\boldsymbol{m}'_{\zeta\xi}\right)\boldsymbol{b} + {}^{\theta\theta}m_{\zeta\xi} = 0
\end{aligned} \tag{32}$$

Therefore, at the position level, we must enforce Equations (31) and (32) (six algebraic equations) in order to keep the ξ–η–ζ frame along the PA of the body.

The velocity constraints are obtained from the time derivative of Equations (31) and (32) as

$$\hat{\mathbf{I}}^{\mathrm{T}\delta\delta}\boldsymbol{M}\dot{\boldsymbol{\delta}} = \mathbf{0} \tag{33}$$

$$\begin{aligned}
&\left(\boldsymbol{b}^{\mathrm{T}\;\delta\delta}\boldsymbol{M}_{\xi\eta}^{*} + {}^{\delta\theta}\boldsymbol{m}_{\xi\eta}^{\mathrm{T}} + {}^{\theta\delta}\boldsymbol{m}'_{\xi\eta}\right)\dot{\boldsymbol{\delta}} = 0\\
&\left(\boldsymbol{b}^{\mathrm{T}\;\delta\delta}\boldsymbol{M}_{\eta\zeta}^{*} + {}^{\delta\theta}\boldsymbol{m}_{\eta\zeta}^{\mathrm{T}} + {}^{\theta\delta}\boldsymbol{m}'_{\eta\zeta}\right)\dot{\boldsymbol{\delta}} = 0\\
&\left(\boldsymbol{b}^{\mathrm{T}\;\delta\delta}\boldsymbol{M}_{\zeta\xi}^{*} + {}^{\delta\theta}\boldsymbol{m}_{\zeta\xi}^{\mathrm{T}} + {}^{\theta\delta}\boldsymbol{m}'_{\zeta\xi}\right)\dot{\boldsymbol{\delta}} = 0
\end{aligned} \tag{34}$$

where the auxiliary matrices of Equation (27) have been used.

The acceleration constraints are obtained from the time derivative of Equations (33) and (34) as

$$\hat{\mathbf{I}}^{\mathrm{T}\delta\delta}\boldsymbol{M}\ddot{\boldsymbol{\delta}} = \mathbf{0} \tag{35}$$

$$\begin{aligned}
\left(\boldsymbol{b}^{\mathrm{T}\,\delta\delta}\boldsymbol{M}^{*}_{\xi\eta} + {}^{\delta\theta}\boldsymbol{m}^{\mathrm{T}}_{\xi\eta} + {}^{\theta\delta}\boldsymbol{m}'_{\xi\eta}\right)\ddot{\boldsymbol{\delta}} &= -\dot{\boldsymbol{\delta}}^{\mathrm{T}\delta\delta}\boldsymbol{M}^{*}_{\xi\eta}\dot{\boldsymbol{\delta}} \\
\left(\boldsymbol{b}^{\mathrm{T}\,\delta\delta}\boldsymbol{M}^{*}_{\eta\zeta} + {}^{\delta\theta}\boldsymbol{m}^{\mathrm{T}}_{\eta\zeta} + {}^{\theta\delta}\boldsymbol{m}'_{\eta\zeta}\right)\ddot{\boldsymbol{\delta}} &= -\dot{\boldsymbol{\delta}}^{\mathrm{T}\delta\delta}\boldsymbol{M}^{*}_{\eta\zeta}\dot{\boldsymbol{\delta}} \\
\left(\boldsymbol{b}^{\mathrm{T}\,\delta\delta}\boldsymbol{M}^{*}_{\zeta\xi} + {}^{\delta\theta}\boldsymbol{m}^{\mathrm{T}}_{\zeta\xi} + {}^{\theta\delta}\boldsymbol{m}'_{\zeta\xi}\right)\ddot{\boldsymbol{\delta}} &= -\dot{\boldsymbol{\delta}}^{\mathrm{T}\delta\delta}\boldsymbol{M}^{*}_{\zeta\xi}\dot{\boldsymbol{\delta}}
\end{aligned} \tag{36}$$

4. Example

An example is provided in this section to assist the reader in better understanding the process presented in the preceding sections. The example is a simple structure made of six rods that form a tetrahedron as shown schematically in Figure 2.The rods can have different lengths and different cross sections. There are four nodes in total; they all have only translational DoFs.

