Kahan Discretizations of Skew-Symmetric Lotka-Volterra Systems and Poisson Maps

C. A. Evripidou[1,2] · P. Kassotakis[1] · P. Vanhaecke[3] 📖

Abstract

The Kahan discretization of the Lotka-Volterra system, associated with any skew-symmetric graph Γ, leads to a family of rational maps, parametrized by the step size. When these maps are Poisson maps with respect to the quadratic Poisson structure of the Lotka-Volterra system, we say that the graph Γ has the Kahan-Poisson property. We show that if Γ is connected, it has the Kahan-Poisson property if and only if it is a cloning of a graph with vertices $1, 2, \ldots, n$, with an arc $i \to j$ precisely when $i < j$, and with all arcs having the same value. We also prove a similar result for augmented graphs, which correspond with deformed Lotka-Volterra systems and show that the obtained Lotka-Volterra systems and their Kahan discretizations are superintegrable as well as Liouville integrable.

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✉ P. Vanhaecke
pol.vanhaecke@math.univ-poitiers.fr

C. A. Evripidou
evripidou.charalambos@ucy.ac.cy

P. Kassotakis
pavlos1978@gmail.com

[1] Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus

[2] Department of Mathematics and Statistics, La Trobe University, Melbourne, Victoria 3086, Australia

[3] Laboratoire de Mathématiques et Applications, UMR 7348 CNRS, Université de Poitiers, 11 Boulevard Marie et Pierre Curie, Téléport 2 - BP 30179, 86 962 Chasseneuil Futuroscope Cedex, France

1 Introduction

With any complex skew-symmetric $n \times n$ matrix $A = (a_{i,j})$ is associated a (skew-symmetric) Lotka-Volterra system, which is the Hamiltonian system, described in terms of the standard coordinates $x_1, \ldots, x_n$ of $\mathbb{C}^n$ by the following system of differential equations:

$$\dot{x}_i = \sum_{j=1}^{n} a_{i,j} x_i x_j , \quad i = 1, 2, \ldots, n . \tag{1.1}$$

The Hamiltonian structure, which is also determined by A, is defined by the basic Poisson brackets $\{x_i, x_j\} = a_{i,j} x_i x_j$, for $1 \leqslant i, j \leqslant n$, with Hamiltonian $H :=$ $x_1 + x_2 + \cdots + x_n$. The matrix A may be viewed as the adjacency matrix of a (skew-symmetric) graph Γ, having the integers $1, 2, \ldots, n$ as vertices and with an arc from i to j with value $a_{i,j}$ when $a_{i,j} \neq 0$ and $i < j$. We often think of the Lotka-Volterra system as being associated with Γ and denote it by LV(Γ). Notice that Γ is determined, up to isomorphism, by LV(Γ), as was shown in [4]. When the entries of A are all real, one may also consider (1.1) on $\mathbb{R}^n$, so in what follows we suppose that the entries of A belong to $\mathbb{F}$, where $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$ and we consider (1.1) on $\mathbb{F}^n$.

For a system of quadratic differential equations, such as (1.1), a natural discretization has been constructed by Kahan [3], leading to a rational map, called its Kahan map (see also [1, 2]). Applied to (1.1), the Kahan map with step size $\varepsilon \in \mathbb{F}^*$ is the rational map, corresponding to the field automorphism $\tilde{} : \mathbb{F}(x_1, \ldots, x_n) \to \mathbb{F}(x_1, \ldots, x_n)$, defined by the following formulas, where the right hand side has been obtained from (1.1) by polarization:

$$\frac{\tilde{x}_i - x_i}{\varepsilon} = \sum_{j=1}^{n} a_{i,j} \left(\tilde{x}_i x_j + x_i \tilde{x}_j \right) , \quad i = 1, 2, \ldots, n .$$

