



Memory-dependent response of the thermoelastic two-dimensional functionally graded rectangular plate

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Abstract

This article uses a memory-dependent derivative (MDD) — which may be better than a fractional derivative — to develop a novel heat conduction problem in a functionally graded material (FGM) layer with a distinct exponential gradient model. A theoretical framework is designed for a functionally graded plate (FGP) incorporating the fractional heat conduction theory that incorporates single-phase-lag (SPL) and two-temperature discrepancy factors to capture the thermoelastic response and the memory-dependent effect. Then, the modified model is used to investigate the thermoelastic response of an FGP subjected to thermal shock at the left surface of the plate, keeping other faces at zero temperature. The temperature change is determined using the integral transform technique, and the solution is obtained in the Laplace transform domain. The transient temperature response in the time domain is evaluated through numerical inversion of the Laplace transform to generate numerical data. The general solutions of the governing equation of stress function are obtained by utilizing material attributes represented by the exponential-law index. The transient responses, namely temperature, displacement, and stress, are graphically depicted. FGP is composed of partially stabilized zirconia (PSZ) particles, and the austenitic stainless steel (SUS304) matrix was used in the analysis. The use of FGM requires careful compositional choices to prevent thermal stresses from being generated in the FGP. The study compares temperature distributions using non-Fourier and classical Fourier models, revealing wave-like phenomena in fractional heat transfer, which are undetected in classical Fourier heat conduction.

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Nomenclature

I^α	Riemann-Liouville fraction integral of the α^{th} order
α	fractional order
h	thickness of the layer, m
τ_q	phase lag of heat flux, s
q	heat transfer rate, $W \cdot m^{-2}$
$\beta_1, \beta_2, \beta_3, \beta_4, \delta$	gradient indices
s	Laplace transform variable
T	conductive temperature, $^\circ C$
k, k_0, k_h	thermal conductivity, $W \cdot m^{-1} \cdot K^{-1}$
Γ	Gamma function
ρ, ρ_0, ρ_h	mass density, $kg \cdot m^{-3}$
c, c_0, c_h	specific heat capacity, $J \cdot kg^{-1} \cdot K^{-1}$
∇	spatial gradient operator
Φ	thermodynamic temperature
b	temperature discrepancy parameter
$\kappa = k/\rho c$	thermal diffusivity in the medium, $m^2 \cdot s^{-1}$
E^e, E_0	Young's modulus, GPa
$\alpha_t^e, \alpha_0, \alpha_h$	coefficient of linear thermal expansion, K^{-1}
ν	Poisson's ratio
χ	stress function
I_η	first-kind modified Bessel function of the η^{th} order
K_η	second-kind Bessel function of the η^{th} order
∇^2	two-dimensional Laplacian operator

1 Introduction

Functionally graded materials (FGMs) are heterogeneous materials with continuous variations in properties along specific axes. They are renowned for their excellent mechanical toughness and heat resistance, making them widely used in advanced industries like aerospace and aviation. Many investigations have been conducted in related fields to better understand the mechanical properties of FGMs at extremely high temperatures. These studies have employed functionally graded layers (Byrd and Birman 2010; Zhou et al. 2011; Wang et al. 2018; Ma and Chen 2011; Ohmichi et al. 2016), cylinders and spheres (Tarn and Wang 2004; Zhao et al. 2007; Hosseini et al. 2007; Asgari and Akhlaghi 2009; Nezhad et al. 2011; Daneshjou et al. 2015), and three-dimensional (Ootao and Tanigawa 2005; Kim and Noda 2001; Li and Wen 2014; Yu et al. 2016) models. The research investigation was performed using techniques such as analytical methods (Delale and Erdogan 1983; Eischen 1987; Erdogan 1995; Bao and Wang 1995; Erdogan and Wu 1997; Choi et al. 1998; Cai and Bao 1998; Long and Delale 2005; Ding et al. 2010), numerical simulations (Santare and Lambros 2000; Kim and Paulino 2002; Comi and Mariani 2007), and experimental methods (Butcher et al. 1998; Abanto-Bueno and Lambros 2006). However, the extensive use of laser heating technology has led to significant interest in studying high-rate heat transfer.

Heat conduction analysis needs to be revised due to the infinite speed at which thermal disturbances propagate in the traditional Fourier law. Several other heat conduction models or generalized heat conduction theories have been proposed to overcome this disadvantage, such as the hyperbolic heat conduction model (Lord and Shulman 1967) and the dual-phase-lag heat conduction model (Tzou 1995a,b). Torabi and Saedodin (2011) examined hyperbolic heat conduction with a heat flux boundary condition. Askarizadeh and Ahmadikia (2016) examined the impact of non-Fourier thermal conduction on convective straight fins with a constant cross-section under periodic boundary conditions. They proposed an exact analytical solution for the dual-phase-lag heat conduction. Rahideh et al. (2012) simulated heat conduction in composite materials and FGMs. They specifically focused on a multi-layered FGM medium that experiences heat generation. The simulation utilized non-Fourier heat transfer equations and accounted for finite heat-wave speed. Babaei and Chen (2010) addressed the problem of transient hyperbolic heat conduction in a functionally graded hollow cylinder of infinite length. Keles and Conker (2011) proposed a hyperbolic heat conduction problem in FGM cylinders and spheres with varied material characteristics. Akbarzadeh and Chen (2012, 2013) examined the transient heat conduction in a functionally graded medium using the dual-phase-lag theory. Peng et al. (2018) employed the hyperbolic heat-moisture coupling model to explore the impact of phase delays between heat flux and moisture flux on the transient hygrothermal response in an elastic cylinder. The fractional calculus, which is an extension of the conventional integer order calculus, has been developed to investigate many problems in science and technology, such as thermoelastic difficulties (Podlubny 1998; Ortigueira 2011). Povstenko (2004) introduced a quasi-static uncoupled theory of thermoelasticity that relies on a time-fractional equation. This equation combined aspects of both heat conduction and wave equations and was also examined by Fujita (1990). In addition, Sherief et al. (2010) and Youssef (2010) expanded the Cattaneo heat conduction equation into fractional versions, respectively. Ezzat (2011, 2011) formulated a novel model for the equation of fractional heat conduction by employing the recently constructed Taylor series expansion of time-fractional order, as proposed by Jumarie (2010). Ezzat et al. (2018, 2016, 2013, 2023, 2019) work on fractional calculus applications to thermoelasticity, including the introduction of new models of the fractional heat conduction equation. The study examined the transient thermal stresses caused by a crack in a hollow elastic cylinder using the fractional thermoelasticity theory and the superposition method (Zhang and Li 2017). Zhang et al. (2018) examined the issue of thermal shock in an elastic half-space with a penny-shaped crack close to the surface. Their investigation was based on a fractional thermoelasticity theory. Ma and He (2016) conducted a study on the fractional order theory of thermoelasticity, examining the transient thermo-piezoelectric reaction of a functionally graded piezoelectric rod under dynamic heat. Recently, Zhang et al. (2019) investigated the problem of FGMs based on the fractional order generalized fractional heat conduction with heat flow phase-lag, examining two typical scenarios of convective heat or temperature movement at boundaries using the Laplace transform method. From the study mentioned above articles, it is evident that most literature pertains to classical, hyperbolic, or phase-lag thermoelasticity theories. However, various authors suggest different formulas.; that is, the thermoelastic theories are not unique in their form.

Over the past decade, fractional calculus has gained significant attention due to its applications in physics and various engineering fields. Wang and Li (2011) presented the memory-dependent derivative (MDD) concept has been introduced as an integral version of a common derivative, including a kernel function across a sliding interval. This definition accurately represents the memory effect, where the previous state influences the pace of instantaneous change. The MDD is more intuitive for understanding the physical significance and has more expressive power than the fractional one. Several seminal studies may

be reviewed in the literature by Ezzat et al. (2014, 2016, 2017) and Sur et al. (2024, 2023a, 2023b, 2023, 2024, 2024) after the development of the MDD philosophy. It is widely understood that time-fractional-order thermoelasticity can be applied to depict memory-dependent derivative (MDD) behaviours. Therefore, the study of thermoelasticity within the framework of the memory-dependent two-temperature theory holds significant importance. As far as the author is aware, there is no existing analysis for a transient thermal conduction problem of a two-dimensional functionally graded plate with a memory-dependent derivative, and this may be due to the extreme difficulty of using the analytical approach.

The purpose of this paper is to modify the conventional Fourier law of heat conduction and establish analytical solutions to two-dimensional problems to capture the thermoelastic response using fractional-order single-phase-lag (FOSPL) by introducing a temperature discrepancy factor in the context of memory-dependent effect. The resulting non-dimensional equations are applied to a specific problem of an FGP subjected to thermal shock on the boundary, which is traction-free. An integral transform approach is introduced to obtain the exact solutions in the Laplace transform domain for different forms of kernel function. The Laplace transforms are inverted using the modified Durbin's numerical inversion method. The innovative theory is analyzed using graphical representations, with FGMs as representative composite materials due to their exceptional heat resistance in non-isothermal conditions. The presence of a memory-dependent derivative also enhances the exceptional predictive capability exhibited. In conclusion, the definition of MDD is more intuitive as compared to the fractional derivative. Some parametric results are established to display the influences of the fractional-order parameter, the phase-lag parameter and the exponential-law index on the considered physical quantities.

The subsequent sections of this paper is organized as follows: The modified mathematical model for the governing equations with FOSPL and two-temperature theory via MDD effect is described in Sect. 2. The practical statement of heat conduction and thermoelastic field is the topic of discussion in Sect. 3. An overview of the formation of analytical solutions for the temperature distribution and the stress that is associated with it is given in Sect. 3. The numerical results are provided, and parametric studies are undertaken in Sect. 4. The findings that were presented are included in Sect. 5.

