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Towards granular calculus of single-valued neutrosophic functions under granular computing

Nguyen Thi Kim Son[1] · Nguyen Phuong Dong[2] · Le Hoang Son[3] · Hoang Viet Long[4,5]

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Abstract

Neutrosophic theory studies objects whose values vary in the sets of elements and are not true or false, but in between, that can be called by neutral, indeterminate, unclear, vague, ambiguous, incomplete or contradictory quantities. In this paper, we firstly introduce preliminaries on granular calculus and analysis related to single-valued neutrosophic functions. Based on horizontal membership functions approach, we establish some basic arithmetic operations of single-valued neutrosophic numbers, that red allow us to directly introduce the terms of neutrosophic function in usual mathematical formulas. Additionally, we build metrics on the space of single-valued neutrosophic numbers induced from Hamming distance. Then, we define some backgrounds on the limit, derivative and integral of single-valued neutrosophic functions. Finally, in order to demonstrate the usable of our theoretical results, we present some applications to well-known problems arising in engineering such as logistic model, the inverted pendulum system, Mass - Spring - Damper model.

Keywords Triangular neutrosophic numbers · Horizontal membership functions · Granular computing · Single-valued neutrosophic functions

✉ Hoang Viet Long
hoangvietlong@tdtu.edu.vn

Nguyen Thi Kim Son
ntkson@daihocthudo.edu.vn

Nguyen Phuong Dong
nguyenphuongdong@hpu2.edu.vn

Le Hoang Son
sonlh@vnu.edu.vn

1 Faculty of Natural Science, Hanoi Metropolitan University, Hanoi, Vietnam

2 Department of Mathematics, Hanoi Pedagogical University 2, Hanoi, Vietnam

3 VNU Information Technology Institute, Vietnam National University, Hanoi, Vietnam

4 Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam

5 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

1 Introduction

1.1 Briefly review the calculus of uncertainty functions

Fuzzy sets were introduced by Zadeh [68] to manipulate data and information possessing nonstatistical uncertainties. After that, Zadeh and numerous researchers from the whole world have promoted fuzzy theory reaching to every aspects of engineering science. Nowaday, based on the Mathematics Subject Classification of American Mathematical Society (MSC2010 database), fuzzy theory has formed many different branches such as fuzzy logic, fuzzy graph theory, fuzzy algebraic structures, fuzzy real analysis, fuzzy measure theory, fuzzy differential equations, fuzzy topology, fuzzy control systems, fuzzy probability, etc. Fuzzy theory have a bright future like today, beside many breakthrough researches in algebraic structures of fuzzy numbers space, there has been many researches on fuzzy calculus and fuzzy analysis. In order to model real world systems containing uncertainty by fuzzy differential equations or dinamic systems, the concept of derivative calculus must be introduced. Derivative calculus of fuzzy valued functions were dependened on the type of difference arithmetic correspondently. The first fuzzy derivative seem to be introduced in 1972 [17]. Not long after that, extensive researches on this issue were conducted, namely by Dubois-Prade derivatives [21], Puri-Ralescu derivative based on Hukuhara distance [39], Goetschel-Voxman derivative [23], Seikkala derivative [42] and Friedman-Ming-Kandel derivative [22]. However, when applying these derivatives into engineering problems, there have been appeared some disadvantage and drawback such as the uncertainty of solution of one engineering problem modeling by fuzzy dinamic systems increases when time tends to infinity. It was not until 2005 [8] that Bede and Gal invented strongly generalized Hukuhara derivative. With slightly different notion, Bede and Stefanini [10, 51] introduced generalized Hukuhara derivative. These concepts of fuzzy derivative have been openning up a period of applied researches of fuzzy mathematics in modeling of control system, dynamic scale of economy, etc, see [9] for example.

The fuzzy set of Zadeh is actually characterized by a membership function with the range of [0, 1],i.e., we measure the uncertainty degree of an object belonging to a fuzzy set via single value in interval [0, 1]. However in actual practice, due to the influence of some margin of hesitation, an element may neither belong to fuzzy set nor do not belong to fuzzy set. In the language of fuzzy set the total degree of membership with non-membership of an element in a fuzzy set is generally not equal to. Therefore, Atanassov [5] introduced Intuitionistic fuzzy sets as an extension of fuzzy set of Zadeh. In the view of intuitionistic fuzzy set, an element has degrees of membership and non-membership, relatively independent. A comprihensive study on intuitionistic fuzzy sets can be referenced from [6, 7]. However, as we know, the up to date researchers on intuitionistic fuzzy sets focus on algebraic structure, rarely studies on analysis and topological structures of intuitionistic fuzzy sets space. That has greatly limited the applications of intuitionistic fuzzy logic in engineering, where systems are often modeled by differential equations or control problems.

Neutrosophic set (NS) and neutrosophic logic were invented by Smarandache [43], which are really extension of appeared earlier logic in the the philosophical and mathematical aspects. Neutrosophy logic orients the study of statement that are not true, nor false, but neutral, indeterminate, contradictory or something in between. On the mathematical side, every field posses its own neutrosophic part, namely indeterminacy part. Thus, engineering studies rise to research topics which the underlying are the neutrosophic set and logic, the neutrosophic probability and statistics, the neutrosophic dynamic system and modeling, etc. Smarandache [46] laid the first attempt to study neutrosophic precalculus and neutrosophic calculus based on the existing calculus of interval analysis. Neutrosophic algebraic structures and neutrosophic cognitive maps were investigated in [14, 16]. Neutrosophic measure, neutrosophic probability and statistics were studied in [13, 44, 45, 52]. Neutrosophic systems application in decision making seem to be very successful. Ye [62, 64] proposed a multi-attribute decision making (MADM) method using the correlation coefficient under single-valued neutrosophic environment. Ye [63] further developed clustering method and decision making methods by similarity measures of SVNS. Meanwhile, Ye [65] presented cross entropy measures of SVNS and applied them to decision making (for more details, see [1–4, 11, 12, 18–20, 24–28, 32–35, 40, 41, 47–50, 53–59, 61, 66, 67]).

In some latest publications, based on horizontal membership function approach and granular computing, Mazandarani et al [29–31] studied fuzzy differential systems and related problems, which can be considered as a particular scenario of neutrosophic dynamic systems. However, neutrosophic set theory in general and neutrosophic dynamic systems in particular are still in the first stage of development. The main achievements focus on algebraic structures of neutrosophic sets. Recently, there are only some literature that have attained the first step in defining the distance between neutrosophic sets and neutrosophic numbers, see [34, 35, 63, 66] or have introduced some most fundamental concepts in neutrosophic calculus, see [44, 46, 52] for example. However, until now, the studies on analysis structures and topological structures on the space of neutrosophic set and neutrosophic numbers have almost never appeared. The reason for this disadvantages comes from the intrinsic nature of space of neutrosophic sets. For more details, due to opposite number law does not make sense in the space of neutrosophic sets, i.e., if $\mathcal{A}$ is a NS and $-\mathcal{A}$ is the opposite element then in general $\mathcal{A} + (-A)$ is not the zero NS. Thus, the subtraction operation defined by $\mathcal{A} - \mathcal{B} = \mathcal{A} + (-\mathcal{B})$ is not the candidate for difference operator in neutrosophic derivative calculus. Hence, it leads to big challenges for researchers if we want to study the analysis properties as well as constructing dynamical models for this object. Furthermore, the multi-coordinate $\mathcal{A}(T_{\mathcal{A}}, I_{\mathcal{A}}, F_{\mathcal{A}})$ of neutrosophic set makes more complicate when studying topological structure of the space. As the best of our knowledge, there does not have any suitable derivative concept defined for the neutrosophic-valued functions yet. Hence, it is one of the dynamics that promotes us in this work.

1.2 Contributions and structure of the paper

As the aforesaid in previous section, the difficulty in defining a suitable difference between neutrosophic sets is the limit of research in analysis calculus of neutrosophic-valued functions. Consequently, this leads to the study of many significant engineering problems related to derivative of a neutrosophic-valued functions such as modeling a systems by neutrosophic differential equations, modeling the evolution of a species by neutrosophic dynamic systems, the control problems of a neutrosophic-valued target or approximation of an underlying Input/Output systems by a neutrosophic systems, having no progress. Hence, the aims of this paper are

1. Throughout this paper, we introduce three types of single valued triangular neutrosophic numbers with triangular memberships functions for each components. The reason is that, the advantage and simply when presenting the parametric metric form as the classical fuzzy numbers. To this ends, we will define the $(\alpha, \beta, \gamma)-$cuts of neutro-

sophic fuzzy numbers and through the linearity of triangular membership functions, we can convert neutrosophic numbers into parametric forms as intervals. This parametric forms have advantage that, we can easily define levels-set wise of the derivatives and integral as well as building numerical algorithms.
2. The concept of arithmetic operations on the set of neutrosophic numbers is defined via horizontal membership functions. This idea original introduced Piegat et al. (see [36–38]) and developed for granular differentiability of fuzzy-valued functions by Mazandarani et al. [29–31]. Especially, we can define the granular difference between neutrosophic numbers - one important step to define further differentiability of neutrosophic-valued functions as well as neutrosophic differential equations and other applications. It can be seen that this approach does not necessitate that the decreased diameter of neutrosophic-valued function or multi-case of solution related to so-call switching points as we often face in fuzzy analysis.
3. We laid the first step in constructing topological structures on the set of neutrosophic numbers by introducing Hamming metric and building complete metric space $(\mathcal{T}, D^{gr})$. Due to the fact that the space $\mathcal{T}$ endowed with the metric D^{gr} ensures the convergent of Cauchy sequence, we can further study the qualitative and quantitative nature of solution to some dynamical systems and processes arising in science and engineering.
4. At last, we demonstrate the effectiveness and significance of our theoretical results by applying them in some engineering problems related to logistic model or some mechanical models such inverted pendulum systems or Mass- Spring- Damper model. Our research will open up many potential applications in applied science and engineering that directly employ derivative and integral calculus as the essential tools such as optimal control of wireless networks, modeling a wires in circuits by a dynamic system of neutrosophic objects, etc.

The organizational structure of this paper is as follows: Section 2 recalls some preliminaries on single valued neutrosophic set and neutrosophic numbers, in which we introduce the levels set notion as the bridge between neutrosophic set with granular computing. Next, we introduce some types of single valued triangular neutrosophic numbers along with their respective parametric form. For more clearly, we give some numerical examples for each subsection. Section 3 is used to present granular representation of single valued triangular neutrosophic numbers, that is the foundation to build some calculus properties such as the neutrosophic gr-derivative and neutrosophic gr-integral before some applications to engineering problems are presented in Section 4. Finally, some conclusions and future works are discussed in Section 5.

2 Single valued triangular neutrosophic number

We call a neutrosophic set (NS) $\mathcal{A}$ defined in the universal of discourse X, denote generally by x, if it is represented by the form

$$\mathcal{A} = \{\langle x, T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x)\rangle : x \in X\}$$

where $T_{\mathcal{A}} : X \to ^{-}]0, 1[^{+}$ is denoted for the truth membership function representing the degree of confidence, $I_{\mathcal{A}} : X \to ^{-}]0, 1[^{+}$ is called the indeterminacy membership function which represents the degree of uncertainty and $F_{\mathcal{A}} : X \rightarrow ^{-}]0, 1[^{+}$ is called the falsity membership function which represents the degree of scepticism such that the following relation holds

$$0^{-} \leq T_{\mathcal{A}}(x) + I_{\mathcal{A}}(x) + F_{\mathcal{A}}(x) \leq 3^{+}.$$

In this paper, we consider single valued NS, which is a NS $\mathcal{A}$ with x is a single valued independent variable (see [15]), whose the truth, indeterminacy and falsity membership functions exhibit the relation

$$0 \leq T_{\mathcal{A}}(x) + I_{\mathcal{A}}(x) + F_{\mathcal{A}}(x) \leq 3.$$

A single valued NS $\mathcal{A}$ defined on the universal set of real numbers $\mathbb{R}$ is said to be single valued neutrosophic number or neutrosophic number (NN) for short if it has following properties

(i) $\mathcal{A}$ is neut-normal, i.e., there exist three points $a_0, b_0, c_0 \in \mathbb{R}$ such that $T_{\mathcal{A}}(a_0) = 1$, $I_{\mathcal{A}}(b_0) = 1$ and $F_{\mathcal{A}}(c_0) = 1$.

(ii) $\mathcal{A}$ is neut-convex, i.e., the following conditions hold

$$\begin{aligned} T_{\mathcal{A}}\left(\lambda x_1 + (1-\lambda)x_2\right) &\geq \min\left\{T_{\mathcal{A}}(x_1), T_{\mathcal{A}}(x_2)\right\}, \\ I_{\mathcal{A}}\left(\lambda x_1 + (1-\lambda)x_2\right) &\leq \max\left\{I_{\mathcal{A}}(x_1), I_{\mathcal{A}}(x_2)\right\}, \\ F_{\mathcal{A}}\left(\lambda x_1 + (1-\lambda)x_2\right) &\leq \max\left\{F_{\mathcal{A}}(x_1), F_{\mathcal{A}}(x_2)\right\}, \end{aligned}$$

for each $\lambda \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}$.

Definition 2.1 The (α, β, γ) - cut (or level set) of a single valued NS $\mathcal{A}$, denoted by $\mathcal{A}_{(\alpha,\beta,\gamma)}$, is defined by $\mathcal{A}_{(\alpha,\beta,\gamma)} = \{x \in X : T_{\mathcal{A}}(x) \geq \alpha, I_{\mathcal{A}}(x) \leq \beta, F_{\mathcal{A}}(x) \leq \gamma\}$, where $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma \leq 3$.

Here, we consider a special type of single valued neutrosophic number, namely single valued triangular neutrosophic number.

Definition 2.2 A single valued triangular NN is given by

$$\mathcal{A} = \langle[(p_1, q_1, r_1); \alpha], [(p_2, q_2, r_2); \beta], [(p_3, q_3, r_3); \gamma]\rangle,$$

where $\alpha, \beta, \gamma \in [0, 1]$ and the truth membership function $T_{\mathcal{A}} : \mathbb{R} \rightarrow [0, \alpha]$, the indeterminacy membership function $I_{\mathcal{A}} : \mathbb{R} \rightarrow [\beta, 1]$ and the falsity membership function $F_{\mathcal{A}} : \mathbb{R} \rightarrow [\gamma, 1]$ satisfy following condition

$$0 \leq T_{\mathcal{A}}(x) + I_{\mathcal{A}}(x) + F_{\mathcal{A}}(x) \leq 3 \text{ for all} x \in \mathcal{A}.$$

We denote by $\mathcal{T}$ the set of all single valued triangular NNs. Then, based on the dependence between quantities: the truth, the indeterminacy and the falsity, we can classify the set $\mathcal{T}$ of single valued triangular NNs into three following types

2.1 Single valued triangular NN of type 1

The quantities of truth, indeterminacy and falsity are not dependent. Then, a single valued triangular NN of type 1 is defined as $\mathcal{A} = \langle p_1, q_1, r_1; p_2, q_2, r_2; p_3, q_3, r_3 \rangle$, with membership functions are defined as follows, respectively

$$T_{\mathcal{A}}(x) = \begin{cases} \dfrac{x - p_1}{q_1 - p_1} & p_1 \le x \le q_1, \\ 1 & x = q_1, \\ \dfrac{r_1 - x}{r_1 - q_1} & q_1 \le x \le r_1, \\ 0 & \text{otherwise}, \end{cases}$$

$$I_{\mathcal{A}}(x) = \begin{cases} \dfrac{x - p_2}{q_2 - p_2} & p_2 \le x \le q_2, \\ 0 & x = q_2, \\ \dfrac{r_2 - x}{r_2 - q_2} & q_2 \le x \le r_2, \\ 1 & \text{otherwise}, \end{cases}$$

$$F_{\mathcal{A}}(x) = \begin{cases} \dfrac{x - p_3}{q_3 - p_3} & p_3 \le x \le q_3, \\ 0 & x = q_3, \\ \dfrac{r_3 - x}{r_3 - q_3} & q_3 \le x \le r_3, \\ 1 & \text{otherwise}, \end{cases}$$

We can easily find the parametric form of $\mathcal{A}$ as

$$\mathcal{A}_{(\alpha,\beta,\gamma)} = \left[T^-_{\mathcal{A}}(\alpha), T^+_{\mathcal{A}}(\alpha); I^-_{\mathcal{A}}(\beta), I^+_{\mathcal{A}}(\beta); F^-_{\mathcal{A}}(\gamma), F^+_{\mathcal{A}}(\gamma)\right], \tag{1}$$

where $\alpha, \beta, \gamma \in [0, 1]$ such that $0 \le \alpha + \beta + \gamma \le 3$ and

$$\begin{aligned} T^-_{\mathcal{A}}(\alpha) &= p_1 + \alpha(q_1 - p_1), \;\; T^+_{\mathcal{A}}(\alpha) = r_1 - \alpha(r_1 - q_1), \\ I^-_{\mathcal{A}}(\beta) &= q_2 - \beta(q_2 - p_2), \;\; I^+_{\mathcal{A}}(\beta) = q_2 + \beta(r_2 - q_2), \\ F^-_{\mathcal{A}}(\gamma) &= q_3 - \gamma(q_3 - p_3), \;\; F^+_{\mathcal{A}}(\gamma) = q_3 + \gamma(r_3 - q_3). \end{aligned}$$

Example 2.1 Let $\mathcal{A} = (8, 14, 20; 12, 16, 22; 10, 15, 24)$ be a single valued triangular NN. Then, from (1) its parametric form is

$$\mathcal{A}_{(\alpha,\beta,\gamma)} = [8 + 6\alpha, 20 - 6\alpha; 16 - 4\beta, 16 + 6\beta; 15 - 5\gamma, 15 + 9\gamma], \quad \alpha, \beta, \gamma \in [0, 1].$$

In Table 1, we give some values of $T^-_{\mathcal{A}}(\alpha), T^+_{\mathcal{A}}(\alpha), I^-_{\mathcal{A}}(\beta), I^+_{\mathcal{A}}(\beta), F^-_{\mathcal{A}}(\gamma), F^+_{\mathcal{A}}(\gamma)$ of the number $\mathcal{A}$ at some concrete levels whose graphical representation is shown Fig. 1.

