



# Conjugate Frobenius Manifold and Inversion Symmetry

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## Abstract

We give a conjugacy relation on certain type of Frobenius manifold structures using the theory of flat pencils of metrics. It leads to a geometric interpretation for the inversion symmetry of solutions to Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations.

**Keywords** Frobenius manifold · Flat pencil of metrics · Poisson bracket of hydrodynamic type · Inversion symmetry · WDVV equations

**Mathematics Subject Classification** 53D45

## 1 Introduction

Boris Dubrovin introduced the notion of a Frobenius manifold as a geometric realization of a potential  $\mathbb{F}$  which satisfies a system of partial differential equations known in topological field theory as Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations. More precisely, a Frobenius algebra is a commutative associative algebra with an identity  $e$  and a nondegenerate bilinear form  $\Pi$  compatible with the product, i.e.,  $\Pi(a \circ b, c) = \Pi(a, b \circ c)$ . A Frobenius manifold is a manifold with a smooth structure of a Frobenius algebra on the tangent space at any point with certain compatibility conditions. Globally, we require the metric  $\Pi$  to be flat and the identity vector field  $e$  to be covariantly constant with respect to the corresponding Levi–Civita connection. Detailed information about Frobenius manifolds and related topics can be found in [7].

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Let  $M$  be a Frobenius manifold. In flat coordinates  $(t^1, \dots, t^r)$  for  $\Pi$  where  $e = \partial_{t^r}$  the compatibility conditions imply that there exists a function  $\mathbb{F}(t^1, \dots, t^r)$  which encodes the Frobenius structure, i.e., the flat metric is given by

$$\Pi_{ij}(t) = \Pi(\partial_{t^i}, \partial_{t^j}) = \partial_{t^r} \partial_{t^i} \partial_{t^j} \mathbb{F}(t) \tag{1.1}$$

and, setting  $\Omega_1(t)$  to be the inverse of the matrix  $\Pi(t)$ , the structure constants of the Frobenius algebra are given by

$$C_{ij}^k(t) = \Omega_1^{kp}(t) \partial_{t^p} \partial_{t^i} \partial_{t^j} \mathbb{F}(t).$$

Here, and in what follows, summation with respect to repeated upper and lower indices is assumed. The definition includes the existence of a vector field  $E$  of the form  $E = (a_i^j t^i + b^j) \partial_{t^j}$  satisfying

$$E\mathbb{F}(t) = (3 - d) \mathbb{F}(t) + \frac{1}{2} A_{ij} t^i t^j + B_i t^i + c \tag{1.2}$$

where  $a_i^j, b_j, c, A_{ij}, B_i$  and  $d$  are constants with  $a_r^r = 1$ . The vector field  $E$  is called the Euler vector field and the number  $d$  is called the charge of the Frobenius manifold. The associativity of Frobenius algebra implies that the potential  $\mathbb{F}(t)$  satisfies the WDVV equations

$$\partial_{t^i} \partial_{t^j} \partial_{t^k} \mathbb{F}(t) \Omega_1^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^n} \mathbb{F}(t) = \partial_{t^n} \partial_{t_j} \partial_{t_k} \mathbb{F}(t) \Omega_1^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^i} \mathbb{F}(t), \quad \forall i, j, q, n. \tag{1.3}$$

Conversely, an arbitrary potential  $\mathbb{F}(t^1, \dots, t^r)$  satisfying Eqs. (1.3) and (1.2) with (1.1) determines a Frobenius manifold structure on its domain [7]. Moreover, there exists a quasihomogenous flat pencil of metrics (QFPM) of degree  $d$  associated to the Frobenius structure on  $M$  which consists of the intersection form  $\Omega_2$  and the flat metric  $\Omega_1$  with the function  $\tau = \Pi_{i1} t^i$  (see Definition 2.3 below). Here

$$\Omega_2^{ij}(t) := E(dt^i \circ dt^j) \tag{1.4}$$

where the product  $dt^i \circ dt^j$  is defined by lifting the product on  $TM$  to  $T^*M$  using the flat metric  $\Omega_1$ . In this article we prove that, when  $d \neq 1, e(\tau) = 0$  and  $E(\tau) = (1-d)\tau$ , we can construct another QFPM of degree  $2 - d$  on  $M$  consisting of the intersection form  $\Omega_2$  and a different flat metric  $\tilde{\Omega}_1$ . We call it the conjugate QFPM. In particular, under a specific regularity condition, we get a conjugation between a certain type of Frobenius manifold structures on a given manifold. Precisely, we prove the following theorem.

**Theorem 1.1** *Let  $M$  be a Frobenius manifold with the Euler vector field  $E$  and the identity vector field  $e$ . Suppose the associated QFPM is regular of degree  $d$  with a function  $\tau$ . Assume that  $e(\tau) = 0$  and  $E(\tau) = (1 - d)\tau$ . Then we can construct another Frobenius manifold structure on  $M \setminus \{\tau = 0\}$  of degree  $2 - d$ . Moreover, we*

can apply the same method to the new Frobenius manifold structure and it leads to the original Frobenius manifold structure.

For a fixed Frobenius manifold the new structure that can be obtained using Theorem 1.1 will be called the conjugate Frobenius manifold structure.

Let us assume  $\Pi_{i,j} = \delta_{i+j}^{r+1}$ , i.e., the potential  $\mathbb{F}$  has the standard form

$$\mathbb{F}(t) = \frac{1}{2}(t^r)^2 t^1 + \frac{1}{2} t^r \sum_{i=2}^{r-1} t^i t^{r-i+1} + G(t^1, \dots, t^{r-1}) \tag{1.5}$$

and the quasihomogeneity condition (1.2) takes the form

$$E = d_i t^i \partial_{t^i}, \quad E\mathbb{F}(t) = (3 - d) \mathbb{F}(t); \quad d_r = 1. \tag{1.6}$$

Here, the numbers  $d_i$  are called the degrees of the Frobenius manifold. Recall that a symmetry of the WDVV equations is a transformation of the form

$$t^i \mapsto z^i, \quad \Pi \mapsto \tilde{\Pi}, \quad \mathbb{F} \mapsto \tilde{\mathbb{F}}$$

such that  $\tilde{\mathbb{F}}$  satisfies the WDVV equations. The inversion symmetry ([7], Appendix B) is an involutive symmetry given by setting

$$z^1 = -\frac{1}{t^1}, \quad z^r = \Pi_{ij}(t) \frac{t^i t^j}{2t^1}, \quad z^k = \frac{t^k}{t^1}, \quad 2 \leq k < r. \tag{1.7}$$

Then

$$\tilde{\mathbb{F}}(z) := (t^1)^{-2} \left( \mathbb{F}(t) - \frac{1}{2} t^r \Pi_{ij} t^i t^j \right) \tag{1.8}$$

is another solution to the WDVV equations with the flat metric  $\tilde{\Pi}_{ij}(z) = \delta_{i+j}^{r+1}$ . The charge of the corresponding Frobenius manifold structure is  $2 - d$  and the degrees are

$$\tilde{d}_1 = -d_1, \quad \tilde{d}_r = 1, \quad \tilde{d}_i = d_i - d_1 \text{ for } 1 < i < r. \tag{1.9}$$

The inversion symmetry is obtained from a special Schlesinger transformation of the system of linear ODEs with rational coefficients associated to the WDVV equations. A geometric relation between Frobenius manifold structures correspond to  $\mathbb{F}(t)$  and  $\tilde{\mathbb{F}}(z)$  was outlined through the sophisticated notion of Givental groups in [13]. In this article, we obtained a simple geometric interpretation and we report that  $\tilde{\mathbb{F}}(z)$  is the potential of the conjugate Frobenius manifold structure. In other words, we prove the following theorem.