The mass matrix for each element (rod), assuming that the mass is uniformly distributed, is constructed as

$$\boldsymbol{M}_{(i)} = \frac{m_{(i)}}{6}\begin{bmatrix} 2\mathbf{I} & \mathbf{I} \\ \mathbf{I} & 2\mathbf{I} \end{bmatrix}$$

where $m_{(i)} = \rho_{(i)}\,A_{(i)}\,\ell_{(i)}$ is the mass; $\rho_{(i)}$, $A_{(i)}$, and $\ell_{(i)}$ are the mass density, the cross-section area, and the length of rod (i), respectively. Considering the connectivity of the nodes by elements, the 12×12 mass matrix for the structure is constructed as

$$\boldsymbol{M} = \frac{1}{6}\begin{bmatrix} 2m_{(2+3+4)}\mathbf{I} & m_{(3)}\mathbf{I} & m_{(2)}\mathbf{I} & m_{(4)}\mathbf{I} \\ m_{(3)}\mathbf{I} & 2m_{(1+3+5)}\mathbf{I} & m_{(1)}\mathbf{I} & m_{(5)}\mathbf{I} \\ m_{(2)}\mathbf{I} & m_{(1)}\mathbf{I} & 2m_{(1+2+6)}\mathbf{I} & m_{(6)}\mathbf{I} \\ m_{(4)}\mathbf{I} & m_{(5)}\mathbf{I} & m_{(6)}\mathbf{I} & 2m_{(4+5+6)}\mathbf{I} \end{bmatrix}$$

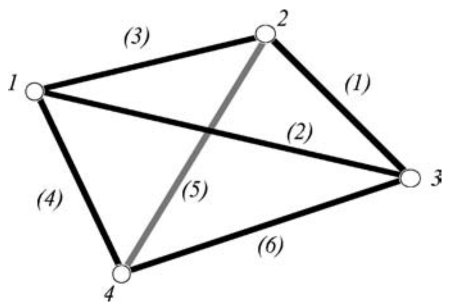

Figure 2. A tetrahedron structure made of six solid rods.

where

$$
\begin{aligned}
m_{(2+3+4)} &= m_{(2)} + m_{(3)} + m_{(4)} \\
m_{(1+3+5)} &= m_{(1)} + m_{(3)} + m_{(5)} \\
m_{(1+2+6)} &= m_{(1)} + m_{(2)} + m_{(6)} \\
m_{(4+5+6)} &= m_{(4)} + m_{(5)} + m_{(6)}
\end{aligned}
$$

We assume the following masses for the rods:

$$m_{(1)} = 10, \quad m_{(2)} = 20, \quad m_{(3)} = 3, \quad m_{(4)} = 4, \quad m_{(5)} = 5, \quad m_{(6)} = 6$$

This results in a total mass of $m = 48$ for the structure. With these element masses, the mass matrix finds the following numerical entries:

$$M = \frac{1}{6}\begin{bmatrix} 54\mathbf{I} & 3\mathbf{I} & 20\mathbf{I} & 4\mathbf{I} \\ 3\mathbf{I} & 36\mathbf{I} & 10\mathbf{I} & 5\mathbf{I} \\ 20\mathbf{I} & 10\mathbf{I} & 72\mathbf{I} & 6\mathbf{I} \\ 4\mathbf{I} & 5\mathbf{I} & 6\mathbf{I} & 30\mathbf{I} \end{bmatrix}$$

We can test the total mass condition by verifying that $\hat{\mathbf{I}}^{\mathrm{T}} M \hat{\mathbf{I}} = m\,\mathbf{I} = 48\mathbf{I}$. The mass matrix in its expanded form is