2 Mathematical model

The generalized fractional single phase-lag (FSPL) heat transfer model (Ezzat and Karmany 2011):

$$q(x, y, t) \left(1 + \frac{\tau_q^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) = -\nabla T(x, y, t, k), 0 < \alpha \leq 1 \quad (1)$$

where the fractional derivative (Sherief et al. 2010) is

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \begin{cases} f(t) - f(0), & \alpha \rightarrow 0 \\ I^{\alpha-1} \frac{\partial^\alpha f(t)}{\partial t^\alpha}, & 0 < \alpha < 1 \\ \frac{\partial f(t)}{\partial t}, & \alpha = 1 \end{cases} \quad (2)$$

and

$$I^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0 \quad (3)$$

As the variable $\alpha \rightarrow 1$, Eq. (1) simplifies to the widely recognized Cattaneo law. Kim-mich (2002) states that the time-fractional diffusion equation, with fractional-order α describes various diffusion cases where $0 < \alpha < 1$ correspond to weak diffusion, $\alpha = 1$, it correspond to normal diffusion, $1 < \alpha < 2$ corresponds to strong diffusion, and $\alpha = 2$, it corresponds to ballistic diffusion.

Consider the energy balance while considering heat flux

$$\rho(x)c(x)\frac{\partial}{\partial t}T(x, y, t, k) = -\nabla q(x, y, t) \tag{4}$$

Chen and Gurtin (1968) proposed dividing real materials into simple and non-simple categories by considering two temperatures, that is, conductive and thermodynamic, and the two temperatures are related by

$$\Phi(x, y, t, k) = (1 - b\nabla^2)T(x, y, t, k), b > 0 \tag{5}$$

The material parameter b is a crucial distinction between the two-temperature and classical theories. As a limiting case, $b \rightarrow 0$, $\Phi \rightarrow T$ give rise to the classical Fourier theory. Combing Eqs. (1), (4), and (5), and multiplying $(1 - b\nabla^2)$ and neglecting differential coefficients of order higher than ∇^2 , and using $\nabla^2 \approx (1/\kappa)(\partial/\partial t)$, one obtains

$$\left(1 + \frac{\tau_q^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha}\right) \rho(x)c(x)\frac{\partial}{\partial t} \left(1 - \frac{b}{\kappa(x)} \frac{\partial}{\partial t}\right) T(x, y, t) = \nabla^2 T(x, y, t, k) \tag{6}$$

Wang and Li (2011) introduced the concept of ‘memory-dependent derivatives’ as a captivating and distinct alternative to fractional order derivatives, which accurately capture the influence of memory. The Caputo and Mainardi (1971) fractional derivative was transformed into an integral representation of a first-order ordinary derivative using the kernel function. The MDD for the first order of a function $f(t)$ can be expressed as an integral using a kernel function over a sliding interval, mathematically as

$$D_\omega f(t) = (1/\omega) \int_{t-\tau}^t K(t - \xi) f'(\xi) d\xi \tag{7}$$

The selection of the kernel function $K(t - \xi)$ and the time-delay parameter $\omega > 0$ is flexible and can be selected freely, as

$$K(t - \xi) = 1 - \frac{2g}{\omega}(t - \xi) + \frac{e^2}{\omega^2}(t - \xi)^2 = \begin{cases} 1; & \text{if } e = 0, g = 0 \\ 1 - (t - \xi)/\omega; & \text{if } e = 0, g = 0.5 \\ (1 - (t - \xi)/\omega)^2; & \text{if } e = 1, g = 1 \end{cases} \tag{8}$$

where $\omega > 0$ is the delay time; e and g are constants, respectively. The kernel can be considered the degree of the past effect on the present. Moreover, if $K \equiv 1$,

$$D_\omega f(t) = \frac{1}{\omega} \int_{t-\omega}^t f'(\xi) d\xi = \frac{f(t) - f(t - \omega)}{\omega} \rightarrow f'(t) \quad \text{as } \omega \rightarrow 0 \tag{9}$$

and it is noted that an ordinary function d/dt is obtained if the limiting case of D_ω is taken as $K = 1$ and $\omega \rightarrow 0$ (Mondal et al. 2021). The right-hand side of Eq. (7) can be interpreted as the mean of $f'(\xi)$ over the previous interval $[t - \omega, t]$, with varying weights. From an applications perspective, memory significantly impacts at $0 \leq K(t - \xi) < 1$ for $\xi \in [t - \omega, t]$,

resulting in memory-dependent derivatives that are often smaller in magnitude compared to the common derivatives $f'(t)$. The possibility of this situation depends on implementing the concept of MDD, as fractional derivatives are insufficient when the lower-end value greatly varies from the top-end value in the definition of fractional order derivatives. The modified partial differential equation of Eq. (6), which expresses memory-dependent characteristics, can be presented as:

$$e^{-\gamma x} \left(1 + \frac{\tau_q^\alpha}{\alpha!} D_\omega^\alpha \right) \rho(x)c(x) \frac{\partial}{\partial t} \left(1 - \frac{b}{\kappa(x)} \frac{\partial}{\partial t} \right) T(x, y, t) = \nabla^2 T(x, y, t, k) \quad (10)$$

In a layered FGM, the characteristics of the material vary depending on its location. However, in practical applications, FGMs are typically engineered to have material properties that change in a specified direction for specific reasons. For instance, combining ceramics with metals can result in materials that can withstand high temperatures and minimize thermal stresses. Ceramics with lower heat conductivity and thermal expansion coefficient are matched by metals, which possess stronger toughness and better heat conductivity. Therefore, for the sake of simplicity, it is logical to assume in this document that the material properties are only dependent on spatial variable x and exhibit an exponential gradient pattern (Noda and Jin 1993) as follows:

$$\rho(x) = \rho_0 e^{\beta_1 x}, c(x) = c_0 e^{\beta_2 x}, k(x) = k_0 e^{\delta x} \quad (11)$$

where $\rho_0(\rho_h)$, $c_0(c_h)$, $k_0(k_h)$ are the corresponding values that vary through the way x axis, and gradient indices are defined by

$$\beta_1 = \frac{1}{h} \ln \frac{\rho_h}{\rho_0}, \beta_2 = \frac{1}{h} \ln \frac{c_h}{c_0}, \delta = \frac{1}{h} \ln \frac{k_h}{k_0} \quad (12)$$

where $\gamma = \beta_1 + \beta_2 - \delta$ is the graded parameter. The limiting case $\gamma = 0$ means that the thermal conductivity and the volumetric heat capacity have the same exponential law and correspond to a homogenous plate. Under the assumption of the exponential gradient (11), Eq. (10) can be written as

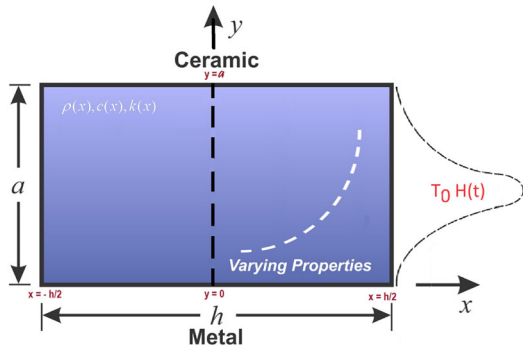
$$e^{-\gamma x} \left(1 + \frac{\tau_q^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \rho(x)c(x) \frac{\partial}{\partial t} \left(1 - \frac{b}{\kappa(x)} \frac{\partial}{\partial t} \right) T(x, y, t) = \nabla^2 T(x, y, t, k) \quad (13)$$

By adjusting the values of fractional order α , and phase-lag τ_q one can derive the limiting cases of the two-temperature FSPL heat conduction, Eq. (10) as

- if we put $\tau_q = 0$, it reduces to classical thermoelasticity (CTE).
- when $\tau_q \geq 0$ and $\alpha = 1$, it reduces to hyperbolic thermoelasticity (HTE) (Lord and Shulman 1967).
- If $\alpha! = 1$ and $\tau_q^\alpha \geq 0$, it reduces to Sherief's fractional model (SFTE) (Sherief et al. 2010)
- Eq. (13) is yielded to Ezzat's fractional model (EFTE) when $\tau_q^\alpha \geq 0$ (Ezzat and Karamany 2011)

Thus, Eq. (10) expresses the partial differential heat conduction in the context of the memory-dependent two-temperature theory the authors are curious about.

Fig. 1 FGP subjected to a thermal shock and boundary conditions



3 Statement of the problem

Let us consider an FGM plate with a thickness a and a width h occupying the space $D : -h/2 \leq x \leq h/2, 0 \leq y \leq a$ as shown in Fig. 1. In this case, the thermal shock $T_0 H(t)$ is applied on the left face surface ($x = h/2$), and the temperature on the other face surface ($x = -h/2$) is kept at zero. Both the top and lower faces are adiabatic. The heat conductivity, Young’s modulus, and the coefficient of the linear thermal expansion may be described using exponential functions that depend on their position.

When a material property changes along x -axis, a two-dimensional transient heat conduction equation based on Eq. (13) is given by

$$e^{-\gamma x} \left(1 + \frac{\tau_q^\alpha}{\alpha!} D_\omega^\alpha \right) \rho(x)c(x) \frac{\partial}{\partial t} \left(1 - \frac{b}{\kappa(x)} \frac{\partial}{\partial t} \right) T = \left(\frac{\partial}{\partial x} k(x) \frac{\partial}{\partial x} + \frac{\partial}{\partial y} k(x) \frac{\partial}{\partial y} \right) T \quad (14)$$

As an example, we have examined the specific boundary condition influenced by thermal shock on one side of a plate using FGP varying according to an exponential law and its thermoelastic response with a memory effect is studied.