**Theorem 1.2** *Let  $M$  be a Frobenius manifold with charge  $d \neq 1$ . Suppose in the flat coordinates  $(t^1, \dots, t^r)$ , the potential  $\mathbb{F}(t)$  has the standard form (1.5) and the quasihomogeneity condition takes the form (1.6) with  $d_i \neq \frac{d_1}{2}$  for every  $i$ . Then we*

can construct the conjugate Frobenius manifold structure on  $M \setminus \{t^1 = 0\}$ . Moreover, flat coordinates for the conjugate Frobenius manifold are

$$s^1 = -t^1, \quad s^i = t^i (t^1)^{\frac{d_1 - 2d_i}{d_1}} \quad \text{for } 1 < i < r, \quad s^r = \frac{1}{2} \sum_{i=1}^r t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1} - 1}. \quad (1.10)$$

In addition, the corresponding potential equals the potential obtained by applying the inversion symmetry to  $\mathbb{F}(t)$  and it is given by

$$\tilde{\mathbb{F}}(s) = (t^1)^{\frac{-4}{d_1}} \left( \mathbb{F}(t^1, \dots, t^r) - \frac{1}{2} t^r \sum_{i=1}^r t^i t^{r-i+1} \right). \quad (1.11)$$

Examples of Frobenius manifolds satisfying the hypotheses of Theorem 1.2 include Frobenius manifold structures constructed on orbits spaces of standard reflection representations of irreducible Coxeter groups in [9, 22] and algebraic Frobenius manifolds constructed using classical  $W$ -algebras [5]. However, the result presented in this article is a consequence of the work [1, 6]. There, we investigated the existence of Frobenius manifold structures on orbits spaces of some non-reflection representations of finite groups and we noticed that certain structures appear in pairs. Analyzing such pairs led us to the notion of conjugate Frobenius manifold.

This article is organized as follows. In Sect. 2, we review the relation between Frobenius manifold, flat pencil of metrics and compatible Poisson brackets of hydrodynamic type. Then we introduce a conjugacy relation between certain class of quasihomogeneous flat pencils of metrics in Sect. 3. It can be interpreted as a conjugacy relation between certain class of compatible Poisson brackets of hydrodynamic type. We prove Theorem 1.1 in Sect. 3 and Theorem 1.2 in Sect. 4. In Sect. 5, we discuss the findings of this article on polynomial Frobenius manifolds. We end the article with some remarks.

## 2 Background

We review in this section the relation between flat pencil of metrics, compatible Poisson brackets of hydrodynamics type and Frobenius manifold. More details can be found in [8].

Let  $M$  be a smooth manifold of dimension  $r$  and fix local coordinates  $(u^1, \dots, u^r)$  on  $M$ .

**Definition 2.1** A symmetric bilinear form  $(\cdot, \cdot)$  on  $T^*M$  is called a contravariant metric if it is invertible on an open dense subset  $M_0 \subseteq M$ . We define the contravariant Christoffel symbols  $\Gamma_k^{ij}$  for a contravariant metric  $(\cdot, \cdot)$  by

$$\Gamma_k^{ij} := -\Omega^{im} \Gamma_{mk}^j$$

where  $\Gamma_{mk}^j$  are the Christoffel symbols of the metric  $\langle \cdot, \cdot \rangle$  defined on  $TM_0$  by the inverse of the matrix  $\Omega^{ij}(u) = (du^i, du^j)$ . We say the metric  $\langle \cdot, \cdot \rangle$  is flat if  $\langle \cdot, \cdot \rangle$  is flat.

Let  $(\cdot, \cdot)$  be a contravariant metric on  $M$  and set  $\Omega^{ij}(u) = (du^i, du^j)$ . Then we will use  $\Omega$  to refer to the metric and  $\Omega(u)$  to refer to its matrix in the coordinates. In particular, the Lie derivative of  $(\cdot, \cdot)$  along a vector field  $X$  will be written  $\text{Lie}_X \Omega$  while  $X\Omega^{ij}$  means the vector field  $X$  acting on the entry  $\Omega^{ij}$ . The Christoffel symbols given in Definition 2.1 determine for  $\Omega$  the contravariant (resp. covariant) derivative  $\nabla^i$  (resp.  $\nabla_i$ ) along the covector  $du^i$  (resp. the vector field  $\partial_{u^i}$ ). They are related by the identity  $\nabla^i = \Omega^{ij}(u)\nabla_j$ .

**Definition 2.2** A flat pencil of metrics (FPM) on  $M$  is a pair  $(\Omega_2, \Omega_1)$  of two flat contravariant metrics  $\Omega_2$  and  $\Omega_1$  on  $M$  satisfying

1.  $\Omega_2 + \lambda\Omega_1$  defines a flat metric on  $T^*M$  for a generic constant  $\lambda$ ,
2. the Christoffel symbols of  $\Omega_2 + \lambda\Omega_1$  are  $\Gamma_{2k}^{ij} + \lambda\Gamma_{1k}^{ij}$ , where  $\Gamma_{2k}^{ij}$  and  $\Gamma_{1k}^{ij}$  are the Christoffel symbols of  $\Omega_2$  and  $\Omega_1$ , respectively.

**Definition 2.3** A flat pencil of metrics  $(\Omega_2, \Omega_1)$  on  $M$  is called quasihomogeneous flat pencil of metrics (QFPM) of degree  $d$  if there exists a function  $\tau$  on  $M$  such that the vector fields  $E$  and  $e$  defined by

$$\begin{aligned} E &= \nabla_2 \tau, & E^i &= \Omega_2^{ij}(u)\partial_{u^j} \tau \\ e &= \nabla_1 \tau, & e^i &= \Omega_1^{ij}(u)\partial_{u^j} \tau \end{aligned} \tag{2.1}$$

satisfy

$$[e, E] = e, \quad \text{Lie}_E \Omega_2 = (d - 1)\Omega_2, \quad \text{Lie}_e \Omega_2 = \Omega_1 \quad \text{and} \quad \text{Lie}_e \Omega_1 = 0. \tag{2.2}$$

Such a QFPM is **regular** if the  $(1,1)$ -tensor

$$R_i^j = \frac{d - 1}{2} \delta_i^j + \nabla_{1i} E^j \tag{2.3}$$

is nondegenerate on  $M$ .

Let  $(\Omega_2, \Omega_1)$  be a QFPM of degree  $d$ . Then according to [8], we can fix flat coordinates  $(t^1, t^2, \dots, t^r)$  for  $\Omega_1$  such that

$$\begin{aligned} \tau &= t^1, & E^i &= \Omega_2^{i1}, & e^i &= \Omega_1^{i1}, & \Gamma_{1,k}^{ij} &= 0, \\ \Gamma_{2,k}^{i1} &= \frac{1 - d}{2} \delta_k^i, & \Gamma_{2,k}^{1j} &= \frac{d - 1}{2} \delta_k^j + \partial_{t^k} E^j, & \partial_{t^1} E^1 &= 1 - d. \end{aligned} \tag{2.4}$$

Moreover, if  $(\Omega_2, \Omega_1)$  is regular then  $d \neq 1$ .

Consider the loop space  $\mathfrak{L}(M)$  of  $M$ , i.e., the space of smooth maps from the circle  $S^1$  to  $M$ . A local Poisson bracket on  $\mathfrak{L}(M)$  is a Lie algebra structure on the space of

local functionals on  $\mathcal{L}(M)$ . Let  $\{., .\}$  be a local Poisson bracket of hydrodynamic type (PBHT), i.e., it has the following form in the local coordinates [8]

$$\{u^i(x), u^j(y)\} = \Omega^{ij}(u(x))\delta'(x-y) + \Gamma_k^{ij}(u(x))u_x^k\delta(x-y), \quad i, j = 1, \dots, r \quad (2.5)$$

where  $\delta(x - y)$  is the Dirac delta function defined by  $\int_{S^1} f(y)\delta(x - y)dy = f(x)$ . Then we say  $\{., .\}$  is nondegenerate if  $\det \Omega^{ij} \neq 0$  and the Lie derivative of  $\{., .\}$  along a vector field  $X := X^i \partial_{u^i}$  reads

$$\begin{aligned} \text{Lie}_X \{., .\}(u^i(x), u^j(y)) &= (X^s \partial_{u^s} \Omega^{ij} - \Omega^{sj} \partial_{u^s} X^i - \Omega^{is} \partial_{u^s} X^j) \delta'(x - y) \\ &\quad (X^s \partial_{u^s} \Gamma_k^{ij} - \Gamma_k^{sj} \partial_{u^s} X^i - \Gamma_k^{is} \partial_{u^s} X^j \\ &\quad + \Gamma_s^{ij} \partial_{u^k} X^s - \Omega^{is} \partial_{u^s} \partial_{u^k} X^j) u_x^k \delta(x - y). \end{aligned}$$

We will use the following two theorems.