$$M = \frac{1}{6}\begin{bmatrix}
54 & 0 & 0 & 3 & 0 & 0 & 20 & 0 & 0 & 4 & 0 & 0 \\
0 & 54 & 0 & 0 & 3 & 0 & 0 & 20 & 0 & 0 & 4 & 0 \\
0 & 0 & 54 & 0 & 0 & 3 & 0 & 0 & 20 & 0 & 0 & 4 \\
3 & 0 & 0 & 36 & 0 & 0 & 10 & 0 & 0 & 5 & 0 & 0 \\
0 & 3 & 0 & 0 & 36 & 0 & 0 & 10 & 0 & 0 & 5 & 0 \\
0 & 0 & 3 & 0 & 0 & 36 & 0 & 0 & 10 & 0 & 0 & 5 \\
20 & 0 & 0 & 10 & 0 & 0 & 72 & 0 & 0 & 6 & 0 & 0 \\
0 & 20 & 0 & 0 & 10 & 0 & 0 & 72 & 0 & 0 & 6 & 0 \\
0 & 0 & 20 & 0 & 0 & 10 & 0 & 0 & 72 & 0 & 0 & 6 \\
4 & 0 & 0 & 5 & 0 & 0 & 6 & 0 & 0 & 30 & 0 & 0 \\
0 & 4 & 0 & 0 & 5 & 0 & 0 & 6 & 0 & 0 & 30 & 0 \\
0 & 0 & 4 & 0 & 0 & 5 & 0 & 0 & 6 & 0 & 0 & 30
\end{bmatrix}$$

With respect to a ξ–η–ζ frame coinciding with the principal axes of the structure, the nodes have the following coordinates (refer to Appendix B):

$$\boldsymbol{s}^1 = \begin{Bmatrix} 0.515 \\ 0.433 \\ -0.085 \end{Bmatrix}, \boldsymbol{s}^2 = \begin{Bmatrix} -0.412 \\ 0.546 \\ -0.839 \end{Bmatrix}, \boldsymbol{s}^3 = \begin{Bmatrix} -0.250 \\ -0.189 \\ 0.762 \end{Bmatrix}, \boldsymbol{s}^4 = \begin{Bmatrix} 0.169 \\ -0.981 \\ -0.668 \end{Bmatrix}$$

We can verify that $\hat{\mathbf{I}}^{\mathrm{T}}\boldsymbol{M}\,\boldsymbol{s} = \mathbf{0}$, meaning that the nodal coordinates are measured correctly from the mass center. Furthermore, we can compute the rotational inertia matrix for the structure as:

$$\hat{\hat{\boldsymbol{s}}}^{\mathrm{T}}\boldsymbol{M}\,\hat{\hat{\boldsymbol{s}}} = \begin{bmatrix} 17.970 & 0 & 0 \\ 0 & 14.477 & 0 \\ 0 & 0 & 10.465 \end{bmatrix}$$

The result shows that the three products of inertia are zero; i.e., the reference frame is indeed the principal axes.

Now we construct only one of the auxiliary matrices as an exercise. For example, we construct the second auxiliary matrix from Equation (7) to find

$$\boldsymbol{M}_{\eta\zeta} = -\frac{1}{6}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 0 & 0 & 3 & 0 & 0 & 20 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 36 & 0 & 0 & 10 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 10 & 0 & 0 & 72 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 5 & 0 & 0 & 6 & 0 & 0 & 30 & 0 \end{bmatrix}$$

The other two auxiliary matrices can be computed in a similar fashion. We note again that these auxiliary matrices are constants and therefore need to be computed only once.

The three algebraic constraints to keep the origin of the reference frame at the instantaneous mass center, as the body deforms, are according to Equation (13); i.e., $\hat{\mathbf{I}}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{\delta} = \mathbf{0}$:

$$\frac{81}{6}\begin{Bmatrix}\delta_\xi^1\\ \delta_\eta^1\\ \delta_\zeta^1\end{Bmatrix} + \frac{54}{6}\begin{Bmatrix}\delta_\xi^2\\ \delta_\eta^2\\ \delta_\zeta^2\end{Bmatrix} + \frac{108}{6}\begin{Bmatrix}\delta_\xi^3\\ \delta_\eta^3\\ \delta_\zeta^3\end{Bmatrix} + \frac{45}{6}\begin{Bmatrix}\delta_\xi^4\\ \delta_\eta^4\\ \delta_\zeta^4\end{Bmatrix} = \mathbf{0}$$