From the point of view of discrete integrability, a natural question is whether the Kahan map is a Poisson map with respect to the above Poisson structure. In formulas, this means that $\{\tilde{x}_i, \tilde{x}_j\} = a_{i,j} \tilde{x}_i \tilde{x}_j$, for $1 \leqslant i < j \leqslant n$. When this is the case, we say that Γ has the Kahan-Poisson property. A general skew-symmetric graph Γ does not have the Kahan-Poisson property. In order to give a more precise answer, let us denote by Γ_n the skew-symmetric graph with vertices $1, \ldots, n$, and with an arc $i \to j$, with value 1, for any $i < j$; for $\gamma \in \mathbb{F}^*$ we denote by $\gamma \Gamma_n$ the graph with the same vertices and arcs as Γ_n, but where the value of every arc is γ.

Theorem 1.1 *Let Γ be a connected skew-symmetric graph. Then Γ has the Kahan-Poisson property if and only if Γ is isomorphic to $\gamma \Gamma_n^{\varpi}$ for some $n \in \mathbb{N}^*$, some $\gamma \in \mathbb{F}^*$ and some weight vector ϖ for Γ_n.*

For a skew-symmetric graph which is not connected, Theorem 1.1 applies to each one of its connected components.

The notions of cloning and decloning of skew-symmetric graphs and Lotka-Volterra systems were introduced in [4] in the study of morphisms and automorphisms of

graphs and Lotka-Volterra systems. For a graph, decloning amounts to identifying two (or more) vertices when they have the same neighborhood (which means that the corresponding lines of its adjacency matrix A are identical); the quotient graph is then said to be irreducible. With this terminology, the theorem can also be stated by saying that the only connected, irreducible skew-symmetric graphs Γ which have the Kahan-Poisson property are the graphs $\gamma\Gamma_n$, where $n \in \mathbb{N}^*$ and $\gamma \in \mathbb{F}^*$.

The fact that Γ_n (and hence $\gamma\Gamma_n$, for all γ) has the Kahan-Poisson property has already been shown in [7, Prop. 3.8]. We show in Proposition 2.3 that the Kahan-Poisson property is preserved by cloning and decloning. This proves the easier, inverse implication in Theorem 1.1. The main result which we prove in this paper is the direct implication, which we first show in dimension 3 (Section 3), then in dimension 4, using the result in dimension 3 (Section 4.2), and finally in dimension $n > 4$, using the result in dimension 4 (Section 4.3).

In Section 5, we prove the following generalization of Theorem 1.1 to augmented graphs, which correspond to deformed Lotka-Volterra systems (see Section 5 for the definition of an augmented graph and the KP property for such graphs):

Theorem 1.2 *Let Δ be an augmented graph of a connected skew-symmetric graph Γ. Then Δ has the Kahan-Poisson property if and only if Δ is isomorphic to an augmented graph of $\gamma\Gamma_n^\varpi$ for some $n \in \mathbb{N}^*$, some $\gamma \in \mathbb{F}^*$ and some weight vector ϖ for Γ_n.*

We show in Section 6 that the (deformed) Lotka-Volterra systems, corresponding to the (augmented) graphs which appear in Theorems 1.1 and 1.2, have a discretization with good integrability properties, namely the Kahan discretization, which is in these cases a Poisson map with respect to the original Poisson structure, is both superintegrable and Liouville integrable. It follows that the (deformed) Lotka-Volterra systems whose Kahan discretization is integrable with respect to their original Poisson structure are characterized by the Kahan-Poisson property.

2 The Kahan-Poisson Property

In this section we introduce the Kahan-Poisson property for skew-symmetric graphs and establish it for a particular family of such graphs. We first recall the basic facts which we will use about the Kahan discretization of systems of quadratic differential equations and about skew-symmetric Lotka-Volterra systems; see [3, 4] for details.