**Theorem 2.4** [21] *Let  $X$  be a vector field on  $M$  and  $\{., .\}$  be a PBHT on  $\mathcal{L}(M)$ . If  $\text{Lie}_X^2 \{., .\} = 0$ , then  $\text{Lie}_X \{., .\}$  is a PBHT and it is compatible with  $\{., .\}$ , i.e.,  $\{., .\} + \lambda \text{Lie}_X \{., .\}$  is a PBHT for every constant  $\lambda$ .*

**Theorem 2.5** [10] *The form (2.5) defines a nondegenerate PBHT  $\{., .\}$  if and only if the matrix  $\Omega^{ij}(u)$  defines a flat contravariant metric on  $M$  and  $\Gamma_k^{ij}(u)$  are its Christoffel symbols.*

From Theorems 2.5 and 2.4, we get the following corollary:

**Corollary 2.6** *Let  $\{., .\}_2$  and  $\{., .\}_1$  be two nondegenerate compatible PBHT on  $\mathcal{L}(M)$  having the form*

$$\{u^i(x), u^j(y)\}_\alpha = \Omega_\alpha^{ij}(u(x))\delta'(x - y) + \Gamma_{\alpha,k}^{ij}(u(x))u_x^k\delta(x - y), \quad \alpha = 1, 2.$$

*Suppose  $\{., .\}_2 + \lambda \{., .\}_1$  is a nondegenerate PBHT for a generic constant  $\lambda$ . Then  $(\Omega_2, \Omega_1)$  is a FPM on  $M$ . Conversely, a FPM on  $M$  determines nondegenerate compatible Poisson brackets of hydrodynamic type on  $\mathcal{L}(M)$ .*

As mentioned in the introduction, if  $M$  is a Frobenius manifold of charge  $d$  then there is an associated QFPM  $(\Omega_2, \Omega_1)$  of degree  $d$  on  $M$ , where  $\Omega_2$  is the intersection form and  $\Omega_1$  is the flat metric. In the flat coordinates  $(t^1, \dots, t^r)$  we have  $\tau = \Pi_{i1} t^i$ . Then the Euler vector field  $E$  and the identity vector field  $e$  of the Frobenius manifold have the form (2.1) and satisfy equations (2.2). The following theorem give a converse statement.

**Theorem 2.7** [8] *Let  $M$  be a manifold carrying a regular QFPM  $(\Omega_2, \Omega_1)$  of degree  $d$ . Then there exists a unique Frobenius manifold structure on  $M$  of charge  $d$  where  $(\Omega_2, \Omega_1)$  is the associated QFPM.*

### 3 Conjugate Frobenius Manifold

We fix a manifold  $M$  with a QFPM  $T = (\Omega_2, \Omega_1)$  of degree  $d \neq 1$ . We fix a function  $\tau$  for  $T$  which determines the vector fields  $E$  and  $e$  (see Definition 2.3). We suppose

$$e(\tau) = 0 \text{ and } E(\tau) = (1 - d)\tau. \tag{3.1}$$

We introduce the function  $f(\tau) := (\tau)^{\frac{2}{1-d}}$  and the vector field  $\tilde{e} := f(\tau)e$ . We define

$$\tilde{\Omega}_1 := \text{Lie}_{\tilde{e}}\Omega_2 = f\Omega_1 - f'(E \otimes e + e \otimes E). \tag{3.2}$$

Then

$$\begin{aligned} \text{Lie}_{\tilde{e}}^2\Omega_2 &= f^2(\text{Lie}_e^2\Omega_2) + (2(f')^2E(\tau) - 4ff')e \otimes e + ff'e(\tau)\Omega_1 \\ &\quad + ((f')^2 - ff'')e(\tau)(E \otimes e + e \otimes E) = 0 \end{aligned} \tag{3.3}$$

We fix flat coordinates  $(t^1, \dots, t^r)$  leading to the identities (2.4). Considering the condition (3.1), we will further assume that  $e = \partial_r$ . Thus

$$\Omega_1^{i1} = \delta_r^i, \quad \partial_{t^r}\Omega_2^{i1} = \partial_{t^r}E^i = \delta_r^i. \tag{3.4}$$

Let  $\{., .\}$  denote the nondegenerate PBHT associated to  $\Omega_2$ . Then by Corollary 2.6,  $\text{Lie}_e\{., .\}$  is the PBHT associated to  $\Omega_1$  and  $\text{Lie}_e^2\{., .\} = 0$ . We have a similar statement for  $\tilde{e}$ .

**Proposition 3.1**  $\text{Lie}_{\tilde{e}}^2\{., .\} = 0$ . In particular,  $\text{Lie}_{\tilde{e}}\{., .\}$  is a PBHT compatible with  $\{., .\}$ .

**Proof** The PBHT associated to  $\Omega_2$  has the form

$$\{t^\alpha(x), t^\beta(y)\} = \Omega_2^{\alpha\beta} \delta'(x - y) + \Gamma_{2,\gamma}^{\alpha\beta} t_x^\gamma \delta(x - y).$$

Here and in what follows, it is to be understood that all functions on the right hand side depend on  $t(x)$ . Note that

$$\text{Lie}_{\tilde{e}}\{., .\}(t^\alpha(x), t^\beta(y)) = \tilde{\Omega}_1^{\alpha\beta} \delta'(x - y) + \tilde{\Gamma}_{2,\gamma}^{\alpha\beta} t_x^\gamma \delta(x - y)$$

where

$$\begin{aligned} \tilde{\Gamma}_{2,\gamma}^{\alpha\beta} &= \tilde{e}^\varepsilon \partial_\varepsilon \Gamma_{2,\gamma}^{\alpha\beta} - \Gamma_{2,\gamma}^{\varepsilon\beta} \partial_\varepsilon \tilde{e}^\alpha - \Gamma_{2,\gamma}^{\alpha\varepsilon} \partial_\varepsilon \tilde{e}^\beta + \Gamma_{2,\varepsilon}^{\alpha\beta} \partial_\gamma \tilde{e}^\varepsilon - \Omega_2^{\alpha\varepsilon} \partial_{\varepsilon\gamma}^2 \tilde{e}^\beta \\ &= -\Gamma_{2,\gamma}^{\varepsilon\beta} \delta_r^\alpha \delta_\varepsilon^1 f' - \Gamma_{2,\gamma}^{\alpha\varepsilon} \delta_r^\beta \delta_\varepsilon^1 f' + \Gamma_{2,\varepsilon}^{\alpha\beta} \delta_r^\varepsilon \delta_\gamma^1 f' - \Omega_2^{\alpha\varepsilon} \delta_r^\beta \delta_\gamma^1 \delta_\varepsilon^1 f''. \end{aligned}$$

From Eq. (3.3), the coefficients of  $\delta'(x - y)$  of  $\text{Lie}_{\tilde{e}}^2\{., .\}$  vanish while the coefficients  $\tilde{\Gamma}_{2,\gamma}^{\alpha\beta}$  of  $\delta(x - y)$  have the form

$$\begin{aligned} \widetilde{\Gamma}_{2,\gamma}^{\alpha\beta} &= -ff''\partial_r\Omega_2^{\alpha\varepsilon}\delta_r^\beta\delta_\gamma^1\delta_\varepsilon^1 + f'^2\delta_r^\alpha\delta_r^\beta\delta_m^1\delta_\varepsilon^1\Gamma_{2,\gamma}^{m\varepsilon} - f'^2\delta_r^\beta\delta_r^m\delta_\gamma^1\delta_\varepsilon^1\Gamma_{2,m}^{\alpha\varepsilon} \\ &\quad + f'^2\delta_r^\beta\delta_r^\alpha\delta_\varepsilon^1\delta_m^1\Gamma_{2,\gamma}^{\varepsilon m} - f'^2\delta_r^\alpha\delta_r^m\delta_\gamma^1\delta_\varepsilon^1\Gamma_{2,m}^{\varepsilon\beta} + f'f''\Omega_2^{\varepsilon m}\delta_r^\alpha\delta_r^\beta\delta_\gamma^1\delta_m^1 \\ &\quad - f'^2\delta_\gamma^1\delta_r^\beta\delta_m^1\delta_r^\varepsilon\Gamma_{2,\varepsilon}^{\alpha m} - f'^2\delta_\gamma^1\delta_r^\alpha\delta_m^1\delta_r^\varepsilon\Gamma_{2,\varepsilon}^{m\beta} - \widetilde{\Omega}_2^{\alpha\varepsilon}\delta_r^\beta\delta_\gamma^1\delta_\varepsilon^1f''. \end{aligned}$$