2.2 Single valued triangular NN of type 2

In this type of number, two quantities: indeterminacy membership function and falsity membership function are dependent. Then, a single valued triangular NN of type 2 is defined

Table 1 Values of $T^-_{\mathcal{A}}(\alpha)$, $T^+_{\mathcal{A}}(\alpha)$, $I^-_{\mathcal{A}}(\beta)$, $I^+_{\mathcal{A}}(\beta)$, $F^-_{\mathcal{A}}(\gamma)$, $F^+_{\mathcal{A}}(\gamma)$

α, β, γ	$T^-_{\mathcal{A}}(\alpha)$	$T^+_{\mathcal{A}}(\alpha)$	$I^-_{\mathcal{A}}(\beta)$	$I^+_{\mathcal{A}}(\beta)$	$F^-_{\mathcal{A}}(\gamma)$	$F^+_{\mathcal{A}}(\gamma)$
0	8	20	16	16	15	15
0.1	8.6	19.4	15.6	16.6	14.5	15.9
0.2	9.2	18.8	15.2	17.2	14	16.8
0.3	9.8	18.2	14.8	17.8	13.5	17.7
0.4	10.4	17.6	14.4	18.4	13	18.6
0.5	11	17	14	19	12.5	19.5
0.6	11.6	16.4	13.6	19.6	12	20.4
0.7	12.2	15.8	13.2	20.2	11.5	21.3
0.8	12.8	15.2	12.8	20.8	11	22.2
0.9	13.4	14.6	12.4	21.4	10.5	23.1
1	14	14	12	22	10	24

as $\mathcal{A} = \langle p_1, q_1, r_1; p_2, q_2, r_2; \beta_{neu}; \gamma_{neu}\rangle$, whose membership functions are defined in compact form as

$$T_{\mathcal{A}}(x) = \begin{cases} \dfrac{x - p_1}{q_1 - p_1} & p_1 \leq x \leq q_1, \\ 1 & x = q_1, \\ \dfrac{r_1 - x}{r_1 - q_1} & q_1 \leq x \leq r_1, \\ 0 & \text{otherwise}, \end{cases}$$

$$I_{\mathcal{A}}(x) = \begin{cases} \dfrac{q_2 - x + \beta_{neu}(x - p_2)}{q_2 - p_2} & p_2 \leq x \leq q_2, \\ \beta_{neu} & x = q_2, \\ \dfrac{x - q_2 + \beta_{neu}(r_2 - x)}{r_2 - q_2} & q_2 \leq x \leq r_2, \\ 1 & \text{otherwise}, \end{cases}$$

$$F_{\mathcal{A}}(x) = \begin{cases} \dfrac{q_2 - x + \gamma_{neu}(x - p_2)}{q_2 - p_2} & p_2 \leq x \leq q_2, \\ \gamma_{neu} & x = q_2, \\ \dfrac{x - q_2 + \gamma_{neu}(r_2 - x)}{r_2 - q_2} & q_2 \leq x \leq r_2, \\ 1 & \text{otherwise}, \end{cases}$$

Similarly, the parametric form of $\mathcal{A}$ is

$$\mathcal{A}_{(\alpha,\beta,\gamma)} = \left[T^-_{\mathcal{A}}(\alpha), T^+_{\mathcal{A}}(\alpha); I^-_{\mathcal{A}}(\beta), I^+_{\mathcal{A}}(\beta); F^-_{\mathcal{A}}(\gamma), F^+_{\mathcal{A}}(\gamma)\right],$$

where

$$\begin{aligned} T^-_{\mathcal{A}}(\alpha) &= p_1 + \alpha(q_1 - p_1), & T^+_{\mathcal{A}}(\alpha) &= r_1 - \alpha(r_1 - q_1), \\ I^-_{\mathcal{A}}(\beta) &= \frac{q_2 - \beta_{neu} p_2 - \beta(q_2 - p_2)}{1 - \beta_{neu}}, & I^+_{\mathcal{A}}(\beta) &= \frac{q_2 - \beta_{neu} r_2 + \beta(r_2 - q_2)}{1 - \beta_{neu}}, \\ F^-_{\mathcal{A}}(\gamma) &= \frac{q_2 - \gamma_{neu} p_2 - \gamma(q_2 - p_2)}{1 - \gamma_{neu}}, & F^+_{\mathcal{A}}(\gamma) &= \frac{q_2 - \gamma_{neu} r_2 + \gamma(r_2 - q_2)}{1 - \gamma_{neu}}. \end{aligned}$$

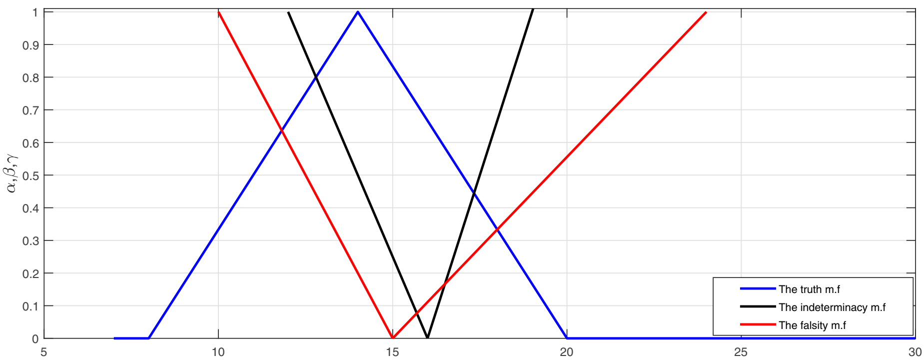

Fig. 1 Membership functions of NN of type 1 $\mathcal{A}$

Here, $\alpha \in [0, 1]$, $\beta \in [\beta_{neu}, 1]$ and $\gamma \in [\gamma_{neu}, 1]$ such that $0 \leq \alpha + \beta + \gamma \leq 3$.

Example 2.2 Let $\mathcal{A} = (8, 14, 20; 12, 16, 22; 0.5; 0.6)$ be a single valued NN. Then, its parametric form is

$$\mathcal{A}_{(\alpha,\beta,\gamma)} = [8 + 6\alpha, 20 - 6\alpha; 20 - 8\beta, 10 + 12\beta; 22 - 10\gamma, 7 + 15\gamma],$$

for $\alpha \in [0, 1]$, $\beta \in [0.5, 1]$, $\gamma \in [0.6, 1]$.

In Table 2, we give some values of $T^-_{\mathcal{A}}(\alpha)$, $T^+_{\mathcal{A}}(\alpha)$, $I^-_{\mathcal{A}}(\beta)$, $I^+_{\mathcal{A}}(\beta)$, $F^-_{\mathcal{A}}(\gamma)$, $F^+_{\mathcal{A}}(\gamma)$ of the number $\mathcal{A}$ whose graphical representation is shown Fig. 2.

Table 2 Values of $T^-_{\mathcal{A}}(\alpha)$, $T^+_{\mathcal{A}}(\alpha)$, $I^-_{\mathcal{A}}(\beta)$, $I^+_{\mathcal{A}}(\beta)$, $F^-_{\mathcal{A}}(\gamma)$, $F^+_{\mathcal{A}}(\gamma)$

α, β, γ	$T^-_{\mathcal{A}}(\alpha)$	$T^+_{\mathcal{A}}(\alpha)$	$I^-_{\mathcal{A}}(\beta)$	$I^+_{\mathcal{A}}(\beta)$	$F^-_{\mathcal{A}}(\gamma)$	$F^+_{\mathcal{A}}(\gamma)$
0	8	20				
0.1	8.6	19.4				
0.2	9.2	18.8				
0.3	9.8	18.2				
0.4	10.4	17.6				
0.5	11	17	16	16		
0.6	11.6	16.4	17.2	19.6	16	16
0.7	12.2	15.8	18.4	20.2	15	17.5
0.8	12.8	15.2	19.6	20.8	14	19
0.9	13.4	14.6	20.8	21.4	13	20.5
1	14	14	12	22	12	22

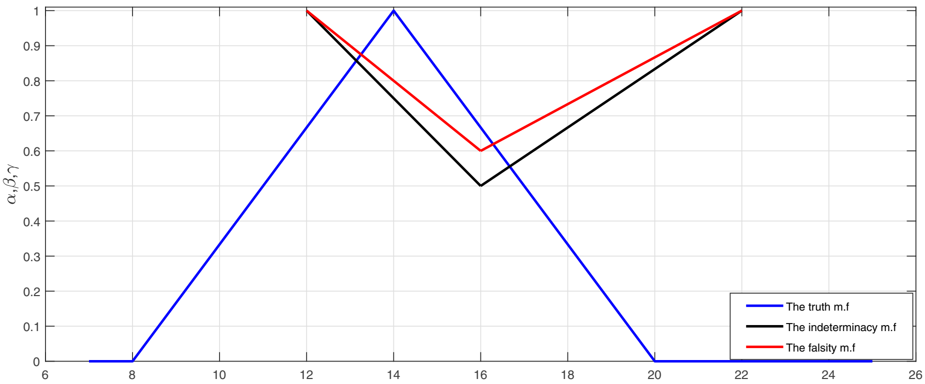

Fig. 2 Membership functions of NN of type 2 $\mathcal{A}$

2.3 Single valued triangular NN of type 3

Here, the quantities: the truth, indeterminacy and falsity membership function are dependent. Then, a single valued triangular NN of type 3 is defined as $\mathcal{A} = \langle p_1, q_1, r_1; \alpha_{neu}; \beta_{neu}; \gamma_{neu} \rangle$, whose membership functions are defined as follows

$$T_{\mathcal{A}}(x) = \begin{cases} \alpha_{neu} \dfrac{x - p_1}{q_1 - p_1} & p_1 \leq x \leq q_1, \\ \alpha_{neu} & x = q_1, \\ \alpha_{neu} \dfrac{r_1 - x}{r_1 - q_1} & q_1 \leq x \leq r_1, \\ 0 & \text{otherwise}, \end{cases}$$

$$I_{\mathcal{A}}(x) = \begin{cases} \dfrac{q_1 - x + \beta_{neu}(x - p_1)}{q_1 - p_1} & p_1 \leq x \leq q_1, \\ \beta_{neu} & x = q_1, \\ \dfrac{x - q_1 + \beta_{neu}(r_1 - x)}{r_1 - q_1} & q_1 \leq x \leq r_1, \\ 1 & \text{otherwise}, \end{cases}$$

$$F_{\mathcal{A}}(x) = \begin{cases} \dfrac{q_1 - x + \gamma_{neu}(x - p_1)}{q_1 - p_1} & p_1 \leq x \leq q_1, \\ \gamma_{neu} & x = q_1, \\ \dfrac{x - q_1 + \gamma_{neu}(r_1 - x)}{r_1 - q_1} & q_1 \leq x \leq r_1, \\ 1 & \text{otherwise}, \end{cases}$$

The parametric form of the number $\mathcal{A}$ is

$$\mathcal{A}_{(\alpha,\beta,\gamma)} = \left[T_{\mathcal{A}}^{-}(\alpha), T_{\mathcal{A}}^{+}(\alpha); I_{\mathcal{A}}^{-}(\beta), I_{\mathcal{A}}^{+}(\beta); F_{\mathcal{A}}^{-}(\gamma), F_{\mathcal{A}}^{+}(\gamma)\right],$$

Table 3 Values of $T^-_{\mathcal{A}}(\alpha)$, $T^+_{\mathcal{A}}(\alpha)$, $I^-_{\mathcal{A}}(\beta)$, $I^+_{\mathcal{A}}(\beta)$, $F^-_{\mathcal{A}}(\gamma)$, $F^+_{\mathcal{A}}(\gamma)$

α, β, γ	$T^-_{\mathcal{A}}(\alpha)$	$T^+_{\mathcal{A}}(\alpha)$	$I^-_{\mathcal{A}}(\beta)$	$I^+_{\mathcal{A}}(\beta)$	$F^-_{\mathcal{A}}(\gamma)$	$F^+_{\mathcal{A}}(\gamma)$
0	14	22				
0.1	14.4	20.8				
0.2	14.8	19.6				
0.3	15.2	18.4				
0.4	15.6	17.2				
0.5	16	16				
0.6						
0.7					16	16
0.8			16	16	15.333	18
0.9			15	19	14.667	20
1			14	22	14	22

where

$$
\begin{aligned}
T^-_{\mathcal{A}}(\alpha) &= p_1 + \frac{\alpha}{\alpha_{neu}}(q_1 - p_1), & T^+_{\mathcal{A}}(\alpha) &= r_1 - \frac{\alpha}{\alpha_{neu}}(r_1 - q_1),\\
I^-_{\mathcal{A}}(\beta) &= \frac{q_1 - \beta_{neu} p_1 - \beta(q_1 - p_1)}{1 - \beta_{neu}}, & I^+_{\mathcal{A}}(\beta) &= \frac{q_1 - \beta_{neu} r_1 + \beta(r_1 - q_1)}{1 - \beta_{neu}},\\
F^-_{\mathcal{A}}(\gamma) &= \frac{q_1 - \gamma_{neu} p_1 - \gamma(q_1 - p_1)}{1 - \gamma_{neu}}, & F^+_{\mathcal{A}}(\gamma) &= \frac{q_1 - \gamma_{neu} r_1 + \gamma(r_1 - q_1)}{1 - \gamma_{neu}}.
\end{aligned}
$$

Here, $\alpha \in [0, \alpha_{neu}]$, $\beta \in [\beta_{neu}, 1]$ and $\gamma \in [\gamma_{neu}, 1]$ such that $0 \leq \alpha + \beta + \gamma \leq 3$.

Example 2.3 Let $\mathcal{A} = (14, 16, 22; 0.5; 0.8; 0.7)$ be a single valued NN. Then, its parametric form is

$$
\mathcal{A}_{(\alpha,\beta,\gamma)} = \left[14 + 4\alpha, 22 - 12\alpha; 24 - 10\beta, -8 + 30\beta; \frac{62}{3} - \frac{20}{3}\gamma, 2 + 20\gamma\right],
$$

for $\alpha \in [0, 0.5]$, $\beta \in [0.8, 1]$, $\gamma \in [0.7, 1]$.

In Table 3, we give some values of $T^-_{\mathcal{A}}(\alpha)$, $T^+_{\mathcal{A}}(\alpha)$, $I^-_{\mathcal{A}}(\beta)$, $I^+_{\mathcal{A}}(\beta)$, $F^-_{\mathcal{A}}(\gamma)$, $F^+_{\mathcal{A}}(\gamma)$ of the number $\mathcal{A}$ whose graphical representation is shown Fig. 3.

