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Micromechanically Established Constitutive Equations for Multiphase Materials with Viscoelastic–Viscoplastic Phases

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Abstract. A model for viscoelastic–viscoplastic solids is incorporated in a micromechanical analysis of composites with periodic microstructures in order to establish closed-form coupled constitutive relations for viscoelastic–viscoplastic multiphase materials. This is achieved by employing the homogenization technique for the establishment of concentration tensors that relate the local elastic and inelastic fields to the externally applied loading. The resulting constitutive equations are sufficiently general such that viscoelastic, viscoplastic and perfectly elastic phases are obtained as special cases by a proper selection of the material parameters the phase. Results show that the viscoelastic and viscoplastic mechanisms have significant effect on the global stress-strain, relaxation and creep behavior of the composite, and that its response is strongly rate-dependent in the reversible and irreversible regimes.

Key words: coupled viscoelastoplasticity, homogenization, micromechanics, multiphase materials, unified viscoplasticity, viscoelasticity, viscoplasticity

1. Introduction

Although there is great amount of literature concerning the behavior of viscoelastic and viscoplastic materials, models that deal with these combined two effects are few. As is discussed by Frank and Brockman (2001), thermoplastics in the glassy region exhibit both viscoelastic (reversible) and viscoplastic (irreversible) response. At low strains these polymers exhibit linear or nearly linear viscoelastic behavior, but at higher strains the viscoelastic response becomes nonlinear. After yielding, the response becomes irreversible exhibiting viscoplastic behavior. In particular, these authors examined the response of polycarbonate which exhibits both viscoelastic followed by viscoplastic behavior with rate dependency. Saleeb et al. (2001) presented an extensive investigation of the behavior at elevated temperature of a titanium alloy (TIMETAL 21S), which is utilized as a matrix in titanium matrix composites. This material exhibits both reversible and irreversible behavior with pronounced rate dependency in each region. At low strains the response of this metal is reversible and time-dependent, but at higher strains its response becomes irreversible and time-dependent as well, exhibiting both viscoelastic and viscoplastic behavior. More recently, Saleeb and Arnold (2004) extended their analysis by considering the hardening saturation of the viscoelastoplastic material at sufficiently large strains and its optimal parametric characterization.

Both Frank and Brockman (2001) and Saleeb et al. (2001) present multiaxial modeling of the coupled reversible and irreversible effects in viscoelastic/viscoplastic (VE/VP) materials. This allows the modeling of a wide spectrum of material responses under various loading conditions.

Frank and Brockman (2001) based their VE/VP model on the concept of intermolecular cooperativity which states that the meshing between polymer chain segments limits the rate at which the segments can move under the influence of applied load. Certain types of motion can occur in small regions and have short relaxation time. Other types require motion of large regions and have a long relaxation time. The total strain is decomposed into VE and VP components every one of which is different in the various domains. The nonlinear viscoelastic mechanism is modeled by hereditary representation, and nonlinearity in the relaxation functions is introduced through the concept of reduced time. The irreversible effects are represented by the unified viscoplasticity theory of Bodner and Partom (1972) equations. The two mechanisms are coupled and include the effects of hydrostatic pressure, strain rate and isotropic strain hardening.

Saleeb et al. (2001) based their modeling of VE/VP materials on the introduction of Gibbs potential function which is decomposed into reversible and irreversible portions, both of which depend on external and internal state variables. The total strain is also decomposed into VE and VP components. In its final form, the reversible effects are modeled by a hereditary representation with several relaxation times, whereas the irreversible effects are based on a single viscoplastic mechanism that includes nonlinear kinematic hardening.

Due to the numerous number of material parameters involved in the VE/VP constitutive equations, Frank and Brockman (2001), Saleeb et al. (2001) and Saleeb and Arnold (2004) employed nonlinear optimization techniques to estimate these parameters. These estimates are based on objective functions that minimize the difference between measured and the correlated responses that are generated under various types of loading.

Micromechanical prediction of the response of rate-dependent inelastic composites have been considered by several investigators as discussed below. The homogenization theory has been employed by Wu and Ohno (1999) to investigate the time-dependent field in viscoplastic composites with periodic microstructure. The analysis of the resulting repeating unit cell of the composite was performed in conjunction with the finite element method. Results for the transverse creep of a continuously reinforced metal matrix composite are presented. Viscoplastic composite materials have been analyzed by Fish and Shek (1998) by employing the homogenization technique in conjunction with their eigenstrain approach according to which all inelastic strains are regarded as eigenstrains in elastic body. The finite element method is used to carry out the computations. The viscous effects were represented either by the power-law or by the Bodner–Partom (Bodner, 2002) unified viscoplastic model. The homogenization method has been also used by Van der Sluis et al. (1999, 2000) to establish the response of elastic–viscoplastic composites, in conjunction with the finite element technique. The Perzyna's viscoplastic constitutive equations (Perzyna, 1966) have been used to represent the inelastic phase behavior.

Walker et al. (1994) and Fotiu and Nemat-Nasser (1996) investigated rate-dependent elastoplastic composites with periodic microstructure by considering the repeating unit cell which was discretized and analyzed by the development of multiple Fourier series in order to obtain the Green's function. The latter investigators employed several strategies in order to reduce the numerical computational effort.

The transformation field analysis, which is based on the assumption of a piecewise uniform strain field, has been developed by Dvorak (1992). This approach has been generalized by Chaboche et al. (2001) and applied to investigate, in conjunction with Mori and Tanaka (1973) procedure, the behavior of viscoplastic metal matrix composites. Damage effects are included and the solution is obtained by the finite element scheme.

A different approach for the analysis elastic–viscoplastic composites has been followed by Li and Weng (1998). It is based on the transition from elasticity to viscoelasticity, in conjunction with the correspondence principle, and then to viscoplasticity by adopting a secant-viscosity approach. The Mori and Tanaka (1973) micromechanical model has been used in this investigation, and results were given in the form of time-dependent creep response of a particulate composite.

Another different approach for the analysis of nonlinear composites has been presented by Ponte Castañeda (1996). This investigator proposed a second-order method which has the capability of generating nonlinear estimates that are exact to second order in the contrast between the phases. This method has been applied to study the response of two-phase composites with power-law constitutive behavior.

Finally, by using variational principles, Talbot and Willis (1996) established bounds for the overall response of a certain class of nonlinear two-phase composites. This approach has been applied on a composite in which every constituent is isotropic, incompressible and is described by a power-law relation between the equivalent stress and the equivalent strain rate.

A micromechanical model, referred to as "high-fidelity generalized method of cells" (HFGMC), that is based on the homogenization procedure for periodic multiphase materials has been recently developed for the prediction of the effective thermoelastic moduli of unidirectional composites (Aboudi et al., 2001). The predicted moduli have been shown to be in excellent agreement with several finite element solutions. In addition, the accuracy of the local field which is predicted by this micromechanical theory, was demonstrated in Aboudi et al. (2001) by comparisons with elasticity solution for an inclusion in an infinite matrix. This micromechanical theory has also been employed by Aboudi (2001) to predict the effective moduli of electro-magneto-thermo-elastic composites, and extensive comparisons with the results of Li and Dunn (1998) exhibit excellent agreement.

More recently, the new micromechanical theory has been extended by Aboudi et al. (2002, 2003) to include inelastic behavior of the constituents, and by Bednarcyk et al. (2004) for the incorporation of the effect of imperfect bonding between the phases. The reliability and accuracy of the predicted inelastic response have been verified by performing extensive comparisons with analytical and finite element solutions under various circumstances (e.g., composites under normal, axial shear and thermal loadings) in the presence and absence of inelastic effects. A recent review by Aboudi (2004) summarizes this micromechanical theory (and its predecessor: the "generalized method of cells" (GMC)) and its applications under various circumstances. It should be mentioned that this micromechanical theory has been incorporated by NASA Glenn Research Center into an extensive micromechanics analysis code referred to as MAC/GMC which has many user-friendly features and significant flexibility (see Bednarcyk and Arnold (2002) for the most recent version of its user guides).