3.1 Single-phase lag two-temperature thermal field

By referring to the dimensionless variables

$$\begin{aligned} (x^*, y^*) &= (x, y)/h, (t^*, \tau_q^{\alpha*}, D_\omega^{\alpha*}) = (t, \tau_q^\alpha, D_\omega^\alpha)k_0/\rho_0c_0h^2, \\ (T^*, \Phi^*) &= (T, \Phi)/T_0, (\beta_1^*, \beta_2^*, \delta^*, \gamma^*) = (\beta_1, \beta_2, \delta, \gamma)h, \\ (u^*, v^*) &= (u, v)/\rho_0c_0h^2, \sigma_{ij}^* = \sigma_{ij}/\rho_0c_0h^2 \end{aligned} \quad (15)$$

Here and in the following, the asterisk $*$ of the dimensionless variables is omitted for simplicity for later stages. The dimensionless two-dimensional transient heat conduction equation of Eq. (14) is given by

$$e^{-\gamma x} \left(1 + \frac{\tau_q^\alpha}{\alpha!} D_\omega^\alpha \right) \frac{1}{\kappa(x)} \frac{\partial}{\partial t} \left(1 - \frac{b}{\kappa(x)} \frac{\partial}{\partial t} \right) T = \frac{\partial^2 T}{\partial x^2} + \frac{k'(x)}{k(x)} \frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial y^2} \quad (16)$$

subjected to mechanical conditions

$$\sigma_{xy}(x, 0, t) = \sigma_{yy}(x, 0, t) = v(x, 0, t) = 0, -h/2 < x < h/2 \quad (17)$$

and initial and associated thermal boundary conditions

$$\Phi(x, y, 0) = T(x, y, 0) = 0, \quad \frac{\partial}{\partial t} T(x, y, 0) = 0 \quad (18)$$

$$T(-h/2, y, t) = 0, \quad T(h/2, y, t) = T_0 H(t) \quad (19)$$

3.2 Thermoelastic field

The equilibrium equation (in the absence of body forces) for the plane problem:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (20)$$

The strain-displacement relations and compatibility equation have the form

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (21)$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad (22)$$

The constitutive equations are

$$\begin{aligned} \varepsilon_{xx} &= (\sigma_{xx} - \nu \sigma_{yy}) / E^e + \alpha_t^e T, \\ \varepsilon_{yy} &= (\sigma_{yy} - \nu \sigma_{xx}) / E^e + \alpha_t^e T, \\ \varepsilon_{xy} &= E^e \sigma_{xy} / (1 + \nu^e) \end{aligned} \quad (23)$$

where

$$\begin{aligned} E^e &= E, \quad \nu^e = \nu, \quad \alpha_t^e = \alpha_t \text{ for plane stress} \\ E^e &= E / (1 - \nu^2), \quad \nu^e = \nu / (1 - \nu), \quad \alpha_t^e = (1 + \nu) \alpha_t \text{ for plane strain} \end{aligned} \quad (24)$$

while the material properties depend on the position.

The stress components in terms of stress function χ are obtained as

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_{xy} = \frac{\partial^2 \chi}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2} \quad (25)$$

The basic equation governing the temperature and the stress function in the FGM plate can be obtained by substituting Eqs. (25) into the compatibility condition (22) through the constitutive law (23), as

$$\begin{aligned} \nabla^2 \left(\frac{1}{E^e} \nabla^2 \chi + \alpha_t T \right) &= \frac{\partial^2}{\partial x^2} \left(\frac{1 + \nu^e}{E^e} \frac{\partial^2 \chi}{\partial y^2} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1 + \nu^e}{E^e} \frac{\partial^2 \chi}{\partial x^2} \right) \\ &\quad - 2 \frac{\partial^2}{\partial x \partial y} \left(\frac{1 + \nu^e}{E^e} \frac{\partial^2 \chi}{\partial x \partial y} \right) \end{aligned} \quad (26)$$

It is supposed that the material possesses the exponential gradient pattern (Noda and Jin 1993):

$$E^e(x) = E_0 e^{\beta_3 x}, \quad \alpha_t^e(x) = \alpha_0 e^{\beta_4 x} \quad (27)$$

where $E_0(E_h)$, $\alpha_0(\alpha_h)$ are the corresponding values that vary through the way x - axis, and $\beta_3 (< 0)$, β_4 are gradient indices defined by

$$\beta_3 = \frac{1}{h} \ln \frac{E_h}{E_0}, \beta_4 = \frac{1}{h} \ln \frac{\alpha_h}{\alpha_0} \tag{28}$$

Substituting Eq. (27) into Eqs. (26) and using Eq. (15), one obtains the following dimensionless forms

$$\nabla^2 \nabla^2 \chi - 2\beta_3 \frac{\partial \nabla^2 \chi}{\partial y} + \beta_3^2 \left[\nabla^2 \chi - (1 + \nu) \frac{\partial^2 \chi}{\partial x^2} \right] + e^{(\beta_3 + \beta_4)x} \left(\nabla^2 T + 2\beta_4 \frac{\partial T}{\partial x} + \beta_4^2 T \right) = 0 \tag{29}$$

The traction-free boundary conditions as

$$\sigma_{xx} = \sigma_{xy} = 0 \quad \text{at } x = \pm h/2 \tag{30}$$

$$\int_{-h/2}^{h/2} \sigma_{yy} dy = \int_{-h/2}^{h/2} \sigma_{yy} y dy = 0 \quad \text{at } y = 0, a \tag{31}$$

Equations (1)-(34) constitute the mathematical formulation of the problem.

4 Solution of the problem

4.1 Solution of the heat conduction problem

Employing the Laplace transform defined by

$$\bar{f}(r, s) = \mathcal{L}[f(r, t)] = \int_0^\infty e^{-st} f(r, t) dt, \text{Re}(s) \geq d > 0 \tag{32}$$

By utilizing the convolution theorem, it becomes possible to employ the Laplace transform to the higher-order memory-derivative D_ω^p , satisfying the property

$$\mathcal{L}[\omega D_\omega^p f(x, t)] = \mathcal{L} \left[\int_{t-\omega}^t K(t-\xi) f^p(x, \xi) d\xi \right] = s^{p-1} G(s, \omega) \mathcal{L}[f(x, t)] \tag{33}$$

where $p \in \mathbb{R}$ and

$$G(s, \omega) = (1 - e^{-s\omega}) \left(1 - \frac{2f}{s\omega} + \frac{2e^2}{s^2\omega^2} \right) - \left(e^2 - 2f + \frac{2e^2}{s\omega} \right) e^{-s\omega} \tag{34}$$

Using Eqs. (11), (18), and applying Laplace transforms to the governing Eqs. (16), and (19), we obtain the transformed equation; one obtains

$$e^{-\gamma x} q^2 \bar{T} = \frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} + \gamma \frac{\partial \bar{T}}{\partial x} \tag{35}$$

$$\bar{T}(-h/2, y, s) = 0, \bar{T}(h/2, y, s) = T_0/s \tag{36}$$

where

$$q^2 = \left(1 + \frac{1}{\alpha!} G(s, \omega) \right) \frac{s}{\kappa_0} \left(1 - \frac{b}{\kappa_0} s \right) \tag{37}$$

We bring in a finite sine-Fourier integral transform in the interval $0 < y \leq \bar{a}$, stated as

$$\mathfrak{F}_s\{g(z)\} = \tilde{g}(n) = \int_0^{\bar{a}} g(z) \sin(\vartheta_n z) dz; 0 \leq z \leq \bar{a}, \vartheta_n = n\pi/\bar{h}, n = 1, 2, 3... \quad (38)$$

together with its inverse transform

$$\mathfrak{F}_s^{-1}\{\tilde{g}(n)\} = g(z) = (2/\bar{a}) \sum_{n=1}^{\infty} \tilde{g}(n) \sin(\vartheta_n z) \quad (39)$$

and the orthogonal property as

$$\mathfrak{F}_s\{\partial^2 g(z)/\partial z^2\} = \vartheta_n[g(0) - (-1)^n g(\bar{a})] - \vartheta_n^2 \tilde{g}(n) \quad (40)$$

Applying the sine-Fourier transform as defined in Eq. (39) to the governing Eqs. (35), one obtains the transformed equation as

$$\frac{\partial^2 \bar{T}}{\partial x^2} + \gamma \frac{\partial \bar{T}}{\partial x} - (e^{-\gamma x} q^2 + \vartheta_n^2) \bar{T} = 0 \quad (41)$$

$$\bar{T}(-h/2, n, s) = 0, \bar{T}(h/2, n, s) = T_0/s \quad (42)$$

Taking the transformation of variables as

$$\xi = (2q/\gamma)e^{-\gamma x/2} \quad (43)$$

and taking

$$\bar{T}(x, n, s) = \xi \bar{\theta}(\xi, n, s) \quad (44)$$

Considering Eqs. (43)-(44) into (41)-(42), one obtains

$$\frac{\partial^2 \bar{\theta}}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial \bar{\theta}}{\partial \xi} - \left(1 + \frac{\eta^2}{\xi^2}\right) \bar{\theta} = 0 \quad (45)$$

$$\bar{\theta}(-h/2, n, s) = 0, \bar{\theta}(h/2, n, s) = \theta_0/s \quad (46)$$