Then from the identities (2.4) and the definition of  $f(\tau)$ , it follows that  $\widetilde{\Gamma}_{2,\gamma}^{\alpha\beta} = 0$ . For example,

$$\begin{aligned} \widetilde{\Gamma}_{2,1}^{rr} &= -f\partial_r\Omega_2^1f'' + f'^2\Gamma_{2,1}^{11} - f'^2\Gamma_{2,r}^{1r} + \Omega_2^{11}f''f' \\ &\quad + f'^2\Gamma_{2,1}^{11} - f'^2\Gamma_{2,r}^{r1} - f'^2\Gamma_{2,r}^{r1} - f'^2\Gamma_{2,r}^{r1} - \widetilde{\Omega}_1^{r1}f'' \\ &= -(d+1)f'^2 + (1-d)\tau f'f'' - ff'' - (-f)f' = 0 \end{aligned}$$

and when  $\gamma = 1, \alpha = r$  and  $\beta \neq r$

$$\widetilde{\Gamma}_{2,1}^{r\beta} = -2f'^2\Gamma_{2,r}^{1\beta} = -2f'^2\left(\frac{d-1}{2}\delta_r^\beta + \partial_{tr}E^\beta\right) = 0.$$

□

**Lemma 3.2** *The pair  $\widetilde{T} = (\widetilde{\Omega}_2, \widetilde{\Omega}_1)$  form a QFPM of degree  $\widetilde{d} = 2 - d$ . Moreover, if  $T$  is regular then  $\widetilde{T}$  is regular.*

**Proof** The second term of the identity

$$\widetilde{\Omega}_1(t) = f\Omega_1 - f'E^i(\partial_{ti} \otimes \partial_{tr} + \partial_{tr} \otimes \partial_{ti})$$

contributes only to entries of the last row and last column of  $\widetilde{\Omega}_1(t)$ . From the normalization of  $\Omega_1$ , we get

$$\widetilde{\Omega}_1^{i1}(t) = (f - f'E(\tau))\delta_r^i = (f - (1-d)\tau f')\delta_r^i = (-f)\delta_r^i.$$

Therefore,

$$\det \widetilde{\Omega}_1(t) = f^r \det \Omega_1(t) \neq 0.$$

Hence, using Proposition 3.1 and Corollary 2.6,  $\widetilde{T}$  is a FPM. Let  $\widetilde{\nabla}$  denote the contravariant (and also the covariant) derivative of  $\widetilde{\Omega}_1$  and set  $\widetilde{\tau} := -\tau = -t^1$ . Then the vector fields

$$\widetilde{e} := \widetilde{\nabla}_1\widetilde{\tau}, \text{ and } \widetilde{E} := \nabla_2\widetilde{\tau} = -E$$

satisfy equations (2.2) and

$$\text{Lie}_{\widetilde{E}}\widetilde{\Omega}_2 = \text{Lie}_{-E}\Omega_2 = -(d-1)\Omega_2 = (\widetilde{d}-1)\Omega_2. \tag{3.5}$$



Hence,  $\tilde{T}$  is a QFPM of degree  $\tilde{d} = 2 - d$ . For the regularity condition (2.3), we have

$$\tilde{R}_i^j(t) = \frac{\tilde{d} - 1}{2} \delta_i^j + \tilde{\nabla}_{1i}(-E^j) = \frac{1 - d}{2} \delta_i^j - \nabla_{1i}(E^j) = -R_i^j(t). \tag{3.6}$$

Therefore,  $\det(\tilde{R}_i^j) \neq 0$  if and only if  $\det(R_i^j) \neq 0$ . □

We keep the definitions  $\tilde{\tau} = -\tau$  and  $\tilde{E} = -E$  given in the proof of Lemma 3.2 and we call  $\tilde{T} = (\Omega_2, \tilde{\Omega}_1)$  the conjugate QFPM of  $T$ . The name is motivated by the following corollary.

**Corollary 3.3**  $\tilde{T}$  has a conjugate and it equals  $T$ .

**Proof** We observe that  $\tilde{d} = 2 - d \neq 1$  and the function  $\tilde{\tau} = -\tau$  satisfies the requirements (3.1) as

$$\tilde{\epsilon}(\tilde{\tau}) = 0 \text{ and } \tilde{E}(\tilde{\tau}) = -E(-\tau) = (1 - d)\tau = (1 - \tilde{d})\tilde{\tau}. \tag{3.7}$$

However, applying Lemma 3.2 to  $\tilde{T}$ , we get a QFPM  $(\Omega_2, \text{Lie}_{\tilde{\epsilon}}\Omega_2)$  where

$$\tilde{\tilde{\epsilon}} = f(\tilde{\tau})\tilde{\epsilon} = \tilde{\tau}^{\frac{2}{1-\tilde{d}}}\tilde{\epsilon} = (t^1)^{\frac{2}{1-d}} \cdot (t^1)^{\frac{2}{1-d}} \partial_{t^1} = e.$$

□

Now we can prove Theorem 1.1.

**Proof of Theorem 1.1** From the work in [8], regularity of the associated QFPM implies that the charge  $d \neq 1$ . Then the proof follows from applying Lemma 3.2, Corollary 3.3 and Theorem 2.7 to the associated regular QFPM. □

For a fixed Frobenius manifold, the new Frobenius manifold structure constructed using Theorem 1.1 will be called the conjugate Frobenius manifold structure.

**Example 3.4** We consider the Frobenius manifold structure of charge -1 defined by the following solution to the WDVV equations.

$$\mathbb{F} = \frac{1}{2}t_2^2t_1 + t_1^2 \log t_1$$

In the examples, we use subscript indices instead of superscript indices for convenience. Here, the identity vector field  $e = \partial_{t_2}$  and the Euler vector field  $E = 2t_1\partial_{t_1} + t_2\partial_{t_2}$ . Note that  $EF = (3 - d)F + 2t_1^2$ . The corresponding regular QFPM consists of

$$\Omega_2(t) = \begin{pmatrix} 2t_1 & t_2 \\ t_2 & 4 \end{pmatrix}, \quad \Omega_1(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.8}$$

The conjugate QFPM  $\tilde{T} = (\Omega_2, \tilde{\Omega}_1)$  is of degree  $\tilde{d} = 3$ . In the coordinates

$$s_1 = -t_1, \quad s_2 = \frac{t_2}{t_1}$$

we have

$$\Omega_2(s) = \begin{pmatrix} -2s_1 & s_2 \\ s_2 & \frac{4}{s_1^2} \end{pmatrix}, \quad \tilde{\Omega}_1(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the potential of the conjugate Frobenius manifold structure has the form

$$\tilde{\mathbb{F}} = \frac{1}{2}s_1s_2^2 - \log s_1.$$

Note that the Euler vector field  $\tilde{E} = -E(s) = -2s_1\partial_{s_1} + s_2\partial_{s_2}$  and  $\tilde{E}\tilde{\mathbb{F}} = (3 - \tilde{d})\tilde{\mathbb{F}} + 2$ . We observe that applying the inversion symmetry to the potential  $\mathbb{F}(t)$ , we get

$$\hat{\mathbb{F}}(z) = \frac{1}{2}z_1z_2^2 - \log z_1 + \text{constant}$$

and  $\hat{\mathbb{F}}(z)$  defines the same conjugate Frobenius manifold structure. We prove this for certain type of Frobenius manifolds in next section.