In the present paper the aforementioned micromechanical analysis is extended and applied to establish the effective constitutive equations of multiphase materials in which any one of the phases can behave as VE/VP material. This has been accomplished by adopting the VE/VP model of Frank and Brockman (2001) in which VE, VP and perfectly elastic materials can be obtained merely as special cases by a proper choice of the material's parameters. The micromechanical analysis is based on the homogenization technique of composite materials with periodic microstructures. The analysis proceeds by identifying a repeating unit cell which is descritized into several subcells. By imposing the equilibrium equations and the continuity of displacement and tractions as well as the periodic boundary conditions, the concentration tensors of the multiphase composite (which relate the local field to the externally applied one) are generated. The latter establish the global inelastic constitutive relations of the multiphase material. The resulting constitutive equations predict, in particular, the behavior of a VE/VP matrix reinforced by unidirectional fibers. As a result, these equations model the behavior of anisotropic VE/VP which forms a further generalization of the monolithic VE/VP isotropic model. The resulting micromechanical analysis is comprehensive since it can handle various types of phases all of which form the multiphase material.

Results are presented at two rates of loading in order to illustrate the VE/VP behavior of a monolithic (unreinforced) material by comparison with the response of a VE material (in which the VP effects have been neglected), and with the response of a VP material (in which the VE effects have been neglected). The response to monotonic and cyclic loadings are shown, as well as relaxation and creep curves. Similarly, response curves are shown for a VE/VP matrix reinforced by elastic boron fibers that is subjected to transverse loading (perpendicular to the fiber direction), and to axial shear loading. Relaxation and creep curves for the composite and the unreinforced matrix are exhibited under various circumstances.

2. Viscoelastic–Viscoplastic Constitutive Relations

The modeling of viscoelastic–viscoplastic (VE/VP) materials that has been presented by Frank and Brockman (2001) (which is briefly summarized here) is based on the assumption that the distribution of domain sizes surrounding a particular point in a medium are characterized by $N+1$ discrete ranges. The stress and strain states in the vicinity of a particular point are assumed to vary gradually enough that all sizes of domains about that point can be considered to have the same state.

The stress tensor $\boldsymbol{\sigma}_{(k)}$ in the k-th domain ($k = 0, \ldots, N$) is decomposed into deviatoric and dilatational parts as follows:

$$\boldsymbol{\sigma}_{(k)} = \hat{\boldsymbol{\sigma}}_{(k)} + p_{(k)}\mathbf{I} \tag{1}$$

where $\mathbf{I}$ is the unit 2nd order tensor and the mean normal stress is given by

$$p_{(k)} = \frac{1}{3}\,\mathrm{trace}\,[\boldsymbol{\sigma}_{(k)}] \tag{2}$$

The total stress at a point is given by the sum of contributions from each of the $N+1$ domains:

$$\boldsymbol{\sigma} = \sum_{k=0}^{N} \boldsymbol{\sigma}_{(k)} \tag{3}$$

The strain tensor $\boldsymbol{\epsilon}$ is the same for each domain and it can be decomposed into deviatoric and dilatational parts as follows:

$$\boldsymbol{\epsilon} = \hat{\boldsymbol{\epsilon}} + \frac{1}{3}e\mathbf{I} \tag{4}$$

where the dilatation is given by

$$e = \mathrm{trace}[\boldsymbol{\epsilon}] \tag{5}$$

It is assumed that at the domain level the strain tensor can be divided into VE and VP portions:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{ve}}_{(k)} + \boldsymbol{\epsilon}^{\mathrm{vp}}_{(k)} \tag{6}$$

for any $k = 0, \ldots, N$. It should be noted that since the viscoplastic strain is incompressible, it follows that:

$$\hat{\epsilon} = \hat{\epsilon}^{\text{ve}}_{(k)} + \hat{\epsilon}^{\text{vp}}_{(k)} \tag{7}$$

Frank and Brockman (2001) present the nonlinear viscoelastic relations in the k-th domain at time t as follows:

$$p_{(k)}(t) = \int_{-\infty}^{t} K_{(k)}[t'(t) - t'(\xi)] \frac{\partial}{\partial \xi} e(\xi)\, d\xi \tag{8}$$

$$\hat{\boldsymbol{\sigma}}_{(k)}(t) = 2 \int_{-\infty}^{t} G_{(k)}[t'(t) - t'(\xi)] \frac{\partial}{\partial \xi} \hat{\epsilon}^{\text{ve}}_{(k)}(\xi)\, d\xi \tag{9}$$

where $K_{(k)}$ and $G_{(k)}$ are the bulk and shear relaxation functions, respectively, and the nonlinearity in the relaxation functions is introduced through the reduced time function

$$t'(t) = \int_{0}^{t} \frac{dx}{\varphi_{(k)}} \tag{10}$$

where $\varphi_{(k)}$ is a non-negative shift function that depends on the stress and strain in the domain.

In order to derive the micromechanical analysis (as presented in the following section), we need to extract the instantaneous response from Equations (8)–(9) by defining the following relaxation functions:

$$\phi_{(k)}(t) = 1 - \frac{K_{(k)}(t)}{K_{(k)}(0)} \tag{11}$$

$$\psi_{(k)}(t) = 1 - \frac{G_{(k)}(t)}{G_{(k)}(0)} \tag{12}$$

where $\phi_{(k)}(t) = 0$ and $\psi_{(k)}(t) = 0$ for $t \leq 0$. Consequently, the constitutive relations (8)–(9) take the form

$$p_{(k)}(t) = K_{(k)}(0) e(t) - p^{I}_{(k)}(t) \tag{13}$$

$$\hat{\boldsymbol{\sigma}}_{(k)}(t) = 2G_{(k)}(0)\hat{\epsilon}(t) - \hat{\boldsymbol{\sigma}}^{I}_{(k)}(t) \tag{14}$$

where the inelastic contributions are given by

$$p^{I}_{(k)}(t) = K_{(k)}(0) \int_{0}^{t} \phi_{(k)}[t'(t) - t'(\xi)] \frac{\partial}{\partial \xi} e(\xi)\, d\xi \tag{15}$$

$$\hat{\boldsymbol{\sigma}}^{I}_{(k)}(t) = 2G_{(k)}(0) \left[\epsilon^{\text{vp}}_{(k)}(t) + \int_{0}^{t} \psi_{(k)}[t'(t) - t'(\xi)] \frac{\partial}{\partial \xi} \hat{\epsilon}^{\text{ve}}_{(k)}(\xi)\, d\xi \right] \tag{16}$$

such that $p^{I}_{(k)}(0) = \hat{\boldsymbol{\sigma}}^{I}_{(k)}(0) = 0$.

Let the bulk and shear relaxation functions be represented in the form of exponents in time:

$$K_{(k)}(t) = K_{(k)} \exp[-t/\tau_k] \tag{17}$$

$$G_{(k)}(t) = G_{(k)}\, g_{(k)}\big(J_{2(k)}, Z_{(k)}\big) \exp[-t/\tau_k] \tag{18}$$

where $K_{(k)}$, $G_{(k)}$, and τ_k are material parameters, and the function $g_{(k)}$ introduces the nonlinear VE effects. It is given, according to Frank and Brockman (2001), by

$$g_{(k)}\big(J_{2(k)}, Z_{(k)}\big) = 1 - C_g \frac{3J_{2(k)}}{\big[Z_{(k)} + C_w W_{p(k)}\big]^2} \tag{19}$$

where $Z_{(k)}$ is the flow resistance of the material, $W_{p(k)}$ is the plastic work, C_g, C_w are material constants, and $J_{2(k)} = \frac{1}{2}\hat{\boldsymbol{\sigma}}_{(k)}\hat{\boldsymbol{\sigma}}_{(k)}$ is the second invariant of the deviatoric stress. The constants τ_k form the relaxation time of the k-th domain with $\tau_0 = 0$.