The general solution of Eq. (18) with the aid of modified Bessel functions

$$\bar{\theta} = C_1 I_\eta(\xi) + C_2 K_\eta(\xi) \quad (47)$$

where $\eta^2 = 1 + (2\vartheta_n/\gamma)^2$, and C_i ($i = 1, 2$) are two parameters independent of ξ , which can be determined by two boundary conditions (46) as

$$C_1 = \frac{\theta_0 K_\eta(-h/2)}{s[I_\eta(h/2)K_\eta(-h/2) - I_\eta(-h/2)K_\eta(h/2)]} \quad (48)$$

$$C_2 = \frac{\theta_0 I_\eta(-h/2)}{s[I_\eta(h/2)K_\eta(-h/2) - I_\eta(-h/2)K_\eta(h/2)]} \quad (49)$$

From Eqs. (44) and (48)-(49), we obtain the solution for Eq. (41)

$$\bar{T}(x, n, s) = \frac{\theta_0[K_\eta(-h/2)I_\eta(\xi) + I_\eta(-h/2)K_\eta(\xi)]\xi}{s[I_\eta(h/2)K_\eta(-h/2) - I_\eta(-h/2)K_\eta(h/2)]} \tag{50}$$

and then using Eq. (39), the temperature Eq. (50) in the Laplace domain as

$$\begin{aligned} \bar{T}(x, y, s) &= \sum_{n=1}^{\infty} \frac{\theta_0[K_\eta(-h/2)I_\eta((2q/\gamma)e^{-\gamma x/2}) + I_\eta(-h/2)K_\eta((2q/\gamma)e^{-\gamma x/2})]}{s[I_\eta(h/2)K_\eta(-h/2) - I_\eta(-h/2)K_\eta(h/2)]} \\ &\times (2q/\gamma)(2/a)e^{-\gamma x/2} \sin(\vartheta_n y) \end{aligned} \tag{51}$$

Equation (51) depicts the temperature at each instant and all positions of the plate with a finite height in the Laplace domain.

4.2 Solution of thermal stress

Using temperature is given by Eq. (51), Eq. (29) reduces to

$$\begin{aligned} \nabla^4 \chi - 2\beta_3 \frac{\partial \nabla^2 \chi}{\partial y} + \beta_3^2 \nabla^2 \chi - (1 + \nu)\beta_3^2 \frac{\partial^2 \chi}{\partial x^2} = \\ \left\{ \begin{aligned} &[-\gamma^2 \beta_4^2 K_\eta(h/2)I_\eta((2q/\gamma)e^{-\gamma x/2}) + I_\eta(-h/2)K_\eta((2q/\gamma)e^{-\gamma x/2})] \\ &\times [e^{\gamma x}(\gamma - 2\beta_4) - \gamma^2 \beta_4^2] + [K_\eta(h/2)I_\eta((2q/\gamma)e^{-\gamma x/2})\beta_4 \\ &+ I_\eta(-h/2)K_\eta((2q/\gamma)e^{-\gamma x/2})\beta_4][e^{\gamma x/2}\gamma(5\gamma - 4\beta_4)\beta_4] \end{aligned} \right\} \\ \times \theta_0 \beta_4 (2q/\gamma)(2/a) e^{(\beta_3 + \beta_4)x - 3\gamma x/2} \frac{a\pi s [I_\eta(-h/2) - I_\eta(h/2)]^2}{\sin(h\pi/2)} \sin(\vartheta_n y) \end{aligned} \tag{52}$$

The general solution can be expressed as the sum of the complementary and particular solutions as $\chi = \chi_c + \chi_p$. The separation of variables can get the complementary part χ_c (Ohmichi and Noda 2006) of Eq. (52)

$$\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \begin{pmatrix} 1 \\ y \\ \beta_3 \\ ye^{\beta_3 y} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} \begin{pmatrix} 1 \\ e^{\beta_3 y} \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} e^{\beta_3 \sqrt{\nu}/2} \\ e^{-\beta_3 \sqrt{\nu}/2} \end{pmatrix} e^{\Upsilon_i x} e^{\wp y} e^{\Psi x} e^{\Delta_i y} \quad i = 1, 2, 3, 4 \tag{53}$$

in which $\beta_3 \neq 0$ and Υ_i is root of the characteristic equation

$$\Upsilon^4 + (2\wp^2 - 2\beta_3 \wp - \nu\beta_3^2)\Upsilon^2 + \wp^2(\wp - \beta_3)^2 = 0 \tag{54}$$

and Δ_i is root of the characteristic equation

$$\Delta^4 - 2\beta_3 \Delta^3 + (\beta_3^2 + 2\Psi^2)\Delta^2 - 2\beta_3 \Psi^2 \Delta + \Psi^2(\Psi^2 - \nu\beta_3^2) = 0 \tag{55}$$

and p_{ni} is root of the characteristic equation

$$p_n^4 - 2\beta_3 p_n^3 + (\beta_3^2 - 2\beta_n^2)p_n^2 + 2\beta_3 \beta_n^2 p_n + \beta_n^2(\beta_n^2 + \nu\beta_3^2) = 0 \tag{56}$$

Therefore, the complementary solution χ_c of the equation

$$\begin{aligned} \chi_c &= A_0 + B_0 y + C_0 e^{\beta_3 y} + D_0 y e^{\beta_3 y} \\ &+ \sum_{n=1}^{\infty} (A_n e^{p_{n1} y} + B_n e^{p_{n2} y} + C_n e^{p_{n3} y} + D_n e^{p_{n4} y}) \cos \vartheta_n x \end{aligned} \tag{57}$$

The particular integral χ_p is given by

$$\chi_p = \frac{4\theta_0\beta_4^2 q \sin(h\pi/2) K_\eta(h/2)}{a^2\pi s [I_\eta(-h/2) - I_\eta(h/2)]^2} \sum_{m=0}^{\infty} \frac{A^{2m+\eta} e^{\Omega x} \sin(\vartheta_n y)}{m! \Gamma(m + \eta + 1)} [\Lambda_1 + \Lambda_2 e^{-\gamma x} + \Lambda_3 e^{-\gamma x/2}] \quad (58)$$

where

$$\begin{aligned} \Lambda_1 &= \frac{-\gamma\beta_4(\gamma-2\beta_4)}{\Omega^4-2\beta_3\Omega^3+\beta_3^2\Omega^2}, \quad \Lambda_2 = \frac{\gamma^3\beta_4^3}{(\Omega-\gamma)^4-2\beta_3(\Omega-\gamma)^3+\beta_3^2(\Omega-\gamma)^2}, \\ \Lambda_3 &= \frac{(5\gamma-4\beta_4)\beta_4^{2m+\eta}}{(\Omega-\gamma/2)^4-2\beta_3(\Omega-\gamma/2)^3+\beta_3^2(\Omega-\gamma/2)^2}, \\ \Omega &= 2k + \eta + \beta_3 + \beta_4 - \gamma/2 \end{aligned} \quad (59)$$

Thermal stresses can be obtained from Eq. (57)-(58) and (25)

$$\begin{aligned} \sigma_{xx} &= (C_0\beta_3^2 + D_0\beta_3^2 y + 2D_0\beta_3) e^{\beta_3 y} \\ &+ \sum_{n=1}^{\infty} (A_n p_{n1}^2 e^{p_{n1} y} + B_n p_{n2}^2 e^{p_{n2} y} + C_n p_{n3}^2 e^{p_{n3} y} + D_n p_{n4}^2 e^{p_{n4} y}) \cos \vartheta_n x \\ &- \vartheta_n^2 \frac{4\theta_0\beta_4^2 q \sin(h\pi/2) K_\eta(h/2)}{a^2\pi s [I_\eta(-h/2) - I_\eta(h/2)]^2} \sum_{m=0}^{\infty} \frac{A^{2m+\eta} e^{\Omega x} \sin(\vartheta_n y)}{m! \Gamma(m + \eta + 1)} \\ &\times [\Lambda_1 + \Lambda_2 e^{-\gamma x} + \Lambda_3 e^{-\gamma x/2}] \end{aligned} \quad (60)$$

$$\begin{aligned} \sigma_{xy} &= - \sum_{n=1}^{\infty} (A_n p_{n1} e^{p_{n1} y} + B_n p_{n2} e^{p_{n2} y} + C_n p_{n3} e^{p_{n3} y} + D_n p_{n4} e^{p_{n4} y}) \vartheta_n \sin \vartheta_n x \\ &+ \frac{4\theta_0\beta_4^2 q \sin(h\pi/2) K_\eta(h/2)}{a^2\pi s [I_\eta(-h/2) - I_\eta(h/2)]^2} \sum_{m=0}^{\infty} \frac{A^{2m+\eta} e^{\Omega x} \beta \cos(\vartheta_n y)}{m! \Gamma(m + \eta + 1)} \\ &\times [\Omega\Lambda_1 + \Lambda_2(\Omega - \gamma) e^{-\gamma x} + \Lambda_3(\Omega - \gamma/2) e^{-\gamma x/2}] \end{aligned} \quad (61)$$

$$\begin{aligned} \sigma_{yy} &= - \sum_{n=1}^{\infty} (A_n e^{p_{n1} y} + B_n e^{p_{n2} y} + C_n e^{p_{n3} y} + D_n e^{p_{n4} y}) \beta_n^2 \cos \vartheta_n x \\ &+ \frac{4\theta_0\beta_4^2 q \sin(h\pi/2) K_\eta(h/2)}{a^2\pi s [I_\eta(-h/2) - I_\eta(h/2)]^2} \sum_{m=0}^{\infty} \frac{A^{2m+\eta} e^{\Omega x} \sin(\vartheta_n y)}{m! \Gamma(m + \eta + 1)} \\ &\times [\Omega^2 \Lambda_1 + \Lambda_2(\Omega - \gamma)^2 e^{-\gamma x} + \Lambda_3(\Omega - \gamma/2)^2 e^{-\gamma x/2}] \end{aligned} \quad (62)$$

The unknown constants A_n , B_n , C_n , D_n can be obtained from Eqs. (60)-(61) and (30); and C_0 , D_0 can be obtained from Eqs. (62) and (31). The lengthy calculation has been omitted for conciseness, but it should still be considered for numerical calculations.

