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Wave propagation and free vibration of a Timoshenko beam mounted on the viscoelastic Pasternak foundation modeled by fraction-order derivatives

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Abstract
There is a second frequency spectrum existing for the classic Timoshenko beam with hinged supports at both ends. However, it is usually assumed that the second frequency spectrum is unphysical. In this paper, the modified Timoshenko beam is studied. The modified Timoshenko beam is mounted upon the viscoelastic foundation. The flexural wave propagation and the free-vibration problem are investigated. The viscoelastic foundation is modeled by the standard solid model or Zenner model with the fraction-order derivative. It is found that there are two kinds of flexural waves that are not only dispersive but also attenuated. In contrast to the classical Timoshenko beam, there is only one frequency spectrum in the modified Timoshenko beam. Furthermore, complex-valued natural frequencies are induced by the viscoelastic foundation. The imaginary part of the complex natural frequency reflects the attenuation properties associated with time. The dispersion and attenuated curves of flexural waves and the complex-valued natural frequency of the first three orders are provided in the numerical results. The influences of the viscoelastic foundation are discussed based on the numerical results.

Keywords Fraction-order viscoelasticity · Pasternak foundation · Modified Timoshenko beam · Flexural waves · Dispersion and attenuation · Complex natural frequency

1 Introduction

The soil–structure interaction problems play an important role in the fields of structural and foundation engineering, e.g., building, geotechnical, highway structures, railroad structures, submerged pipes, etc. The static and dynamic analyses of beams are usually carried out by using Bernoulli–Euler beam theory (Yu et al. 2017; Bogdanoff 2015; Awodola 2012; Gürgze

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1998; Gra and Ab 2020), for the case of slender beams, or Timoshenko theory for stocky beams (Ozgumus and Kaya 2010; Han et al. 1999; Xiang and Yang 2008; Simsek 2010; Krawczuk et al. 2003). Whether a Bernoulli–Euler beam or a Timoshenko beam is being considered, the hypothesis of a plane cross section is adopted. However, the warping deformation of a cross section is inevitable and the hypothesis of a plane cross section does not hold, especially for stocky beams and laminated composite beams. Accordingly, the high-order shear beam models have been proposed successively (Reddy 1985; Reddy et al. 1997; Wang et al. 2000; Voyiadjis and Shi 1991; Shi 2007). These models realistically describe the static and dynamic mechanical behavior of the beams, but the description of the subsoil and its interaction with a beam resting on it is not easy.

For over a century, various physical and mathematical models about foundations were formulated. The most frequently used model is the Winkler model. In the Winkler model, the beam-supporting soil is modeled as a series of closely spaced, mutually independent, linear elastic springs that provide the supporting force in direct proportion to the deflection of the beam. The static and dynamical mechanical responses of finite and infinite beams mounted on the Winkler foundation were also investigated (Thambiratnam and Zbuge 1996; Chen 2002; De Rosa 1989). The main drawback of the Winkler foundation is the noncontinuity of deformation of the foundation soils, which is clearly inconsistent with the actual situation. Therefore, many researchers have devoted work to generalize and improve the Winkler model. The most well-known and commonly used refined foundation model is the Pasternak model. In the Pasternak model, there are two parameters. The first parameter represents the stiffness of the vertical spring, as in the Winkler model, while the second parameter is introduced to account for the coupling effect of the linear springs. It can be assumed that there is a shear layer on the top of the vertical springs and the second parameter can also be considered as the shear stiffness of the shear layer. Naidu and Rao (1995a,b) studied the buckling and the vibration behavior of the Bernoulli–Euler beam resting on the Pasternak foundation. The effect of the elastic foundation on buckling loads for various end boundaries was examined. The finite-element technique was also used by Yokoyama (1995) to investigate the vibration behavior of a Bernoulli–Euler and a Timoshenko beam mounted on the Pasternak foundation. Mously (1999) derived an explicit formula for the natural vibration frequency of Timoshenko beams mounted on the Pasternak foundation.

Usually, the foundation soil exhibits viscoelastic behavior. In order to reflect the history-dependent mechanical behavior, the classic Winkler model and Pasternak model are improved with consideration of viscoelasticity. There are two approaches to describe the viscoelastic behavior, i.e., the integral-type constitutive relation and the differential-type constitutive relation. In the differential model of viscoelasticity, the elasticity is reflected by the spring elements, while the viscosity is introduced by the dashpot. The elastic element abides by Hooke's law, while the dashpot follows Newton's law. The commonly used viscoelastic models include the Maxwell model, the Kelvin model and the Zenner model. These models are obtained by the different combinations of the elastic elements and the dashpots. Recently, the static and dynamic behavior of a Bernoulli–Euler beam or a Timoshenko beam resting on the viscoelastic foundation were also investigated (Li et al. 2019; Syngellakis et al. 2020). However, the rheology behavior of the foundation soil is usually complicated. In order to realistically describe the rheology behavior of the foundation soil, many model parameters are introduced that result in difficulty of characterization of these model parameters from experiment data.