The shift function $\varphi_{(k)}$ in Equation (10) has been chosen as follows:

$$\varphi_{(k)} = \exp\left[b\left(\frac{1}{f_{(k)}(t)} - \frac{1}{f_0}\right)\right] \tag{20}$$

with

$$f_{(k)}(t) = f_0 + C_v \frac{3J_{2(k)}}{\big[Z_{(k)} + C_w W_{p(k)}\big]^2} \tag{21}$$

where b, f_0, and C_v are material constants.

Frank and Brockman (2001) used the Bodner and Partom (1972) power-law model to represent the viscoplastic effects in the material. Accordingly, the rate of the viscoplastic strain is given as follows:

$$\dot{\epsilon}^{\mathrm{vp}}_{(k)}(t) = \frac{D_0}{\sqrt{J_{2(k)}}}\left[\frac{3J_{2(k)}}{Z^2_{(k)}}\right]^n \hat{\boldsymbol{\sigma}}_{(k)} \tag{22}$$

where D_0 and n are material constants. The power-law given by Equation (22) models a viscoplastic material with isotropic hardening. However, it is possible to generalize this law by introducing an exponential law with directional hardening in order to account for the Bauschinger effect (Bodner, 2002).

To accommodate the effect of pressure on the flow resistance, the following relation is employed:

$$Z_{(k)} = Z'_{(k)} \exp[-p_{(k)}/P_{0(k)}] \tag{23}$$

with

$$P_{0(k)} = P_0 \frac{K_{(k)}}{\sum_{k=0}^{N} K_{(k)}} \tag{24}$$

where P_0 is a parameter. The evolution equation of $Z'_{(k)}$ is taken as

$$\dot{Z}'_{(k)}(t) = m\left[\frac{Z'_{(k)}(t) - (1-\alpha)Z_{0(k)}}{Z_{0(k)}}\right]\dot{W}_{p(k)} \tag{25}$$

where m, α are additional parameters,

$$Z_{0(k)} = Z_0 \frac{G_{(k)}}{\sum_{k=0}^{N} G_{(k)}} \tag{26}$$

and Z_0 is a constant.

The rate of plastic work is determined from

$$\dot{W}_{p(k)} = \boldsymbol{\sigma}_{(k)}\dot{\boldsymbol{\epsilon}}^{\mathrm{vp}}_{(k)} \tag{27}$$

Extensive discussion of the viscoelastic and viscoplastic mechanisms and the physical interpretation of the aforementioned parameters is given by Frank and Brockman (2001).

3. The Recursive Form of the Constitutive Relations

In order to compute the field quantities at a given time $t + \Delta t$ (Δt being a time increment), it appears from Equations (15)–(16) that it is necessary to store all their values from $t = 0$ to t. However, recursive formulas are developed herein that necessitate the storage of some field quantities just at time t.

From Equation (10) we have

$$t'(t + \Delta t) = t'(t) + \theta_{(k)} \tag{28}$$

where

$$\theta_{(k)} = \int_t^{t+\Delta t} \exp\left[b\left(\frac{1}{f_0} - \frac{1}{f_{(k)}(x)}\right)\right]dx \tag{29}$$

Equation (15) provides in conjunction with (11) and (17)

$$p^I_{(k)}(t + \Delta t) = K_{(k)}e(t + \Delta t) - I_{(k)}(t + \Delta t) \tag{30}$$

where

$$I_{(k)}(t+\Delta t) = K_{(k)} \int_0^{t+\Delta t} \exp\left[-\frac{t'(t+\Delta t)-t'(\xi)}{\tau_k}\right] \dot{e}(\xi)\, d\xi$$
$$= \exp\left(-\frac{\theta_{(k)}}{\tau_k}\right) I_{(k)}(t) + K_{(k)} \int_t^{t+\Delta t}$$
$$\exp\left[-\frac{t'(t)-t'(\xi)+\theta_{(k)}}{\tau_k}\right] \dot{e}(\xi)\, d\xi \tag{31}$$

The integral on the right-hand side of (31) can be approximated as follows:

$$\int_t^{t+\Delta t} \exp\left[-\frac{t'(t)-t'(\xi)+\theta_{(k)}}{\tau_k}\right] \dot{e}(\xi)\, d\xi$$
$$\approx \frac{1}{2}[e(t+\Delta t)-e(t)]\left[1+\exp\left(-\frac{\theta_{(k)}}{\tau_k}\right)\right] \tag{32}$$

Consequently, the following recursive formula can be established:

$$I_{(k)}(t+\Delta t) = \exp\left(-\frac{\theta_{(k)}}{\tau_k}\right) I_{(k)}(t)$$
$$+\frac{1}{2} K_{(k)}[e(t+\Delta t)-e(t)]\left[1+\exp\left(-\frac{\theta_{(k)}}{\tau_k}\right)\right] \tag{33}$$

Similarly, Equation (16) provides in conjunction with (12) and (18)

$$\hat{\boldsymbol{\sigma}}^I_{(k)}(t+\Delta t) = 2G_{(k)}\hat{\boldsymbol{\epsilon}}(t+\Delta t) - \mathbf{J}_{(k)}(t+\Delta t) \tag{34}$$

where

$$\mathbf{J}_{(k)}(t+\Delta t) = 2G_{(k)} \int_0^{t+\Delta t} g_{(k)} \exp\left[-\frac{t'(t+\Delta t)-t'(\xi)}{\tau_k}\right] \dot{\hat{\boldsymbol{\epsilon}}}^{\mathrm{ve}}_{(k)}(\xi)\, d\xi \tag{35}$$

The recursive formula for $\mathbf{J}_{(k)}$ is given by

$$\mathbf{J}_{(k)}(t+\Delta t) = \exp\left(-\frac{\theta_{(k)}}{\tau_k}\right) \mathbf{J}_{(k)}(t)$$
$$+ G_{(k)}\, g_{(k)}\left[\hat{\boldsymbol{\epsilon}}^{\mathrm{ve}}_{(k)}(t+\Delta t) - \hat{\boldsymbol{\epsilon}}^{\mathrm{ve}}_{(k)}(t)\right]\left[1+\exp\left(-\frac{\theta_{(k)}}{\tau_k}\right)\right] \tag{36}$$

Once $I_{(k)}(t+\Delta t)$ and $\mathbf{J}_{(k)}(t+\Delta t)$ have been determined at time $t+\Delta t$, the inelastic pressure $p^I_{(k)}$ and the inelastic stress deviator $\hat{\boldsymbol{\sigma}}^I_{(k)}$ can be readily obtained from Equations (30) and (34), respectively. The latter quantities determine $p_{(k)}$ and $\hat{\boldsymbol{\sigma}}_{(k)}$ at this time by employing Equations (13)–(14). The stress $\boldsymbol{\sigma}_{(k)}(t+\Delta t)$ is determined from Equation (1). Finally, the total stress $\boldsymbol{\sigma}(t+\Delta t)$ is computed from Equation (3).

The above incremental procedure establishes the constitutive relation of the VE/VP material at any time. These relations can be presented at time t in the compact form

$$\boldsymbol{\sigma}(t) = \lambda\, e(t)\, \mathbf{I} + 2G\boldsymbol{\epsilon}(t) - \boldsymbol{\sigma}^I(t) \tag{37}$$

where

$$\lambda = K - \frac{2}{3}G, \quad K = \sum_{k=0}^{N} K_{(k)}, \quad G = \sum_{k=0}^{N} G_{(k)} \tag{38}$$

and

$$\boldsymbol{\sigma}^I(t) = \sum_{k=0}^{N} \left[\hat{\boldsymbol{\sigma}}^I_{(k)}(t) + p^I_{(k)}(t)\mathbf{I}\right] \tag{39}$$

which represents the total inelastic stress, with $\boldsymbol{\sigma}^I(0) = 0$. The special cases of modeling VE, VP and perfectly elastic materials can be obtained from Equation (37) by a proper selection of the material constants.