3 Granular presentation of single valued triangular neutrosophic number

3.1 Horizontal membership function

Definition 3.1 Let $\mathcal{A} = (a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3)$ be a single valued triangular neutrosophic number whose parametric form is

$$
\mathcal{A}_{(\alpha,\beta,\gamma)} = \left[T^-_{\mathcal{A}}(\alpha), T^+_{\mathcal{A}}(\alpha); I^-_{\mathcal{A}}(\beta), I^+_{\mathcal{A}}(\beta); F^-_{\mathcal{A}}(\gamma), F^+_{\mathcal{A}}(\gamma)\right].
$$

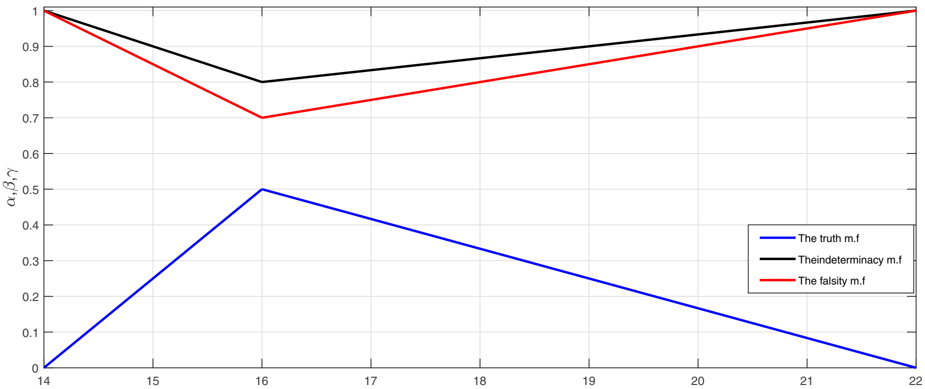

Fig. 3 Membership functions of NN of type 3 $\mathcal{A}$

Then, we can represent the horizontal membership function (HMF) of $\mathcal{A}$ as an element $A^{gr}(\alpha, \beta, \gamma, \mu)$, which is given by

$$A^{gr} : [0, \alpha_{neu}] \times [\beta_{neu}, 1] \times [\gamma_{neu}, 1] \times [0, 1]^3 \to [a_1, c_1] \times [a_2, c_2] \times [a_3, c_3],$$

and maps $(\alpha, \beta, \gamma, \mu)$ into $\left(x_\alpha(\mu_1), x_\beta(\mu_2), x_\gamma(\mu_3)\right) \in \mathbb{R}^3$, where the notion "gr" represents for the granular information that are contained in $\left(x_\alpha, x_\beta, x_\gamma\right) \in [a_1, c_1] \times [a_2, c_2] \times [a_3, c_3]$ and $\mu \in [0, 1]^3$ standing for μ_1, μ_2, μ_3 is called relative-distance-measure (RDM for short) variables. In particular, we have $A^{gr}(\alpha, \beta, \gamma, \mu) = \left(x_\alpha(\mu_1), x_\beta(\mu_2), x_\gamma(\mu_3)\right)$, where

$$\begin{aligned} x_\alpha(\mu_1) &:= T^{gr}_{\mathcal{A}}(\alpha, \mu_1) = T^-_{\mathcal{A}}(\alpha) + \left(T^+_{\mathcal{A}}(\alpha) - T^-_{\mathcal{A}}(\alpha)\right)\mu_1, \\ x_\beta(\mu_2) &:= I^{gr}_{\mathcal{A}}(\beta, \mu_2) = I^-_{\mathcal{A}}(\beta) + \left(I^+_{\mathcal{A}}(\beta) - I^-_{\mathcal{A}}(\beta)\right)\mu_2, \\ x_\gamma(\mu_3) &:= F^{gr}_{\mathcal{A}}(\gamma, \mu_3) = F^-_{\mathcal{A}}(\gamma) + \left(F^+_{\mathcal{A}}(\gamma) - F^-_{\mathcal{A}}(\gamma)\right)\mu_3. \end{aligned}$$

Proposition 3.1 *The HMF of a number $\mathcal{A} \in \mathcal{T}$ is denoted by $\mathcal{H}(\mathcal{A}) \triangleq A^{gr}(\alpha, \beta, \gamma, \mu)$. Moreover, the $(\alpha, \beta, \gamma)-$ cuts of $\mathcal{A}$ can be obtained by using following inverse transformation*

$$\begin{aligned} \mathcal{A}_{(\alpha,\beta,\gamma)} &:= \mathcal{H}^{-1}(A^{gr}(\alpha, \beta, \gamma, \mu)) \\ &= \left\{ \left[\inf_{\xi \geq \alpha} \min_{\mu_1} T^{gr}_{\mathcal{A}}(\xi, \mu_1), \sup_{\xi \geq \alpha} \max_{\mu_1} T^{gr}_{\mathcal{A}}(\xi, \mu_1) \right], \right. \\ &\qquad \left[\inf_{\xi \geq \beta} \min_{\mu_2} I^{gr}_{\mathcal{A}}(\xi, \mu_2), \sup_{\xi \geq \beta} \max_{\mu_2} I^{gr}_{\mathcal{A}}(\xi, \mu_2) \right], \\ &\qquad \left. \left[\inf_{\xi \geq \gamma} \min_{\mu_3} F^{gr}_{\mathcal{A}}(\xi, \mu_3), \sup_{\xi \geq \gamma} \max_{\mu_3} F^{gr}_{\mathcal{A}}(\xi, \mu_3) \right] \right\}. \end{aligned}$$

Definition 3.2 Two elements $\mathcal{A}$ and $\tilde{\mathcal{A}} \in \mathcal{T}$ are said to be equal and written by $\mathcal{A} = \tilde{\mathcal{A}}$ if and only if $\mathcal{H}(\mathcal{A}) = \mathcal{H}(\tilde{\mathcal{A}})$ for all triplet $(\mu_1, \mu_2, \mu_3) = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3) \in [0, 1] \times [0, 1] \times [0, 1]$, i.e.,

$$\begin{cases} T^{gr}_{\mathcal{A}}(\alpha, \mu_1) = T^{gr}_{\tilde{\mathcal{A}}}(\alpha, \mu_1) \\ I^{gr}_{\mathcal{A}}(\beta, \mu_2) = I^{gr}_{\tilde{\mathcal{A}}}(\beta, \mu_2) \\ F^{gr}_{\mathcal{A}}(\gamma, \mu_3) = F^{gr}_{\tilde{\mathcal{A}}}(\gamma, \mu_3) \end{cases}$$

for each $(\alpha, \beta, \gamma) \in [0, \alpha_{neu}] \times [\beta_{neu}, 1] \times [\gamma_{neu}, 1]$ and for all $(\mu_1, \mu_2, \mu_3) \in [0, 1] \times [0, 1] \times [0, 1]$.

3.2 Arithmetic operations

Definition 3.3 Denote "$\otimes$" by one of three arithmetic operations in $\mathcal{T}$: addition, subtraction or multiplication operation. Then $\mathcal{H}(\mathcal{A}_1 \otimes \mathcal{A}_2) \triangleq \mathcal{H}(\mathcal{A}_1) \otimes \mathcal{H}(\mathcal{A}_2)$. It should be noted that the difference in this sense is called granular difference (gr-difference), denoted by $\ominus^{gr}$.

Example 3.1 Let $\mathcal{A} = (5, 10, 15; 3, 6, 9; 10, 16, 22)$ and $\tilde{\mathcal{A}} = (4, 6, 8; 1, 3, 5; 9, 11, 13)$ be two triangular neutrosophic numbers of type 1 whose parametric representations are given by

$$\begin{aligned} \mathcal{A}_{(\alpha,\beta,\gamma)} &= \{[5 + 5\alpha, 15 - 5\alpha], [6 - 3\beta, 6 + 3\beta], [16 - 6\gamma, 16 + 6\gamma]\}, \\ \tilde{\mathcal{A}}_{(\alpha,\beta,\gamma)} &= \{[4 + 2\alpha, 8 - 2\alpha], [3 - 2\beta, 3 + 2\beta], [11 - 2\gamma, 11 + 2\gamma]\}. \end{aligned}$$

From Definition 3.1, it implies that

$$\begin{aligned} A^{gr}(\alpha, \beta, \gamma, \mu) &= (5 + 5\alpha + (10 - 10\alpha)\mu_1; 6 - 3\beta + 6\beta\mu_2; 16 - 6\gamma + 12\gamma\mu_3), \\ \tilde{A}^{gr}(\alpha, \beta, \gamma, \tilde{\mu}) &= (4 + 2\alpha + (4 - 4\alpha)\tilde{\mu}_1; 3 - 2\beta + 4\beta\tilde{\mu}_2; 11 - 2\gamma + 4\gamma\tilde{\mu}_3). \end{aligned}$$

In addition, employing Definition 3.3, we immediately obtain that

- $A^{gr}(\alpha, \beta, \gamma, \mu) + \tilde{A}^{gr}(\alpha, \beta, \gamma, \mu) = (9 + 7\alpha + (14 - 14\alpha)\mu_1; 9 - 5\beta + 10\beta\mu_2; 27 - 8\gamma + 16\gamma\mu_3)$,
- $A^{gr}(\alpha, \beta, \gamma, \mu) - \tilde{A}^{gr}(\alpha, \beta, \gamma, \mu) = (1 + 2\alpha + (6 - 6\alpha)\mu_1; 3 - \beta + 2\beta\mu_2; 5 - 4\gamma + 8\gamma\mu_3)$,
- $0.5 \cdot A^{gr}(\alpha, \beta, \gamma, \mu) = (2.5 + 2.5\alpha + (5 - 5\alpha)\mu_1; 3 - 1.5\beta + 3\beta\mu_2; 8 - 3\gamma + 6\gamma\mu_3)$,
- $$\begin{aligned} A^{gr}(\alpha, \beta, \gamma, \mu) \times \tilde{A}^{gr}(\alpha, \beta, \gamma, \mu) = \Big(&20 + 30\alpha + 10\alpha^2 + 20(1 - \alpha)(2 + 3\alpha)\mu_1 + 40(1 - \alpha)^2\mu_1^2; \\ &18 - 21\beta + 6\beta^2 + 6\beta(7 - 4\beta)\mu_2 + 24\beta^2\mu_2^2; \\ &176 - 98\gamma + 12\gamma^2 + 4\gamma(49 - 12\gamma)\mu_3 + 48\gamma^2\mu_3^2\Big). \end{aligned}$$

Then, by using Proposition 3.1 and (α, β, γ) −cuts representation theorem, we obtain

- The addition

$$\mathcal{A} + \tilde{\mathcal{A}} = (9, 17, 24; 4, 9, 14; 19, 27, 35).$$

- The subtraction

$$\mathcal{A} \ominus^{gr} \tilde{\mathcal{A}} = (-3, 4, 11; -2, 3, 8; -3, 5, 13).$$

- The multiplication by the scalar $\lambda = 0.5$

$$\lambda\mathcal{A} = (2.5, 5, 7.5; 1.5, 3, 4.5; 5, 8, 11).$$

- The multiplication

$$\mathcal{A} \cdot \tilde{\mathcal{A}} = (20, 60, 120; 3, 18, 45; 90, 176, 286).$$

Especially, we have the spare of $\mathcal{A}$ is given by

$$\mathcal{A}^2 := \mathcal{A} \cdot \mathcal{A} = (25, 100, 225; 9, 36, 81; 100, 256, 484).$$

Definition 3.4 Let $f : [a, b] \subset \mathbb{R} \to \mathcal{T}$ be a $\mathcal{T}$-valued function including n distinct single valued triangular neutrosophic numbers $\mathcal{A}_1, , \mathcal{A}_2, \ldots, \mathcal{A}_n$. Then, the HMF of f at $t \in [a, b]$, denoted by $\mathcal{H}(f(t)) \triangleq f^{gr}(t, \alpha, \beta, \gamma, \mu_f)$, is as

$$f^{gr} : [a, b] \times [0, \alpha_{neu}] \times [\beta_{neu}, 1] \times [\gamma_{neu}, 1] \times [0, 1]^{3n} \to \mathbb{R}^3,$$

where $\mu_f \triangleq (\mu_{1,\mathcal{A}_1}, \ldots, \mu_{1,\mathcal{A}_n}, \mu_{2,\mathcal{A}_1}, \ldots, \mu_{2,\mathcal{A}_n}, \mu_{3,\mathcal{A}_1}, \ldots, \mu_{3,\mathcal{A}_n})$.

Remark 3.1 Let f be a $\mathcal{T}-$ valued function defined in closed interval $[a, b] \subset \mathbb{R}$. Then, since the value $f(t) \in \mathcal{T}$, we can write $f(t)$ as $f(t) = \left\{\langle x, T_{f(t)}(x), I_{f(t)}(x), F_{f(t)}(x)\rangle : x \in \mathbb{R}\right\}$ for each $t \in [a, b]$, whose HMF can be written as

$$f^{gr}(t, \alpha, \beta, \gamma, \mu_f) = \left(T^{gr}_{f(t)}(\alpha, \mu_{1,f}), I^{gr}_{f(t)}(\beta, \mu_{2,f}), F^{gr}_{f(t)}(\gamma, \mu_{3,f})\right).$$

Example 3.2 Let $\mathcal{A} = (5, 10, 15; 3, 6, 9; 10, 16, 22)$ and $\tilde{\mathcal{A}} = (4, 6, 8; 1, 3, 5; 9, 11, 13)$ be two triangular neutrosophic numbers of type 1 with respective HMF is as follows

$$\begin{aligned} A^{gr}(\alpha, \beta, \gamma, \mu) &= (5 + 5\alpha + (10 - 10\alpha)\mu_1; 6 - 3\beta + 6\beta\mu_2; 16 - 6\gamma + 12\gamma\mu_3), \\ \tilde{A}^{gr}(\alpha, \beta, \gamma, \tilde{\mu}) &= (4 + 2\alpha + (4 - 4\alpha)\tilde{\mu}_1; 3 - 2\beta + 4\beta\tilde{\mu}_2; 11 - 2\gamma + 4\gamma\tilde{\mu}_3), \end{aligned}$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\mu = (\mu_1, \mu_2, \mu_3)$, $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3) \in [0, 1] \times [0, 1] \times [0, 1]$. Consider a $\mathcal{T}$-valued function $f(t) = \mathcal{A}t + \tilde{\mathcal{A}} \sin 2t$ on the interval $[0, 7]$. The horizontal membership function of f is given by

$$\begin{aligned} \mathcal{H}(f(t)) &= tA^{gr}(\alpha, \beta, \gamma, \mu) + \tilde{A}^{gr}(\alpha, \beta, \gamma, \tilde{\mu}) \sin 2t \\ &= \big([5 + 5\alpha + (10 - 10\alpha)\mu_1]\, t + \left[4 + 2\alpha + (4 - 4\alpha)\tilde{\mu}_1\right] \sin 2t; \\ &\qquad [6 - 3\beta + 6\beta\mu_2]\, t + \left[3 - 2\beta + 4\beta\tilde{\mu}_2\right] \sin 2t; \\ &\qquad [16 - 6\gamma + 12\gamma\mu_3]\, t + \left[11 - 2\gamma + 4\gamma\tilde{\mu}_3\right] \sin 2t\big). \end{aligned}$$

and the graphical representation of $\mathcal{T}-$ valued function $f(t)$ is shown in Fig. 4

3.3 Neutrosophic metric space

Definition 3.5 Let $\mathcal{A}, \tilde{\mathcal{A}} \in \mathcal{T}$. The function $D^{gr} : \mathcal{T} \times \mathcal{T} \to \mathbb{R}^+ \cup \{0\}$ defined by

$$\begin{aligned} D^{gr}(\mathcal{A}, \tilde{\mathcal{A}}) = \sup_{\alpha,\beta,\gamma} \max_{\mu_i,\tilde{\mu}_i} \frac{1}{3} \Big\{ &\left|T^{gr}_{\mathcal{A}}(\alpha, \mu_1) - T^{gr}_{\tilde{\mathcal{A}}}(\alpha, \tilde{\mu}_1)\right| + \left|I^{gr}_{\mathcal{A}}(\beta, \mu_2) - I^{gr}_{\tilde{\mathcal{A}}}(\beta, \tilde{\mu}_2)\right| \\ &+ \left|F^{gr}_{\mathcal{A}}(\gamma, \mu_3) - F^{gr}_{\tilde{\mathcal{A}}}(\gamma, \tilde{\mu}_3)\right| \Big\}, \end{aligned} \tag{2}$$

is a distance between two type 1 single valued triangular neutrosophic numbers $\mathcal{A}$ and $\tilde{\mathcal{A}}$.