2.1 The Kahan Discretization

Consider a system of differential equations on $\mathbb{F}^n$:

$$\dot{x}_i = Q_i(x) \,, \quad i = 1, 2, \ldots, n \,. \tag{2.1}$$

Here, $x = (x_1, \ldots, x_n)$ and Q_i is assumed to be a quadratic form, whose corresponding symmetric bilinear form is denoted by B_i, so that $Q_i(x) = B_i(x, x)$. The *Kahan discretization* of (2.1) is by definition given by

$$\frac{\tilde{x}_i - x_i}{2\varepsilon} = B_i(\tilde{x}, x) , \quad i = 1, 2, \ldots, n , \tag{2.2}$$

where $\varepsilon \in \mathbb{F}^*$ is a non-zero parameter, the *step size*. When (linearly!) solved for $\tilde{x}_1, \ldots, \tilde{x}_n$, one gets a family of birational maps from $\mathbb{F}^n$ to itself, parametrized by $\varepsilon \in \mathbb{F}^*$. Thinking of ε as being fixed, it is called the *Kahan map*. We will mostly work with the corresponding endomorphism K of the field of rational functions $\mathbb{F}(x) = \mathbb{F}(x_1, x_2, \ldots, x_n)$, defined by $K(x_i) := \tilde{x}_i$, for $i = 1, \ldots, n$; we call it the *Kahan morphism*.

2.2 Lotka-Volterra Systems

We are interested in the Kahan discretization of skew-symmetric Lotka-Volterra systems. As recalled in the introduction, a Lotka-Volterra system is associated with any skew-symmetric $n \times n$ matrix A; we also view $A = (a_{i,j})$ as the adjacency matrix of a skew-symmetric graph $\Gamma = (S, A)$, with vertex set $S = \{1, 2, \ldots, n\}$, and think of the Lotka-Volterra system as being associated with the graph Γ, denoted LV(Γ). The Poisson structure of LV(Γ), which we consider here as a Poisson bracket on $\mathbb{F}(x)$, is the quadratic bracket, given by $\{x_i, x_j\} = a_{i,j} x_i x_j$, for $i, j = 1, 2, \ldots, n$. It makes $(\mathbb{F}(x), \{\cdot, \cdot\})$ into a Poisson field. The Hamiltonian vector field $\mathcal{X}_H = \{\cdot, H\}$ on $\mathbb{F}^n$, associated with the Hamiltonian $H := x_1 + x_2 + \cdots + x_n$, is given by the following quadratic differential equations:

$$\dot{x}_i = \sum_{j=1}^{n} a_{i,j} x_i x_j , \quad i = 1, 2, \ldots, n .$$

Its Kahan map is defined by the following specialisation of (2.2):

$$\frac{\tilde{x}_i - x_i}{\varepsilon} = \tilde{x}_i \sum_{j=1}^{n} a_{i,j} x_j + x_i \sum_{j=1}^{n} a_{i,j} \tilde{x}_j , \quad i = 1, 2, \ldots, n . \tag{2.3}$$

An important example for this paper is the Lotka-Volterra system LV(Γ_n), whose underlying graph Γ_n has vertices $1, 2, \ldots, n$ and has an arc from i to j with value 1 when $i < j$ (see Fig. 1).

2.3 The Kahan-Poisson Property

The property of the Kahan map in which we are interested in this paper is its preservation of the Poisson structure of the associated Lotka-Volterra system, i.e., that the Kahan map, which is a birational automorphism of $\mathbb{F}^n$, is also a Poisson map (and hence a birational Poisson automorphism). Before giving the definition, let us clarify the independence on ε: when the Kahan map is a Poisson map for some value of ε then it is a Poisson map for all values of ε. Indeed, (2.3) is homogeneous when ε is given weight -1, while giving a weight 1 to all x_i; the claim then follows from the