In the presence/absence of viscoplasticity and for very fast loading the constitutive relation (37) describes a viscoplastic/elastic material whose stiffness is given by λ and G, whereas in the limit of long time it describes a viscoplastic/elastic material whose stiffness is given by $\lambda_{(0)}$ and $G_{(0)}$, where $\lambda_{(0)} = K_{(0)} - 2G_{(0)}/3$.

It can be easily shown that in the special case of $g_{(k)} = 1$, $\varphi_{(k)} = 1$ and the assumption that the viscoplastic strain tensor is the same in all regions, Equation (37) can be represented in the form

$$\boldsymbol{\sigma}(t) = \lambda_{(0)}\, e(t)\, \mathbf{I} + 2G_{(0)}\left[\boldsymbol{\epsilon}(t) - \boldsymbol{\epsilon}^{\mathrm{vp}}(t)\right] + \sum_{k=1}^{N} \left[I_{(k)}(t)\mathbf{I} + \mathbf{J}_{(k)}(t)\right]$$

which coincides in this special case with Equations (22) and (25) of Saleeb et al. (2001).

4. Homogenization of Periodic Composites

Consider a multiphase composite in which the microstructures are distributed periodically in the plane $x_2 - x_3$ that is given with respect to the global coordinates (x_2, x_3), see Figure 1. In the framework of the homogenization method the displacement is asymptotically expanded as follows:

$$\mathbf{u}(\mathbf{x}, \mathbf{y}) = \mathbf{u}_0(\mathbf{x}, \mathbf{y}) + \delta\, \mathbf{u}_1(\mathbf{x}, \mathbf{y}) + \cdots \tag{40}$$

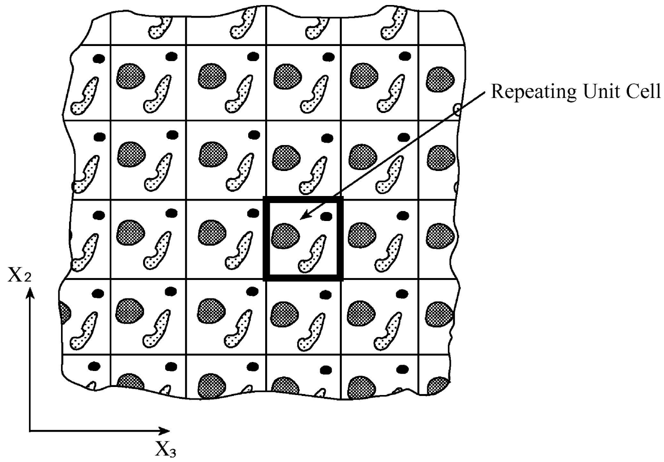

Figure 1. A multiphase composite with a periodic microstructure in the $x_2 - x_3$ plane characterized by a repeating unit cell (highlighted).

where $\mathbf{x} = (x_1, x_2, x_3)$ are the macroscopic (global) coordinate system, and $\mathbf{y} = (y_1, y_2, y_3)$ are the microscopic (local) coordinates that are defined with respect to the repeating unit cell. The material's periodicity imposes the constraint $\mathbf{u}_\alpha(\mathbf{x}, \mathbf{y}) = \mathbf{u}_\alpha(\mathbf{x}, \mathbf{y} + n_\mathrm{p}\mathbf{d}_\mathrm{p})$ on the different-order terms $\mathbf{u}_\alpha(\alpha = 0, 1, \ldots)$ in Equation (40), where n_p are arbitrary integers and the constant vectors $\mathbf{d}_\mathrm{p}$ characterize the material's periodicity. The size of the unit cell is further assumed to be much smaller than the size of the body so that the relation between the global and local systems is $\mathbf{y} = \mathbf{x}/\delta$ where δ is a small scaling parameter characterizing the size of the unit cell. This implies that a movement of order unity on the local scale corresponds to a very small movement on the global scale.

Due to the change of coordinates from the global to the local systems the following relation must be employed in evaluating the derivative of a field quantity

$$\frac{\partial}{\partial x_i} \rightarrow \frac{\partial}{\partial x_i} + \frac{1}{\delta}\frac{\partial}{\partial y_i} \tag{41}$$

The quantities $\mathbf{u}_0$ are the displacements in the homogenized region and hence they are not a function of $\mathbf{y}$. Let

$$\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \equiv \bar{\mathbf{u}} \tag{42}$$

and

$$\mathbf{u}_1 \equiv \tilde{\mathbf{u}}(\mathbf{x}, \mathbf{y}) \tag{43}$$

where the latter are the fluctuating displacements which are unknown periodic functions with respect to **y**, and they arise due to the heterogeneity of the medium.

The strain components are determined from the displacement expansion (40) yielding, in conjunction with Equation (41), the following expression

$$\boldsymbol{\epsilon} = \bar{\boldsymbol{\epsilon}}(\mathbf{x}) + \tilde{\boldsymbol{\epsilon}}(\mathbf{x}, \mathbf{y}) + O(\delta) \tag{44}$$

where

$$\bar{\boldsymbol{\epsilon}}(\mathbf{x}) = \frac{1}{2}(\nabla\bar{\mathbf{u}} + \bar{\mathbf{u}}\nabla) \tag{45}$$

and

$$\tilde{\boldsymbol{\epsilon}}(\mathbf{x}) = \frac{1}{2}(\nabla\tilde{\mathbf{u}} + \tilde{\mathbf{u}}\nabla) \tag{46}$$

This shows that the strain components can be represented as a sum of the average strain $\bar{\boldsymbol{\epsilon}}(\mathbf{x})$ in the composite and a fluctuating strain $\tilde{\boldsymbol{\epsilon}}(\mathbf{x}, \mathbf{y})$. It can be easily shown that

$$\frac{1}{V_y}\int \boldsymbol{\epsilon}\, dV_y = \frac{1}{V_y}\int (\bar{\boldsymbol{\epsilon}} + \tilde{\boldsymbol{\epsilon}})\, dV_y = \bar{\boldsymbol{\epsilon}}$$

where V_y is the volume of the repeating unit cell. This follows directly from the periodicity of the fluctuating strain, implying that the average of the fluctuating strain taken over the unit repeating cell vanishes. For a homogeneous material it is obvious that the fluctuating displacements and strains identically vanish.

For a composite that is subjected to homogeneous deformation, one can use Equation (44) to represent the displacement in the form

$$\mathbf{u}(\mathbf{x}, \mathbf{y}) = \bar{\boldsymbol{\epsilon}} \cdot \mathbf{x} + \tilde{\mathbf{u}} + O(\delta^2) \tag{47}$$

where $\bar{\boldsymbol{\epsilon}} \cdot \mathbf{x}$ represents the contribution of the average strain to the total displacement field.