The fraction-order derivative is a natural generalization of the integer-order derivative. The fraction-order derivative is usually defined by a generalized integral. Due to the nonlocal properties, the physical phenomena associated with the space nonlocality or the time history-dependence are especially suitable to be described by the fraction-order derivative (Bagley and Torvik 1986; Mainardi 2010). In the fraction-order viscoelastic model, the history-dependence behavior of a dashpot described by the fraction-order derivative with Newton's law is only a special case. Zhang et al. (2019) proposed a fractional-order three-element mode that accurately described the viscoelastic dynamic mechanical properties of soil during vibratory compaction. However, investigations of the static and dynamic behavior of a Bernoulli–Euler beam or a Timoshenko beam resting on the fraction-order viscoelastic foundation are still rare.

In the present work, we aim to study the wave propagation and the free-vibration behavior of a Timoshenko beam mounted on the fraction-order viscoelastic Pasternak foundation. The modified Timoshenko beam and the classic Timoshenko beam are both considered and compared. The dispersion and attenuation features of the traveling flexural waves are discussed. Moreover, the complex natural frequency due to the viscoelastic foundation and the elimination of the second frequency spectrum of the classic Timoshenko beam by the modified Timoshenko beam are also discussed.

2 Fraction-order viscoelastic Pasternak foundation

The sinking and deformation of a natural foundation under load has an obvious time dimension. The mechanical behavior of foundation soil is actually between an ideal solid and an ideal fluid. The integer-order viscoelastic model has limitations in describing the complex rheological behavior of foundation soil. The fractional derivative is a natural extension of the integer-order derivative and can realize a smooth transition between two adjacent integer derivatives. The viscoelastic model with fractional derivatives can describe the historical-dependent behavior of a viscoelastic foundation more accurately. The fractional-order standard solid viscoelastic model or Zenner model is adopted in this paper.

The fraction-order derivative of Riemann–Liouville type is defined as:

$$D^{\alpha}\big[f(t)\big]=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\tau)^{-\alpha}f(\tau)d\tau, & 0<\alpha<1,\\ \frac{d}{dt}f(t), & \alpha=1\end{cases} \tag{1}$$

where α $(0<\alpha<1)$ denotes the fractional order, and Γ is the *Gamma* function, i.e., $\Gamma(z)=\int_0^{+\infty}t^{z-1}e^{-t}dt$. The constitutive equation of the viscoelastic Zenner model can be expressed as

$$\sigma+\frac{\eta}{E_1+E_2}\dot{\sigma}=\frac{E_1E_2}{E_1+E_2}\varepsilon+\frac{\eta E_1}{E_1+E_2}\dot{\varepsilon}, \tag{2}$$

where σ and ε are the stress and strain, respectively, E_1 and E_2 are the elastic moduli of the elastic elements, and η is the viscosity coefficient of the viscous element. The constitutive equation of the fraction-order viscoelastic Zenner model can be expressed as

$$\sigma+\frac{E_1}{E_1+E_2}(\tau_\sigma)^{\alpha}D^{\alpha}\big[\sigma(t)\big]=\frac{E_1E_2}{E_1+E_2}\varepsilon+\frac{E_1^2}{E_1+E_2}(\tau_\varepsilon)^{\alpha}D^{\alpha}\big[\varepsilon(t)\big], \tag{3}$$

where τ_σ and τ_ε are the relaxation times of stress and strain, respectively. It is assumed that $\tau_\sigma=\tau_\varepsilon=\eta/E_1$ in the present work.

The Fourier transformation of a function $f(t)$ is defined as

$$\mathcal{F}\big\{f(t)\big\}=\bar{f}(\omega)=\int_{-\infty}^{+\infty}e^{-i\omega\tau}f(\tau)d\tau. \tag{4}$$

For the integer-order derivative and the convolution integral, there are the following properties

$$\mathcal{F}\{f^{(n)}(t)\} = (i\omega)^n \mathcal{F}\{f(t)\}, \tag{5}$$

$$\mathcal{F}\{f(t) * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}, \tag{6}$$

where $f^{(n)}(t)$ denotes the integer-order derivative and the symbol $*$ denotes the convolution integral.

The definition of a fractional-order derivative of Riemann–Liouville type can also be expressed as

$$D^\alpha f(t) = \left[h_+(t) * \frac{d}{dt} f(t)\right], \tag{7}$$

where $h_+(t)$ is defined as

$$h_+(t) = \begin{cases} \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, & t > 0, \\ 0, & t \le 0. \end{cases} \tag{8}$$

The Fourier transformation of $h_+(t)$ is

$$\mathcal{F}\{h_+(t)\} = \int_{-\infty}^{+\infty} h_+(t) e^{-i\omega t} dt = \int_0^{+\infty} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} e^{-i\omega t} dt. \tag{9}$$

Let $i\omega t = x$, then, $t = x/(i\omega)$. Inserting these into Eq. (9) leads to

$$\begin{aligned} \mathcal{F}\{h_+(t)\} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} \left(\frac{x}{i\omega}\right)^{-\alpha} e^{-x} d\left(\frac{x}{i\omega}\right) = \frac{(i\omega)^{\alpha-1}}{\Gamma(1-\alpha)} \int_0^{+\infty} (x)^{-\alpha} e^{-x} dx \\ &= \frac{(i\omega)^{\alpha-1}}{\Gamma(1-\alpha)} \Gamma(1-\alpha) = (i\omega)^{\alpha-1}. \end{aligned} \tag{10}$$