Proposition 3.2 *Such real-valued function D^{gr} is said to be a metric on $\mathcal{T}$.*

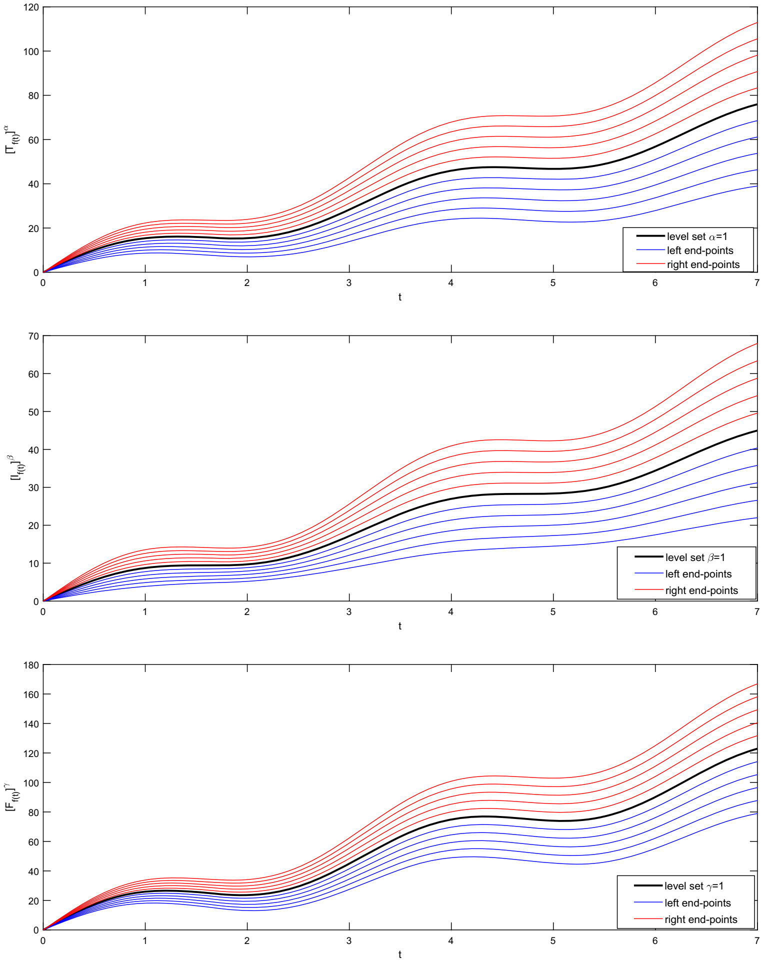

Fig. 4 The (α, β, γ)- cuts of $\mathcal{T}$-valued function f in Example 3.2, where the black curve corresponds to the certain values, the blue curves show the left end-points, while the red show the right end-points

Proof Let $\mathcal{A}$ and $\tilde{\mathcal{A}}$ be two numbers in $\mathcal{T}$ with respective horizontal membership functions

$$A^{gr}(\alpha, \beta, \gamma, \mu) = (T^{gr}_{\mathcal{A}}(\alpha, \mu_1), I^{gr}_{\mathcal{A}}(\beta, \mu_2), F^{gr}_{\mathcal{A}}(\gamma, \mu_3)),$$
$$\tilde{A}^{gr}(\alpha, \beta, \gamma, \tilde{\mu}) = (T^{gr}_{\tilde{\mathcal{A}}}(\alpha, \tilde{\mu}_1), I^{gr}_{\tilde{\mathcal{A}}}(\beta, \tilde{\mu}_2), F^{gr}_{\tilde{\mathcal{A}}}(\gamma, \tilde{\mu}_3)).$$

From the formula (2), it implies that $D^{gr}(\mathcal{A}, \tilde{\mathcal{A}}) \geq 0$ for all $\mathcal{A}, \tilde{\mathcal{A}} \in \mathcal{T}$ and if $D^{gr}(\mathcal{A}, \tilde{\mathcal{A}}) = 0$ then we deduce that

$$\begin{cases} \max\limits_{\mu_1, \tilde{\mu}_1} \left| T^{gr}_{\mathcal{A}}(\alpha, \mu_1) - T^{gr}_{\tilde{\mathcal{A}}}(\alpha, \tilde{\mu}_1) \right| = 0 \\ \max\limits_{\mu_2, \tilde{\mu}_2} \left| I^{gr}_{\mathcal{A}}(\beta, \mu_2) - I^{gr}_{\tilde{\mathcal{A}}}(\beta, \tilde{\mu}_2) \right| = 0 \\ \max\limits_{\mu_3, \tilde{\mu}_3} \left| F^{gr}_{\mathcal{A}}(\gamma, \mu_3) - F^{gr}_{\tilde{\mathcal{A}}}(\gamma, \tilde{\mu}_3) \right| = 0. \end{cases}$$

Equivalently, it implies

$$\begin{cases} T^{gr}_{\mathcal{A}}(\alpha,\mu_1) = T^{gr}_{\tilde{\mathcal{A}}}(\alpha,\tilde{\mu}_1) \\ I^{gr}_{\mathcal{A}}(\beta,\mu_2) = I^{gr}_{\tilde{\mathcal{A}}}(\beta,\tilde{\mu}_2) \\ F^{gr}_{\mathcal{A}}(\gamma,\mu_3) = F^{gr}_{\tilde{\mathcal{A}}}(\gamma,\tilde{\mu}_3), \end{cases}$$

for all $\mu_i = \tilde{\mu}_i \in [0,1]$ $(i = 1,2,3)$ which follows that $\mathcal{A} = \tilde{\mathcal{A}}$.

Since symmetry of D^{gr} is obvious, the rest of proof is to show that

$$D^{gr}(\mathcal{A}_1,\mathcal{A}_2) \le D^{gr}(\mathcal{A}_1,\mathcal{A}_3) + D^{gr}(\mathcal{A}_3,\mathcal{A}_2) \quad \text{for all} \mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3 \in \mathcal{T}. \tag{3}$$

Indeed, since the following inequality

$$\left|T^{gr}_{\mathcal{A}_1}(\alpha,\mu_1) - T^{gr}_{\mathcal{A}_2}(\alpha,\tilde{\mu}_1)\right| \le \left|T^{gr}_{\mathcal{A}_1}(\alpha,\mu_1) - T^{gr}_{\mathcal{A}_3}(\alpha,\overline{\mu}_1)\right| + \left|T^{gr}_{\mathcal{A}_3}(\alpha,\overline{\mu}_1) - T^{gr}_{\mathcal{A}_2}(\alpha,\tilde{\mu}_1)\right|$$

holds for each $\alpha \in [0,\alpha_{neu}]$, $\mu_1,\tilde{\mu}_1,\overline{\mu}_1 \in [0,1]$, we deduce that

$$\left|T^{gr}_{\mathcal{A}_1}(\alpha,\mu_1) - T^{gr}_{\mathcal{A}_2}(\alpha,\tilde{\mu}_1)\right| \le \max_{\mu_1,\tilde{\mu}_1,\overline{\mu}_1} \left\{\left|T^{gr}_{\mathcal{A}_1}(\alpha,\mu_1) - T^{gr}_{\mathcal{A}_3}(\alpha,\overline{\mu}_1)\right| + \left|T^{gr}_{\mathcal{A}_3}(\alpha,\overline{\mu}_1) - T^{gr}_{\mathcal{A}_2}(\alpha,\tilde{\mu}_1)\right|\right\}.$$

Therefore,

$$\begin{aligned} \sup_{\alpha} \max_{\mu_1,\tilde{\mu}_1} \left|T^{gr}_{\mathcal{A}_1}(\alpha,\mu_1) - T^{gr}_{\mathcal{A}_2}(\alpha,\tilde{\mu}_1)\right| &\le \sup_{\alpha} \max_{\mu_1,\overline{\mu}_1} \left|T^{gr}_{\mathcal{A}_1}(\alpha,\mu_1) - T^{gr}_{\mathcal{A}_3}(\alpha,\overline{\mu}_1)\right| \\ &+ \sup_{\alpha} \max_{\overline{\mu}_1,\tilde{\mu}_1} \left|T^{gr}_{\mathcal{A}_3}(\alpha,\overline{\mu}_1) - T^{gr}_{\mathcal{A}_2}(\alpha,\tilde{\mu}_1)\right|. \end{aligned}$$

By similar arguments, we also obtain

$$\begin{aligned} \sup_{\beta} \max_{\mu_2,\tilde{\mu}_2} \left|I^{gr}_{\mathcal{A}_1}(\beta,\mu_2) - I^{gr}_{\mathcal{A}_2}(\beta,\tilde{\mu}_2)\right| &\le \sup_{\beta} \max_{\mu_2,\overline{\mu}_2} \left|I^{gr}_{\mathcal{A}_1}(\beta,\mu_2) - I^{gr}_{\mathcal{A}_3}(\beta,\overline{\mu}_2)\right| \\ &+ \sup_{\beta} \max_{\overline{\mu}_2,\tilde{\mu}_2} \left|I^{gr}_{\mathcal{A}_3}(\beta,\overline{\mu}_2) - I^{gr}_{\mathcal{A}_2}(\beta,\tilde{\mu}_2)\right|, \\ \sup_{\gamma} \max_{\mu_3,\tilde{\mu}_3} \left|F^{gr}_{\mathcal{A}_1}(\gamma,\mu_3) - F^{gr}_{\mathcal{A}_2}(\gamma,\tilde{\mu}_3)\right| &\le \sup_{\gamma} \max_{\mu_3,\overline{\mu}_3} \left|F^{gr}_{\mathcal{A}_1}(\gamma,\mu_3) - F^{gr}_{\mathcal{A}_3}(\gamma,\overline{\mu}_3)\right| \\ &+ \sup_{\gamma} \max_{\overline{\mu}_3,\tilde{\mu}_3} \left|F^{gr}_{\mathcal{A}_3}(\gamma,\overline{\mu}_3) - F^{gr}_{\mathcal{A}_2}(\gamma,\tilde{\mu}_3)\right|. \end{aligned}$$

Finally, by adding both sides of three above inequalities, we obtain the inequality (3). □

Remark 3.2 Such metric D^{gr} is said to be the granular metric on the space $\mathcal{T}$ of all single valued triangular neutrosophic numbers. Hence, the space $\mathcal{T}$ equipped with the metric D^{gr} is a metric space.

Theorem 3.1 *The metric space $(\mathcal{T}, D^{gr})$ is complete space.*

Proof Let $\{\mathcal{A}_n\}_{n\ge1} \subset \mathcal{T}$ be a Cauchy sequence in $\mathcal{T}$, that means

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N}^* \text{ such that} \forall n \ge N,\ p \ge 1,\ \text{we have} D^{gr}(\mathcal{A}_{n+p},\mathcal{A}_n) < \epsilon,$$

or equivalently,

$$\sup_{\alpha,\beta,\gamma} \max_{\mu_1,\tilde{\mu}_1} \left| T^{gr}_{\mathcal{A}_{n+p}}(\alpha, \mu_1) - T^{gr}_{\mathcal{A}_n}(\alpha, \tilde{\mu}_1) \right| < \epsilon,$$

$$\sup_{\alpha,\beta,\gamma} \max_{\mu_2,\tilde{\mu}_2} \left| I^{gr}_{\mathcal{A}_{n+p}}(\beta, \mu_2) - I^{gr}_{\mathcal{A}_n}(\beta, \tilde{\mu}_2) \right| < \epsilon,$$

$$\sup_{\alpha,\beta,\gamma} \max_{\mu_3,\tilde{\mu}_3} \left| F^{gr}_{\mathcal{A}_{n+p}}(\gamma, \mu_3) - F^{gr}_{\mathcal{A}_n}(\gamma, \tilde{\mu}_3) \right| < \epsilon.$$

As a result, we directly obtain that

$$\left| T^{gr}_{\mathcal{A}_{n+p}}(\alpha, \mu_1) - T^{gr}_{\mathcal{A}_n}(\alpha, \tilde{\mu}_1) \right| < \epsilon,$$

$$\left| I^{gr}_{\mathcal{A}_{n+p}}(\beta, \mu_2) - I^{gr}_{\mathcal{A}_n}(\beta, \tilde{\mu}_2) \right| < \epsilon,$$

$$\left| F^{gr}_{\mathcal{A}_{n+p}}(\gamma, \mu_3) - F^{gr}_{\mathcal{A}_n}(\gamma, \tilde{\mu}_3) \right| < \epsilon.$$

Thus, we deduce that $\left\{T^{gr}_{\mathcal{A}_n}(\alpha, \mu_1)\right\}_{n\geq 1}$, $\left\{I^{gr}_{\mathcal{A}_n}(\beta, \mu_2)\right\}_{n\geq 1}$, $\left\{F^{gr}_{\mathcal{A}_n}(\gamma, \mu_3)\right\}_{n\geq 1}$ are Cauchy sequences in the space of real numbers $\mathbb{R}$, and then, these sequences are convergent in $\mathbb{R}$.

Particularly, let us consider the sequence $\left\{T^{gr}_{\mathcal{A}_n}(\alpha, \mu_1)\right\}_{n\geq 1}$. By Definition 3.1, we can rewrite $T^{gr}_{\mathcal{A}_n}(\alpha, \mu_1) = T^{-}_{\mathcal{A}_n}(\alpha) + \left(T^{+}_{\mathcal{A}_n}(\alpha) - T^{-}_{\mathcal{A}_n}(\alpha)\right)\mu_1$.

Since $\left\{T^{gr}_{\mathcal{A}_n}(\alpha, \mu_1)\right\}_{n\geq 1}$ is a convergent sequence and $0 \leq \mu_1 \leq 1$, it follows that the sequences $\left\{T^{-}_{\mathcal{A}_n}(\alpha)\right\}$ and $\left\{T^{+}_{\mathcal{A}_n}(\alpha)\right\}$ are also convergent. No loss generality, we assume that $\lim_{n\to\infty} T^{-}_{\mathcal{A}_n}(\alpha) = T^{-}_{\mathcal{A}}(\alpha)$, $\lim_{n\to\infty} T^{+}_{\mathcal{A}_n}(\alpha) = T^{+}_{\mathcal{A}}(\alpha)$ and due to the fact that $T^{-}_{\mathcal{A}_n}(\alpha) \leq T^{+}_{\mathcal{A}_n}(\alpha)$, $\forall n \geq 1$, we obtain that $T^{-}_{\mathcal{A}}(\alpha) \leq T^{+}_{\mathcal{A}}(\alpha)$. At last, by employing analogous method as in proof of Theorem 8.5 in [9], we deduce that the interval $[T^{-}_{\mathcal{A}}(\alpha), T^{+}_{\mathcal{A}}(\alpha)]$ is the $\alpha -$ cuts of a fuzzy number. As a consequence, the similar results are also obtained for the sequences $\left\{I^{gr}_{\mathcal{A}_n}(\beta, \mu_2)\right\}_{n\geq 1}$ and $\left\{F^{gr}_{\mathcal{A}_n}(\gamma, \mu_3)\right\}_{n\geq 1}$.

Therefore, we can see that if $\{\mathcal{A}_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{T}$ then $\mathcal{A}_n$ converges to an element $\mathcal{A} \in \mathcal{T}$. Hence, this achieves the proof. □

3.4 The continuity

Definition 3.6 Let $f : [a, b] \subset \mathbb{R} \to \mathcal{T}$ be a $\mathcal{T}$-valued function and $t_0 \in [a, b]$. An element $\ell \in \mathcal{T}$ is called the limit of $f(t)$ as t tends to t_0 and written by $\lim_{t\to t_0} f(t) = \ell$ iff for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in [a, b]$ satisfying $0 < |t - t_0| < \delta$ then $D^{gr}(f(t), \ell) < \epsilon$.