For the rotational constraint, here we show only one of the constraints from Equation (14). The second constraint; i.e., $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{M}_{\eta\zeta}\boldsymbol{b} = 0$, is

$$-\frac{1}{6}\begin{Bmatrix}0.515+\delta_\xi^1\\ 0.433+\delta_\eta^1\\ -0.085+\delta_\zeta^1\\ -0.412+\delta_\xi^2\\ 0.546+\delta_\eta^2\\ -0.839+\delta_\zeta^2\\ -0.250+\delta_\xi^3\\ -0.189+\delta_\eta^3\\ 0.762+\delta_\zeta^3\\ 0.169+\delta_\xi^4\\ -0.981+\delta_\eta^4\\ -0.668+\delta_\zeta^4\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&54&0&0&3&0&0&20&0&0&4&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&3&0&0&36&0&0&10&0&0&5&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&20&0&0&10&0&0&72&0&0&6&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&4&0&0&5&0&0&6&0&0&30&0\end{bmatrix}$$

$$\times\begin{Bmatrix}0.515+\delta_\xi^1\\ 0.433+\delta_\eta^1\\ -0.085+\delta_\zeta^1\\ -0.412+\delta_\xi^2\\ 0.546+\delta_\eta^2\\ -0.839+\delta_\zeta^2\\ -0.250+\delta_\xi^3\\ -0.189+\delta_\eta^3\\ 0.762+\delta_\zeta^3\\ 0.169+\delta_\xi^4\\ -0.981+\delta_\eta^4\\ -0.668+\delta_\zeta^4\end{Bmatrix} = 0$$

If we eliminate from this equation all the entities that get multiplied by zeros, we obtain the following compact form for this constraint:

$$-\frac{1}{6}\begin{Bmatrix}-0.085+\delta_\zeta^1\\-0.839+\delta_\zeta^2\\0.762+\delta_\zeta^3\\-0.668+\delta_\zeta^4\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}54&3&20&4\\3&36&10&5\\20&10&72&6\\4&5&6&30\end{bmatrix}\begin{Bmatrix}0.433+\delta_\eta^1\\0.546+\delta_\eta^2\\-0.189+\delta_\eta^3\\-0.981+\delta_\eta^4\end{Bmatrix}=0$$

Although the preceding step is not necessary for computational purposes, the simplification has been done to show more clearly the variables that are involved in this constraint. We note that this equation is nonlinear in the η and ζ components of the deflection vectors. The other two rotational constraints can be found in a similar fashion.

For the constraints at the velocity level, the translational constraints in Equation (15); i.e., $\hat{\mathbf{I}}^{\mathrm{T}}\boldsymbol{M}\dot{\boldsymbol{\delta}}=\mathbf{0}$, are expressed as

$$\frac{81}{6}\begin{Bmatrix}\dot\delta_\xi^1\\\dot\delta_\eta^1\\\dot\delta_\zeta^1\end{Bmatrix}+\frac{54}{6}\begin{Bmatrix}\dot\delta_\xi^2\\\dot\delta_\eta^2\\\dot\delta_\zeta^2\end{Bmatrix}+\frac{108}{6}\begin{Bmatrix}\dot\delta_\xi^3\\\dot\delta_\eta^3\\\dot\delta_\zeta^3\end{Bmatrix}+\frac{45}{6}\begin{Bmatrix}\dot\delta_\xi^4\\\dot\delta_\eta^4\\\dot\delta_\zeta^4\end{Bmatrix}=\mathbf{0}$$

The second rotational velocity constraint in Equation. (16); i.e., $\mathbf{b}^{\mathrm{T}}\boldsymbol{M}_{\eta\zeta}^{*}\dot{\boldsymbol{\delta}}=0$, becomes

$$-\frac{1}{6}\begin{Bmatrix}0.515+\delta_\xi^1\\0.433+\delta_\eta^1\\-0.085+\delta_\zeta^1\\-0.412+\delta_\xi^2\\0.546+\delta_\eta^2\\-0.839+\delta_\zeta^2\\-0.250+\delta_\xi^3\\-0.189+\delta_\eta^3\\0.762+\delta_\zeta^3\\0.169+\delta_\xi^4\\-0.981+\delta_\eta^4\\-0.668+\delta_\zeta^4\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0&0&0\\0&0&54&0&0&3&0&0&20&0&0&4\\0&54&0&0&3&0&0&20&0&0&4&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&3&0&0&36&0&0&10&0&0&5\\0&3&0&0&36&0&0&10&0&0&5&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&20&0&0&10&0&0&72&0&0&6\\0&20&0&0&10&0&0&72&0&0&6&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&4&0&0&5&0&0&6&0&0&30\\0&4&0&0&5&0&0&6&0&0&30&0\end{bmatrix}\begin{Bmatrix}\dot\delta_\xi^1\\\dot\delta_\eta^1\\\dot\delta_\zeta^1\\\dot\delta_\xi^2\\\dot\delta_\eta^2\\\dot\delta_\zeta^2\\\dot\delta_\xi^3\\\dot\delta_\eta^3\\\dot\delta_\zeta^3\\\dot\delta_\xi^4\\\dot\delta_\eta^4\\\dot\delta_\zeta^4\end{Bmatrix}=0$$