**Example 3.5** We consider Frobenius manifold structures found recently in [3] on the orbits space of the reflection group of type  $B_4$ . It is provided to us by the anonymous reviewer of this article as an example of Frobenius manifold structure whose associated QFPM has a conjugate but it is not regular. The potential of this Frobenius manifold reads

$$\mathbb{F} = \frac{1}{2}t_4^2t_1 + t_2t_3t_4 - \frac{1}{72}t_1^4 + \frac{1}{2}t_3t_1^2 + \frac{1}{6}t_2^2t_3t_1 - \frac{9}{4}t_3^2 + \frac{1}{108}t_4^4t_3 + \frac{3}{2}t_3^2 \log t_3.$$

where the charge and degrees given by

$$d = \frac{1}{3}, \quad d_1 = \frac{2}{3}, \quad d_2 = \frac{1}{3}, \quad d_3 = \frac{4}{3}, \quad d_4 = 1.$$

The action of the Euler vector field reads

$$E \mathbb{F}(t) = (3 - d) \mathbb{F}(t) + \frac{1}{2}A_{ij}t^i t^j = (3 - d) \mathbb{F}(t) + 2t_3^2 \tag{3.9}$$

and the intersection metric  $\Omega_2$  will be

$$\dot{\Omega}_2^{ij}(t) = \Omega_2^{ij}(t) + A^{ij}, \quad A^{ij} = \Omega_1^{i\alpha}(t)\Omega_1^{j\beta}(t)A_{\alpha\beta}. \tag{3.10}$$

The associated QFPM  $T = (\Omega_2, \Omega_1)$  is not regular. However, it has a conjugate QFPM  $\tilde{T} = (\Omega_2, \tilde{\Omega}_1)$ . Flat coordinates  $(s_1, s_2, s_3, s_4)$  for  $\tilde{\Omega}_1$  are defined by

$$t_1 = -s_1, \quad t_2 = s_2, \quad t_3 = -s_1^3s_3, \quad t_4 = -s_4s_1^3 - s_2s_3s_1^2$$

Note that one can still apply the inversion symmetry to the potential  $\mathbb{F}$  to get a Frobenius manifold structure with a potential  $\widehat{\mathbb{F}}(z)$  [7]. We checked that the QFPM obtained from  $\widehat{\mathbb{F}}(z)$  agrees with  $\widetilde{T}$ . We do not consider this type of Frobenius manifolds in the next section as we will assume regularity condition (2.3) of the quasihomogeneous flat pencils of metrics.

Let us assume  $E$  has the form  $E = d_i t^i \partial_{t^i}$ . Then  $d_1 = 1 - d$  and we have the following standard results.

**Corollary 3.6** *T is regular QFPM if and only if  $d_i \neq \frac{d_1}{2}$  for all  $i$ .*

**Proof** Applying the definition 2.3 to the matrix  $R_i^j(t) = (\frac{d-1}{2} + d_i)\delta_i^j = (-\frac{d_1}{2} + d_i)\delta_i^j$ . □

**Lemma 3.7** *If  $\Omega_1^{ij} \neq 0$ , then  $d_i + d_j = 2 - d$ . Thus, if the numbers  $d_i$  are all distinct then we can choose the coordinates  $(t^1, \dots, t^r)$  such that  $\Omega_1^{ij} = \delta_{r+1}^{i+j}$ .*

**Proof** Notice that using  $[e, E] = e$ , we get  $\text{Lie}_E \Omega_1 = (d - 2)\Omega_1$ . Then the statement follows from the equation

$$(2 - d)\Omega_1^{ij}(t) = \text{Lie}_E \Omega_1^{ij}(dt^i, dt^j) = -d_i \Omega_1(dt^i, dt^j) - d_j \Omega_1(dt^i, dt^j).$$

□

### 4 Relation with Inversion Symmetry

We continue using notations and assumptions given in the previous section, but we suppose that  $T$  is regular. Consider the Frobenius manifold structure defined on  $M$  by Theorem 2.7 and let  $\mathbb{F}(t)$  be the corresponding potential. We assume  $\Omega_1^{ij}(t) = \delta_{r+1}^{i+j}$  which is equivalent to requiring that  $\mathbb{F}(t)$  has the standard form (1.5). We suppose further that the quasihomogeneity condition for  $\mathbb{F}(t)$  takes the form (1.6). In this case the intersection form  $\Omega_2$  satisfies [8]

$$\Omega_2^{ij}(t) = (d - 1 + d_i + d_j)\Omega_1^{i\alpha}\Omega_1^{j\beta}\partial_{t^\alpha}\partial_{t^\beta}\mathbb{F}. \tag{4.1}$$

Note that at this stage we are working under the hypothesis of Theorem 1.2.

Let us consider the coordinates (1.10) on  $M \setminus \{t^1 = 0\}$ . Then the nonzero entries of the Jacobian matrix are

$$\begin{aligned} \frac{\partial s^i}{\partial t^1} &= \frac{d_1 - 2d_i}{d_1} t^i (t^1)^{\frac{-2d_i}{d_1}}, \quad \frac{\partial s^r}{\partial t^1} = \left(\frac{-2 - d_1}{2d_1}\right) \sum_2^{r-1} t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1}-2} \\ &\quad - \frac{2}{d_1} t^r (t^1)^{\frac{-2}{d_1}-1}, \\ \frac{\partial s^i}{\partial t^i} &= (t^1)^{\frac{d_1-2d_i}{d_1}}, \quad \frac{\partial s^r}{\partial t^i} = t^{r-i+1} (t^1)^{\frac{-2}{d_1}-1}, \quad \frac{\partial s^r}{\partial t^r} = (t^1)^{\frac{-2}{d_1}}. \end{aligned}$$

**Proposition 4.1** Consider the conjugate QFPM  $\tilde{T} = (\Omega_2, \tilde{\Omega}_1)$ . Then  $\tilde{\tau} = s^1$ ,  $\tilde{\Omega}_1^{ij}(s) = \delta_{r+1}^{i+j}$ ,  $\tilde{e} = \partial_{s^r}$  and  $\tilde{E} = \tilde{d}_i s^i \partial_{s^i}$  where the numbers  $\tilde{d}_i$  are given in (1.9).

**Proof** Using the duality between the degrees outlined in Lemma 3.7, we calculate the entries  $\tilde{\Omega}_1^{ij}(s)$  as follows.

(I) For  $i = 1$

$$\tilde{\Omega}_1^{1j}(s) = -\frac{\partial s^j}{\partial t^\alpha} \tilde{\Omega}_1^{1\alpha} = -\frac{\partial s^r}{\partial t^r} \tilde{\Omega}_1^{1r} = -\frac{\partial s^r}{\partial t^r} (- (t^1)^{\frac{2}{d_1}}) \delta^{1r} = \delta_r^1.$$

(II) For  $1 < i < r$  and  $1 < j < r$

$$\begin{aligned} \tilde{\Omega}_1^{ij}(s) &= \frac{\partial s^i}{\partial t^k} \frac{\partial s^j}{\partial t^k} \tilde{\Omega}_1^{kl} \\ &= \frac{\partial s^i}{\partial t^1} \frac{\partial s^j}{\partial t^1} \tilde{\Omega}_1^{11} + \frac{\partial s^i}{\partial t^i} \frac{\partial s^j}{\partial t^1} \tilde{\Omega}_1^{i1} + \frac{\partial s^i}{\partial t^1} \frac{\partial s^j}{\partial t^j} \tilde{\Omega}_1^{1j} + \frac{\partial s^i}{\partial t^i} \frac{\partial s^j}{\partial t^j} \tilde{\Omega}_1^{ij} \\ &= \frac{\partial s^i}{\partial t^i} \frac{\partial s^j}{\partial t^j} \tilde{\Omega}_1^{ij} \delta^{i+j,r+1} \\ &= (t^1)^{\frac{2d_1 - 2d_i - 2d_j - i + 1 + 2}{d_1}} \delta^{i+j,r+1} \\ &= \delta^{i+j,r+1}. \end{aligned}$$