Especially, we have

- If $t_0 = a$ then $\lim_{t\to a^+} f(t) = \ell$ means that for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in [a, b]$ satisfying $0 < t - a < \delta$ then $D^{gr}(f(t), \ell) < \epsilon$.
- If $t_0 = b$ then $\lim_{t\to b^-} f(t) = \ell$ means that for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in [a, b]$ satisfying $0 < b - t < \delta$ then $D^{gr}(f(t), \ell) < \epsilon$.

Definition 3.7 $\mathcal{T} -$ valued function $f : (a, b) \subset \mathbb{R} \to \mathcal{T}$ is said to be continuous on (a, b) if for all $t_0 \in (a, b)$, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (a, b)$ satisfying $|t - t_0| < \delta$ then $D^{gr}(f(t), f(t_0) < \epsilon$.

3.5 The neutrosophic derivatives

Definition 3.8 Let $f : U \subset \mathbb{R} \rightarrow \mathcal{T}$ be a $\mathcal{T}-$ valued function. Then, f is called granular differentiable (gr-differentiable for short) at a point $t_0 \in U$ if there exists an element $f'_{gr}(t_0) \in \mathcal{T}$ such that the following limit

$$\lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus^{gr} f(t_0)}{h} = f'_{gr}(t_0),$$

exists for h sufficiently near 0 and then, the value $f'_{gr}(t_0)$ is called the granular derivative (gr-derivative) of $\mathcal{T}$-valued function f at the point t_0. The function f is called gr-differentiable on U if the gr-derivative of f exists for all points $t_0 \in U$ and mapping $t \in U \mapsto f'_{gr}(t)$ is then called gr-derivative of f and denoted by f'_{gr} or $\dot{f}_{gr}$.

Proposition 3.3 *A necessary and sufficient condition for a function $f : U \subset \mathbb{R} \rightarrow \mathcal{T}$ is gr-differentiable at a point $t_0 \in U$ is the differentiability of its horizontal membership function at that point. Moreover, we have $\mathcal{H}\left(f'_{gr}(t_0)\right) = \dfrac{\partial f^{gr}(t_0, \alpha, \beta, \gamma, \mu_f)}{\partial t}$.*

Proof Since the assumption $\mathcal{T}-$ valued function f is gr-differentiable at $t_0 \in U$, we have

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that} \forall h : 0 < h < \delta \Rightarrow D^{gr}\left(\frac{f(t_0 + h) \ominus^{gr} f(t_0)}{h}, f'_{gr}(t_0)\right) < \epsilon.$$

For simplicity in presentation, let us denote $\frac{f(t_0+h)\ominus^{gr} f(t_0)}{h}$ and $f'_{gr}(t_0)$ by $\mathcal{A}$ and $\mathcal{A}'$, respectively. Then, by employing the concept of granular metric, the above statement can be rewritten as follows

$$\begin{aligned}
&\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that} \forall h : 0 < h < \delta \Rightarrow \\
&\sup_{\alpha,\beta,\gamma} \max_{\mu_i,\tilde{\mu}_i} \frac{1}{3}\left\{\left|T^{gr}_{\mathcal{A}}(\alpha, \mu_1) - T^{gr}_{\mathcal{A}'}(\alpha, \tilde{\mu}_1)\right| + \left|I^{gr}_{\mathcal{A}}(\beta, \mu_2) - I^{gr}_{\mathcal{A}'}(\beta, \tilde{\mu}_2)\right|\right. \\
&\left. + \left|F^{gr}_{\mathcal{A}}(\gamma, \mu_3) - F^{gr}_{\mathcal{A}'}(\gamma, \tilde{\mu}_3)\right|\right\} < \epsilon,
\end{aligned}$$

that is equivalent to

$$\left\| \frac{1}{h}\left[f^{gr}(t_0 + h, \alpha, \beta, \gamma, \mu_f) - f^{gr}(t_0, \alpha, \beta, \gamma, \mu_f)\right] - \left(f'_{gr}\right)^{gr}(t_0, \alpha, \beta, \gamma, \mu_{f'}) \right\|$$

is getting as small as h tends to 0. Here, we denote

$$\frac{1}{h}\left[f^{gr}(t_0 + h, \alpha, \beta, \gamma, \mu_f) - f^{gr}(t_0, \alpha, \beta, \gamma, \mu_f)\right] = \left(T^{gr}_{\mathcal{A}}(\alpha, \mu_1), I^{gr}_{\mathcal{A}}(\beta, \mu_2), F^{gr}_{\mathcal{A}}(\gamma, \mu_3)\right),$$

$$\left(f'_{gr}\right)^{gr}(t_0, \alpha, \beta, \gamma, \mu_{f'}) = \left(T^{gr}_{\mathcal{A}'}(\alpha, \tilde{\mu}_1), I^{gr}_{\mathcal{A}'}(\beta, \tilde{\mu}_2), F^{gr}_{\mathcal{A}'}(\gamma, \tilde{\mu}_3)\right).$$

Therefore, this follows that the gr-differentiability of f implies the differentiability of its horizontal membership function. By analogous arguments, we also prove the converse statement. The proof is complete. □

Proposition 3.4 *Let $f,\ g : [a, b] \rightarrow \mathcal{T}$ be differentiable $\mathcal{T}-$ valued functions. Then, based on the horizontal membership function approach, the following statements are fulfilled:*

(i) $(\mathcal{A})'_{gr} = \tilde{0}$, *where $\mathcal{A} \in \mathcal{T}$ and $\tilde{0}$ is zero neutrosophic number.*
(ii) $(\alpha f(t) \pm \beta g(t))'_{gr} = \alpha f'_{gr}(t) + \beta g'_{gr}(t)$, *where $t \in [a, b]$ and $\alpha, \beta \in \mathbb{R}$.*
(iii) $(fg)'_{gr}(t) = f'_{gr}(t)g(t) + f(t)g'_{gr}(t)$, *where $t \in [a, b]$.*

Example 3.3 Let $\mathcal{A} = (5, 10, 15; 3, 6, 9; 10, 16, 22)$ and $\tilde{\mathcal{A}} = (4, 6, 8; 1, 3, 5; 9, 11, 13)$ be two triangular neutrosophic numbers of type 1. Consider the $\mathcal{T}$-valued function $f(t) = \mathcal{A}t + \tilde{\mathcal{A}}\sin 2t$ with $t \in [0, 7]$. Then, the horizontal membership function of f is given in Example 3.2 and its derivative is as

$$\begin{aligned}\frac{\partial f^{gr}(t, \alpha, \beta, \gamma, \mu, \tilde{\mu})}{\partial t} = &\Big([5 + 5\alpha + (10 - 10\alpha)\mu_1] + \big[8 + 4\alpha + (8 - 8\alpha)\tilde{\mu}_1\big]\cos 2t;\\ &[6 - 3\beta + 6\beta\mu_2] + \big[6 - 4\beta + 8\beta\tilde{\mu}_2\big]\cos 2t;\\ &[16 - 6\gamma + 12\gamma\mu_3] + \big[22 - 4\gamma + 8\gamma\tilde{\mu}_3\big]\cos 2t\Big),\end{aligned}$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\mu = (\mu_1, \mu_2, \mu_3),\ \tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3) \in [0, 1]^3$. Thus, it follows that the function f is gr-differentiable and the (α, β, γ) - cuts of its derivative is given by

$$\begin{aligned}[f'_{gr}(t)]_{(\alpha,\beta,\gamma)} &= \mathcal{H}^{-1}\left(\frac{\partial f^{gr}(t, \alpha, \alpha_u, \alpha_v)}{\partial t}\right)\\ &= ([5 + 5\alpha, 15 - 5\alpha] + [8 + 4\alpha, 16 - 4\alpha]\cos 2t;\\ &\quad [6 - 3\beta + 6 + 3\beta] + [6 - 4\beta, 6 + 4\beta]\cos 2t;\\ &\quad [16 - 6\gamma, 16 + 6\gamma] + [22 - 4\gamma, 22 + 4\gamma]\cos 2t).\end{aligned}$$

By using $(\alpha, \beta, \gamma)-$cuts representation theorem, we have

$$\begin{aligned}f'_{gr}(t) = &\Bigg(\bigcup_{\alpha}\{[5 + 5\alpha, 15 - 5\alpha] + [8 + 4\alpha, 16 - 4\alpha]\cos 2t\};\\ &\bigcup_{\beta}\{[6 - 3\beta, 6 + 3\beta] + [6 - 4\beta, 6 + 4\beta]\cos 2t\};\\ &\bigcup_{\gamma}\{[16 - 6\gamma, 16 + 6\gamma] + [22 - 4\gamma, 22 + 4\gamma]\cos 2t\}\Bigg)\\ = &(5, 10, 15; 3, 6, 9; 10, 16, 22) + (8, 12, 16; 2, 6, 10; 18, 22, 26)\cos 2t\end{aligned}$$

Therefore, we obtain that the gr-derivative $f'_{gr}(t)$ is $\mathcal{A} + 2\tilde{\mathcal{A}}\cos 2t$ which graphical representation is shown in Fig. 5

3.6 The neutrosophic integral

Definition 3.9 Let $f : [a, b] \to \mathcal{T}$ be a continuous $\mathcal{T}-$ function whose HMF $\mathcal{H}(f(t))$ is integrable on $[a, b]$. If there exists a number $\upsilon \in \mathcal{T}$ such that $\mathcal{H}(\upsilon) = \int_a^b \mathcal{H}(f(t))\,dt$ then such number m is said to be the granular integral (gr-integral for short) of f on $[a, b]$ and denoted by $\upsilon = \int_a^b f(t)dt$.

Remark 3.3 Similar to Proposition 3.3, we can easily prove that the integrability of the function f is equivalent to the integrability of its horizontal membership function.

Remark 3.4 Let a function $f : [a, b] \to \mathcal{T}$ and a point $x \in [a, b]$. Then, if f is integrable on $[a, b]$, f is also integrable on the sub-interval $[a, x]$.

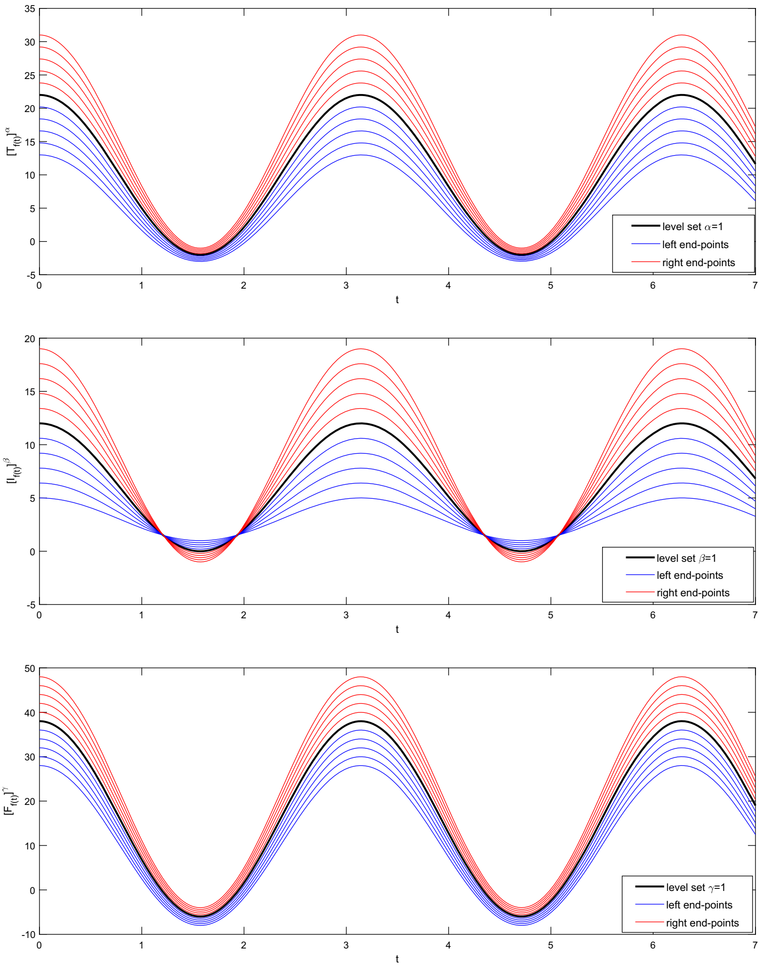

Fig. 5 The (α, β, γ)- cuts of the gr-derivative of $\mathcal{T}$-valued function f in Example 3.3, where the black curve corresponds to the certain values, while the blue and red curves represent for the left and right end-points

Lemma 3.1 *If function $f : [a, b] \to \mathcal{T}$ is continuous on $[a, b]$ then for each $t \in [a, b]$, the function $\Psi(t) = \displaystyle\int_a^t f(s)ds$ is an anti-derivative of the function f.*

Proof Let $t_0 \in [a, b]$ be arbitrary. Due to the continuity of f at point t_0, we have that for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in [a, b]$ satisfying $|t - t_0| < \delta$ then $D^{gr}(f(t), f(t_0)) < \epsilon$, i.e., $\lim_{t \to t_0} f(t) = f(t_0)$.

For h sufficiently near 0, let us consider the following quotient

$$\frac{\Delta\Psi}{\Delta t_0} = \frac{1}{h}\left[\Psi(t_0 + h) \ominus^{gr} \Psi(t_0)\right] = \frac{1}{h}\left[\int_a^{t_0+h} f(s)ds \ominus^{gr} \int_a^{t_0} f(s)ds\right],$$

whose horizontal membership function is given by

$$\begin{aligned}\mathcal{H}\left(\frac{\Delta\Psi}{\Delta t_0}\right) &= \frac{1}{h}\left[\int_a^{t_0+h}\mathcal{H}(f(s))ds \ominus^{gr} \int_a^{t_0}\mathcal{H}(f(s))ds\right]\\ &= \frac{1}{h}\left[\int_a^{t_0+h} f^{gr}(s_0+h,\alpha,\beta,\gamma,\mu_f)ds - \int_a^{t_0} f^{gr}(s_0,\alpha,\beta,\gamma,\mu_f)ds\right]\\ &= \frac{1}{h}\int_{t_0}^{t_0+h} f^{gr}(s_0+h,\alpha,\beta,\gamma,\mu_f)ds.\end{aligned}$$

Next, by applying mean value theorem for integrals, we obtain that

$$\begin{aligned}\mathcal{H}\left(\frac{\Delta\Psi}{\Delta t_0}\right) &= \frac{1}{h}\int_{t_0}^{t_0+h} f^{gr}(s_0+h,\alpha,\beta,\gamma,\mu_f)ds\\ &= f^{gr}(t_0+\theta h,\alpha,\beta,\gamma,\mu_f),\end{aligned}$$

in which $\theta \in (0, 1)$. Since the fact that $t_0 + \theta h$ tends to t_0 as $h \to 0$ then we have

$$\mathcal{H}\left(\Psi'_{gr}(t_0)\right) = \lim_{h\to 0}\mathcal{H}\left(\frac{\Delta\Psi}{\Delta t_0}\right) = \lim_{h\to 0} f^{gr}(t_0+\theta h,\alpha,\beta,\gamma,\mu_f) = f^{gr}(t_0,\alpha,\beta,\gamma,\mu_f),$$

which implies that $[\Psi'_{gr}(t_0)]_{(\alpha,\beta,\gamma)} = \mathcal{H}^{-1}(f^{gr}(t_0,\alpha,\beta,\gamma,\mu_f)) = [f(t_0)]_{(\alpha,\beta,\gamma)}$ for each $\alpha,\beta,\gamma \in [0, 1]$. Since the point $t_0 \in [a, b]$ is chosen arbitrarily, this achieves the proof. □

Theorem 3.2 (Newton-Leibniz Formula) *Assume that* $\Phi : [a, b] \subseteq \mathbb{R} \to \mathcal{T}$ *be a gr-differentiable* $\mathcal{T}$*-valued function and function* $f(t) := \Phi'_{gr}(t)$ *is continuous on* $[a, b]$*. Then* f *is gr-integrable and*

$$\int_a^b f(t)dt = \Phi(b) \ominus^{gr} \Phi(a).$$

Proof By Lemma 3.1, we obtain that function $\Psi(t) = \int_a^t f(s)ds$ is an anti-derivative of function f on $[a, b]$ whose horizontal membership function is given as follows

$$\Psi^{gr}(t,\alpha,\beta,\gamma,\mu_\Psi) = \int_a^t f^{gr}(s,\alpha,\beta,\gamma,\mu_f)ds,$$

that means $\Psi^{gr}(t,\alpha,\beta,\gamma,\mu_\Psi)$ is also an anti-derivative of $f^{gr}(t,\alpha,\beta,\gamma,\mu_f)$ on $[a, b]$. Hence, if $\Phi^{gr}(t,\alpha,\beta,\gamma,\mu_\Phi)$ is another anti-derivative of $f^{gr}(t,\alpha,\beta,\gamma,\mu_f)$ on $[a, b]$ then

$$\begin{aligned}\Phi^{gr}(t,\alpha,\beta,\gamma,\mu_\Phi) &= \Psi^{gr}(t,\alpha,\beta,\gamma,\mu_\Psi) + \mathcal{C}\\ &= \int_a^t f^{gr}(s,\alpha,\beta,\gamma,\mu_f)ds + \mathcal{C},\end{aligned}$$

where $\mathcal{C}$ is a constant.