5. Method of Solution

In this section we present a solution methodology of the inelastic field in the repeating unit cell. For two-dimensional multiphase composites, the repeating unit cell extends over $0 \leq y_2 \leq H$, $0 \leq y_3 \leq L$ in terms of the local coordinates (y_2, y_3) as stated above. The microstructure in the $y_2 - y_3$ plane of the composite is modeled by discretizing the cross section of the repeating cell into N_q and N_r cells in the intervals $0 \leq y_2 \leq H$ and $0 \leq y_3 \leq L$, respectively, see Figure 2(a). In addition,

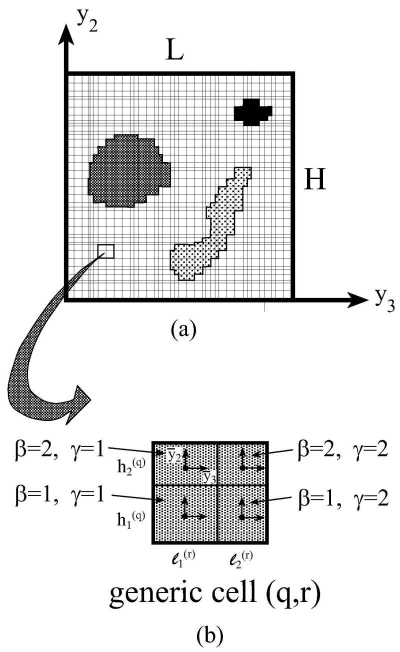

Figure 2. (a) Discretization of the repeating unit cell employed in the present model, (b) generic cell within the repeating unit cell.

every cell consists of four subcells designated by the pair $(\beta\gamma)$ where each index takes the values 1 or 2 which indicate the relative position of the given subcell along the y_2 and y_3 axis, respectively, see Figure 2(b). The indices q and r, whose ranges are $q = 1, 2, \ldots, N_q$ and $r = 1, 2, \ldots, N_r$, identify the cell in the $y_2 - y_3$ plane. The dimensions of the repeating unit cell along the y_2 and y_3 axes are $h_1^{(q)}$, $h_2^{(q)}$ and $l_1^{(r)}$, $l_2^{(r)}$, such that

$$H = \sum_{q=1}^{N_q} \left(h_1^{(q)} + h_2^{(q)}\right)$$

$$L = \sum_{r=1}^{N_r} \left(l_1^{(r)} + l_2^{(r)}\right)$$

Given an applied mechanical loading, an approximate solution for the displacements field is constructed based on volumetric averaging of the field equations together with the imposition of the periodic boundary conditions and continuity conditions in an average sense between the subcells used to characterize the materials' microstructure. This is accomplished by approximating the fluctuating displacements in each subcell using a quadratic expansion in terms of local coordinates

$\bar{\mathbf{y}}^{(\beta)}$, $\bar{\mathbf{y}}^{(\gamma)}$ centered at the subcell's midpoint. A higher order representation of the fluctuating displacement field is necessary in order to capture the local effects created by the mechanical field gradients and the microstructure of the composite.

Equation (37) represents the constitutive relation of the VE/VP material filling subcell $(\beta\gamma)$. Let us write this equation in the following compact form:

$$\boldsymbol{\sigma}^{(\beta\gamma)} = \mathbf{C}^{(\beta\gamma)}\boldsymbol{\epsilon}^{(\beta\gamma)} - \boldsymbol{\sigma}^{I(\beta\gamma)} \tag{48}$$

where $\mathbf{C}^{(\beta\gamma)}$ is the elastic stiffness tensor of the phase filling subcell $(\beta\gamma)$. The elements of $\mathbf{C}^{(\beta\gamma)}$ can be expressed in terms of the Lame' constants $\lambda^{(\beta\gamma)}$ and $G^{(\beta\gamma)}$ given by Equation (38). For a perfectly elastic anisotropic phase the inelastic term in this equation should be omitted, while $\mathbf{C}^{(\beta\gamma)}$ represents its anisotropic stiffness tensor. Thus the present analysis is quite general as it can consider elastic anisotropic, as well as VE, VE/VP, or VP isotropic phases. It should be noted that no summation is implied by repeated Greek letters in the above and henceforth.

The equilibrium equations of the material occupying the subcell $(\beta\gamma)$ in the region $|\bar{y}_2^{(\beta)}| \leq h_\beta^{(q)}/2$, $|\bar{y}_3^{(\gamma)}| \leq l_\gamma^{(r)}/2$ can be written in the form

$$\nabla \cdot \boldsymbol{\sigma}^{(\beta\gamma)} = 0 \tag{49}$$

where $\partial_1 = 0$, $\partial_2 = \partial/\partial\bar{y}_2^{(\beta)}$ and $\partial_3 = \partial/\partial\bar{y}_3^{(\gamma)}$.

As stated before, the fluctuating displacement field in the subcell $(\beta\gamma)$ of the (q, r)th cell is approximated by a second-order expansion in the local coordinates system. Thus (the cell label (q, r) has been omitted)

$$\begin{aligned}\mathbf{u}^{(\beta\gamma)} &= \bar{\boldsymbol{\epsilon}} \cdot \mathbf{x} + \mathbf{W}_{(00)}^{(\beta\gamma)} + \bar{y}_2^{(\beta)}\mathbf{W}_{(10)}^{(\beta\gamma)} + \bar{y}_3^{(\gamma)}\mathbf{W}_{(01)}^{(\beta\gamma)} \\ &\quad + \frac{1}{2}\left(3\bar{y}_2^{(\beta)2} - \frac{h_\beta^{(q)2}}{4}\right)\mathbf{W}_{(20)}^{(\beta\gamma)} + \frac{1}{2}\left(3\bar{y}_3^{(\gamma)2} - \frac{l_\gamma^{(r)2}}{4}\right)\mathbf{W}_{(02)}^{(\beta\gamma)}\end{aligned} \tag{50}$$

where $\mathbf{W}_{(00)}^{(\beta\gamma)}$, which are the fluctuating volume-averaged displacements, and the higher-order terms $\mathbf{W}_{(mn)}^{(\beta\gamma)}$ must be determined from the equilibrium Equations (49) and the periodic boundary conditions that the fluctuating displacements must fulfill, as well as the interfacial continuity conditions of displacements and tractions between subcells. The number of unknowns that describe the fluctuating displacements in the cell (q, r) is 60.

In the perfectly elastic case, the quadratic displacement expansion, Equation (50), produces linear variations in strains and stresses at each point within the subcell. In the presence of inelastic effects, however, a linear strain generated by Equation (50) does not imply the linearity of the stress field due to the path-dependent deformation. Thus, the displacement field microvariables must depend implicitly on the inelastic strain distributions, giving rise to a higher-order stress field than the linear strain field generated from the assumed displacement field representation. In the presence of inelastic effects, this higher-order stress field is represented by a higher-order Legendre polynomial expansion in the local coordinates. Therefore, the strain field generated from the assumed displacement field, and the resulting mechanical field, must also be expressed in terms of Legendre polynomials:

$$\boldsymbol{\epsilon}^{(\beta\gamma)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sqrt{(1+2m)(1+2n)}\boldsymbol{\eta}^{(\beta\gamma)}_{(m,n)}P_m\big(\zeta_2^{(\beta)}\big)P_n\big(\zeta_3^{(\gamma)}\big) \tag{51}$$

$$\boldsymbol{\sigma}^{(\beta\gamma)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sqrt{(1+2m)(1+2n)}\boldsymbol{\tau}^{(\beta\gamma)}_{(m,n)}P_m\big(\zeta_2^{(\beta)}\big)P_n\big(\zeta_3^{(\gamma)}\big) \tag{52}$$

where the non-dimensional variables $\zeta_i^{(\cdot)}$, defined in the interval $-1 \leq \zeta_i^{(\cdot)} \leq 1$, are given in terms of the local subcell coordinates as $\zeta_2^{(\beta)} = \bar{y}_2^{(\beta)}/(h_\beta^{(q)}/2)$, and $\zeta_3^{(\gamma)} = \bar{y}_3^{(\gamma)}/(l_\gamma^{(r)}/2)$. For the given displacement field representation, Equation (50), the upper limits on the summations in Equation (51) become 1. The upper limits on the summations in Equation (52) are chosen so that an accurate representation of the stress field (which depends on the amount of the inelastic flow) is obtained within each subcell. The coefficients $\boldsymbol{\eta}^{(\beta\gamma)}_{(m,n)}$, $\boldsymbol{\tau}^{(\beta\gamma)}_{(m,n)}$ in the above expansions are determined as described below.