(III) For  $1 < i < r$

$$\begin{aligned} \tilde{\Omega}_1^{ir}(s) &= (t^1)^{\frac{2}{d_1}} \frac{\partial s^i}{\partial t^i} \frac{\partial s^r}{\partial t^{r-i+1}} + \left( - (t^1)^{\frac{2}{d_1}} \frac{\partial s^i}{\partial t^1} + \frac{-2d_i}{d_1} t^i (t^1)^{\frac{2}{d_1}-1} \frac{\partial s^i}{\partial t^i} \right) \cdot \frac{\partial s^r}{\partial t^r} \\ &= (t^1)^{\frac{2}{d_1}} (t^1)^{\frac{d_1 - 2d_i}{d_1}} \cdot t^i (t^1)^{\frac{-2}{d_1}-1} + \left( -\frac{d_1 - 2d_i}{d_1} (t^1)^{\frac{2}{d_1}} t^i (t^1)^{\frac{-2d_i}{d_1}} \right. \\ &\quad \left. + \frac{-2d_i}{d_1} t^i (t^1)^{\frac{2}{d_1}-1} (t^1)^{\frac{d_1 - 2d_i}{d_1}} \right) (t^1)^{\frac{-2}{d_1}} \\ &= (t^1)^{\frac{-2d_i}{d_1}} t^i + \left( -\frac{d_1 - 2d_i}{d_1} (t^1)^{\frac{-2d_i}{d_1}} t^i + \frac{-2d_i}{d_1} (t^1)^{\frac{-2d_i}{d_1}} t^i \right) \\ &= (t^1)^{\frac{-2d_i}{d_1}} t^i - (t^1)^{\frac{-2d_i}{d_1}} t^i \\ &= 0. \end{aligned}$$

(IV) Finally,

$$\begin{aligned} \tilde{\Omega}_1^{rr}(s) &= - (t^1)^{\frac{2}{d_1}} \frac{\partial s^r}{\partial t^r} \cdot \frac{\partial s^r}{\partial t^1} + \sum_{i=2}^{r-1} \left( (t^1)^{\frac{2}{d_1}} \frac{\partial s^r}{\partial t^{r-i+1}} - \frac{2d_i}{d_1} t^i (t^1)^{\frac{2}{d_1}-1} \frac{\partial s^r}{\partial t^r} \right) \cdot \frac{\partial s^r}{\partial t^i} \\ &\quad + \left( - (t^1)^{\frac{2}{d_1}} \frac{\partial s^r}{\partial t^1} + \sum_{i=2}^{r-1} -\frac{2d_i}{d_1} t^i (t^1)^{\frac{2}{d_1}-1} \frac{\partial s^r}{\partial t^i} + \frac{-4}{d_1} t^r (t^1)^{\frac{2}{d_1}-1} \frac{\partial s^r}{\partial t^r} \right) \cdot \frac{\partial s^r}{\partial t^r} \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{2}{d_1} + 1\right) \sum_2^{r-1} t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1}-2} + \frac{4}{d_1} t^r (t^1)^{\frac{-2}{d_1}-1} + \sum_2^{r-1} t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1}-2} \\
 &\quad - \sum_2^{r-1} \frac{2d_i}{d_1} t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1}-2} - \sum_2^{r-1} \frac{2d_{r-i+1}}{d_1} t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1}-2} - \frac{4}{d_1} t^r (t^1)^{\frac{-2}{d_1}-1} \\
 &= \sum_2^{r-1} \left(\frac{2}{d_1} + 2 - \frac{2d_i}{d_1} - \frac{2d_{r-i+1}}{d_1}\right) t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1}-2} \\
 &= 0.
 \end{aligned}$$

It is straightforward to show that  $\tilde{e} = \partial_{s,r}$ . The vector field  $\tilde{E} = \Omega_2^{1j}(s)\partial_{s_j}$  while

$$\begin{aligned}
 \Omega_2^{1j}(s) &= \left(d_1 t^1 - d_1 t^1 \frac{\partial s^2}{\partial t^1} - d_2 t^2 \frac{\partial s^2}{\partial t^2} \quad -d_1 t^1 \frac{\partial s^3}{\partial t^1} - d_3 t^3 \frac{\partial s^3}{\partial t^3} \quad \dots \quad -d_1 t^1 \frac{\partial s^r}{\partial t^1} - d_2 t^2 \frac{\partial s^r}{\partial t^2} + \dots - t^r \frac{\partial s^r}{\partial t^r}\right) \\
 &= \left(d_1 t^1 (d_2 - d_1) t^2 (t^1)^{\frac{d_1-2d_2}{d_1}} \quad (d_3 - d_1) t^3 (t^1)^{\frac{d_1-2d_3}{d_1}} \quad \dots \quad \sum_{i=1}^r (-d_1(-\frac{2-d_1}{2d_1}) - d_i) t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1}-1}\right) \\
 &= \left(d_1 t^1 (d_2 - d_1) t^2 (t^1)^{\frac{d_1-2d_2}{d_1}} \quad (d_3 - d_1) t^3 (t^1)^{\frac{d_1-2d_3}{d_1}} \quad \dots \quad \frac{1}{2} \sum_{i=1}^r t^i t^{r-i+1} (t^1)^{\frac{-2}{d_1}-1}\right) \\
 &= (-d_1 s^1 \quad (d_2 - d_1) s^2 \quad (d_3 - d_1) s^3 \quad \dots \quad s^r).
 \end{aligned} \tag{4.2}$$

□

We observe that the inverse transformation of the inversion symmetry (1.7) is given by

$$t^1 = \frac{-1}{z^1}, \quad t^r = z^r + \frac{1}{2} \sum_2^{r-1} \frac{z^i z^{r-i+1}}{z^1}, \quad t^k = \frac{-z^k}{z^1}, \quad 2 \leq k \leq r.$$

Thus, the potential (1.8) obtained from applying the inversion symmetry to  $\mathbb{F}(t)$  has the form

$$\tilde{\mathbb{F}}(z) = (z^1)^2 \mathbb{F}\left(\frac{-1}{z^1}, \frac{-z^2}{z^1}, \dots, \frac{-z^{r-1}}{z^1}, \frac{1}{2} \sum_1^r \frac{z^i z^{r-i+1}}{z^1}\right) + \frac{1}{2} z^r \sum_1^r z^i z^{r-i+1}.$$

**Lemma 4.2** *The potential  $\tilde{\mathbb{F}}(z)$  has the form*

$$\tilde{\mathbb{F}}(s) = (t^1)^{\frac{-4}{d_1}} \left( \mathbb{F}(t^1, \dots, t^r) - \frac{1}{2} t^r \sum_1^r t^i t^{r-i+1} \right), \quad z^i \leftrightarrow s^i. \tag{4.3}$$

**Proof** We use the identities

$$\begin{aligned}
 t^1 &= -s^1 = (s^1)^2 \left(\frac{-1}{s^1}\right), \quad t^r = (s^1)^{\frac{2}{d_1}} \left(\frac{1}{2} \sum_1^r \frac{s^i s^{r-i+1}}{s^1}\right), \\
 t^i &= (s^1)^{\frac{2d_i}{d_1}} \left(\frac{-s^i}{s^1}\right), \quad 1 < i < r,
 \end{aligned}$$

and the quasihomogeneity of the potential  $\mathbb{F}(t)$ , i.e.,

$$\left(\frac{2}{d_1} E\right) \mathbb{F}(t) = \frac{2(3-d)}{d_1} \mathbb{F}(t) = \left(\frac{4}{d_1} + 2\right) \mathbb{F}(t). \tag{4.4}$$