By the application of Eqs. (5), (6), (8), the Fourier transformation of the fraction-order derivative can be obtained:

$$\mathcal{F}\{D^\alpha f(t)\} = (i\omega)\mathcal{F}\{h_+(t)\}\mathcal{F}\{f(t)\} = (i\omega)^\alpha \mathcal{F}\{f(t)\}. \tag{11}$$

Performing the Fourier transformation of Eq. (3), we obtain the complex modulus of the fractional-order Zenner model

$$\begin{aligned} K^* &= \frac{\bar{\sigma}(\omega)}{\bar{\varepsilon}(\omega)} = \frac{q_0 + q_1 (i\omega)^\alpha}{1 + q_2 (i\omega)^\alpha} \\ &= \frac{q_0 + q_1 q_2 \omega^{2\alpha} + (q_1 + q_0 q_2)\omega^\alpha \cos\frac{\alpha\pi}{2}}{1 + 2q_2\omega^\alpha \cos\frac{\alpha\pi}{2} + q_2^2\omega^{2\alpha}} + \frac{(q_1 - q_0 q_2)\omega^\alpha \sin\frac{\alpha\pi}{2}}{1 + 2q_2\omega^\alpha \cos\frac{\alpha\pi}{2} + q_2^2\omega^{2\alpha}} i, \end{aligned} \tag{12}$$

where $q_0 = \frac{E_1 E_2}{E_1+E_2}$, $q_1 = \frac{E_1^2}{E_1+E_2}(\tau_\varepsilon)^\alpha$, $q_2 = \frac{E_1}{E_1+E_2}(\tau_\sigma)^\alpha$.

It is assumed that the pressure p on a certain point on the foundation surface is proportional to the settlement displacement w at that point, namely,

$$p(x,t) = Kw(x,t) \tag{13}$$

in the Winkler foundation model. The deformation of soil under the action of a beam is discontinuous according to the Winkler foundation model. In order to make up for the shortcomings of the Winkler foundation model, the Pasternak foundation model considers the existence of a shear layer on the basis of the Winkler foundation model. A compressible layer and a shear layer act together to provide a constrained reaction force on the beam, namely,

$$p(x,t) = Kw(x,t) - G_p \cdot \nabla^2 w(x,t), \tag{14}$$

where K is the foundation reaction coefficient, also known as the foundation compressibility coefficient, G_P is the foundation shear coefficient, and $\nabla^2 = \frac{\partial^2}{\partial x^2}$ is the Laplace differential operator. If the compressible layer of the foundation is regarded as viscoelastic, the shear layer of the foundation is still elastic. Then, the supporting force of the fractional viscoelastic Pasternak foundation can be expressed as

$$p(x,t) = \frac{K^*}{H} w(x,t) - G_P \cdot \nabla^2 w(x,t). \tag{15}$$

3 Modified Timoshenko beam mounted on the fraction-order viscoelastic Pasternak foundation

The motion equations of a classic Timoshenko beam are

$$\kappa AG\left(\frac{\partial w}{\partial x} - \varphi\right) = -EI\frac{\partial^2 \varphi}{\partial x^2} + \rho I\frac{\partial^2 \varphi}{\partial t^2}, \tag{16a}$$

$$\kappa AG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x}\right) - p(x,t) = \rho A\frac{\partial^2 w}{\partial t^2}. \tag{16b}$$

The modified Timoshenko beam ignores the moment of inertia due to shear deformation and the motion equations can be modified as:

$$\kappa AG\left(\frac{\partial w}{\partial x} - \varphi\right) = -EI\frac{\partial^2 \varphi}{\partial x^2} + \rho I\frac{\partial^3 w}{\partial x \partial t^2}, \tag{17a}$$

$$\kappa AG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x}\right) - p(x,t) = \rho A\frac{\partial^2 w}{\partial t^2}, \tag{17b}$$

where, κ is the cross section correction coefficient. φ is the rotation angle of the cross section, I is the moment of inertia of the cross section, A is the cross-sectional area, E is the elastic modulus, G is the shear modulus, ρ is the volume density, and $p(x,t)$ is the supporting force of the foundation. After elimination of φ, the motion equations with respect to transverse displacement w can be obtained as

$$EI\frac{\partial^4 w}{\partial x^4} - \rho I\frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{EI}{\kappa AG}\frac{\partial^2}{\partial x^2}\left(p(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}\right) = -p(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}, \tag{18}$$

$$\begin{aligned} &EI\frac{\partial^4 w}{\partial x^4} - \rho I\frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{EI}{\kappa AG}\frac{\partial^2}{\partial x^2}\left(p(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}\right) \\ &+ \frac{\rho I}{\kappa AG}\frac{\partial^2}{\partial t^2}\left(p(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}\right) = -p(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{19}$$

Comparing the equations of motion of the classical Timoshenko beam (corresponding to Eq. (19)) with those of the modified Timoshenko beam (corresponding to Eq. (18)), it is noted that the first three terms on the left of the equals sign are the same. The first term corresponds to the basic theory of Euler beams. The second term is the additional shear stress corresponding to the moment of inertia caused by bending deformation. The third term is the additional shear stress caused by shear deformation. It can be seen that the fundamental difference between the two models is whether the moment of inertia caused by the shear deformation of the beam element is considered. The extra term in Eq. (19) results from replacing the term $\rho I \frac{\partial^2 \varphi}{\partial t^2}$ in Eq. (16a) by the term $\rho I \frac{\partial^3 w}{\partial x \partial t^2}$.