The strain coefficients $\boldsymbol{\eta}^{(\beta\gamma)}_{(m,n)}$ in the subcell of cell (q, r) are explicitly determined in terms of the displacement field (50), using the orthogonal properties of Legendre polynomials. The non-zero components are given as follows (omitting (q, r))

$$\eta^{(\beta\gamma)}_{11(0,0)} = \bar{\epsilon}_{11}$$

$$\eta^{(\beta\gamma)}_{22(0,0)} = \bar{\epsilon}_{22} + W^{(\beta\gamma)}_{2(10)}$$

$$\eta^{(\beta\gamma)}_{22(1,0)} = \frac{\sqrt{3}}{2}h_\beta^{(q)}W^{(\beta\gamma)}_{2(20)}$$

$$\eta^{(\beta\gamma)}_{33(0,0)} = \bar{\epsilon}_{33} + W^{(\beta\gamma)}_{3(01)}$$

$$\eta^{(\beta\gamma)}_{33(0,1)} = \frac{\sqrt{3}}{2}l_\gamma^{(r)}W^{(\beta\gamma)}_{3(02)}$$

$$\eta^{(\beta\gamma)}_{23(0,0)} = \bar{\epsilon}_{23} + \frac{1}{2}\big(W^{(\beta\gamma)}_{2(01)} + W^{(\beta\gamma)}_{3(10)}\big)$$

$$\eta^{(\beta\gamma)}_{23(1,0)} = \frac{\sqrt{3}}{4}h_\beta^{(q)}W^{(\beta\gamma)}_{3(20)}$$

$$\eta^{(\beta\gamma)}_{23(0,1)} = \frac{\sqrt{3}}{4}l_\gamma^{(r)}W^{(\beta\gamma)}_{2(02)}$$

$$\eta^{(\beta\gamma)}_{13(0,0)} = \bar{\epsilon}_{13} + \frac{1}{2}W^{(\beta\gamma)}_{1(01)}$$

$$\eta^{(\beta\gamma)}_{13(0,1)} = \frac{\sqrt{3}}{4}l_\gamma^{(r)}W^{(\beta\gamma)}_{1(02)}$$

$$\eta_{12(0,0)}^{(\beta\gamma)} = \bar{\epsilon}_{12} + \frac{1}{2} W_{1(10)}^{(\beta\gamma)}$$
$$\eta_{12(1,0)}^{(\beta\gamma)} = \frac{\sqrt{3}}{4} h_\beta^{(q)} W_{1(20)}^{(\beta\gamma)} \tag{53}$$

The stress coefficients $\boldsymbol{\tau}_{(m,n)}^{(\beta\gamma)}$ in the subcell of cell (q, r) are expressed in terms of the strain coefficients and the unknown inelastic strain distributions, by first substituting the Legendre polynomial representations for the total strain and stress into the constitutive equations, Equation (48), and then utilizing the orthogonality of Legendre polynomials:

$$\boldsymbol{\tau}_{(m,n)}^{(\beta\gamma)} = \mathbf{C}^{(\beta\gamma)} \boldsymbol{\eta}_{(m,n)}^{(\beta\gamma)} - \mathbf{R}_{(m,n)}^{(\beta\gamma)} \tag{54}$$

The $\mathbf{R}_{(m,n)}^{(\beta\gamma)}$ terms represent the inelastic stress distributions calculated in the following manner

$$\mathbf{R}_{(m,n)}^{(\beta\gamma)} = \frac{1}{2}\sqrt{(2m+1)(2n+1)} \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{\sigma}^{I(\beta\gamma)} P_m\big(\zeta_2^{(\beta)}\big) P_n\big(\zeta_3^{(\gamma)}\big) d\zeta_2^{(\beta)} d\zeta_3^{(\gamma)} \tag{55}$$

Note that in both Equations (54) and (55) the cell labeling (q, r) has been omitted.

By imposing on an average basis (integral sense):

(1) the equilibrium equations (49),
(2) the continuity of tractions between subcells and neighboring cells,
(3) the continuity of displacements between subcells and neighboring cells,
(4) the periodic boundary conditions of tractions and displacements, a system of $60N_qN_r$ algebraic equations in the unknown coefficients $W_{i(mn)}^{(\beta\gamma)}$ is obtained, Aboudi et al. (2003). This system of equations is symbolically represented by

$$\mathbf{KU} = \mathbf{f} + \mathbf{g} \tag{56}$$

where the structural stiffness matrix $\mathbf{K}$ contains information on the geometry and properties of the materials within the individual subcells $(\beta\gamma)$ within the cells comprising the multiphase periodic composite. The displacement vector $\mathbf{U}$ contains the unknown displacement coefficients in each subcell, i.e.,

$$\mathbf{U} = \big[\mathbf{U}_{11}^{(11)}, \ldots, \mathbf{U}_{N_qN_r}^{(22)}\big] \tag{57}$$

where in subcell $(\beta\gamma)$ of cell (q, r) these coefficients are

$$\mathbf{U}_{qr}^{(\beta\gamma)} = \big[\mathbf{W}_{(00)}, \mathbf{W}_{(10)}, \mathbf{W}_{(01)}, \mathbf{W}_{(20)}, \mathbf{W}_{(02)}\big]_{qr}^{(\beta\gamma)}$$

The mechanical force **f** contains information on the applied average strain $\bar{\epsilon}$, and the inelastic force vector **g** appearing on the right-hand side of Equation (56) contains the inelastic effects given in terms of the integrals of the inelastic stress distributions that are represented by the coefficients $\mathbf{R}^{(\beta\gamma)}_{(m,n)}$. These integrals depend implicitly on the elements of the displacement coefficient vector **U**, requiring an incremental procedure for the solution of Equation (56) at each point along the loading path.

6. Global Constitutive Relations

Once the solution $\mathbf{U}(t)$ for a given set of average strains $\bar{\epsilon}(t)$ has been established, we can determine, in particular, the average strains $[\boldsymbol{\eta}^{(\beta\gamma)}_{(0,0)}]^{(q,r)}$ in subcell $(\beta\gamma)$ of the cell (q, r) given by (53).

The average stress $[\bar{\boldsymbol{\sigma}}^{(\beta\gamma)}]^{(q,r)}$ in subcell $(\beta\gamma)$ of the cell (q, r) is given by

$$\left[\bar{\boldsymbol{\sigma}}^{(\beta\gamma)}\right]^{(q,r)} = \left[\mathbf{C}^{(\beta\gamma)}\boldsymbol{\eta}^{(\beta\gamma)}_{(0,0)} - \mathbf{R}^{(\beta\gamma)}_{(0,0)}\right]^{(q,r)} \tag{58}$$

The equation relating the average total strains and inelastic strains in the subcells to the macroscopically applied strains is given by Aboudi et al. (2003),

$$\left[\boldsymbol{\eta}^{(\beta\gamma)}_{(0,0)}\right]^{(q,r)} = \left[\mathbf{A}^{(\beta\gamma)}\bar{\epsilon} + \mathbf{D}^{(\beta\gamma)}\right]^{(q,r)} \tag{59}$$

where $[\mathbf{A}^{(\beta\gamma)}]^{(q,r)}$ is the strain concentration tensor of the subcell $(\beta\gamma)$, and $[\mathbf{D}^{(\beta\gamma)}]^{(q,r)}$ is a vector that involves the current inelastic effects in the subcell. In the absence of inelastic effects this vector vanishes, and we can readily determine from (59) the strain concentration tensor $[\mathbf{A}^{(\beta\gamma)}]^{(q,r)}$ by solving the system (56) six consecutive times at everyone of which a single non-zero component of $\bar{\epsilon}$ is imposed.

The VE/VP analysis is performed in conjunction with an incremental procedure according to which the applied loading $\bar{\epsilon}$ is imposed in a stepwise manner. Thus for a given value of applied loading, the average strains $[\boldsymbol{\eta}^{(\beta\gamma)}_{(0,0)}]^{(q,r)}$ in the subcell can be obtained from the solution of Equation (56). Hence from the already known concentration tensors $[\mathbf{A}^{(\beta\gamma)}]^{(q,r)}$, we can determine $[\mathbf{D}^{(\beta\gamma)}]^{(q,r)}$ from (59) at the current loading level.