Then

$$\begin{aligned} & (t^1)^{\frac{-4}{d_1}} \left[ \mathbb{F}(t^1, \dots, t^r) - \frac{1}{2} t^r \sum_1^r t^i t^{r-i+1} \right] \\ &= (t^1)^{\frac{-4}{d_1}} \left[ \mathbb{F}(t^1, \dots, t^r) + (-t^1(t^r)^2) - \frac{1}{2} t^r \sum_2^{r-1} t^i t^{r-i+1} \right] \\ &= (s^1)^{\frac{-4}{d_1}} \left[ \mathbb{F}\left((s^1)^2 \left(\frac{-1}{s^1}\right), (s^1)^{\frac{2d_2}{d_1}} \left(\frac{-s^2}{s^1}\right), \dots, (s^1)^{\frac{2d_{r-1}}{d_1}} \left(\frac{-s^{r-1}}{s^1}\right), \right. \right. \\ &\quad \left. \left. (s^1)^{\frac{2}{d_1}} \left(\frac{1}{2} \sum_1^r \frac{s^i s^{r-i+1}}{s^1}\right)\right) \right. \\ &\quad \left. + ((s^r)^2 (s^1)^{\frac{4}{d_1}+1} + s^r \sum_2^{r-1} s^i s^{r-i+1} (s^1)^{\frac{4}{d_1}} + s^1 \left(\frac{1}{2} \sum_2^{r-1} (s^1)^{\frac{2}{d_1}-1} s^i s^{r-i+1}\right)^2\right) \right. \\ &\quad \left. - \frac{1}{2} s^r (s^1)^{\frac{4}{d_1}} \sum_2^{r-1} s^i s^{r-i+1} - s^1 \left(\frac{1}{2} \sum_2^{r-1} (s^1)^{\frac{2}{d_1}-1} s^i s^{r-i+1}\right)^2 \right] \\ &= (s^1)^{\frac{-4}{d_1}} \left[ (s^1)^{\frac{4}{d_1}+2} \mathbb{F}\left(\frac{-1}{s^1}, -\frac{s^2}{s^1}, -\frac{s^3}{s^1}, \dots, \frac{1}{2} \sum_{i=1}^n \frac{-s^i s^{n-i+1}}{s^1}\right) + (s^r)^2 (s^1)^{\frac{4}{d_1}+1} \right. \\ &\quad \left. + \frac{1}{2} s^r \sum_2^{r-1} s^i s^{r-i+1} (s^1)^{\frac{4}{d_1}} \right] \\ &= (s^1)^2 \mathbb{F}\left(\frac{-1}{s^1}, \frac{-s^2}{s^1}, \dots, \frac{-s^{r-1}}{s^1}, \frac{1}{2} \sum_1^r \frac{s^i s^{r-i+1}}{s^1}\right) + \frac{1}{2} s^r \sum_1^r s^i s^{r-i+1} \end{aligned}$$

which is the potential of the inversion symmetry by setting  $s^i = z^i$ . □

Now we prove Theorem 1.2 stated in the introduction.

**Proof of Theorem 1.2** By Corollary 3.6 and Theorem 1.1, we use the above notations and assume  $T = (\Omega_2, \Omega_1)$  is the associated QFPM. We need to show that the conjugate QFPM  $\tilde{T} = (\Omega_2, \tilde{\Omega}_1)$  equals the QFPM associated to the potential  $\tilde{\mathbb{F}}(s)$  given in (4.3). This leads to verifying that  $\Omega_2(s)$  equals the intersection form  $\tilde{\Omega}_2(s)$  defined by  $\tilde{\mathbb{F}}(s)$ . It is straightforward to show that  $\tilde{F}(s)$  is a quasihomogenous function, i.e.,  $\tilde{E}\tilde{F} = (3 - \tilde{d})\tilde{F}$ . Hence

$$\tilde{\Omega}_2^{ij}(s) := (\tilde{d} - 1 + \tilde{d}_i + \tilde{d}_j) \Omega_1^{i\alpha} \Omega_1^{j\beta} \partial_{s^\alpha} \partial_{s^\beta} \tilde{\mathbb{F}}.$$

After long calculations we find that  $\widetilde{\Omega}_2^{ij}(s) = \widehat{\Omega}_2^{ij}(s)$ . For examples, we obtained the first row of  $\Omega_2^{ij}(s)$  in (4.2) and for even  $r$  and  $1 < i, j < r$ , we get by denoting  $\partial_{t^i} \partial_{t^j} G(t)$  as  $G_{i,j}$

$$\begin{aligned} \Omega_2^{ij}(s) &= \frac{\partial s^i}{\partial t^1} \frac{\partial s^j}{\partial t^1} \Omega_2^{1,1} + \frac{\partial s^i}{\partial t^i} \frac{\partial s^j}{\partial t^1} \Omega_2^{i,1} + \frac{\partial s^i}{\partial t^1} \frac{\partial s^j}{\partial t^j} \Omega_2^{1,j} + \frac{\partial s^i}{\partial t^i} \frac{\partial s^j}{\partial t^j} \Omega_2^{i,j} \\ &= d_1 \left(1 - \frac{2d_i}{d_1}\right) \left(1 - \frac{2d_j}{d_1}\right) t^i t^j (t^1)^{1 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} \\ &\quad + d_i \left(1 - \frac{2d_j}{d_1}\right) t^i t^j (t^1)^{1 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} \\ &\quad + d_j \left(1 - \frac{2d_i}{d_1}\right) t^i t^j (t^1)^{1 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} \\ &\quad + (d - 1 + d_i + d_j) (t^1)^{2 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} (G_{r-i+1, r-j+1} + t^r \delta^{r, i+j}) \\ &= (d_1 - d_i - d_j) t^i t^j (t^1)^{1 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} \\ &\quad + (-d_1 + d_i + d_j) (t^1)^{2 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} (G_{r-i+1, r-j+1} + t^r \delta^{r, i+j}) \\ &= (d_1 - d_i - d_j) (t^1)^{1 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} \left(t^i t^j - t^1 G_{r-i+1, r-j+1} - t^1 t^r \delta^{r, i+j}\right). \end{aligned} \tag{4.5}$$

On the other hand

$$\begin{aligned} &\frac{\partial^2 \widetilde{\mathbb{F}}}{\partial s^{r-i+1} \partial s^{r-j+1}} \\ &= \left(t^r \delta_{r, i+j} (t^1)^{1 - \frac{2}{d_1} - \frac{2d_{r-i+1}}{d_1}} + G_{r-i+1, r-j+1} (t^1)^{-1 - \frac{4}{d_1} + \frac{2d_{r-i+1}}{d_1}}\right) \\ &\quad \times \left(-s^1\right)^{\frac{2d_{r-j+1}}{d_1} - 1} + \left(t^i (t^1)^{1 - \frac{2}{d_1} - \frac{2d_i}{d_1}}\right) \left(s^i (s^1)^{\frac{2}{d_1} - 1}\right) \\ &= \left(t^r \delta_{r, i+j} (t^1)^{2 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} + G_{r-i+1, r-j+1} (t^1)^{-2 - \frac{4}{d_1} + \frac{2d_{r-i+1}}{d_1} + \frac{2d_{r-j+1}}{d_1}}\right) \\ &\quad - \left(t^i t^j (t^1)^{2 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}}\right) \\ &= (t^1)^{1 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} \left(t^r \delta^{r, i+j} t^1 + G_{r-i+1, r-j+1} t^1 - t^i t^j\right). \end{aligned} \tag{4.6}$$

Therefore,

$$\begin{aligned} \widehat{\Omega}_2^{ij}(s) &= (d_i + d_j - d_1) (t^1)^{1 - \frac{2d_i}{d_1} - \frac{2d_j}{d_1}} \left(t^r t^1 \delta^{r, i+j} + G_{r-i+1, r-j+1} t^1 - t^i t^j\right) \\ &= \Omega_2^{ij}(s). \end{aligned} \tag{4.7}$$

□

**Example 4.3** Consider the following solution to WDVV equations

$$\mathbb{F} = \frac{t_1^3}{6} - \frac{1}{2}t_2^2t_1 + \frac{1}{2}t_2^2t_3 + \frac{1}{2}t_1t_3^2. \tag{4.8}$$

It corresponds to a trivial Frobenius manifold structure, i.e., Frobenius algebra structure does not depend on the point. Here the charge  $d = 0$ , the Euler vector field  $E = \sum t_i \partial_{t_i}$  and identity vector field  $e = \partial_{t_3}$ . The intersection form is

$$\Omega_2(t) = \begin{pmatrix} t_1 & t_2 & t_3 \\ t_2 & t_3 - t_1 & -t_2 \\ t_3 & -t_2 & t_1 \end{pmatrix}$$