Inserting Eq. (15) into Eqs. (18) and (19) leads to the motion equations of the classic and modified Timoshenko beam mounted on a fraction-order viscoelastic Pasternak foundation:

$$\left(1+\frac{G_P}{\kappa AG}\right)\frac{\partial^4 w}{\partial x^4}-\frac{\rho}{E}\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}-\left(\frac{K^*}{H\kappa AG}+\frac{G_P}{HEI}\right)\frac{\partial^2 w}{\partial x^2}+\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}+\frac{K^*}{HEI}w=0, \tag{20}$$

$$\left(1+\frac{G_P}{\kappa AG}\right)\frac{\partial^4 w}{\partial x^4}+\frac{\rho^2}{\kappa EG}\frac{\partial^4 w}{\partial t^4}-\frac{\rho}{E}\left(1+\frac{E}{\kappa G}+\frac{G_P}{\kappa AG}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}-\left(\frac{K^*}{H\kappa AG}+\frac{G_P}{EI}\right)\frac{\partial^2 w}{\partial x^2}+\frac{\rho A}{EI}\left(1+\frac{K^* I}{H\kappa A^2 G}\right)\frac{\partial^2 w}{\partial t^2}+\frac{K^* w}{HEI}=0. \tag{21}$$

4 Wave solutions of Timoshenko beams on a fractional viscoelastic Pasternak foundation

Assume the wave solution of a beam

$$w(x,t)=A_1 e^{i(kx-\omega t)}, \tag{22}$$

where k is the wavenumber, and A_1 is the amplitude of polarization. Inserting Eq. (22) into Eq. (20) leads to

$$ak^4+\left(b-d\omega^2\right)k^2-s\omega^2+g=0, \tag{23}$$

where $a=(1+\frac{G_P}{\kappa AG})$, $b=(\frac{K^*}{H\kappa AG}+\frac{G_P}{EI})$, $d=\frac{\rho}{E}(1+\frac{E}{\kappa G})$, $s=\frac{\rho A}{EI}$, $g=\frac{K^*}{HEI}$.

From Eq. (23), two groups of complex wavenumbers for the modified Timoshenko beam are obtained as

$$k_1=-k_2=\sqrt{\frac{-(b-d\omega^2)+\sqrt{(b-d\omega^2)^2-4a(g-s\omega^2)}}{2a}}=k_1^r+ik_1^m, \tag{24a}$$

$$k_3=-k_4=\sqrt{\frac{-(b-d\omega^2)-\sqrt{(b-d\omega^2)^2-4a(g-s\omega^2)}}{2a}}=k_3^r+ik_3^m. \tag{24b}$$

The wave solution can be expressed as

$$w(x,t)=\left(C_1 e^{-k_1^m x}e^{ik_1^r x}+C_2 e^{k_1^m x}e^{-ik_1^r x}+C_3 e^{-k_3^m x}e^{ik_3^r x}+C_4 e^{k_3^m x}e^{-ik_3^r x}\right)e^{-i\omega t}. \tag{25}$$

The wave speeds of the two traveling flexural waves are

$$c_1 = \frac{\omega}{k_1^r}, \qquad c_2 = \frac{\omega}{k_3^r}. \tag{26}$$

The real part of the complex wavenumber is actually the wavenumber, while the imaginary part of the complex wavenumber reflects the attenuation properties, namely, the attenuation coefficient associated with distance.

Similarly, the dispersion equation of a classic Timoshenko beam is obtained as

$$ak^4 + \left(b - d_1\omega^2\right)k^2 + m\omega^4 - s_1\omega^2 + g = 0. \tag{27}$$

The complex wavenumbers are

$$k_1 = -k_2 = \sqrt{\frac{-(b - d_1\omega^2) + \sqrt{(b - d_1\omega^2)^2 - 4a(m\omega^4 + g - s_1\omega^2)}}{2a}}, \tag{28a}$$

$$k_3 = -k_4 = \sqrt{\frac{-(b - d_1\omega^2) - \sqrt{(b - d_1\omega^2)^2 - 4a(m\omega^4 + g - s_1\omega^2)}}{2a}}, \tag{28b}$$

where, $d_1 = \frac{\rho}{E}(1 + \frac{E}{\kappa G} + \frac{G_P}{\kappa AG})$, $s_1 = \frac{\rho A}{EI}(1 + \frac{K^* I}{H\kappa A^2 G})$, $m = \frac{\rho^2}{\kappa EG}$.