Again, in order to show more clearly the elements that are involved in this constraints, we can express this constraint in the followwing compact form:

$$
-\frac{1}{6}\left(\left\{\begin{array}{c} 0.433+\delta_\eta^1 \\ 0.546+\delta_\eta^2 \\ -0.189+\delta_\eta^3 \\ -0.981+\delta_\eta^4 \end{array}\right\}^{\mathrm{T}} \begin{bmatrix} 54 & 3 & 20 & 4 \\ 3 & 36 & 10 & 5 \\ 20 & 10 & 72 & 6 \\ 4 & 5 & 6 & 30 \end{bmatrix} \left\{\begin{array}{c} \dot{\delta}_\zeta^1 \\ \dot{\delta}_\zeta^2 \\ \dot{\delta}_\zeta^3 \\ \dot{\delta}_\zeta^4 \end{array}\right\} + \left\{\begin{array}{c} -0.085+\delta_\zeta^1 \\ -0.839+\delta_\zeta^2 \\ 0.762+\delta_\zeta^3 \\ -0.668+\delta_\zeta^4 \end{array}\right\}^{\mathrm{T}} \right.
$$

$$
\left. \times \begin{bmatrix} 54 & 3 & 20 & 4 \\ 3 & 36 & 10 & 5 \\ 20 & 10 & 72 & 6 \\ 4 & 5 & 6 & 30 \end{bmatrix} \left\{\begin{array}{c} \dot{\delta}_\eta^1 \\ \dot{\delta}_\eta^2 \\ \dot{\delta}_\eta^3 \\ \dot{\delta}_\eta^4 \end{array}\right\}\right) = 0
$$

Note that the velocity constraints are linear in the time derivative of the deflections. The acceleration constraints of Equations (17) and (18) can be constructed in a similar fashion.

In this example, since the nodes only exhibit translational deflections, we only used the constraints from Section 2. When both translational and rotational deflections are present, more number of auxiliary matrices must be computed according to the formulas in Section 3.

5. Conclusion and Discussion

In this paper the conditions to use the principal axes of a deformable body as its floating reference frame is presented. The conditions are first provided for deformable bodies when only nodal translational DoFs are assumed and then for deformable bodies when both nodal translational and rotational DoFs are present. Obviously, these conditions are only applicable to deformable bodies that are modeled for three-dimensional and not for planar formulation. The principal axes conditions require construction of auxiliary matrices that are used in the constraint equations. These constraints enforce the floating reference frame to remain along the instantaneous principal axes as the body deforms. These conditions are holonomic; i.e., they can be enforced at the position, velocity, and acceleration levels.

The auxiliary matrices, especially for the general case of nodal translational and rotational DoFs, may give the impression that the conditions for principal axes could be computationally inefficient. However, we must realize that these matrices are constructed only once prior to the start of an analysis. Construction of these matrices requires trivial matrix manipulation between the system mass matrix and other matrices containing unit vectors. Furthermore, the inclusion of six constaints, considering the large number of nodal DoFs, does not cause significant increase in the size of a problem.

Although it has not been discussed in this paper, the equations of motion for a multibody system can be represented in many different forms. The generalized coordinates used in these equations can be described in either absolute or relative sense. Since the equations of motion are usually in the form of second-order differential or mixed differential-algebraic equations, the second time derivative of constraints, if any, should be incorporated in them. Regardless of the form of the equations of motion, we can always use the principal axes conditions as the moving reference frame for a deformable body. We must append the second time derivative of the principal axes conditions, either Equations (17) and (18) or Equations (35) and (36), to the equations of motion using six Lagrange multipliers.