In addition, by substituting $t = a$, we have $\Phi^{gr}(a,\alpha,\beta,\gamma,\mu_\Phi) = \int_a^a f^{gr}(s,\alpha,\beta,\gamma,\mu_f)$ $ds + \mathcal{C}$, or equivalently, $\Phi^{gr}(a,\alpha,\beta,\gamma,\mu_\Phi) = \mathcal{C}$. Thus, it implies that

$$\begin{aligned}&\Phi^{gr}(t,\alpha,\beta,\gamma,\mu_\Phi) = \int_a^t f^{gr}(s,\alpha,\beta,\gamma,\mu_f)ds + \Phi^{gr}(a,\alpha,\beta,\gamma,\mu_\Phi)\\ \Longleftrightarrow\ &\int_a^t f^{gr}(s,\alpha,\beta,\gamma,\mu_f)ds = \Phi^{gr}(t,\alpha,\beta,\gamma,\mu_\Phi) - \Phi^{gr}(a,\alpha,\beta,\gamma,\mu_\Phi).\end{aligned}$$

Letting $t = b$ then we obtain the following formula

$$\int_a^b f^{gr}(s, \alpha, \beta, \gamma, \mu_f)ds = \Phi^{gr}(b, \alpha, \beta, \gamma, \mu_\Phi) - \Phi^{gr}(a, \alpha, \beta, \gamma, \mu_\Phi),$$

whose $(\alpha, \beta, \gamma) -$ cuts can be given as follows

$$\int_a^b \mathcal{H}^{-1}\left(f^{gr}(s, \alpha, \beta, \gamma, \mu_f)\right) ds = \mathcal{H}^{-1}\left(\Phi^{gr}(b, \alpha, \beta, \gamma, \mu_\Phi) - \Phi^{gr}(a, \alpha, \beta, \gamma, \mu_\Phi)\right),$$

that means the following integral equality holds

$$\int_a^b f(t)dt = \Phi(b) \ominus^{gr} \Phi(a).$$

□

Example 3.4 Consider a function $\Phi : [0, 5] \to \mathcal{T}$ given by $\Phi(t) = \mathcal{A}_1 e^{-t} + \mathcal{A}_2 t^2$ in which $\mathcal{A}_1 = (4, 7, 10; 0, 1, 2; 3, 5, 7)$ and $\mathcal{A}_2 = (2, 4, 6; 1, 2, 3; 1, 3, 5) \in \mathcal{T}$. Then, the horizontal membership function of $\Phi(t)$ is given by

$$\begin{aligned}
\mathcal{H}(\Phi(t)) &= \Phi^{gr}(t, \alpha, \beta, \gamma, \mu, \tilde{\mu}) \\
&= A_1^{gr}(\alpha, \beta, \gamma, \mu)e^{-t} + A_2^{gr}(\alpha, \beta, \gamma, \tilde{\mu})t^2 \\
&= \Big([4 + 3\alpha + (6 - 6\alpha)\mu_1]\, e^{-t} + \big[2 + 2\alpha + (4 - 4\alpha)\tilde{\mu}_1\big]\, t^2; \\
&\qquad [1 - \beta + 2\beta\mu_2]\, e^{-t} + \big[2 - \beta + 2\beta\tilde{\mu}_2\big]\, t^2; [5 - 2\gamma + 4\gamma\mu_3]\, e^{-t} \\
&\quad + \big[3 - 2\gamma + 4\gamma\tilde{\mu}_3\big]\, t^2\Big).
\end{aligned}$$

By using similar arguments as in Example 3.2, we have that the $\mathcal{T} -$ valued function $\Phi(t)$ is gr-differentiable on [0, 5] and its derivative is $\Phi'_{gr}(t) = f(t)$ whose horizontal membership function is given by

$$\begin{aligned}
f^{gr}(t, \alpha, \beta, \gamma, \mu, \tilde{\mu}) &= \frac{\partial \Phi^{gr}(t, \alpha, \beta, \gamma, \mu, \tilde{\mu})}{\partial t} \\
&= -A_1^{gr}(\alpha, \beta, \gamma, \mu)e^{-t} + 2A_2^{gr}(\alpha, \beta, \gamma, \tilde{\mu})t \\
&= \big([-4 - 3\alpha - (6 - 6\alpha)\mu_1]\, e^{-t} + \big[4 + 4\alpha + (8 - 8\alpha)\tilde{\mu}_1\big]\, t; \\
&\qquad [-1 + \beta - 2\beta\mu_2]\, e^{-t} + \big[4 - 2\beta + 4\beta\tilde{\mu}_2\big]\, t; \\
&\qquad [-5 + 2\gamma - 4\gamma\mu_3]\, e^{-t} + \big[6 - 4\gamma + 8\gamma\tilde{\mu}_3\big]\, t\big),
\end{aligned}$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\mu = (\mu_1, \mu_2, \mu_3)$, $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3) \in [0, 1]^3$. Then, employing $(\alpha, \beta, \gamma) -$ level sets representation theorem, we obtain that

$$f(t) = \mathcal{A}_3 e^{-t} + \mathcal{A}_4 t,$$

in which $\mathcal{A}_3 = (-10, -7, -4; -2, -1, 0; -7, -5, -3)$, $\mathcal{A}_4 = (4, 8, 12; 2, 4, 6; 2, 6, 10)$ are single valued triangular neutrosophic numbers. In addition, we can see that f is continuous function on [0, 10]. Then, by applying Theorem 3.2, it follows that

$$\int_0^5 f(t)dt = \Phi(5) \ominus^{gr} \Phi(0) = \left(\mathcal{A}_3 e^{-5} + 5\mathcal{A}_4\right) \ominus^{gr} \mathcal{A}_3 = \left(1 - e^{-5}\right) \mathcal{A}_1 + 5\mathcal{A}_4.$$

4 Applications to $\mathcal{T}-$ valued differential equations

4.1 $\mathcal{T}-$ valued differential equations

In the following, based on the HMF approach, we will investigate some classes of $\mathcal{T}-$ valued differential equations. Indeed, let us consider following initial problem to $\mathcal{T}-$ valued differential equations

$$\begin{cases} x'_{gr}(t) &= f(t, x(t)) \\ x(t_0) &= x_0 \end{cases} \qquad t \in [t_0, T], \tag{4}$$

where $f : [t_0, T] \to \mathcal{T}$ is a $\mathcal{T}-$ valued function including n distinct neutrosophic numbers $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, $x'_{gr}(\cdot)$ represents for the gr-derivative of $x(\cdot)$ w.r.t t and $x_0 \in \mathcal{T}$ is initial condition. According to Definition 3.2, the initial problem (4) can be transformed into following form

$$\begin{cases} \mathcal{H}(x'_{gr}(t)) &= \mathcal{H}(f(t, x(t))) \\ \mathcal{H}(x(t_0)) &= \mathcal{H}(x_0) \end{cases} \qquad t \in [t_0, T],$$

Then, by using Proposition 3.3, we obtain that

$$\begin{cases} \dfrac{\partial x^{gr}(t, \alpha, \beta, \gamma, \mu_x)}{\partial t} &= f^{gr}(t, x^{gr}(t, \alpha, \beta, \gamma, \mu_x), \alpha, \beta, \gamma, \mu_f) \\ x^{gr}(t_0, \alpha, \beta, \gamma, \mu_x) &= x_0^{gr}(\alpha, \beta, \gamma, \mu_x) \end{cases} \qquad t \in [t_0, T], \tag{5}$$

where $\alpha, \beta, \gamma \in [0, 1]$, $\mu_x \in [0, 1]^3$ and $\mu_f \triangleq (\mu_{1,\mathcal{A}_1}, \ldots, \mu_{1,\mathcal{A}_n}, \mu_{2,\mathcal{A}_1}, \ldots, \mu_{2,\mathcal{A}_n}, \mu_{3,\mathcal{A}_1}, \ldots, \mu_{3,\mathcal{A}_n})$.

Thus, under the HMF approach, we can see that the use of gr-differentiability help us only need to solve just one differential equation that is equivalent to the given equation and we call this equivalent equation is granular differential equation. Moreover, we can see that if $\mathcal{T}-$ valued differential equation (4) doesn't have solution then the corresponding granular differential equation also does not. Conversely, if $\tilde{x}^{gr}(t, \alpha, \beta, \gamma, \mu_x)$ is a solution of problem (5) then it is also the solution of problem (4).

Remark 4.1 Some important results such the well-posedness or the existence and uniqueness of solution to Cauchy problems (4) for $\mathcal{T}-$ valued differential equations correspond to those of Cauchy problem (5) for granular differential equations.

Example 4.1 Consider following $\mathcal{T}-$ valued differential equations

$$\begin{cases} \dot{x}_{gr}(t) &= y(t) - \tilde{u} \\ \dot{y}_{gr}(t) &= -4x(t) \end{cases} \tag{6}$$

subject to the initial condition

$$x(0) = y(0) = \tilde{0},$$

where $\tilde{u} = (-1, 0, 1; 0, 1, 2; -2, 0, 2)$, $\tilde{0} = (0, 0, 0; 0, 0, 0; 0, 0, 0)$ are single valued triangular neutrosophic numbers and $t \in [0, 10]$. By using the similar method to obtain the system (5), it follows that the corresponding granular system of differential equations of (6) is

$$\begin{cases} \dfrac{\partial x^{gr}(t,\alpha,\beta,\gamma,\mu_x)}{\partial t} &= y^{gr}(t,\alpha,\beta,\gamma,\mu_y) - \tilde{u}^{gr}(\alpha,\beta,\gamma,\mu_{\tilde{u}}) \\ \dfrac{\partial y^{gr}(t,\alpha,\beta,\gamma,\mu_y)}{\partial t} &= -x^{gr}(t,\alpha,\beta,\gamma,\mu_x) \\ x^{gr}(0,\alpha,\beta,\gamma,\mu_x) &= y^{gr}(0,\alpha,\beta,\gamma,\mu_y) = 0, \end{cases}$$

in which $\tilde{u}^{gr}(\alpha,\beta,\gamma,\mu_{\tilde{u}}) = \big(\alpha - 1 + (2-2\alpha)\mu_{\tilde{u},1};\, 1-\beta+2\beta\mu_{\tilde{u},2};\, 4\gamma\mu_{\tilde{u},3}\big)$ and the triplets $\mu_x = \mu_y = \mu_{\tilde{u}} = \overline{\mu}$. Then, the solution of the above system is given as

$$\begin{cases} x^{gr}(t,\alpha,\beta,\gamma,\overline{\mu}) &= -\tilde{u}^{gr}(\alpha,\beta,\gamma,\overline{\mu})\sin 2t \\ y^{gr}(t,\alpha,\beta,\gamma,\overline{\mu}) &= \dfrac{1}{2}\,\tilde{u}^{gr}(\alpha,\beta,\gamma,\overline{\mu})\,(1-\cos 2t)\,, \end{cases}$$

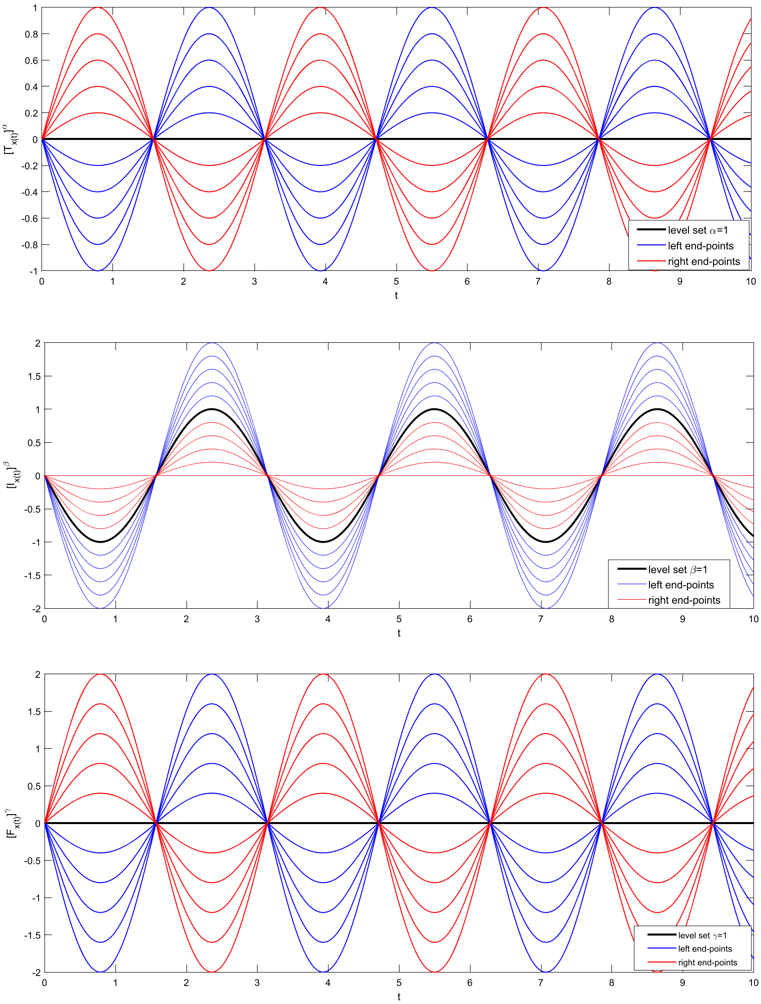

Fig. 6 The $(\alpha,\beta,\gamma)-$ cuts of function $x(t)$ that corresponds to the solution of system (6)

whose $(\alpha, \beta, \gamma)-$ cuts can be given as

$$\begin{cases} [x(t)]_{(\alpha,\beta,\gamma)} = ([-1+\alpha, 1-\alpha]; [-1-\beta, -1+\beta]; [-2\gamma, 2\gamma]) \sin 2t \\ [y(t)]_{(\alpha,\beta,\gamma)} = \left(\left[-\frac{1}{2}+\frac{\alpha}{2}, \frac{1}{2}-\frac{\alpha}{2}\right]; \left[\frac{1}{2}-\frac{\beta}{2}, \frac{1}{2}+\frac{\beta}{2}\right]; [-\gamma, \gamma]\right)(1-\cos 2t) \end{cases}$$

Figures 6 and 7 show the $(\alpha, \beta, \gamma)-$ level sets of solution of the system (6).