5 Vibration solution of a Timoshenko beam on a fractional viscoelastic Pasternak foundation

Assume the vibration solution

$$w = W(x)e^{-i\omega t}. \tag{29}$$

Consider the simply supported boundary conditions at both ends, namely,

$$w(x)|_{x=0} = 0, \qquad \left(\frac{\partial^2 w(x)}{\partial x^2}\right)\bigg|_{x=0} = 0 \tag{30a, b}$$

$$w(x)|_{x=l} = 0, \qquad \left(\frac{\partial^2 w(x)}{\partial x^2}\right)\bigg|_{x=l} = 0. \tag{30c, d}$$

Let $W_n(x) = \sin\frac{n\pi}{l}x$ $(n = 1, 2, 3, \ldots)$. In order to reflect the attenuation properties of free vibration of a Timoshenko beam on a viscoelastic foundation, assume the natural frequency is complex valued, i.e., $\omega_n = \omega_n^r + i\omega_n^m$. The real part of the complex-valued natural frequency is actually the natural frequency, while the imaginary part is the attenuation coefficient associated with time. Therefore, the vibration solution can be expressed as

$$w(x,t) = \sin\left(\frac{n\pi}{l}x\right)e^{-i\omega_n t}. \tag{31}$$

Inserting Eq. (31) into Eq. (20) leads to the frequency equation

$$\left[-d\left(\frac{n\pi}{l}\right)^2 - s\right]\omega_n^2 + \left[a\left(\frac{n\pi}{l}\right)^2 + b\right]\left(\frac{n\pi}{l}\right)^2 + g = 0. \tag{32}$$

From Eq. (32), we obtain the natural frequency of a modified Timoshenko beam on a viscoelastic Pasternak foundation

$$\omega_n = \sqrt{\frac{a(\frac{n\pi}{l})^4 + b(\frac{n\pi}{l})^2 + g}{d(\frac{n\pi}{l})^2 + s}}. \tag{33}$$

Similarly, the natural frequency of a classic Timoshenko beam can be obtained as

$$\omega_n = \sqrt{\frac{[d_1(\frac{n\pi}{l})^2 + s_1] \pm \sqrt{[d_1(\frac{n\pi}{l})^2 + s_1]^2 - 4m[a(\frac{n\pi}{l})^4 + b(\frac{n\pi}{l})^2 + g]}}{2m}}. \tag{34}$$

By comparison of the expressions of natural frequency of modified and classic Timoshenko beams, i.e., Eqs. (33) and (34), it is noted that the classic Timoshenko beam has the second frequency spectrum, while the modified Timoshenko beam has only one frequency spectrum. Therefore, the appearance of the second frequency spectrum is due to the fact that the moment of inertia caused by the shear deformation is taken into account. In fact, there are two kinds of vibration mode, i.e., longitudinal shear mode and bending mode, for the hinged–hinged Timoshenko beam. The two vibration modes interfere and result in two distinct spectra of frequency, where the bending mode dominates in the first spectrum of frequency, while the longitudinal shear mode dominates in the second spectrum of frequency. Furthermore, deformations due to shear and bending are of the same phase for the first spectrum but antiphase for the second spectrum. The modified Timoshenko beam model can eliminate the second frequency spectrum. It should be pointed out that only the hinged–hinged boundary condition is considered in the present work. The end-boundary condition can be hinged, clamped, elastic and free and different combinations of them at the two ends. Moreover, the boundary conditions from the foundation soil are also not taken into account. The boundary conditions from the foundation soil are much more complicated and in general, include the contact region, the lift-off region and free soil region, and were studied in Nobili (2012) in detail.

6 Results and discussions

Consider a steel beam whose mechanical and geometric parameters are as follows: cross-sectional area $A = 7.64 \times 10^{-3}$ m^2, elastic modulus $E = 2.06 \times 10^{11}$ N/m^2, shear modulus $G = 0.75 \times 10^{11}$ N/m^2, mass density $\rho = 7.85 \times 10^3$ Kg/m^2, shear correction coefficient $\kappa = 5/6$, moment of inertia $I = 1.579 \times 10^{-4}$ m^2, and thickness of foundation $H = 1$ m. The viscoelastic coefficients of the foundation are: $E_1 = 3.5 \times 10^7$ N/m^2, $E_2 = 2.5 \times 10^7$ N/m^2, $\eta = 2.0 \times 10^4$ N $\cdot$ s/m^2, and the shear coefficient of the foundation is $G_P = 1 \times 10^7$ N/m^2.

6.1 The complex modulus of the fraction-order viscoelastic foundation

It is noted from Fig. 1 that the viscoelastic foundation has different moduli in the lower and higher frequency regions. The modulus is obviously dependent upon the frequency in a special frequency region. Outside of this frequency region, the modulus is nearly independent of the frequency. We can call the special frequency region the frequency-sensitive region. The attenuation coefficient shows a dramatic increase within the frequency-sensitive region and can largely be ignored outside of the frequency-sensitive region. The fraction

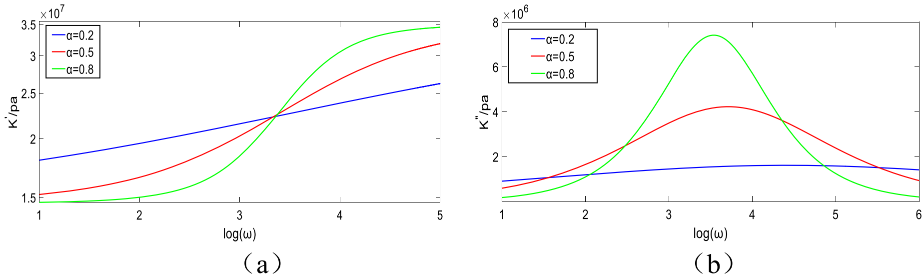

Fig. 1 The real part and imaginary part of the complex modulus of a fractional-order viscoelastic foundation (fractional-order Zenner model) for different fraction orders. (**a**) The real part; (**b**) The imaginary part