5 Inversion of the transformed functions

The inversion formula for Laplace transform for Eq. (32) can be written as

$$f(x, y, t) = \mathcal{L}^{-1}[f(x, y, t)] = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st} \bar{f}(x, y, s) ds \quad (63)$$

Taking $s = d + iz$ in Eq. (63), one obtains

$$f(x, y, t) = \frac{e^{dt}}{2\pi} \int_{-\infty}^{\infty} e^{itz} \bar{f}(x, y, d + iz) dz \tag{64}$$

Now, taking Fourier series expansion of the function $h(x, y, t) = e^{-dt} f(x, y, t)$ in the interval $[0, 2L]$, one obtains the approximate formula (Durbin 1974) summed up to a finite number N_0 of terms as

$$\begin{aligned} f_N(x, y, t) &= \frac{e^{dt}}{L} \left\{ \frac{\bar{f}(x, y, d)}{2} + \operatorname{Re} \sum_{k=1}^N [e^{i\varphi} \bar{f}(x, y, d + i\varphi)] \right\} + E_r \\ &= \frac{1}{2} C_0 + \sum_{k=1}^N C_k + E_r \end{aligned} \tag{65}$$

where $\varphi = k\pi/L$, N_0 is a finite integer, d parameter has a value of $5 \leq dt \leq 10$, and E_r is discretization error is added to produce the total approximate error, and

$$C_k = \frac{\exp(dt)}{T} \left\{ \exp \left[ik\pi t/T \bar{f} \left(x, d + i \frac{k\pi t}{T} \right) \right] \right\} \tag{66}$$

The ‘‘Korrektur’’ method allows a reduction of the discretization error without enlarging the truncation error. Therefore, using Eq. (65) can be expressed as

$$f_N(x, y, t) = f_N(x, y, t) - e^{-2dL} f_{N'}(x, y, 2L + t), N < N' \tag{67}$$

we shall now describe the ε -algorithm that is used to accelerate the convergence of the series in Eq. (65). Let $N = 2q + 1$, where q is a natural number and let $S_m = \sum_{k=1}^m C_k$ be the sequence of partial sums of series in (65). We define the ε -sequence by $\varepsilon_{0,m} = 0, \dots, \varepsilon_{1,m} = S_m$ and $\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + 1/(\varepsilon_{p,m+1} - \varepsilon_{p,m})$, $p = 1, 2, 3, \dots$. It can be shown (Honig and Hirdes 1984) that the sequence $\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \dots, \varepsilon_{N,1}$ converges to $f(x, y, t) + E_r - C_0/2$ faster than the sequence of partial sums s_m ($m = 1, 2, 3, \dots$). The actual procedure used to invert the Laplace transform consists of using Eq. (79) together with the ε -algorithm. The actual procedure used to invert the Laplace transform consists of using Eq. (66). The values of $d = 7.5$ and L are chosen according to the criterion outlined in Durbin (1974).

6 Numerical results and discussion

This section conducts numerical computations to demonstrate the impact of phase delays of heat flux, fractional order, temperature discrepancy factor, and material attributes on the transient temperature field and its associated stress under memory-dependent derivatives. For numerical calculations, the following physical parameters were considered: $h = 1$, $a = 1$, and the reference temperature as 150°C . The thermomechanical properties composed of partially stabilized zirconia (PSZ), and austenitic stainless steel (SUS304) are given in Table 1.

In this session, numerical calculations were conducted to analyze the impact of heating on the plate. The results of these calculations are shown in the following figures using the MATHEMATICA program.

Table 1 Thermo-mechanical properties of PSZ and SUS 304

Material properties	PSZ	SUS304
Thermal conductivity	1.67	15.97
Thermal diffusivity ($\times 10^{-6}$)	6.24	5.58
Coefficient of thermal expansion ($\times 10^{-6}$)	2.93	14.87
Density	5730	7930
Specific heat	0.467	0.361
Young's Modulus	211	193
Poisson's ratio	0.3	0.3

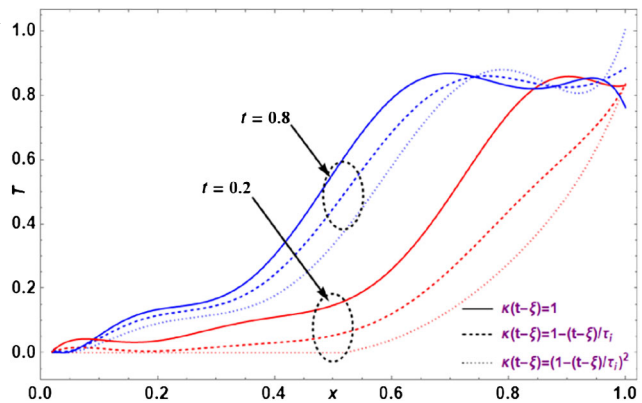
Fig. 2 Temperature profile when $\alpha = 0.8$, $\tau_q = 0.3$, $b = 0.8$, $\delta = -1$, $\gamma = 1$


Figure 2 illustrates the spatial arrangement of the temperature distribution at different dimensionless time intervals using the MDD kernel function. At the beginning, the dimensionless temperature change is zero on the initial surface, as anticipated. However, the x -axis along the far end surface has a greater magnitude in dimensionless temperature. The temperature change in the FGM layer is time-dependent and observed with propagation characteristics of wave-like phenomena. For instance, the temperature up to position $x = 0.5$ remains constant or slightly gradual change for $t = 0.2$ and 0.8 , respectively. It suggests that thermal waves do not propagate to these positions within such short time intervals. However, when the value of x exceeds 0.5 , the temperature shift can be observed over the entire FGM layer.

Figure 3 illustrates the impact of the FGM parameter γ along the y -axis on the temperature distributions represented by Eq. (51) with $x = 0.5$. The temperatures slightly fall along the y -axis in proportion to the γ , since the heat flow from the boundary $x = 1$ also rises with the parameter γ . Table 2 displays the disparity in temperature distributions between homogeneous materials (HM) and functionally graded materials (FGM) along the y -axis, as shown by Eq. (51). For $\gamma = 1.2$, the thermal conductivities of FGM are greater than those of HM. As a result, the heat input of FGM from the boundary $x = 0.5$ is bigger than that of HM.

Following Akbarzadeh and Chen (2013), Ortigueira (2011), fixing $\alpha = 1$ in Eq. (16), the heat flux propagation velocity can be obtained as $V = \sqrt{1/\tau_q D_\omega}$. As the kernel function in Eq. (34) is 1 and taking $\tau_q = 0.0146$, one obtains $V = \{68.49\}^{1/2}$. The dimensionless propagation distance $\Delta x = Vt = \{68.49\}^{1/2}(0.06) = 0.5$, is in agreement with the numerical prediction, as shown in Fig. 4. It is reflected that if the time is taken as $t = 0.05$, then the hyperbolic temperature increases with the increase in position is less than $\Delta x = 0.5$. By

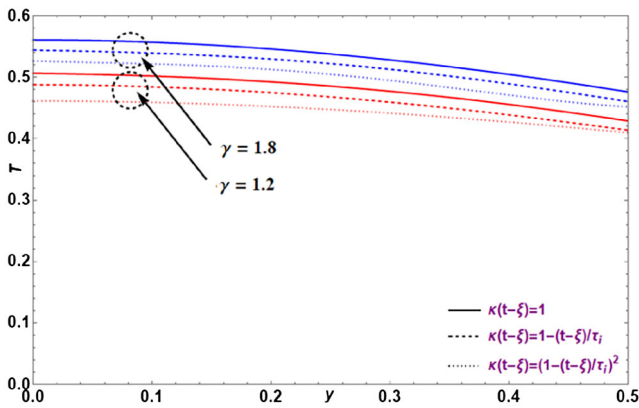


Fig. 3 Temperature profile when $x = 0.5, \alpha = 0.8, \tau_q = 0.3, b = 0.8, \delta = -1$

Table 2 Comparison of temperature profile at $x = 0.5$ along the x -axis

Conditions	Material	Ohmichi et al. (2016)	Present study
steady-state	HM	0.3314	0.3327
	FGM	0.4271	0.4284

setting $\eta = 1$ in Eq. (45), we observe that the above modified Bessel equation is mathematically equivalent to the one described in reference Zhou et al. (2011). To verify the validity of the numerical inversion of the Laplace transform, an exact solution for the classical Fourier heat conduction with kernel $K(t - \xi) = 1$ was obtained. The exact temperature, denoted as T_{exact} , is provided in reference Zhou et al. (2011), as shown in Figure 6. The temperature T_{num} may be determined using Eq. (51). The error ratio is defined as $(T_{num} - T_{exact})/T_{exact} \times 100$ [%]. The error ratio reaches its highest value of 0.611 when x is equal to 0.8 and y is equal to 0.2. Figures 5 and 7 show the effects of gradient indices γ and δ on the distribution of temperature fields, respectively. From Fig. 5, it can be seen that with a fixed fractional order α with gradient indices γ varying from -1 to 1 , the wave-like behaviour of the fractional heat conduction model becomes more evident. Therefore, the fractional heat conduction model can capture not only the essence of hyperbolic heat conduction but also the diffusion characteristic of classical Fourier heat conduction, irrespective of MDD kernels. Figure 7 also shows a wave-like distribution with gradient indices δ varying from -2 to 2 for different MDD kernel functions. Initially it is noticed that the temperature rises slightly, then decrease till to attain minimum value and then increases exponentially with the increase of x until finished.