Depending on the form of the equations of motion, the principal axes conditions may be required to undergo some form of transformation; e.g., their components may need to be transformed from the body frame to the inertial frame. In some other situations, in addition to the acceleration constraints, we may also need to use the conditions at the position and velocity levels. Therefore, the position and velocity constraints should also undergo the necessary transformation.

It has not been within the scope of this paper to show possible advantages or disadvantages of the use of principal axes over other forms of body reference frames. The paper simply provides the formulation for those who might be interested in experimenting with the principal axes as a floating reference frame. Interested readers may refer to reference [5] for a discussion on some features of the principal axes.

Appendix A

In this paper matrix notation is used. The reader should find the notation very effective in multibody formulation of the equations of motion, especially when deformable bodies are involved. Understanding the notation will assist the reader in following the concepts and derivations that are presented in the paper. The following nomenclature is used:

Reference frames

$\mathrm{x-y-z}$	Inertial
$\xi - \eta - \zeta$	Body attached

Vectors and arrays

Lower case characters, boldface, italic (contains ξ–η–ζ components)

Matrices

Upper case characters, boldface, italic (described in ξ–η–ζ frame)

Right superscripts

i	i-th node; $i = 1, \ldots, n$
n	Total number of nodes
T	Transpose

Right subscripts

ξ, η or ζ Refers to a particular component of a vector

Left superscripts

0 Initial or undeformed state
δ Translational deflection
θ Rotational deflection

Over-scores:

$\sim$ (tilde) transforms a 3-vector to a skew-symmetric matrix
$\wedge$ (hat) stacks vertically 3-vectors or 3×3 skew-symmetric matrices
– (bar) repeats a 3×3 matrix to form a block-diagonal matrix

Null vector and matrix

The following zero 3-vector and 3×3 matrix are defined:

$$\mathbf{0} \equiv \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \mathbf{O} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A1}$$

Unit vectors and identity matrix

Three unit vectors along the three reference axes are defined as

$$\boldsymbol{u}_\xi \equiv \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \boldsymbol{u}_\eta \equiv \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad \boldsymbol{u}_\zeta \equiv \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{A2}$$

A 3×3 identity matrix is defined as

$$\mathbf{I} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_\xi & \boldsymbol{u}_\eta & \boldsymbol{u}_\zeta \end{bmatrix} \tag{A3}$$

3-Vectors and skew-symmetric matrices

A typical 3-vector and its corresponding 3×3 skew-symmetric matrix referring to a typical node i are described as

$$\boldsymbol{b}^i \equiv \begin{Bmatrix} b^i_\xi \\ b^i_\eta \\ b^i_\zeta \end{Bmatrix} \quad \tilde{\boldsymbol{b}}^i \equiv \begin{bmatrix} 0 & -b^i_\zeta & b^i_\eta \\ b^i_\zeta & 0 & -b^i_\xi \\ -b^i_\eta & b^i_\xi & 0 \end{bmatrix} \tag{A4}$$

Stacked vectors or arrays

The following arrays are constructed from stacking 3-vectors:

$$\hat{\mathbf{0}} \equiv \begin{Bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{Bmatrix} \quad \hat{\boldsymbol{u}}_\xi \equiv \begin{Bmatrix} \boldsymbol{u}_\xi \\ \vdots \\ \boldsymbol{u}_\xi \end{Bmatrix} \quad \hat{\boldsymbol{u}}_\eta \equiv \begin{Bmatrix} \boldsymbol{u}_\eta \\ \vdots \\ \boldsymbol{u}_\eta \end{Bmatrix} \quad \hat{\boldsymbol{u}}_\zeta \equiv \begin{Bmatrix} \boldsymbol{u}_\zeta \\ \vdots \\ \boldsymbol{u}_\zeta \end{Bmatrix} \quad \boldsymbol{b} = \begin{Bmatrix} \boldsymbol{b}^1 \\ \vdots \\ \boldsymbol{b}^n \end{Bmatrix} \tag{A5}$$

Note that in $\hat{\mathbf{0}}$, $\hat{\boldsymbol{u}}_\xi$, $\hat{\boldsymbol{u}}_\eta$, and $\hat{\boldsymbol{u}}_\zeta$ each array is composed of the same 3-vectors; however, in $\boldsymbol{b}$ the stack is made from the 3-vectors corresponding to all the nodes. Therefore, if an array does not carry a superscript or a "^", it indicates that the array is a stack of different 3-vectors.