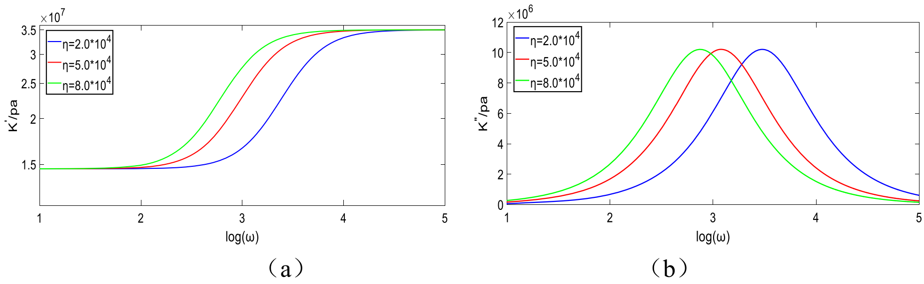

Fig. 2 The real part and imaginary part of the complex modulus of a fractional-order viscoelastic foundation (fractional-order Zenner model) for different viscosity coefficients. (**a**) The real part; (**b**) The imaginary part

order has an obvious influence not only on the amplitude of the modulus and attenuation but also on the range of the frequency-sensitive region. This is why the fraction-order viscoelastic model is better than the integer-order viscoelastic model. Hence, the fraction-order viscoelastic model can model the history-dependent mechanical behavior more elaborately by adjusting the fraction order. The integer-order viscoelastic model is only a limiting case of the fraction-order viscoelastic model and can be recovered by the fraction-order model. From Fig. 2, it is observed that the viscosity coefficient mainly influences the central location of the frequency-sensitive region. Therefore, we will mainly discuss the influences of the fraction order and the viscosity coefficient on the wave propagation and vibration characteristic in the following sections.

6.2 Dispersion and attenuation of flexural waves

There are two traveling flexural waves in the Timoshenko beam. The propagation speed of the first traveling flexural wave is not dependent upon the frequency monotonously, while the propagation speed of the second traveling flexural wave is nearly monotonously dependent upon the frequency. There is a critical frequency existing for the first traveling flexural wave. The propagation speed of the first traveling flexural wave reaches a local peak at the critical frequency. The critical frequency can be estimated by $n\omega = \sqrt{g/s}$. It is noted from Fig. 3 that the fraction order of the viscoelastic model has an obvious influence upon the local peak of the first traveling flexural wave. After the critical frequency, the propagation speed of the second traveling flexural wave becomes clearly dependent upon the fraction order. The influences of the viscosity coefficient are shown in

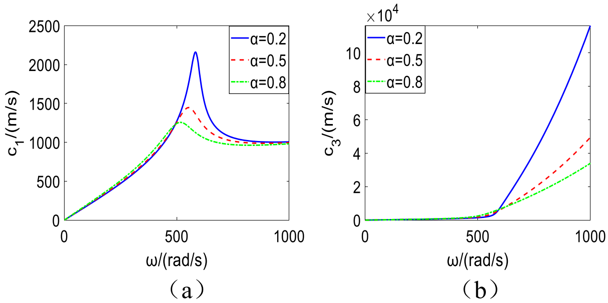

Fig. 3 Dispersion curves of traveling flexural waves in the modified Timoshenko beam mounted on a fraction-order viscoelastic Pasternak foundation with different fractional orders. (**a**) The first traveling wave; (**b**) The second traveling wave

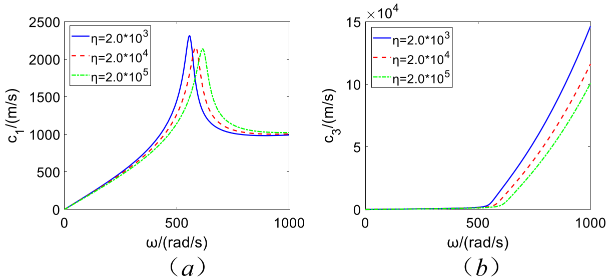

Fig. 4 Dispersion of traveling flexural waves in the modified Timoshenko beam on a fractional-order viscoelastic Pasternak foundation with different viscosity coefficients. (**a**) The first traveling wave; (**b**) The second traveling wave

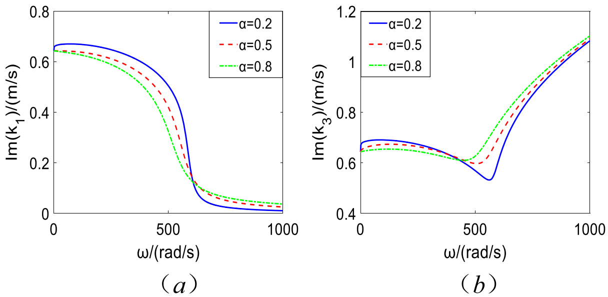

Fig. 5 Attenuation coefficients of traveling flexural waves in the modified Timoshenko beam on a fraction-order viscoelastic Pasternak foundation with different fractional orders. (**a**) The first traveling wave; (**b**) The second traveling wave