Fig. 1 The graph Γ_n
corresponds to the
skew-symmetric $n \times n$ matrix
whose upper-trangular entries
are all equal to 1. The pictured
graph is Γ_6
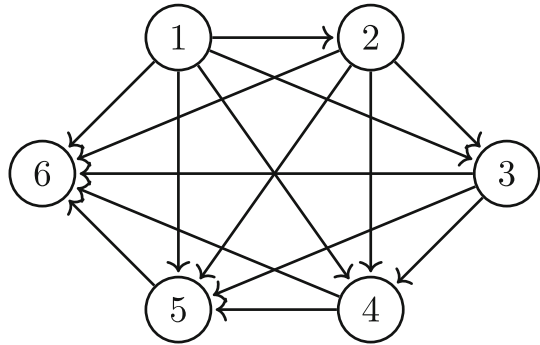
fact that homotheties of quadratic Poisson structures are Poisson maps (see [8, Proposition 8.16]). It also shows that if the Kahan map of a Lotka-Volterra system LV(Γ) is a Poisson map then the Kahan map of the Lotka-Volterra system LV($\gamma\Gamma$), where $\gamma\Gamma$ is Γ with all of its values scaled by $\gamma \in \mathbb{F}^*$, is also a Poisson map. We therefore set $\varepsilon = 1$ and when we speak of the Kahan map or the Kahan morphism of a Lotka-Volterra system, it is implictly assumed that $\varepsilon = 1$. Notice that when $\gamma = -1$, then $\gamma\Gamma = -\Gamma$ is the graph Γ with the direction of all its arcs reversed.

Definition 2.1 Let Γ be a skew-symmetric graph. Then Γ is said to have the *Kahan-Poisson property* (or *KP property*) if the Kahan map of its associated Lotka-Volterra system LV(Γ) is a Poisson map.

In algebraic terms, this means that the Kahan morphism K is an automorphism of Poisson fields, i.e., $\{K(x_i), K(x_j)\} = K\{x_i, x_j\}$, for all $i, j = 1, \ldots, n$, which can also be written as $\{\tilde{x}_i, \tilde{x}_j\} = \widetilde{\{x_i, x_j\}}$.

A first family of skew-symmetric graphs which have the KP property is given by the following proposition:

Proposition 2.2 [7] *For any $n \in \mathbb{N}^*$, the graph Γ_n has the KP property.*

Proof For the proof, we refer to [7, Proposition 3.8]. Yet, we point out the crucial fact that makes the computation feasible. In terms of the variables $u_1, \ldots, u_n$, and $\tilde{u}_1, \ldots, \tilde{u}_n$, defined by $u_i := x_1 + x_2 + \cdots + x_i$, and $\tilde{u}_i := \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_i$, the Kahan map (with $\varepsilon = 1$) takes the simple separated form

$$\tilde{u}_i = u_i \frac{1 + H}{1 - H + 2u_i} \, , \quad i = 1, 2, \ldots, n \, , \tag{2.4}$$

while the Poisson bracket takes the form

$$\{u_i, u_j\} = u_i(u_j - u_i) \, , \quad \text{for } 1 \leqslant i < j \leqslant n \, . \tag{2.5}$$

It therefore suffices to verify that $\{\tilde{u}_i, \tilde{u}_j\} = \tilde{u}_i(\tilde{u}_j - \tilde{u}_i)$, for $1 \leqslant i < j \leqslant n$, with $\tilde{u}_i$ given by (2.4), which is easily done by direct computation, using (2.5). $\qquad\square$

As we mentioned earlier, if we multiply the adjacency matrix A of Γ by any non-zero scalar, the property for the corresponding graph of having the KP property is not affected. In particular, Proposition 2.2 implies that $\gamma \Gamma_n$ has the KP property for any $n \in \mathbb{N}^*$ and for any $\gamma \in \mathbb{F}^*$.