4.2 Some real-life models

Example 4.2 (Logistic equations) In this example, we consider dynamics of a single population model. We denote by $x = \Phi(t)$ the number of individuals of a given species at the time t and r by the percent change of the population. If r is not impacted by the limitation of space and food then we can assume it as a constant. However, in real world, this assumption is unrealistic. Thus, in modeling models of population by dynamic system, we often

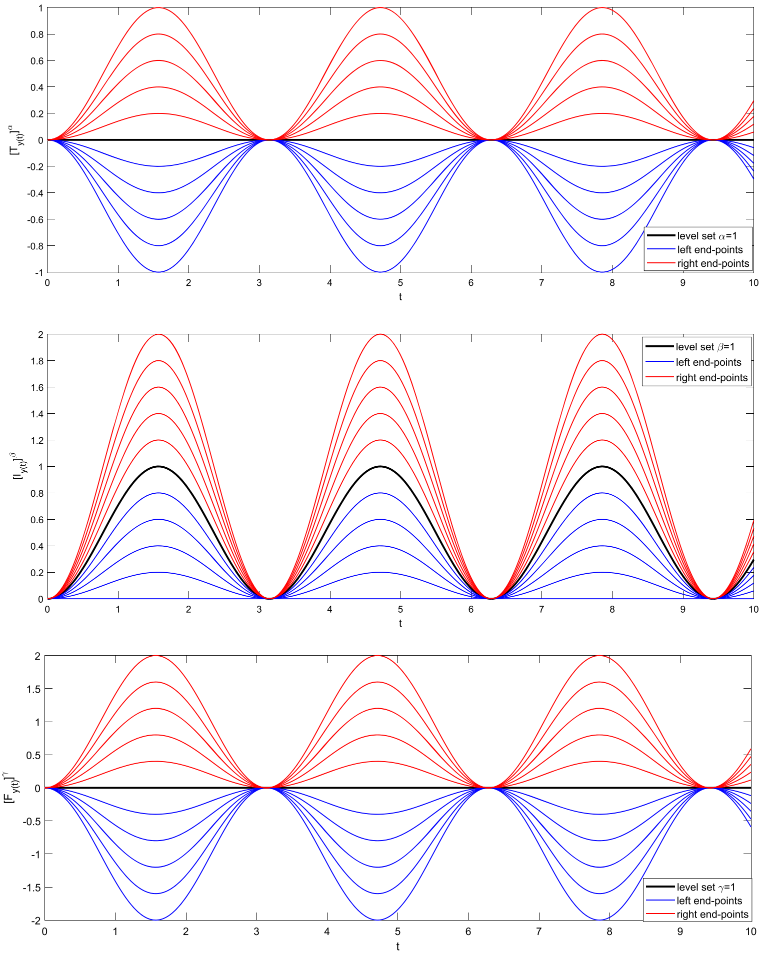

Fig. 7 The $(\alpha, \beta, \gamma)-$ cuts of function $y(t)$ that corresponds to the solution of system (6)

modify the unrestricted growth rate r to ensure that the environment can only support a certain number of the species, denoted by $\mathcal{K}$, namely the carrying capacity of the environment with populations living in. If $x > \mathcal{K}$ then it cause consequences the lack of food and space available to support x, more species will be die than will be born, which leads to the negative growth rate. Conversely, if $x < \mathcal{K}$ then the population growth should be positive. Using the above model of the population growth, we consider the following differential equation that is known as the Verhulst equations or logistic equations

$$\begin{cases} \dot{\Phi}_{gr}(t) &= r\Phi(t)\cdot(1.5 \ominus^{gr} \Phi(t)) \\ \Phi(0) &= \mathcal{A}, \end{cases} \qquad t \in [0,7],$$

where $r = 0.8$ and $\mathcal{A} = (0.1, 0.3, 0.5; 0.1, 0.2, 0.3; 0, 0.1, 0.2)$. Here, due to the uncertainty of available information about the initial population of the species when modeling this real-world problems, neutrosophic value presentation has been considered as a better description in the formulation of the mathematical model.

In addtion, based on the approach mentioned in previous section, we have

$$\frac{\partial \Phi^{gr}(t,\alpha,\beta,\gamma,\mu)}{\partial t} = 0.8\Phi^{gr}(t,\alpha,\beta,\gamma,\mu)\left(1.5 - \Phi^{gr}(t,\alpha,\beta,\gamma,\mu)\right) \tag{7}$$

subject to the initial condition

$$\Phi^{gr}(0,\alpha,\beta,\gamma,\mu) = (0.3 + 0.2\alpha + 0.4(1-\alpha)\mu_1; 0.2 - 0.1\beta + 0.2\beta\mu_2; 0.1 - 0.1\gamma + 0.2\gamma\mu_3),$$

where $\alpha, \beta, \gamma \in [0,1]$ and $\mu_i \in [0,1]$ $(i = 1, 2, 3)$.

The solution of the granular differential (1) is

$$\Phi^{gr}(t,\alpha,\beta,\gamma,\mu) = \frac{1.5\Phi^{gr}(0,\alpha,\beta,\gamma,\mu)}{\Phi^{gr}(0,\alpha,\beta,\gamma,\mu) + [1.5 - \Phi^{gr}(0,\alpha,\beta,\gamma,\mu)]\, e^{-1.2t}},$$

whose $(\alpha,\beta,\gamma)-$ cuts is given as follows

$$\begin{aligned}[\Phi(t)]_{(\alpha,\beta,\gamma)} = \Bigg[& \frac{0.15 + 0.3\alpha}{0.1 + 0.2\alpha + (1.4 - 0.2\alpha)\, e^{-1.2t}}, \frac{0.75 - 0.3\alpha}{0.5 - 0.2\alpha + (1 + 0.2\alpha)\, e^{-1.2t}} \\ & \frac{0.3 - 0.15\beta}{0.2 - 0.1\beta + (1.3 + 0.1\beta)\, e^{-1.2t}}, \frac{0.3 + 0.15\beta}{0.2 + 0.1\beta + (1.3 - 0.1\beta)\, e^{-1.2t}} \\ & \frac{0.15 - 0.15\gamma}{0.1 - 0.1\gamma + (1.4 + 0.1\gamma)\, e^{-1.2t}}, \frac{0.15 + 0.15\gamma}{0.1 + 0.1\gamma + (1.4 - 0.1\gamma)\, e^{-1.2t}} \Bigg].\end{aligned}$$

The $(\alpha,\beta,\gamma)-$ level sets of the solution of the logistic (3.3) with respect to the initial condition $\Phi(0) = (0.1, 0.3, 0.5; 0.1, 0.2, 0.3; 0, 0.1, 0.2)$ is presented in Fig. 8

Remark 4.2 From the above figure, we see that if at the initial time, the population of species is in the carrying capacity of the environment then the population will approach to the carrying capacity value as time increases.

Example 4.3 The inverted pendulum system is a popular demonstration of using feedback control to stabilize an open-loop unstable system. In this example, we consider the following mechanical system which model an inverted pendulum on the cart.

By applying Newton's second law to mechanical system including two masses m_1, m_2, we have following nonlinear model

$$\begin{cases} (m_1 + m_2)\ddot{y} + m_2\ell\ddot{\theta}\cos\theta - m_2\ell\dot{\theta}^2\sin\theta + \mu\dot{y} &= u \\ \ell\ddot{\theta} - g\sin\theta + \ddot{y}\cos\theta &= 0. \end{cases}$$

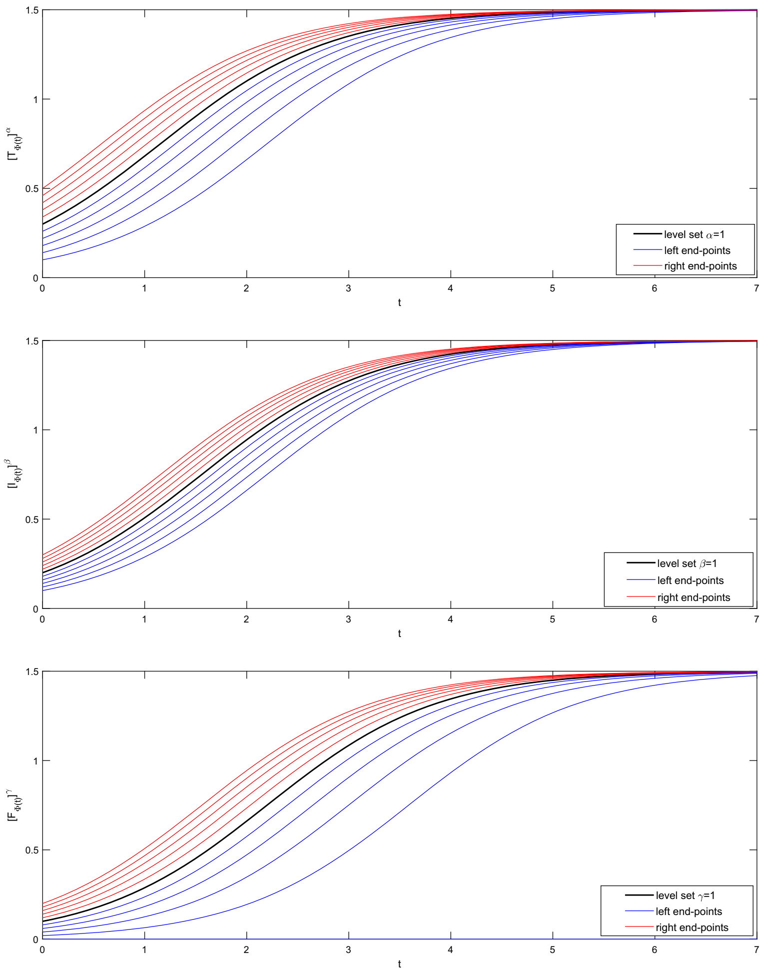

Fig. 8 The $(\alpha, \beta, \gamma)-$ cuts of the solution $\Phi(t)$ of the logistic (3.3).

Let $x_1 = y$, $x_2 = \theta$, $x_3 = \dot{y}$, $x_4 = \dot{\theta}$. Then, we obtain the state equation corresponding to the mechanical system in Fig. 9

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \begin{bmatrix} m_1 + m_2 & m_2 \ell \cos x_2 \\ \cos x_2 & \ell \end{bmatrix}^{-1} \left(\begin{bmatrix} m_2 \ell x_4^2 \sin x_2 - \mu x_3 \\ g \sin x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \right) \end{bmatrix}.$$

Next, by using linearization method, we obtain the linearized system of inverted pendulum model as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m_2}{m_1} g & -\frac{\mu}{m_1} & 0 \\ 0 & \frac{m_1 + m_2}{m_1 \ell} g & \frac{\mu}{m_1 \ell} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ -\frac{1}{m_1 \ell} \end{bmatrix} u, \tag{8}$$

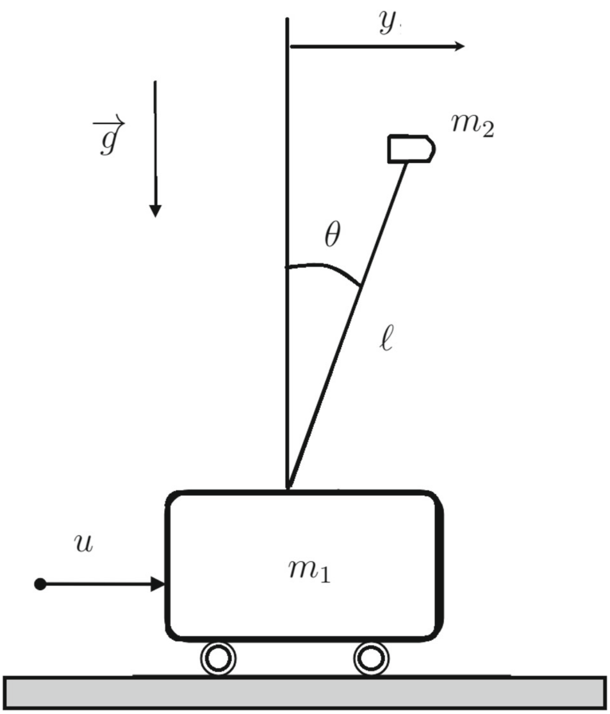

Fig. 9 Inverted pendulum on cart

in which the parameters and their values are given in Table 4.

Here, we consider the acceleration of gravity $g = (9.6, 9.8, 10; 0.5, 1.5, 2.5; 1.5, 2, 2.5) \in \mathcal{T}$ is an uncertain quantity due to errors in measurement and influence of environmental factors such temperature, humidity, meteorology, etc. From the uncertainty of g, it follows that the matrix's coefficients of (8) is also uncertain, that is equivalent to the uncertainty in the form of solution.

Table 4 Parameter's value

m_1	mass of the cart	3 kg
m_2	mass of the pendulum	1 kg
ℓ	length of the pendulum	2 m
μ	friction coefficient	$\frac{166}{39}$ N.s/m
y	position of the cart	
θ	angular rotation	
u	force on the cart	
g	acceleration of gravity	

Table 5 Truth membership function of Re$\lambda(A)$

μ \ α	0	0.2	0.4	0.6	0.8	1.0
0	2.41	2.415	2.42	2.425	2.43	2.436
0.25	2.423	2.425	2.428	2.43	2.433	2.436
0.5	2.436	2.436	2.436	2.436	2.436	2.436
0.75	2.448	2.446	2.443	2.441	2.439	2.436
1.0	2.461	2.456	2.451	2.446	2.441	2.436

In this example, our main aim is to show that the open-loop system is unstable, i.e., the system (8) is considered under assumption that external force $u \equiv 0$. Indeed, the system (8) then becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{g}{3} & -\frac{166}{117} & 0 \\ 0 & \frac{2g}{3} & \frac{83}{117} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{9}$$

Based on the concepts of gr-differentiability and horizontal membership function approach, the differential system (9) can be transformed into following form

$$\begin{bmatrix} \frac{\partial x_1^{gr}(t,\alpha,\beta,\gamma,\mu_{x_1})}{\partial t} \\ \frac{\partial x_2^{gr}(t,\alpha,\beta,\gamma,\mu_{x_2})}{\partial t} \\ \frac{\partial x_3^{gr}(t,\alpha,\beta,\gamma,\mu_{x_3})}{\partial t} \\ \frac{\partial x_4^{gr}(t,\alpha,\beta,\gamma,\mu_{x_4})}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{g^{gr}(\alpha,\beta,\gamma,\mu_g)}{3} & -\frac{166}{117} & 0 \\ 0 & \frac{2g^{gr}(\alpha,\beta,\gamma,\mu_g)}{3} & \frac{83}{117} & 0 \end{bmatrix} \begin{bmatrix} x_1^{gr}(t,\alpha,\beta,\gamma,\mu_{x_1}) \\ x_2^{gr}(t,\alpha,\beta,\gamma,\mu_{x_2}) \\ x_3^{gr}(t,\alpha,\beta,\gamma,\mu_{x_3}) \\ x_4^{gr}(t,\alpha,\beta,\gamma,\mu_{x_4}) \end{bmatrix}, \tag{10}$$

Let us choose $\mu_{x_1} = \mu_{x_2} = \mu_{x_3} = \mu_{x_4} = \mu_g = \mu$ and denote

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{g^{gr}(\alpha,\beta,\gamma,\mu)}{3} & -\frac{166}{117} & 0 \\ 0 & \frac{2g^{gr}(\alpha,\beta,\gamma,\mu)}{3} & \frac{83}{117} & 0 \end{bmatrix}.$$

Then, since the stability of system (9) is equivalent to the stability of corresponding granular linear differential system, the rest of proof is to show that the linear system (10) is unstable for each $(\alpha, \beta, \gamma)-$ cuts and $\mu \in [0, 1]$. By using MATLAB's tool, we obtain following tables about the truth, indeterminacy and falsity membership function of Re$\lambda(A)$ (Tables 5, 6 and 7).

Table 6 Indeterminacy membership function of Re$\lambda(A)$

μ \ β	0	0.2	0.4	0.6	0.8	1.0
0	0.921	0.855	0.784	0.707	0.621	0.522
0.25	0.921	0.889	0.855	0.821	0.784	0.747
0.5	0.921	0.921	0.921	0.921	0.921	0.921
0.75	0.921	0.952	0.983	1.012	1.041	1.069
1.0	0.921	0.983	1.041	1.097	1.15	1.2

Table 7 Falsity membership function of Re$\lambda(A)$

μ \ γ	0	0.2	0.4	0.6	0.8	1.0
0	1.069	1.041	1.012	0.983	0.952	0.921
0.25	1.069	1.055	1.041	1.027	1.012	0.998
0.5	1.069	1.069	1.069	1.069	1.069	1.069
0.75	1.069	1.083	1.097	1.11	1.123	1.137
1.0	1.069	1.097	1.123	1.15	1.175	1.2

As a result, we can see that Re$\lambda(A)$ is always positive for each $\alpha, \beta, \gamma \in [0, 1]$ and $\mu \in [0, 1]$, that means the granular linear differential system (10) is unstable. Therefore, it implies that the open-loop system of inverted pendulum model is an unstable system.