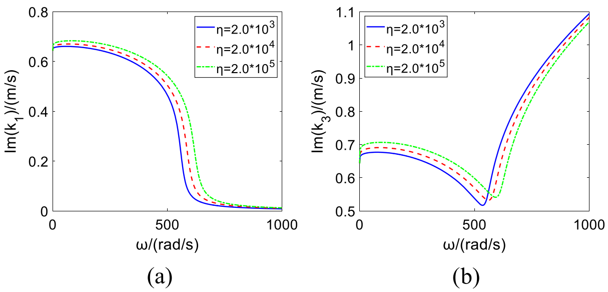

Fig. 6 Attenuation coefficients of traveling flexural waves in the modified Timoshenko beam on a fraction-order viscoelastic Pasternak foundation with different viscosity coefficients. (**a**) The first traveling wave; (**b**) The second traveling wave

Fig. 4. It is noted that the viscosity coefficient mainly affects the location of the local peak. In other words, the critical frequency is mainly determined by the viscosity coefficient. Figures 5 and 6 show the influences of the fraction order and the viscosity coefficient upon the attenuation coefficient. It is noted that the fraction order has different influences compared with the viscosity coefficient. This observation indicates that the

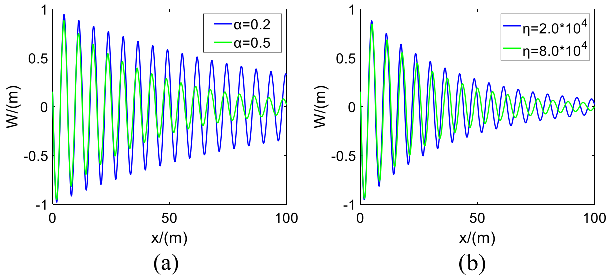

Fig. 7 Waveforms of the first traveling wave in the modified Timoshenko beam mounted on a fractional-order viscoelastic Pasternak foundation with different fractional orders and viscosity coefficients. (**a**) The influences of fractional orders; (**b**) The influences of viscosity coefficients

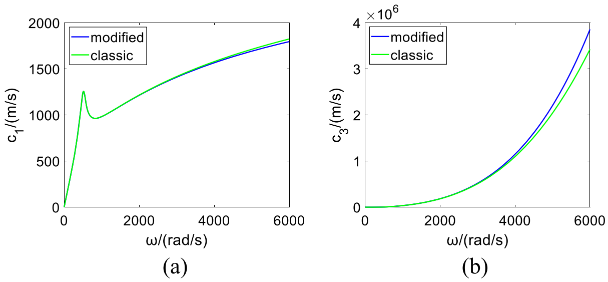

Fig. 8 Comparison of dispersion curves of the classical and modified Timoshenko beams on the fractional-order viscoelastic Pasternak foundations. (**a**) The first traveling wave; (**b**) The second traveling wave

attenuation feature can also be described elaborately by adjusting the fraction order. In order to illustrate the attenuation feature of wave propagation, Fig. 7 shows the waveform of the first traveling flexural wave at different fraction order and viscosity coefficients.

Figure 8 shows the comparison of propagation speeds of two traveling flexural waves in the modified and classic Timoshenko cases. It is noted that the differences are evident only at higher frequencies. At lower frequencies, the propagation speeds predicted by the two beam models are nearly the same. There is a local peak for the first traveling wave for the two beam models. The corresponding critical frequencies are determined by $\omega = \sqrt{g/s}$ for the modified Timoshenko beam and by $\omega = \sqrt{(s_1 + \sqrt{s_1^2 - 4mg})/(2m)}$ for the classic Timoshenko beam.

6.3 Complex natural frequency of a Timoshenko beam

Tables 1 and 2 show the complex-valued natural frequencies of modified and classic Timoshenko beams with simple support at both ends. It is observed that the real part decreases, while the imaginary part increases, when the fraction order increases. This implies that the natural frequency decreases and the attenuation coefficient increases for increasing fraction order. There is a second frequency spectrum existing for the classic Timoshenko beam. It is found that the natural frequencies of the second spectrum are not sensitive to the fraction order but the attenuation coefficients are still sensitive to the fraction order.

Tables 3 and 4 show the influences of viscosity coefficients on the complex-valued natural frequency of modified and classic Timoshenko beams with simple supports at both ends. It is observed that both the natural frequencies and the attenuation coefficients increase when the viscosity coefficients increase. The natural frequencies of the second spectrum are also not sensitive to the viscosity coefficients.

Table 1 Complex-valued natural frequency of simply supported modified Timoshenko beams on the fractional-order viscoelastic Pasternak foundation (rad/s)

Fraction order	First order	Second order	Third order
$\alpha = 0.2$	616.4 + 19.6i	709.3 + 16.9i	946.0 + 12.6i
$\alpha = 0.5$	601.1 + 48.5i	695.7 + 41.6i	935.6 + 30.7i
$\alpha = 0.8$	577.2 + 76.2i	674.6 + 64.8i	919.6 + 47.1i