Stacked matrices

The following matrices are constructed from stacking 3×3 matrices:

$$\hat{\mathbf{O}} \equiv \begin{bmatrix} \mathbf{O} \\ \vdots \\ \mathbf{O} \end{bmatrix} \quad \hat{\mathbf{I}} \equiv \begin{bmatrix} \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix} \quad \hat{\tilde{\boldsymbol{b}}} \equiv \begin{bmatrix} \tilde{\boldsymbol{b}}^1 \\ \vdots \\ \tilde{\boldsymbol{b}}^n \end{bmatrix} \tag{A6}$$

Diagonal and block-diagonal matrices

The following matrices are constructed from 3×3 matrices:

$$\bar{\mathbf{I}} \equiv \begin{bmatrix} \mathbf{I} & \cdots & \mathbf{O} \\ & \ddots & \vdots \\ \mathbf{O} & \cdots & \mathbf{I} \end{bmatrix} \quad \bar{\tilde{\boldsymbol{b}}} \equiv \begin{bmatrix} \tilde{\boldsymbol{b}}^1 & \cdots & \mathbf{O} \\ \vdots & \ddots & \vdots \\ \mathbf{O} & \cdots & \tilde{\boldsymbol{b}}^n \end{bmatrix} \tag{A7}$$

Useful transformations

The following identities are used in this paper to transform an expression from one form to another. These identities can be verified easily by the reader:

$$\hat{\tilde{\boldsymbol{b}}} = \begin{bmatrix} \tilde{\boldsymbol{b}}^1 \\ \vdots \\ \tilde{\boldsymbol{b}}^n \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{b}}^1\boldsymbol{u}_\xi & \tilde{\boldsymbol{b}}^1\boldsymbol{u}_\eta & \tilde{\boldsymbol{b}}^1\boldsymbol{u}_\zeta \\ \vdots & \vdots & \vdots \\ \tilde{\boldsymbol{b}}^n\boldsymbol{u}_\xi & \tilde{\boldsymbol{b}}^n\boldsymbol{u}_\eta & \tilde{\boldsymbol{b}}^n\boldsymbol{u}_\zeta \end{bmatrix} = -\begin{bmatrix} \tilde{\boldsymbol{u}}_\xi\boldsymbol{b}^1 & \tilde{\boldsymbol{u}}_\eta\boldsymbol{b}^1 & \tilde{\boldsymbol{u}}_\zeta\boldsymbol{b}^1 \\ \vdots & \vdots & \vdots \\ \tilde{\boldsymbol{u}}_\xi\boldsymbol{b}^n & \tilde{\boldsymbol{u}}_\eta\boldsymbol{b}^n & \tilde{\boldsymbol{u}}_\zeta\boldsymbol{b}^n \end{bmatrix}$$

or

$$\hat{\tilde{\boldsymbol{b}}} = -[\bar{\tilde{\boldsymbol{u}}}_\xi\boldsymbol{b} \quad \bar{\tilde{\boldsymbol{u}}}_\eta\boldsymbol{b} \quad \bar{\tilde{\boldsymbol{u}}}_\zeta\boldsymbol{b}] \tag{A8}$$

The transpose of Equation (A8) is expressed as

$$\hat{\boldsymbol{b}}^{\mathrm{T}} = -\begin{bmatrix} \boldsymbol{b}^{\mathrm{T}}\bar{\tilde{\boldsymbol{u}}}_{\xi}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}}\bar{\tilde{\boldsymbol{u}}}_{\eta}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}}\bar{\tilde{\boldsymbol{u}}}_{\zeta}^{\mathrm{T}} \end{bmatrix} \tag{A9}$$

Appendix B

In this appendix we show how the nodal data for the tetrahedron (with rod elements on its edges) used in the example are prepared. This process can be applied to other structures [4].