Substitution of (59) into (58) yields

$$\left[\bar{\boldsymbol{\sigma}}^{(\beta\gamma)}\right]^{(q,r)} = \left[\mathbf{C}^{(\beta\gamma)}(\mathbf{A}^{(\beta\gamma)}\bar{\epsilon} + \mathbf{D}^{(\beta\gamma)}) - \mathbf{R}^{(\beta\gamma)}_{(0,0)}\right]^{(q,r)} \tag{60}$$

The average stress in the multiphase periodic composite is determined from

$$\bar{\boldsymbol{\sigma}} = \frac{1}{HL}\sum_{q=1}^{N_q}\sum_{r=1}^{N_r}\sum_{\beta,\gamma=1}^{2} h^{(q)}_{\beta} l^{(r)}_{\gamma}\left[\bar{\boldsymbol{\sigma}}^{(\beta\gamma)}\right]^{(q,r)} \tag{61}$$

Consequently, Equations (60)–(61) establish the effective closed-form constitutive law of the multiphase VE/VP composite at time t as follows

$$\bar{\boldsymbol{\sigma}}(t) = \mathbf{C}^* \, \bar{\boldsymbol{\epsilon}}(t) - \bar{\boldsymbol{\sigma}}^I(t) \tag{62}$$

where $\mathbf{C}^*$ is the effective elastic stiffness tensor of the composite which is given by

$$\mathbf{C}^* = \frac{1}{HL} \sum_{q=1}^{N_q} \sum_{r=1}^{N_r} \sum_{\beta,\gamma=1}^{2} h_\beta^{(q)} l_\gamma^{(r)} \big[\mathbf{C}^{(\beta\gamma)} \mathbf{A}^{(\beta\gamma)}\big]^{(q,r)} \tag{63}$$

and $\bar{\boldsymbol{\sigma}}^I$ denotes the overall (macroscopic) inelastic stress in the composite. It is given by

$$\bar{\boldsymbol{\sigma}}^I = \frac{-1}{HL} \sum_{q=1}^{N_q} \sum_{r=1}^{N_r} \sum_{\beta,\gamma=1}^{2} h_\beta^{(q)} l_\gamma^{(r)} \big[\mathbf{C}^{(\beta\gamma)} \mathbf{D}^{(\beta\gamma)} - \mathbf{R}_{(0,0)}^{(\beta\gamma)}\big]^{(q,r)} \tag{64}$$

7. Computational Procedure

The incremental procedure for carrying out the micromechanical analysis is described as follows. Suppose that the material filling subcell $(\beta\gamma)$ is VE/VP which is governed by Equation (37). It is assumed that at time t the following field quantities $\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}_{(k)}^{\mathrm{vp}}$, $W_{p(k)}$, $Z'_{(k)}$, $I_{(k)}$, and $\mathbf{J}_{(k)}$, $\boldsymbol{\sigma}_{(k)}^I$, and $\mathbf{R}_{(m,n)}$ are available (superscript $(\beta\gamma)$ has been omitted), and it is required to compute all field variables at time $t + \Delta t$.

At time $t + \Delta t$ either all components of $\bar{\boldsymbol{\epsilon}}$ are known, or some have been determined from the conditions of uniaxial stress loading (say).

1. The system of equations (56) is solved and its solution provides the strain coefficients $\boldsymbol{\eta}_{(m,n)}$ by using Equations (53).
2. The stress coefficients $\boldsymbol{\tau}_{(m,n)}$ are computed from Equation (54).
3. The stress $\boldsymbol{\sigma}$ is determined from (52).
4. The total strain $\boldsymbol{\epsilon}$ is determined from Equation (48).
5. The stress $\boldsymbol{\sigma}_{(k)}$ is computed by employing Equations (13)–(14).
6. The evolution equations (22), (25), and (27) are integrated to calculate $\boldsymbol{\epsilon}_{(k)}^{\mathrm{vp}}$, $Z'_{(k)}$ and $W_{p(k)}$, respectively, at time $t + \Delta t$.
7. Functions $g_{(k)}$, $f_{(k)}$ and $\varphi_{(k)}$ are computed from Equations (19), (21) and (20), respectively.
8. The quantities $I_{(k)}$, and $\mathbf{J}_{(k)}$ are calculated from Equations (33) and (36), respectively, at the new time.
9. Compute $p_{(k)}^I$ and $\hat{\boldsymbol{\sigma}}_{(k)}^I$ by using Equation (30) and (34), respectively.

10. The latter are employed to determine the inelastic stress $\boldsymbol{\sigma}^I$ by using Equation (39).
11. The inelastic stress coefficients $\mathbf{R}_{(m,n)}$ are computed by performing the integration in Equation (55).
12. The micromechanics analysis described in Section 6 is employed to determine the global inelastic stress from Equation (64). This establishes the constitutive equations (62) at the current time.

This procedure can be repeated at the next time step.

8. Applications

In the following, we apply the developed micromechanical constitutive relations (62) to investigate the behavior of a VP/VP matrix reinforced by unidirectional circular elastic fibers oriented in the one-direction. The properties of the inelastic matrix are based on those reported by Frank and Brockman (2001) for polycarbonate at 22 °C and are summarized in Table I, whereas the elastic fibers are characterized by a Young's modulus of 379 GPa and Poisson's ratio 0.1, simulating boron fibers. In all cases, the volume ratio of the fibers was taken as 0.25.

The accuracy and reliability of the computational procedure has been checked and verified in the following cases:

1. When the matrix is linearly viscoelastic and subjected to a uniaxial strain loading, its response (given by Equation (37)) can be expressed by closed-form expressions without the need to employ recursion formulas (33) and (36). It was verified that the response calculated from these closed-form expressions coincides with aforementioned computational procedure in which the recursion formulas are involved.
2. By neglecting the VE effects and employing an exponential evolution law for the viscoplastic strain (Bodner, 2002), the response of the unidirectional composite computed according to the above computational procedure (in which recursion formulas (33) and (36) are operative) coincides with the results based on the procedure that was presented by Aboudi et al. (2003) for inelastic composites, whose accuracy and reliability were extensively checked and verified by comparison with several analytical and finite element solutions.
3. Convergence of the VE/VP response computed with sufficiently small Δt has been verified.

Figure 3 exhibits the response of the VE/VP matrix to a uniaxial stress loading at two values of applied strain rates: $\dot{\epsilon} = 5\,\mathrm{s}^{-1}$ and $0.005\,\mathrm{s}^{-1}$. By neglecting the VP mechanism effects in the matrix, a linear VE matrix is obtained. Similarly, by neglecting the VE effects a VP matrix is obtained. The responses in both these two

Table I. Material constants for the VE/VP matrix.

Property	Value
N	24
D_0	1×10^{11} s^{-1}
n	6.5
m	50
α	9.14×10^{-6}
Z_0	186 MPa
P_0	914 MPa
C_g	0.99
C_w	0.125
f_0	0.2
b	2.64
C_v	0.301
G	1.92 GPa
$G_{(0)}$	0.581 GPa
K	5.96 GPa
$K_{(0)}$	1.64 GPa
τ_k	2×10^{-13}–1×10^{8} s

Note. The detailed values of $G_{(k)}$, $K_{(k)}$ and τ_k are given in Frank and Brockman (2001).