Table 2 Complex-valued natural frequency of simply supported classic Timoshenko beams on the fractional-order viscoelastic Pasternak foundation (rad/s)

n		α		
		$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$
1	First spectrum	616.4 + 19.7i	601.1 + 48.5i	577.4 + 76.2i
	Second spectrum	1.97e+04 + 1.24e−03i	1.97e+04 + 2.98e−03i	1.97e+04 + 4.50e−03i
2	First spectrum	709.4 + 17.0i	695.8 + 41.6i	674.7 + 64.8i
	Second spectrum	2.00e+04 + 4.67e−03i	2.00e+04 + 1.13e−02i	2.00e+04 + 1.70e−02i
3	First spectrum	946.5 + 12.6i	936.1 + 30.7i	920.0 + 47.1i
	Second spectrum	2.04e+04 + 9.59e−03i	2.04e+04 + 2.31e−02i	2.04e+04 + 3.49e−02i

Table 3 Complex-valued natural frequency of simply supported modified Timoshenko beams on the fractional-order viscoelastic Pasternak foundation (rad/s)

Viscosity coefficient ($N \cdot s/m^2$)	First order	Second order	Third order
$\eta = 2 \times 10^4$	516.1 + 21.8i	624.5 + 17.9i	884.8 + 12.5i
$\eta = 5 \times 10^4$	525.6 + 52.4i	631.8 + 43.3i	889.5 + 30.5i
$\eta = 8 \times 10^4$	541.3 + 78.1i	644.1 + 65.3i	897.6 + 46.4i

Table 4 Complex-valued natural frequency of simply supported classic Timoshenko beams on the fractional-order viscoelastic Pasternak foundation (rad/s)

n		η		
		$\eta = 2 \times 10^4$	$\eta = 5 \times 10^4$	$\eta = 8 \times 10^4$
1	First spectrum	516.1 + 21.8i	525.6 + 52.4i	541.3 + 78.1i
	Second spectrum	1.97e+04 + 1.15e−03i	1.97e+04 + 2.82e−03i	1.97e+04 + 4.32e−03i
2	First spectrum	624.6 + 17.9i	631.9 + 43.3i	644.2 + 65.3i
	Second spectrum	2.00e+04 + 4.35e−03i	2.00e+04 + 1.06e−02i	2.00e+04 + 1.63e−02i
3	First spectrum	885.2 + 12.6i	890.0 + 30.5i	898.0 + 46.4i
	Second spectrum	2.04e+04 + 8.94e−03i	2.04e+04 + 2.18e−02i	2.04e+04 + 3.35e−02i

In order to illuminate the influences of the fraction order and viscosity coefficients on the natural frequency and attenuation coefficient more clearly, Fig. 9 shows the first three natural frequencies at different fraction orders and viscosity coefficients.

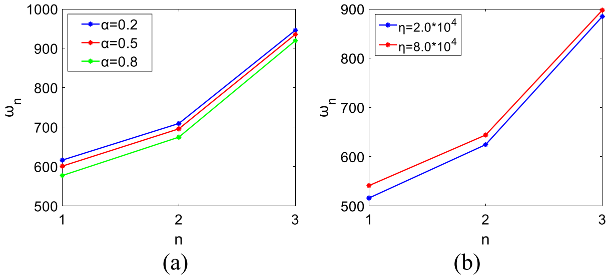

Fig. 9 Natural frequencies of the first three orders of the Timoshenko beam on a fractional viscoelastic Pasternak foundation with different fractional orders and viscosity coefficients. (**a**) Influences of fraction order; (**b**) Influences of viscosity coefficients

7 Conclusions

The rheological behavior of foundation soil is usually very complicated due to the history-dependent feature and we need to introduce many parameters to describe the mechanical behavior of the foundation. The fraction-order viscoelastic model is more flexible than the integer-order viscoelastic model because the definition of the fraction-order derivative includes the convolution integral of time, which is consistent with the history-dependent feature of rheological behavior. The fraction-order Zenner model is used in the present work to study the wave propagation and the free vibration of a Timoshenko beam on the fractional viscoelastic Pasternak foundation. Numerical examples are also provided and parameter studies are performed to show the influences of the fraction order. Based upon the analytic formulation and the numerical results, the following conclusions can be drawn:

(1) There is a frequency-sensitive region for the fraction-order Zenner model. When the frequency falls within the frequency-sensitive region, the imaginary part of the complex modulus increases drastically. The fraction-order viscoelastic foundation will show evident dissipation behavior. Outside of the frequency-sensitive region, the dissipation effects can be ignored.
(2) There are two traveling flexural waves existing in both classic and modified Timoshenko beams. The complex-valued wavenumbers are caused by the dissipation effects of the viscoelastic foundation and the two traveling flexural waves display dispersion and attenuation features.
(3) The dissipation effects of the viscoelastic foundation also result in the complex-valued natural frequency. The real part of the complex natural frequency is the real natural frequency, while the imaginary part is actually the attenuation coefficient associated with time.
(4) There is a second frequency spectrum existing in the classic Timoshenko beam with simple supports at both ends. The second frequency spectrum is caused by the moment of inertial due to shear deformation and can be removed by the modified Timoshenko beam model that ignores the moment of inertial due to shear deformation and retains only the moment of inertial due to bending deformation.
(5) Except for the viscosity coefficient, the fraction order has evident influences upon the dispersion and the attenuation of flexural waves and the complex-valued natural frequency. This indicates that the fraction-order viscoelastic foundation model is more flexible than the integer-order model to describe the viscoelastic behavior.

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Data Availability All data included in this study are available upon request by contact with the corresponding author.

Declarations

Declaration of Competing Interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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