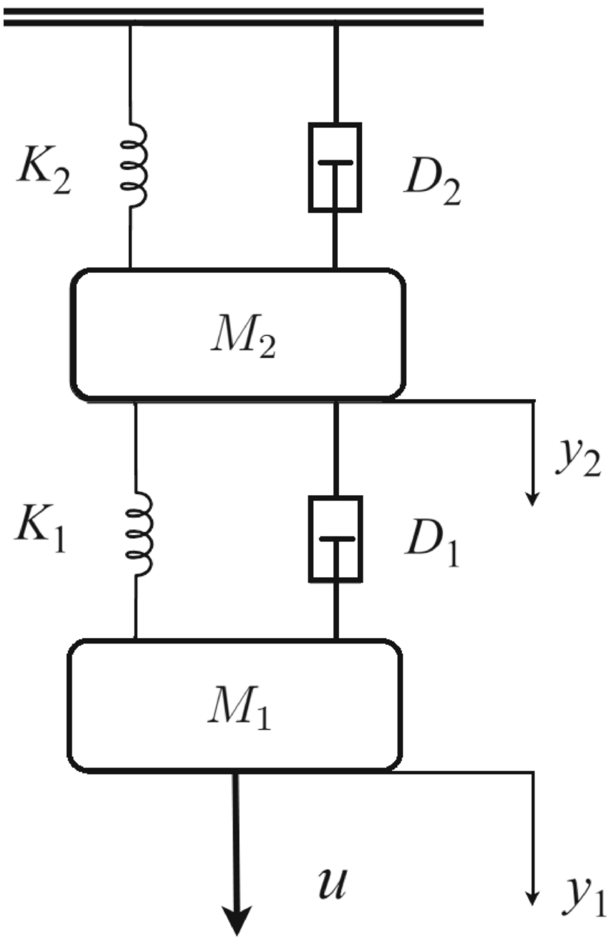

Fig. 10 Mass - Spring - Damper model

Table 8 Parameter values

K_1	the spring constant 1	150 N/m
K_2	the spring constant 2	300 N/m
D_1	the friction coefficient 1	100 N.s/m
D_2	the friction coefficient 2	$\frac{550}{3}$ N.s/m
M_1	mass 1	10 kg
M_2	mass 2	25 kg
y_1, y_2	the displacements	

Example 4.4 (Mass - Spring - Damper) Consider a mechanical system containing two masses that are hung from the ceiling by two strings. Here, each string can be modeled as a combination of a spring and a dashpot for friction (see Fig. 10). If we act to the system an

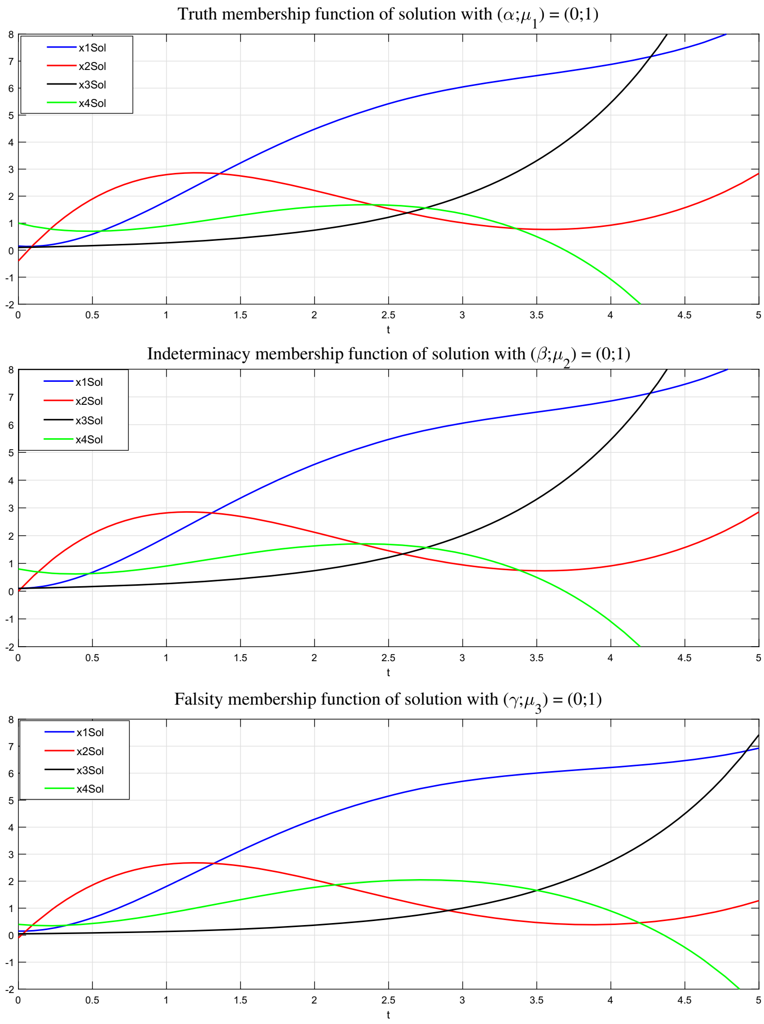

Fig. 11 The (α, β, γ) −cuts of solution of the problem (11)-(12) with $\alpha = \beta = \gamma = 0$ and $\mu = 1$

external force u then by Hook's law, we can deduce that the forces are linearly proportional to the corresponding displacements, while the forces due to the frictions depend on both displacements and velocities. By applying Newton's second law to two masses m_1 and m_2, we obtain that

$$\begin{cases} M_1\ddot{y}_1 = u - K_1(y_1 - y_2) - D_1(\dot{y}_1 - \dot{y}_2) \\ M_2\ddot{y}_2 = K_1(y_1 - y_2) + D_1(\dot{y}_1 - \dot{y}_2) - K_2 y_2 - D_2\dot{y}_2, \end{cases}$$

where $\dot{y}_i$, $\ddot{y}_i$ represent for gr-derivative and second gr-derivative of y_i, respectively.

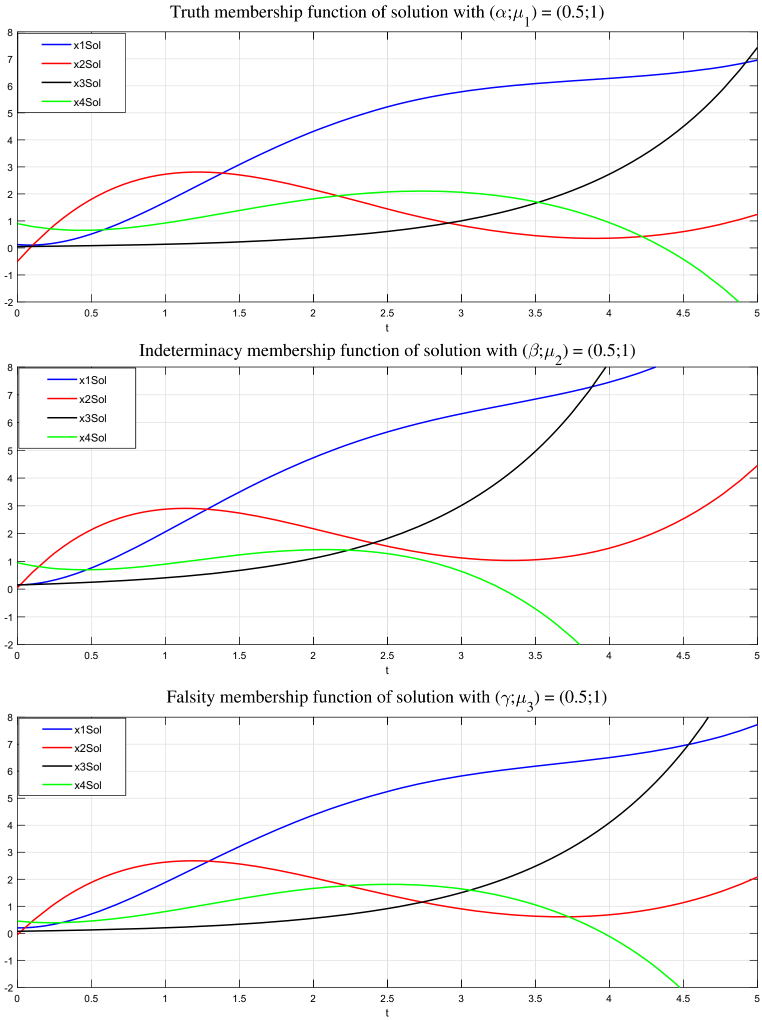

Fig. 12 The (α, β, γ) −cuts of solution of the problem (11)-(12) with $\alpha = \beta = \gamma = 0.5$ and $\mu = 1$

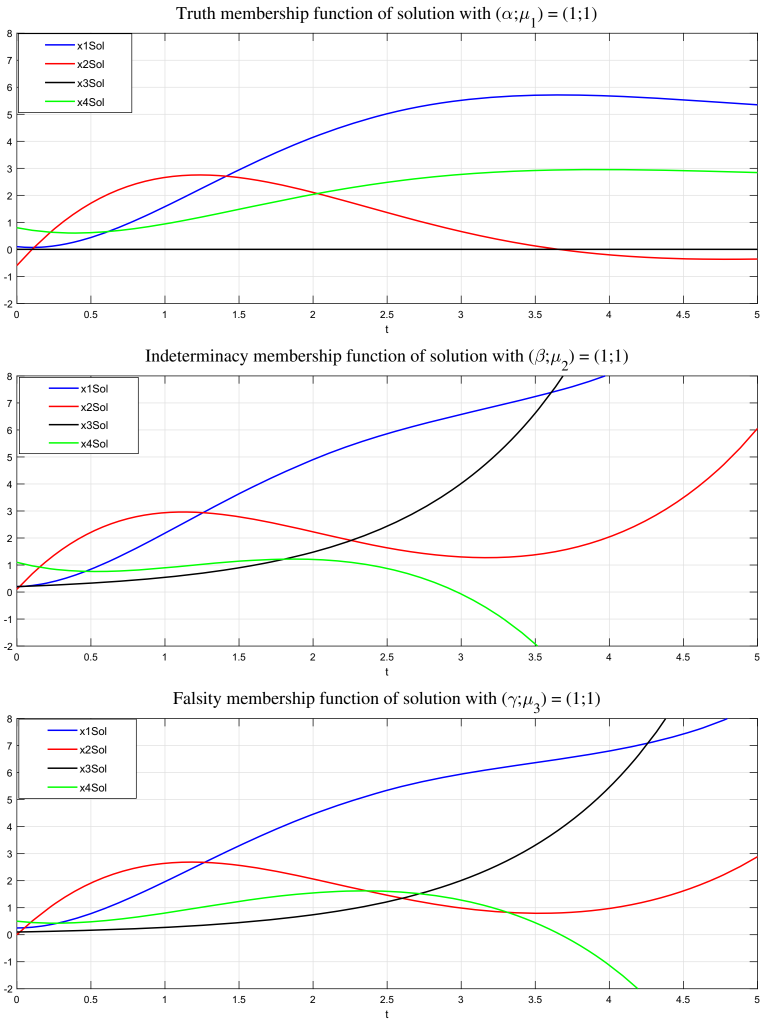

Fig. 13 The (α, β, γ) −cuts of solution of the problem (11)-(12) with $\alpha = \beta = \gamma = 1$ and $\mu = 1$

To obtain the state equations, let us denote $x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = y_2$, $x_4 = \dot{y}_2$. Then, the state equations of the system can be represented by following matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_1}{M_1} & -\frac{D_1}{M_1} & \frac{K_1}{M_1} & \frac{D_1}{M_1} \\ 0 & 0 & 1 & 0 \\ \frac{K_1}{M_2} & \frac{D_1}{M_2} & -\frac{K_1+K_2}{M_2} & -\frac{D_1+D_2}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \end{bmatrix} u, \tag{11}$$

where u is external force and the coefficients D_i, K_i, m_i are determined in Table 8:

In addition, the initial state of this mechanic system is given as

$$\begin{cases} x_1(0) = (0.05, 0.1, 0.15; 0, 0.1, 0.2; 0.05, 0.15, 0.25)\,, \\ x_2(0) = (-0.8, -0.6, -0.4; -0.1, 0, 0.1; -0.2, -0.1, 0), \\ x_3(0) = (-0.1, 0, 0.1; 0, 0.1, 0.2; 0, 0.05, 0.1)\,, \\ x_4(0) = (0.6, 0.8, 1; 0.5, 0.8, 1.1; 0.3, 0.4, 0.5). \end{cases} \tag{12}$$

Since the initial states and the external force acting to the mechanical system cannot be certain values due to the lack of specialized measure equipment and the errors in experiment and computation, it follows that the mechanical system becomes a complex system containing uncertainties in both coefficients and conditions and hence, it is necessary to introduce uncertainty in the solution.

For the initial problem to the system (11) subject to the conditions (12), by using MATLAB's program for Runge Kutta numerical method, we obtain that Figs. 11, 12 and 13 show the graphical representation of solution of mechanical system (11) with initial state (12) with respect to some different values of $(\alpha, \beta, \gamma) -$ cuts.

5 Conclusions

In this work, by using horizontal membership functions approach, a new representation of triangular neutrosophic number is introduced. Additionally, the metric on space of single valued triangular neutrosophic numbers and the continuity of neutrosophic valued functions are also presented. Especially, the concept of derivative of neutrosophic valued function, namely granular derivative, is firstly defined based on granular difference beside the foundation of the concept granular integral. Under these concepts, the neutrosophic differential equations have been investigated. To solve this kind of equations, the horizontal membership function approach is used. The next step of our future research, we will study the controllability and stabilizability for some classes of linear time-invariant neutrosophic systems, neutrosophic dynamic system of fractional order with applications to signal processing.

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Compliance with Ethical Standards

Conflict of interests The authors declare that there is no conflict of interest regarding the publication of this paper.

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Dr. Nguyen Thi Kim Son obtained the PhD degree on Mathematics at Hanoi National University of Education in 2010. She currently worked as the teacher and researcher of Faculty of Natural Science, Hanoi Metropolitan University, Hanoi, Vietnam. Her areas of interest include fuzzy differential equations with modeling, neutrosophic sets and systems with applications, granular computing, partial differential equations with applications. She published more than 20 papers in ISI covered journals. She serves as the reviewer for some international journal related to fuzzy theory such as IEEE Transaction on Fuzzy Systems, Fuzzy Sets and Systems, Journal of Intelligent & Fuzzy Systems, etc.

Nguyen Phuong Dong is the PhD student at Hanoi National University of Education. His research direction related to fuzzy computing with application to some control problem of engineering systems. He has published 4 papers in ISI-covered journals.

Le Hoang Son obtained the PhD degree on Mathematics? Informatics at VNU University of Science, Vietnam National University (VNU) in 2013. He has been promoted to Associate Professor in Information Technology since 2017. Dr. Son worked as senior researcher and Vice Director at the Center for High Performance Computing, VNU University of Science, Vietnam National University during 2007 - 2018. From August 2018, he is Head of Department of Multimedia and Virtual Reality, VNU Information Technology Institute, VNU. His major fields include Artificial Intelligence, Data Mining, Soft Computing, Fuzzy Computing, Fuzzy Recommender Systems, and Geographic Information System. He is a member of International Association of Computer Science and Information Technology (IACSIT), Vietnam Society for Applications of Mathematics (Vietsam), and Key Laboratory of Geotechnical Engineering and Artificial Intelligence in University of Transport Technology (Vietnam). Dr. Son serves as Editorial Board of Applied Soft Computing (ASOC, in SCIE), International Journal of Ambient Computing and Intelligence (IJACI, in SCOPUS), and Vietnam Journal of Computer Science and Cybernetics (JCC). He is an Associate Editor of Journal of Intelligent & Fuzzy Systems (JIFS, in SCIE), IEEE Access (in SCIE), Neutrosophic Sets and Systems (NSS), Vietnam Research and Development on Information and Communication Technology (RD-ICT), VNU Journal of Science: Computer Science and Communication Engineering (JCSCE), and Frontiers in Artificial Intelligence.

Dr. Hoang Viet Long is the Head of Faculty of Information Technology at People's Police University of Technology and Logistics, Bac Ninh, Vietnam. He is currently working as the researcher of Institute for Computational Science at Ton Duc Thang University, Ho Chi Minh City, Vietnam. He obtained PhD diploma in Computer Science at Hanoi University of Science and Technology in 2011, where he defensed his thesis in fuzzy analysis with application to electrical engineering. He has been promoted to Associate Professor in Information Technology since 2017. Recently, he has been concerning in neutrosophic theory and granular computing and published more than 20 papers in ISI-covered journals.