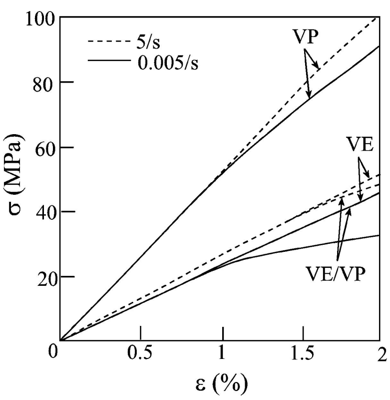

Figure 3. Response of the unreinforced VE/VP matrix to uniaxial stress loading generated at two strain rates. The responses in the special cases of VE and VP materials are shown for comparison.

special cases are included in the figure for comparison. As is expected, the initial slopes of VE/VP and VE curves coincide. The initial slope of the VP curves is the time-independent Young's modulus (5.21 GPa) of the material, and the decreasing response is due merely to the VP effects. It is noted that the rate of loading

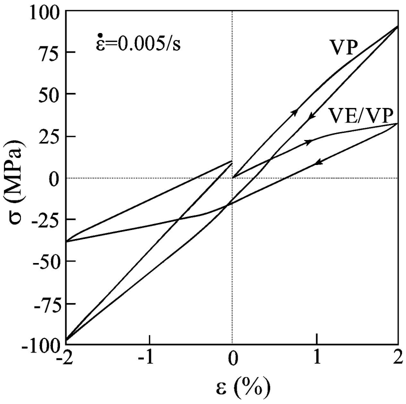

Figure 4. Response of the unreinforced VE/VP and VP matrix to uniaxial stress cyclic loading.

has a significant effect on the response, and the presence/absence of the various mechanisms is pronounced.

Figure 4 presents the response of the matrix to a uniaxial stress cyclic loading. Here the rate of strain loading has been kept constant: $\dot{\epsilon} = 0.005\ \mathrm{s}^{-1}$, and response of the VE/VP matrix is compared to a VP matrix in which the VE effects have been neglected. Again, the presence of the VE effects is seen to be significant.

So far the behavior of the VE/VP matrix has been investigated. In Figure 5 the response of the unidirectional composite is shown when it is uniaxially loaded in the transverse two-direction (perpendicular to the fibers). Just like Figure 3, the strain loading has been applied at two rates, and the effect of including/discarding the VE and VP mechanisms is shown. Due to the present type of loading, the matrix

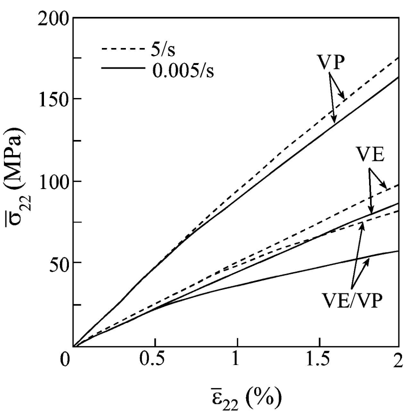

Figure 5. Response of the VE/VP composite to uniaxial stress loading applied in a direction perpendicular to the fiber direction, generated at two strain rates. The responses in the special cases of VE and VP materials are shown for comparison.

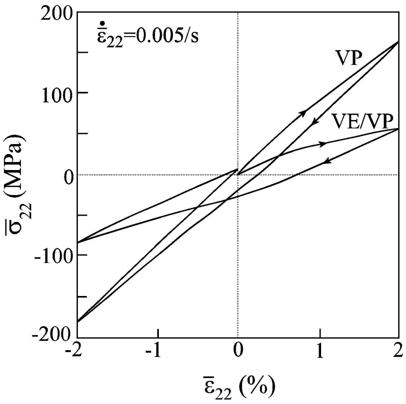

Figure 6. Response of the VE/VP and VP composites to uniaxial stress cyclic loading applied in the transverse 2-direction.

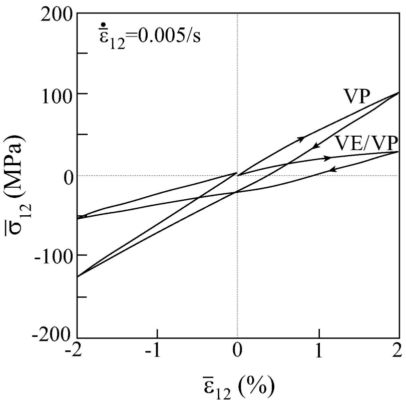

Figure 7. Response of the VE/VP and VP composites to axial shear stress cyclic loading.

dominates the response of the composite. Figures 6 and 7 , just like Figure 4 show the composite response to a uniaxial transverse and axial shear cyclic loadings applied at a constant rate of $0.005\,\mathrm{s}^{-1}$. Both figures contrast the VE/VP and VP responses.

The relaxation curves of the VE/VP matrix and composite are shown in Figure 8. These curves have been generated by the sudden application of a constant strain $\bar{\epsilon}_{22} = 0.005$ in the transverse direction to the fibers. Also shown in this figure are the corresponding relaxation behavior when the VP effects are ignored.

Similarly, Figure 9 exhibits the creep curves of the composite and the unreinforced matrix due to the application of a constant stress $\bar{\sigma}_{22} = 25\,\mathrm{MPa}$ in the transverse direction. Both Figures 8 and 9 indicate that the combined effect of the VE and VP mechanisms is significant. The effect of the fibers in reducing the strain in the matrix is pronounced.

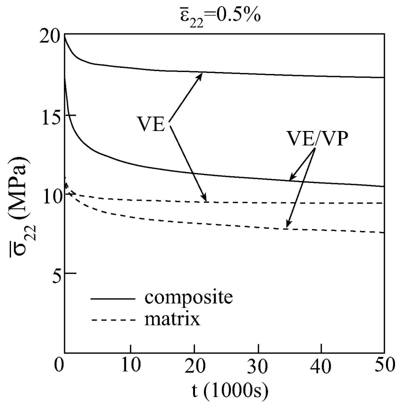

Figure 8. Relaxation curves of the unreinforced VE/VP and VE matrix and unidirectional VE/VP and VE composites caused by the sudden application of a constant strain of 0.5%. In the latter two cases the constant strain is applied in the transverse 2-direction.

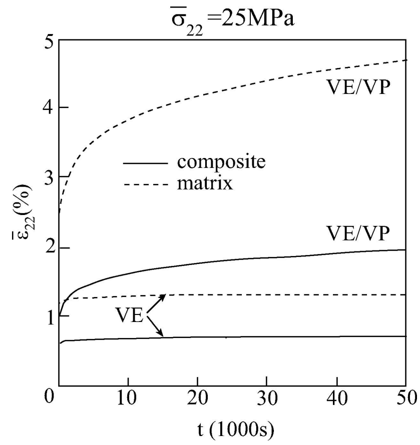

Figure 9. Creep curves of the unreinforced VE/VP and VE matrix and unidirectional VE/VP and VE composites caused by the sudden application of a stress of 25 MPa. In the latter two cases the constant stress is applied in the transverse 2-direction.

9. Conclusions

A VE/VP model that has been recently developed by Frank and Brockman (2001) for polymeric materials, is implemented for the establishment of the global (effective) closed-form constitutive relations of VE/VP multiphase composites in which any constituent can behave as a VE/VP solid. This has been achieved by a micromechanical analysis that is based on the homogenization procedure of composites with periodic microstructure. In the framework of this micromechanics analysis, the concentration tensors that relate the local strain and inelastic strain in a phase to the externally applied strain are established by imposing the equilibrium, interfacial and periodic boundary conditions. The resulting micromechanics analysis is sufficiently general since viscoelastic, viscoplastic and perfectly elastic constituents are obtained as special cases by a proper selection of material parameters. Anisotropic constitutive laws of VE/VP materials can be directly established by employing the proposed micromechanical modeling. The present micromechanical model has been developed for VE/VP composites with continuous reinforcement (doubly-periodic). Extension to the modeling of VE/VP composites with discontinuous reinforcement (triply-periodic) can be performed by adopting the homogenization procedure that has been presented by Aboudi (2001).

It should be mentioned that the periodicity of microstructures assumption is not a limitation, since repeating unit cells with relatively small size (with respect to the heterogeneity) provide reasonable estimates of the behavior of the composite, even if the medium does not have actual geometrical periodicity, Terada et al. (2000).

Alternatively, by adopting the VE/VP material modeling approach of Saleeb et al. (2001) which has been employed to investigate the VE/VP behavior of a metallic alloy, the present micromechanical analysis can be easily modified in order to establish the corresponding global constitutive laws of VE/VP composites.

The derived constitutive relations for VE/VP multiphase materials can be readily employed as a “driver” to analyze VE/VP composite structures.

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