Setting

$$s_1 = -t_1, \quad s_2 = \frac{t_2}{t_1}, \quad s_3 = \frac{t_2^2}{2t_1^3} + \frac{t_3}{t_1^2}$$

the conjugate QFPM has  $\tilde{\Omega}_1^{ij}(s) = \delta_3^{i+j}$  and

$$\Omega_2(s) = \begin{pmatrix} -s_1 & 0 & s_3 \\ 0 & s_3 + \frac{3s_2^2}{2s_1} + \frac{1}{s_1} & -\frac{s_2^3}{s_1^2} - \frac{2s_2}{s_1^2} \\ s_3 & -\frac{s_2^3}{s_1^2} - \frac{2s_2}{s_1^2} & \frac{3s_2^4}{4s_1^3} + \frac{3s_2^2}{s_1^3} - \frac{1}{s_1^3} \end{pmatrix}$$

The potential of the conjugate Frobenius manifold structure reads

$$\tilde{\mathbb{F}}(s) = \frac{-1}{6s_1} + \frac{s_2^2}{2s_1} + \frac{s_2^4}{8s_1} + \frac{1}{2}s_2^2s_3 + \frac{1}{2}s_1s_3^2.$$

One can check that this is the same potential obtained by applying the inversion symmetry to  $\mathbb{F}(t)$ . Note that  $\tilde{E} = -s_1 \partial_{s_1} + s_3 \partial_{s_3}$  and  $\tilde{E}\tilde{\mathbb{F}} = \tilde{\mathbb{F}}$ .

### 5 The Conjugate of a Polynomial Frobenius Manifold

In this section, we recall the construction of Frobenius manifolds on the space of orbits of Coxeter groups given in [9] and we apply the results of this article.

We fix an irreducible Coxeter group  $\mathcal{W}$  of rank  $r$ . We consider the standard real reflection representation  $\psi : \mathcal{W} \rightarrow GL(V)$ , where  $V$  is a complex vector space of dimension  $r$ . Then the orbits space  $M = V/\mathcal{W}$  is a variety whose coordinate ring is the ring of invariant polynomials  $\mathbb{C}[V]^{\mathcal{W}}$ . Using the Shephard-Todd-Chevalley theorem, the ring  $\mathbb{C}[V]^{\mathcal{W}}$  is generated by  $r$  algebraically independent homogeneous polynomials. Moreover, the degrees of a complete set of generators are uniquely specified by the group [16].



We fix a complete set of homogeneous generators  $u^1, u^2, \dots, u^r$  for  $\mathbb{C}[V]^{\mathcal{W}}$ . Let  $\eta_i$  be the degree of  $u^i$ . Here, we have

$$2 = \eta_1 < \eta_2 \leq \eta_3 \leq \dots \leq \eta_{r-1} < \eta_r.$$

It is known that  $\eta_i + \eta_{r-i+1} = \eta_r + \eta_1$ . Consider the invariant bilinear form on  $V$  under the action of  $\mathcal{W}$ . Then it defines a contravariant flat metric  $\Omega_2$  on  $M$  and we let  $u^1$  equals its quadratic form. We fix the vector field  $e := \partial_{u^r}$ . There is another flat contravariant metric  $\Omega_1 := \text{Lie}_e \Omega_2$  on  $M$ , which was initially studied by K. Saito [19, 20] and it is called the Saito flat metric. Then  $T := (\Omega_2, \Omega_1)$  is a FPM and Dubrovin proved the following theorem.

**Theorem 5.1** [8]  *$T = (\Omega_2, \Omega_1)$  is a regular QFPM of charge  $\frac{\eta_r-2}{\eta_r}$  and leads to a polynomial Frobenius manifold structure on  $M$ , i.e., the corresponding potential is a polynomial function in the flat coordinates.*

We observe that the polynomial Frobenius structure defined by  $T$  has  $\tau = \frac{1}{\eta_r} u^1$ , the Euler vector field  $E = \frac{1}{\eta_r} \sum_i \eta_i u^i \partial_{u^i}$ , the identity vector field  $e$  and degrees  $\frac{\eta_i}{\eta_r}$ . Note that  $E$  is independent of the choice of generators but  $e$  is defined up to a constant factor. Thus, changing the set of generators will lead to an equivalent Frobenius manifold structure [9]. The following theorem was conjectured by Dubrovin and proved by C. Hertling.

**Theorem 5.2** [15] *Any semisimple polynomial Frobenius manifold with positive degrees is isomorphic to a polynomial Frobenius structure constructed on the orbits space of the standard real reflection representation of a finite irreducible Coxeter group.*

Clearly,  $T$  satisfies the hypotheses of Theorem 1.1 and we have a conjugate regular QFPM  $\tilde{T} := (\Omega_2, \text{Lie}_{\tilde{e}} \Omega_2)$ , where  $\tilde{e} = (\tau)^{\eta_r} e$ . Moreover, from the work of K. Saito and his collaborators (see also [9]), we can fix  $u^1, \dots, u^r$  to be flat with respect to  $\Omega_1$  and the potential of the polynomial Frobenius manifold will have the standard form (1.5). In particular  $\tilde{T}$  is the regular QFPM of the Frobenius manifold structure obtained by applying inversion symmetry to the polynomial Frobenius manifold on  $M$ . Considering Theorem 5.2, we wonder what is the intrinsic description for the conjugate Frobenius manifold as this may help in the classification of Frobenius manifolds.

In [1], we give a similar discussion for the  $r$  Frobenius manifold structures constructed in [22] on the orbits space  $M$  when  $\mathcal{W}$  is of type  $B_r$  or  $D_r$ .

### 6 Remarks

It is important to mention that the inversion symmetry of the WDVV equation can be applied to a solution  $\mathbb{F}(t)$  in the standard form (1.5) under more general quasi-homogeneity condition than condition (1.6) and without the regularity condition (2.3) of the associated QFPM. In this case, if the conjugate Frobenius manifold structure exists, we believe that it will be equivalent to Frobenius manifold structure obtained

by applying the inversion symmetry, we confirm this by Example 3.4 and Example 3.5.

Note that Frobenius manifold structures which are invariant under inversion symmetry were studied in [18]. We did not consider these cases as the charge will equal 1.

It will be interesting to study the consequences of Theorem 1.2 on the interpretation of the inversion symmetry in terms of the action of the Givental groups obtained in [13] and the relation found in [17] between the principle hierarchies and tau functions of the two solutions to the WDVV equations related by the inversion symmetry. We also believe that the findings in this article can be generalized to the theory of bi-flat  $F$ -manifolds [2].

It is known that the leading term of a certain class of compatible local Poisson structures leads to a regular QFPM and thus to a Frobenius structure [8, 11]. Polynomial Frobenius manifolds obtained in [4] are constructed by fixing the regular nilpotent orbit in a simple Lie algebra and uses compatible local Poisson brackets obtained by Drinfeld-Sokolov reduction. In these cases, the Poisson brackets form an exact Poisson pencil, and thus their central invariants are constants [14]. If the Lie algebra is simply-laced, then the central invariants are equal [12] which means the Poisson structures are consistent with the principle hierarchy associated with the Frobenius manifold [11]. Fix one of these polynomial Frobenius structures and denote the associated local Poisson brackets by  $\mathbb{B}_2$  and  $\mathbb{B}_1$  (here  $\mathbb{B}_2$  is the classical  $W$ -algebra). In the flat coordinates, these local Poisson brackets form an exact Poisson pencil under the identity vector field  $e$ , i.e.,  $\text{Lie}_e \mathbb{B}_2 = \mathbb{B}_1$  and  $\text{Lie}_e \mathbb{B}_1 = 0$ . Let us denote the leading term of  $\mathbb{B}_2$  by  $B_2$  and  $\tilde{e}$  is the vector field associated with the conjugate Frobenius manifold structure. We proved in this article that  $\text{Lie}_{\tilde{e}}^2 B_2 = 0$ . Then it is natural to ask if  $\tilde{e}$  also leads to an exact Poisson pencil, i.e.,  $\text{Lie}_{\tilde{e}}^2 \mathbb{B}_2 = 0$ . Our calculations for the simple Lie algebra of type  $A_3$ , shows that this is not true.

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## Declarations

**Conflict of interest** The authors have no relevant financial or non-financial interests to disclose.

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