The through-the-length variation of the longitudinal normal stress σ_{xx} , the transverse shear stress σ_{xy} , and the transverse normal stress σ_{yy} change significantly along the x -axis. For example, in Fig. 8, at $x = 0.4 \sim 0.5$, the magnitude of the longitudinal stress σ_{xx} is maximum at a point throughout the length due to high tensile area during the fixed value of $\alpha = 0.8, \tau_q = 0.0146, b = 0.6, \delta = -1, \gamma = 1$. Figures 8 and 10 show that the normal stress σ_{xx} and shear stress σ_{xy} have zero temperature at both ends, thus satisfying Eq. (30). Similar comments apply to the transverse shear stress σ_{xy} whose value is about 33% less than that of the longitudinal normal stress σ_{xx} . The through-the-width variation of the normal stress σ_{yy} is a non-linear sinusoidal nature since material properties and the temperature change vary through the thickness, as illustrated in Fig. 9.

Fig. 4 Temperature when $\alpha = 0.8$, $\tau_q = 0.0146$, $b = 0.6$, $\delta = -1$, $\gamma = 1$

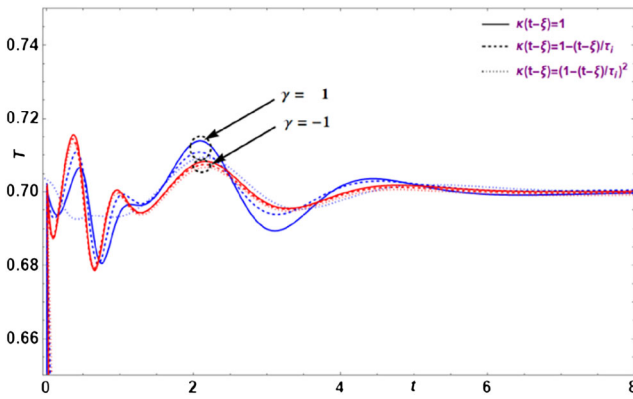
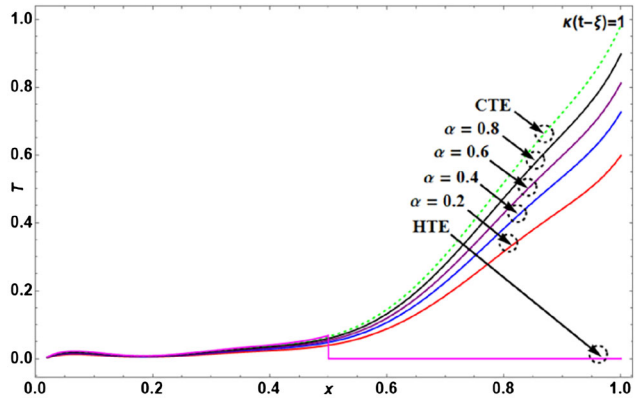
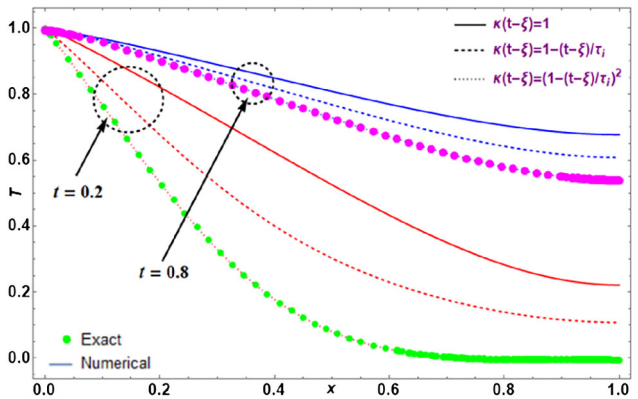


Fig. 5 Effect of gradient index γ when $x = 0.75$, $\alpha = 0.8$, $\tau_q = 0.3$, and $\delta = -1$

Fig. 6 Exact Vs numerical inversion Laplace transform-based temperature



7 Conclusion

The study develops a comprehensive thermal uncoupled model for memory-dependent differential, examining thermal flow under rapid temperature increase with a two-dimensional

Fig. 7 Temperature for various gradient indices δ when $\alpha = 0.8$, $\tau_q = 0.3$, $\lambda = 1$

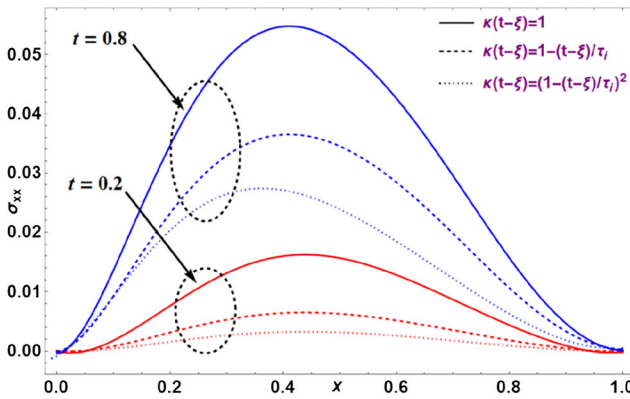
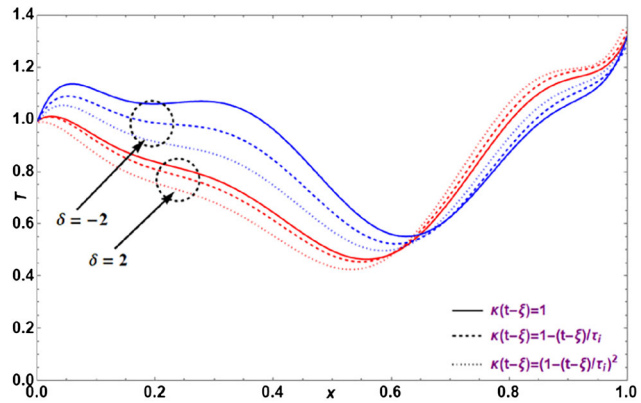


Fig. 8 Normal stress σ_{xx} when $\alpha = 0.8$, $\tau_q = 0.0146$, $b = 0.6$, $\delta = -1$, $\gamma = 1$

FG plate. Appropriately selecting the volumetric ratio of ceramics and metal components can greatly lower the heat stresses in the plate. An increase in the coefficient of linear thermal expansion reduces both maximum and lowest thermal stresses, while an increase in Young’s modulus increases both. Using appropriate compositional materials can minimize thermal stresses, potentially reaching negligible levels. The numerical results yield several inferences:

- Memory-dependent derivatives’ non-Fourier effects significantly impact thermal field response history and distribution, with energy dissipation potentially causing temperature decrease without heat transfer.
- Revised categorization system for materials based on memory-dependent derivative parameters evaluates heat conduction capacity, considering thermoelasticity with two temperatures.
- The phase-lag heat and temperature gradient significantly impact thermal field variables in memory-dependent derivatives time. The temperature change rate increase is influenced by the fractional order and relaxation time during the process. Thus, different theories like CTE, HTE, SFTE, and EFTE models can be derived.
- The PSZ/SUS304 FGP material experiences the maximum compressive stress on the heating surface. It experiences a maximum tensile stress, which is lower than the maximum

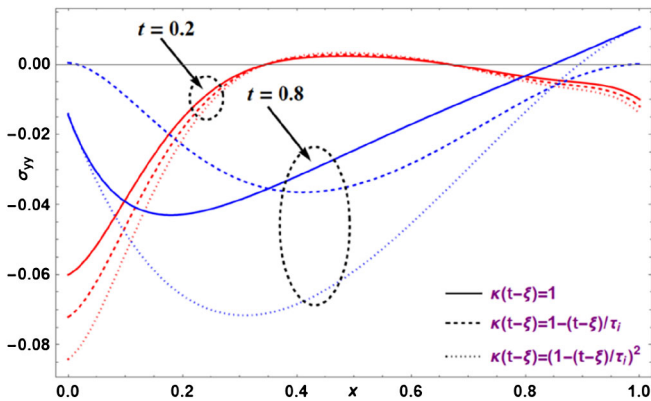


Fig. 9 Normal stress σ_{yy} when $\alpha = 0.8$, $\tau_q = 0.0146$, $b = 0.6$, $\delta = -1$, $\gamma = 1$

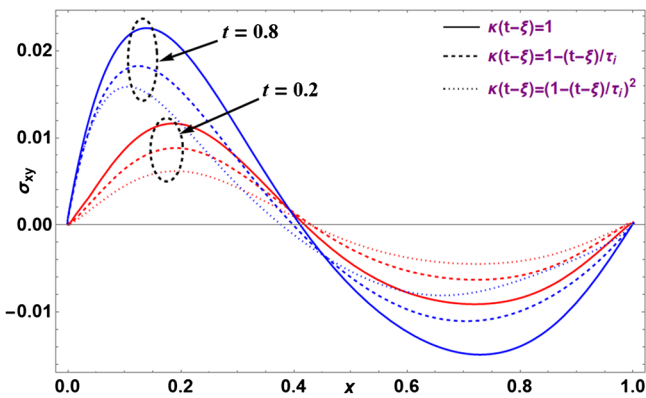


Fig. 10 Shear stress σ_{xy} when $\alpha = 0.8$, $\tau_q = 0.0146$, $b = 0.6$, $\delta = -1$, $\gamma = 1$

compressive stress. The maximum stresses occur along the normal stress σ_{yy} , whereas they are less in normal stress σ_{xx} and shear stress σ_{xy} .

The study suggests that future research could effectively utilize micro- and nano-functionally graded resonators with different compositional materials like ZrO₂/Ti-6Al-4V, ZrO₂/Ti-6Al-4V and ZrO₂/SUS304.