2.4 Cloning and decloning

The cloning of Lotka-Volterra systems, which was introduced in [4], is the inverse operation to decloning, which we already recalled in the introduction. Let $\Gamma = (S, A)$ be a skew-symmetric graph with vertex set $S = \{1, 2, \ldots, n\}$. Let ϖ be a *weight vector* for Γ, i.e., ϖ is a function $\varpi : S \to \mathbb{N}^*$. The *cloning* of (Γ, ϖ) is the skew-symmetric graph $\Gamma^\varpi = (S^\varpi, A^\varpi)$, constructed as follows: on the one hand, every vertex $i \in S$ gives rise to $\varpi(i)$ vertices in S^ϖ, which we denote by $(i, 1), (i, 2), \ldots, (i, \varpi(i))$. On the other hand, the entries $a^\varpi_{(i,k),(j,\ell)}$ of the (skew-symmetric) adjacency matrix A^ϖ of Γ^ϖ are defined by $a^\varpi_{(i,k),(j,\ell)} := a_{i,j}$, for $i, j \in S$ and $1 \leqslant k \leqslant \varpi(i)$, $1 \leqslant \ell \leqslant \varpi(j)$. By definition, the *cloning* of $(\mathrm{LV}(\Gamma), \varpi)$ is $\mathrm{LV}(\Gamma^\varpi)$. We will denote the linear coordinates corresponding to the vertices (i, k) by $y_{i,k}$ and write, as above, $\mathbb{F}(y)$ for the field of rational functions $\mathbb{F}(y_{1,1}, y_{1,2}, \ldots, y_{n,\varpi(n)})$. The Poisson bracket on $\mathbb{F}(y)$ associated with A^ϖ is given by $\{y_{i,k}, y_{j,\ell}\}^\varpi = a_{i,j} y_{i,k} y_{j,\ell}$, as follows from the definition of Γ^ϖ. For $1 \leqslant i \leqslant n$ and $1 \leqslant k, m \leqslant \varpi(i)$ the functions $y_{i,m}/y_{i,k}$ are *Casimir functions* of $\{\cdot, \cdot\}^\varpi$: they belong to the center of the Poisson bracket. As Hamiltonian, we again take the sum of all variables, $H^\varpi := \sum_{i=1}^n \sum_{k=1}^{\varpi(i)} y_{i,k}$. The Hamiltonian vector field $\mathcal{X}_{H^\varpi}$ of $\mathrm{LV}(\Gamma^\varpi)$ is then given by the following differential equations:

$$\dot{y}_{i,k} = y_{i,k} \sum_{j=1}^n \sum_{\ell=1}^{\varpi(j)} a_{i,j} y_{j,\ell}, \qquad i = 1, \ldots, n, \ k = 1, \ldots, \varpi(i).$$

Its Kahan discretization is implicitly defined, as in (2.3), by the following equations:

$$\frac{\bar{y}_{i,k} - y_{i,k}}{\varepsilon} = \bar{y}_{i,k} \sum_{j=1}^n \sum_{\ell=1}^{\varpi(j)} a_{i,j} y_{j,\ell} + y_{i,k} \sum_{j=1}^n \sum_{\ell=1}^{\varpi(j)} a_{i,j} \bar{y}_{j,\ell},$$

$$i = 1, \ldots, n, \ k = 1, \ldots, \varpi(i). \tag{2.6}$$

The corresponding automorphism K^ϖ of $\mathbb{F}(y)$ is defined by $K^\varpi(y_{i,k}) := \bar{y}_{i,k}$, for $i = 1, \ldots, n$ and $k = 1, \ldots, \varpi(i)$. We view $\mathbb{F}(y)$ as a field extension of $\mathbb{F}(x)$ using the *decloning morphism* $\Sigma : \mathbb{F}(x) \to \mathbb{F}(y)$, defined by

$$\Sigma(x_i) := \sum_{k=1}^{\varpi(i)} y_{i,k}. \tag{2.7}$$

We show in the following proposition that the decloning morphism commutes with the Kahan morphism and that the KP property is preserved under cloning and decloning:

Proposition 2.3 *With the above definitions and notations, the following diagram of fields and field morphisms is commutative:*

$$
\begin{array}{ccc}
(\mathbb{F}(x), \{\cdot, \cdot\}) & \xrightarrow{\quad K \quad} & (\mathbb{F}(x), \{\cdot, \cdot\}) \\
\Sigma \downarrow & & \downarrow \Sigma \\
(\mathbb{F}(y), \{\cdot, \cdot\}^{\varpi}) & \xrightarrow{\quad K^{\varpi} \quad} & (\mathbb{F}(y), \{\cdot, \cdot\}^{\varpi})
\end{array}
\tag{2.8}
$$

The vertical arrow Σ is a Poisson morphism, while K is a Poisson morphism if and only if K^{ϖ} is a Poisson morphism.