Initially the structure, in this case the terahedron, is constructed in an inertial x–y–z frame. The coordinates of the nodes in this frame are

$$\mathbf{d}^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{d}^2 = \begin{Bmatrix} 0 \\ 1.2 \\ 0 \end{Bmatrix}, \quad \mathbf{d}^3 = \begin{Bmatrix} 1.3 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{d}^4 = \begin{Bmatrix} 0.5 \\ 0.5 \\ 1.4 \end{Bmatrix}$$

The position of the mass center is obtained from the following equation (refer to Figure B.1):

$$\mathbf{r} = \frac{1}{m}\hat{\mathbf{I}}^{\mathrm{T}}\boldsymbol{M}\,\mathbf{d} \tag{B1}$$

Knowing the mass matrix and the total mass, the position of the mass center is found to be

$$\mathbf{r} = \begin{Bmatrix} 0.566 \\ 0.303 \\ 0.219 \end{Bmatrix}$$

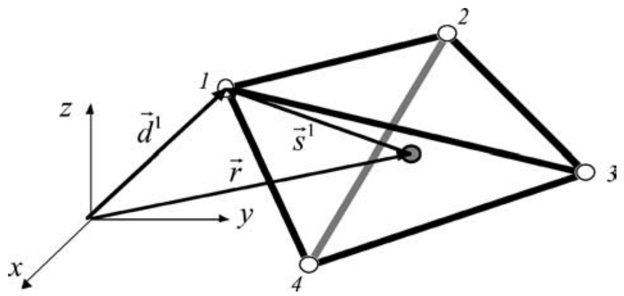

Figure B.1. Schematic presentation of the tetrahedron in the inertial frame.

Next, the position of the nodes with respect to the mass center are computed as ($\mathbf{s}^i = \mathbf{d}^i - \mathbf{r}$)

$$\mathbf{s}^1 = \begin{Bmatrix} -0.566 \\ -0.303 \\ -0.219 \end{Bmatrix}, \ \mathbf{s}^2 = \begin{Bmatrix} -0.566 \\ 0.897 \\ -0.219 \end{Bmatrix}, \ \mathbf{s}^3 = \begin{Bmatrix} 0.734 \\ -0.303 \\ -0.219 \end{Bmatrix}, \ \mathbf{s}^4 = \begin{Bmatrix} -0.066 \\ 0.197 \\ 1.181 \end{Bmatrix}$$

Note that these components are described with respect to a reference frame positioned at the mass center but parallel to the inertial frame.

With respect to this reference frame, the rotational inertia matrix can be computed as

$${}^0\mathbf{J} = \hat{\hat{\mathbf{s}}}^{\mathrm{T}} \boldsymbol{M} \hat{\hat{\mathbf{s}}} = \begin{bmatrix} 13.983 & 3.230 & 0.620 \\ 3.230 & 14.976 & -1.717 \\ 0.620 & -1.717 & 13.953 \end{bmatrix}$$

Note that the mass matrix is independent of the reference frame; i.e., $\mathbf{M} = \boldsymbol{M}$. From normalized eigenvectors of this matrix, the rotational transformation matrix between the principal axes and the inertial frame can found to be

$$\mathbf{A} = \begin{bmatrix} -0.588 & -0.479 & 0.652 \\ -0.772 & 0.094 & -0.628 \\ 0.239 & -0.873 & -0.425 \end{bmatrix}$$

The rotational inertia matrix with respect to the principal axes is determined as

$${}^0\boldsymbol{J} = \mathbf{A}^{\mathrm{T}\,0}\mathbf{J}\mathbf{A} = \begin{bmatrix} 17.970 & 0 & 0 \\ 0 & 14.477 & 0 \\ 0 & 0 & 10.465 \end{bmatrix}$$

The nodal position vectors in the principal axes frame are computed as ($\boldsymbol{s}^i = \mathbf{A}^{\mathrm{T}}\mathbf{s}^i$)

$$\boldsymbol{s}^1 = \begin{Bmatrix} 0.515 \\ 0.433 \\ -0.085 \end{Bmatrix}, \ \boldsymbol{s}^2 = \begin{Bmatrix} -0.412 \\ 0.546 \\ -0.839 \end{Bmatrix}, \ \boldsymbol{s}^3 = \begin{Bmatrix} -0.250 \\ -0.189 \\ 0.762 \end{Bmatrix}, \ \boldsymbol{s}^4 = \begin{Bmatrix} 0.169 \\ -0.981 \\ -0.668 \end{Bmatrix}$$

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