Appendix A: The kernel function

In this context, Laplace transform of Eq. (8)

$$G(s, \omega) = \begin{cases} 1 - e^{-s\omega}; \kappa(t - \xi) = 1 \\ 1 - \frac{1 - e^{-s\omega}}{s}; \kappa(t - \xi) = 1 - (t - \xi) \\ \left(1 - \frac{2}{s\omega}\right) + \frac{2(1 - e^{-s\omega})}{s^2\omega^2}; \kappa(t - \xi) = \left(1 - \frac{t - \xi}{\omega}\right)^2 \end{cases} \quad (\text{A.1})$$

Author contributions All author's combinedly wrote the main manuscript text and prepared figures . Also all authors reviewed the manuscript.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

References

- Abanto-Bueno, J., Lambros, J.: An experimental study of mixed mode crack initiation and growth in functionally graded materials. *Exp. Mech.* **46**, 179–196 (2006)
- Akbarzadeh, A.H., Chen, Z.T.: Transient heat conduction in a functionally graded cylindrical panel based on the dual phase lag theory. *J. Thermophys. Heat Transf.* **33**(6), 1100–1125 (2012). <https://doi.org/10.1007/s10765-012-1204-2>
- Akbarzadeh, A.H., Chen, Z.T.: Heat conduction in one-dimensional functionally graded media based on the dual-phase-lag theory. *Proc. Inst. Mech. Eng. C* **227**(4), 744–759 (2013). <https://doi.org/10.1177/0954406212456651>
- Asgari, M., Akhlaghi, M.: Transient heat conduction in two-dimensional functionally graded hollow cylinder with finite length. *Heat Mass Transf.* **45**(11), 1383–1392 (2009). <https://doi.org/10.1007/s00231-009-0515-8>
- Askarizadeh, H., Ahmadi, H.: Periodic heat transfer in convective fins based on dual-phase-lag theory. *J. Thermophys. Heat Transf.* **30**(2), 359–368 (2016). <https://doi.org/10.2514/1.T4602>
- Babaei, M.H., Chen, Z.: Transient hyperbolic heat conduction in a functionally graded hollow cylinder. *J. Thermophys. Heat Transf.* **24**(2), 325–330 (2010). <https://doi.org/10.2514/1.41368>
- Bao, G., Wang, B.L.: Multiple cracking in functionally graded ceramic/metal coatings. *Int. J. Solids Struct.* **32**, 2853–2871 (1995)
- Butcher, R.J., Rousseau, C.E., Tippur, H.V.: A functionally graded particulate composite: preparation, measurements and failure analysis. *Acta Mater.* **47**, 259–268 (1998)
- Byrd, L., Birman, V.: An investigation of numerical modeling of transient heat conduction in a one-dimensional functionally graded material. *Heat Transf.* **31**(3), 212–221 (2010). <https://doi.org/10.1080/01457630903304384>
- Cai, H., Bao, G.: Crack bridging in functionally graded coatings. *Int. J. Solids Struct.* **35**, 701–717 (1998)
- Caputo, M., Mainardi, F.: A new dissipation model based on memory mechanism. *Pure Appl. Geophys.* **91**(1), 134–147 (1971)
- Chen, P.J., Gurtin, M.E.: On a theory of heat conduction involving two temperatures. *Z. Angew. Math. Phys.* **19**, 559–577 (1968). <https://doi.org/10.1007/BF01594969>
- Choi, H.J., Lee, K.Y., Jin, T.E.: Collinear cracks in a layered half-plane with a graded nonhomogeneous interfacial zone – Part A: mechanical response. *Int. J. Fract.* **94**, 103–122 (1998)
- Comi, C., Mariani, S.S.: Extended finite element simulation of quasi-brittle fracture in functionally graded materials. *Comput. Methods Appl. Mech. Eng.* **196**, 4013–4026 (2007)
- Daneshjou, K., Bakhtiari, M., Alibakhshi, R., Fakoor, M.: Transient thermal analysis in 2D orthotropic F.G. hollow cylinder with heat source. *Int. J. Heat Mass Transf.* **89**, 977–984 (2015). <https://doi.org/10.1016/j.ijheatmasstransfer.2015.05.104>
- De, A., Purkait, P., Das, P., Kanoria, M.: Memory dependent magneto-thermoelastic interaction in a rotating medium based on refined multi-phase-lag model (2024). Preprint (Version 1) available at Research Square <https://doi.org/10.21203/rs.3.rs-3977998/v1>
- Delale, F., Erdogan, F.: The crack problem for a nonhomogeneous plane. *J. Appl. Mech.* **50**, 609–614 (1983)
- Ding, S.H., Li, X., Zhou, Y.T.: Dynamic stress intensity factors of mode I crack problem for functionally graded layered structures. *Comput. Model. Eng. Sci.* **56**, 43–84 (2010)
- Durbin, F.: Numerical inversion of Laplace transforms: an efficient improvement to Dubner and Abate's method. *Comput. J.* **17**(4), 371–376 (1974). <https://doi.org/10.1093/comjnl/17.4.371>
- Eischen, J.W.: Fracture of nonhomogeneous materials. *Int. J. Fract.* **34**, 3–22 (1987)
- Erdogan, F.: Fracture mechanics of functionally graded materials. *Compos. Eng.* **5**, 753–770 (1995)
- Erdogan, F., Wu, B.H.: The surface crack problem for a plate with functionally graded properties. *J. Appl. Mech.* **64**, 449–456 (1997)
- Ezzat, M.A.: Magneto-thermoelasticity with thermoelectric properties and fractional derivative heat transfer. *Physica B, Condens. Matter* **406**(1), 30–35 (2011). <https://doi.org/10.1016/j.physb.2010.10.005>
- Ezzat, M.A.: Analytical study of two-dimensional thermo-mechanical responses of viscoelastic skin tissue with temperature-dependent thermal conductivity and rheological properties. *Mech. Based Des. Struct. Mach.* **51**(5), 2776–2793 (2023). <https://doi.org/10.1080/15397734.2021.1907757>

- Ezzat, M.A., El-Bary, A.A.: Effects of variable thermal conductivity and fractional order of heat transfer on a perfect conducting infinitely long hollow cylinder. *Int. J. Therm. Sci.* **108**, 62–69 (2016). <https://doi.org/10.1016/j.ijthermalsci.2016.04.020>
- Ezzat, M.A., El-Bary, A.A.: Magneto-thermoelectric viscoelastic materials with memory-dependent derivative involving two-temperature. *Int. J. Appl. Electromagn. Mech.* **50**, 549–567 (2016). <https://doi.org/10.3233/JAE-150131>
- Ezzat, M.A., Karamany, A.S.: Theory of fractional order in electro-thermoelasticity. *Eur. J. Mech. A, Solids* **30**(4), 491–500 (2011). <https://doi.org/10.1016/j.euromechsol.2011.02.004>
- Ezzat, M.A., El-Bary, A.A., Fayik, M.A.: Fractional Fourier law with three-phase lag of thermoelasticity. *Mech. Adv. Mat. Struct.* **20**(8), 593–602 (2013). <https://doi.org/10.1080/15376494.2011.643280>
- Ezzat, M.A., Karamany, A.S., El-Bary, A.A.: Generalized thermo-viscoelasticity with memory-dependent derivatives. *Int. J. Mech. Sci.* **89**, 470–475 (2014). <https://doi.org/10.1016/j.ijmecsci.2014.10.006>
- Ezzat, M.A., Karamany, A.S., El-Bary, A.A.: On dual phase-lag thermoelasticity theory with memory-dependent derivative. *Mech. Adv. Mat. Struct.* **24**(11), 908–916 (2017). <https://doi.org/10.1080/15376494.2016.1196793>
- Ezzat, M.A., Karamany, A.S., El-Bary, A.A.: Two-temperature theory in Green-Naghdi thermoelasticity with fractional phase-lag heat transfer. *Microsyst. Technol.* **24**, 951–961 (2018). <https://doi.org/10.1007/s00542-017-3425-6>
- Fujita, Y.: Integrodifferential equation which interpolates the heat equation and the wave equation. *Osaka J. Math.* **27**(2), 309–321 (1990). <https://doi.org/10.18910/4060>
- Hendy, M.H., Amin, M.M., Ezzat, M.A.: Two-dimensional problem for thermoviscoelastic materials with fractional order heat transfer. *J. Therm. Stresses* **42**(10), 1298–1315 (2019). <https://doi.org/10.1080/01495739.2019.1623734>
- Honig, G., Hirdes, U.: A method for the numerical inversion of the Laplace transform. *J. Comput. Appl. Math.* **10**, 113–132 (1984)
- Hosseini, S.M., Akhlaghi, M., Shakeri, M.: Transient heat conduction in functionally graded thick hollow cylinders by analytical method. *Heat Mass Transf.* **43**(7), 669–675 (2007). <https://doi.org/10.1007/s00231-006-0158-y>
- Jumarie, G.: Derivation and solutions of some fractional Black-Scholes equations in coarse-grained space and time. Application to Merton's optimal portfolio. *Comput. Math. Appl.* **59**(3), 1142–1164 (2010). <https://doi.org/10.1016/j.camwa.2009.05.015>
- Keles, I., Conker, C.: Transient hyperbolic heat conduction in thick-walled FGM cylinders and spheres with exponentially-varying properties. *Eur. J. Mech. A, Solids* **30**(3), 449–455 (2011). <https://doi.org/10.1016/j.euromechsol.2010.12.018>
- Kim, K.S., Noda, N.: Green's function approach to three-dimensional heat conduction equation of functionally graded materials. *J. Therm. Stresses* **24**(5), 457–477 (2001). <https://doi.org/10.1080/01495730151126113>
- Kim, J., Paulino, G.: Mixed-mode fracture of orthotropic functionally graded materials using finite elements and the modified crack closure method. *Eng. Fract. Mech.* **69**, 1557–1586 (2002)
- Kimmich, R.: Strange kinetics, porous media, and N.M.R. *J. Chem. Phys.* **284**, 243 (2002)
- Li, M., Wen, P.H.: Finite block method for transient heat conduction analysis in functionally graded media. *Int. J. Numer. Methods Eng.* **99**(5), 372–390 (2014). <https://doi.org/10.1002/nme.v99.5>
- Long, X., Delale, F.: The mixed mode crack problem in an FGM layer bonded to a homogeneous half-plane. *Int. J. Solids Struct.* **42**, 3897–3917 (2005)
- Lord, H.W., Shulman, Y.: A generalized dynamical theory of thermoelasticity. *J. Mech. Phys. Solids* **15**(5), 299–309 (1967). [https://doi.org/10.1016/0022-5096\(67\)90024-5](https://doi.org/10.1016/0022-5096(67)90024-5)
- Ma, C.C., Chen, Y.T.: Theoretical analysis of heat conduction problems of nonhomogeneous functionally graded materials for a layer sandwiched between two half-planes. *Acta Mech.* **221**(3–4), 223–237 (2011). <https://doi.org/10.1007/s00707-011-0498-7>
- Ma, Y., He, T.: The transient response of a functionally graded piezoelectric rod subjected to a moving heat source under fractional order theory of thermoelasticity. *Mech. Adv. Mat. Struct.* **24**(9), 789–796 (2016). <https://doi.org/10.1080/15376494.2016.1196783>
- Mondal, S., Sur, A.: Field equations and memory effects in a functionally graded magneto-thermoelastic rod. *Mech. Based Des. Struct. Mach.* **51**(3), 1408–1430 (2023). <https://doi.org/10.1080/15397734.2020.1868320>
- Mondal, S., Sur, A.: Nonlocal effects in a functionally graded thermoelastic layer due to volumetric absorption laser. *Waves Random Complex Media* **34**(3), 1368–1388 (2024). <https://doi.org/10.1080/17455030.2021.1938286>
- Mondal, S., Sur, A., Kanoria, M.: Thermoelastic response of fiber-reinforced epoxy composite under continuous line heat source. *Waves Random Complex Media* **31**(6), 1749–1779 (2021). <https://doi.org/10.1080/17455030.2019.1699675>

- Nezhad, Y.R., Asemi, K., Akhlaghi, M.: Transient solution of temperature field in functionally graded hollow cylinder with finite length using multi layered approach. *Int. J. Mech. Mater. Des.* **7**(1), 71–82 (2011). <https://doi.org/10.1007/s10999-011-9151-9>
- Noda, N., Jin, Z.: Thermal stress intensity factors for a crack in a strip of a functionally gradient material. *Int. J. Solids Struct.* **30**(8), 1039–1056 (1993). [https://doi.org/10.1016/0020-7683\(93\)90002-0](https://doi.org/10.1016/0020-7683(93)90002-0)
- Ohmichi, M., Noda, N.: Plane thermal stresses in a functionally graded plate subjected to a partial heating. *J. Therm. Stresses* **29**(12), 1127–1142 (2006). <https://doi.org/10.1080/01495730600712683>
- Ohmichi, M., Noda, N., Sumi, N.: Plane heat conduction problems in functionally graded orthotropic materials. *J. Therm. Stresses* **40**(6), 747–764 (2016). <https://doi.org/10.1080/01495739.2016.1249989>
- Ootao, Y., Tanigawa, Y.: Three-dimensional solution for transient thermal stresses of functionally graded rectangular plate due to nonuniform heat supply. *Int. J. Mech. Sci.* **47**(11), 1769–1788 (2005). <https://doi.org/10.1016/j.ijmecsci.2005.06.003>
- Ortigueira, M.D.: *Fractional Calculus for Scientists and Engineers*, vol. 84. Springer, Dordrecht (2011). <https://doi.org/10.1007/978-94-007-0747-4>
- Peng, Y., Zhang, X.Y., Xie, Y.J., Li, X.F.: Transient hygrothermoelastic response in a cylinder considering non-Fourier hyperbolic heat-moisture coupling. *Int. J. Heat Mass Transf.* **126**, 1094–1103 (2018). <https://doi.org/10.1016/j.ijheatmasstransfer.2018.05.084>
- Podlubny, I.: *Fractional Differential Equations*, vol. 198. Academic Press, New York (1998). [https://doi.org/10.1016/S0076-5392\(99\)80021-6](https://doi.org/10.1016/S0076-5392(99)80021-6)
- Povstenko, Y.Z.: Fractional heat conduction equation and associated thermal stress. *J. Therm. Stresses* **28**(1), 83–102 (2004). <https://doi.org/10.1080/014957390523741>
- Rahideh, H., Malekzadeh, P., Haghighi, M.R.G.: Heat conduction analysis of multilayered FGMs considering the finite heat wave speed. *Energy Convers. Manag.* **55**, 14–19 (2012). <https://doi.org/10.1016/j.enconman.2011.09.020>
- Santare, M.H., Lambros, J.: Use of graded finite elements to model the behavior of nonhomogeneous materials. *J. Appl. Mech.* **67**, 819–822 (2000)
- Sherief, H.H., El-Sayed, A.M.A., El-Latif, A.M.A.: Fractional order theory of thermoelasticity. *Int. J. Solids Struct.* **47**(2), 269–275 (2010). <https://doi.org/10.1016/j.ijsolstr.2009.09.034>
- Sur, A.: Magneto-photo-thermoelastic interaction in a slim strip characterized by hereditary features with two relaxation times. *Mech. Time-Depend. Mater.* (2023a). <https://doi.org/10.1007/s11043-023-09658-0>
- Sur, A.: Elasto-thermodiffusive nonlocal responses for a spherical cavity due to memory effect. *Mech. Time-Depend. Mater.* (2023b). <https://doi.org/10.1007/s11043-023-09626-8>
- Sur, A.: Moore–Gibson–Thompson generalized heat conduction in a thick plate. *Indian J. Phys.* **98**, 1715–1726 (2024). <https://doi.org/10.1007/s12648-023-02931-5>
- Tarn, J.Q., Wang, Y.M.: End effects of heat conduction in circular cylinders of functionally graded materials and laminated composites. *Int. J. Heat Mass Transf.* **47**(26), 5741–5747 (2004). <https://doi.org/10.1016/j.ijheatmasstransfer.2004.08.003>
- Torabi, M., Saedodin, S.: Analytical and numerical solutions of hyperbolic heat conduction in cylindrical coordinates. *J. Thermophys. Heat Transf.* **25**(2), 239–253 (2011). <https://doi.org/10.2514/1.51395>
- Tzou, D.Y.: A unified field approach for heat conduction from macro- to micro-scales. *J. Heat Transf.* **117**(1), 8–16 (1995a). <https://doi.org/10.1115/1.2822329>
- Tzou, D.Y.: Experimental support for the lagging behavior in heat propagation. *J. Thermophys. Heat Transf.* **9**(4), 686–693 (1995b). <https://doi.org/10.2514/3.725>
- Wang, J.L., Li, H.F.: Surpassing the fractional derivative: concept of the memory-dependent derivative. *Comput. Math. Appl.* **62**(3), 1562–1567 (2011). <https://doi.org/10.1016/j.camwa.2011.04.028>
- Wang, X., Wang, Z., Zeng, T., Cheng, S., Yang, F.: Exact analytical solution for steady-state heat transfer in functionally graded sandwich slabs with convective-radiative boundary conditions. *Compos. Struct.* **192**, 379–386 (2018). <https://doi.org/10.1016/j.compstruct.2018.03.006>
- Youssef, H.M.: Theory of fractional order generalized thermoelasticity. *J. Heat Transf.* **132**(6), 061301 (2010). <https://doi.org/10.1115/1.4000705>
- Yu, B., Zhou, H.L., Yan, J., Meng, Z.: A differential transformation boundary element method for solving transient heat conduction problems in functionally graded materials. *Numer. Heat Transf. A* **70**(3), 293–309 (2016). <https://doi.org/10.1080/10407782.2016.1173471>
- Zhang, X.Y., Li, X.F.: Transient thermal stress intensity factors for a circumferential crack in a hollow cylinder based on generalized fractional heat conduction. *Int. J. Therm. Sci.* **121**, 336–347 (2017). <https://doi.org/10.1016/j.ijthermalsci.2017.07.015>
- Zhang, X.Y., Chen, Z.T., Li, X.F.: Thermal shock fracture of an elastic half-space with a subsurface penny-shaped crack via fractional thermoelasticity. *Acta Mech.* **229**(12), 4875–4893 (2018). <https://doi.org/10.1007/s00707-018-2252-x>
- Zhang, X.Y., Chen, Z.T., Li, X.F.: Generalized fractional heat conduction in a one-dimensional functionally graded material layer. *J. Thermophys. Heat Transf.* **33**(2), 1–11 (2019). <https://doi.org/10.2514/1.T5667>

Zhao, J., Ai, X., Li, Y.Z.: Transient temperature fields in functionally graded materials with different shapes under convective boundary conditions. *Heat Mass Transf.* **43**(12), 1227–1232 (2007). <https://doi.org/10.1007/s00231-006-0135-5>

Zhou, Y.T., Lee, K.Y., Yu, D.H.: Transient heat conduction in a functionally graded strip in contact with well stirred fluid with an outside heat source. *Int. J. Heat Mass Transf.* **54**(25–26), 5438–5443 (2011). <https://doi.org/10.1016/j.ijheatmasstransfer.2011.07.047>

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