Proof We first show that the diagram is commutative. Let us set $\varepsilon = 1$, as before. Using Σ, (2.6) can be rewritten as

$$
\bar{y}_{i,k} - y_{i,k} = \bar{y}_{i,k} \sum_{j=1}^{n} a_{i,j} \Sigma(x_j) + y_{i,k} \sum_{j=1}^{n} a_{i,j} \overline{\Sigma(x_j)} \, .
\tag{2.9}
$$

For fixed i, summing up (2.9) for $k = 1, \ldots, \varpi(i)$, we get

$$
\overline{\Sigma(x_i)} - \Sigma(x_i) = \overline{\Sigma(x_i)} \sum_{j=1}^{n} a_{i,j} \Sigma(x_j) + \Sigma(x_i) \sum_{j=1}^{n} a_{i,j} \overline{\Sigma(x_j)} \, .
\tag{2.10}
$$

Recall that $\tilde{x}_1, \ldots, \tilde{x}_n$ is the unique solution to (2.3); if we write this solution by making explicit its dependency on $x_1, \ldots, x_n$ as $\tilde{x}_i = R_i(x_1, \ldots, x_n)$, then it follows from comparing (2.3) and (2.10) that $\overline{\Sigma(x_i)} = R_i(\Sigma(x_1), \ldots, \Sigma(x_n))$. Since Σ is an algebra homomorphism and R_i is a rational function of its arguments,

$$
\overline{\Sigma(x_i)} = R_i(\Sigma(x_1), \ldots, \Sigma(x_n)) = \Sigma(R_i(x_1, \ldots, x_n)) = \Sigma(\tilde{x}_i) \, ,
$$

showing the commutativity of the diagram. For $1 \leqslant i < j \leqslant n$ we have

$$
\left\{ \Sigma(x_i), \Sigma(x_j) \right\}^{\varpi} = \sum_{k=1}^{\varpi(i)} \sum_{\ell=1}^{\varpi(j)} \left\{ y_{i,k}, y_{j,\ell} \right\}^{\varpi} = a_{i,j} \sum_{k=1}^{\varpi(i)} \sum_{\ell=1}^{\varpi(j)} y_{i,k} y_{j,\ell}
$$

$$
= a_{i,j} \Sigma(x_i) \Sigma(x_j) = \Sigma \left\{ x_i, x_j \right\} \, ,
$$

as follows from the definitions of the Poisson brackets and of Σ. As a consequence, the vertical arrows in the diagram (2.8) are morphisms of Poisson fields. In order to show that the two horizontal arrows in that diagram are at the same time morphisms of Poisson fields, we first show that

$$
\Sigma(x_i)/y_{i,k} \text{ is a Casimir of } \{\cdot, \cdot\}^{\varpi} \text{ and is an invariant of } K^{\varpi} \, ,
\tag{2.11}
$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant \varpi(i)$. The first statement in (2.11) follows at once from the fact that $y_{i,m}/y_{i,k}$ is for any $1 \leqslant k, m \leqslant \varpi(i)$ a Casimir function of $\{\cdot, \cdot\}^{\varpi}$, as we already recalled. The second statement means that $\Sigma(x_i)/y_{i,k} = \overline{\Sigma(x_i)}/\bar{y}_{i,k}$. To

prove the latter, divide (2.9) by $y_{i,k}$, to see that $\bar{y}_{i,k}/y_{i,k}$ is independent of k; then $\bar{y}_{i,k}/y_{i,k} = \bar{y}_{i,m}/y_{i,m}$, for k and m as above, so that $y_{i,m}/y_{i,k}$ is an invariant, and hence also $\Sigma(x_i)/y_{i,k} = \sum_{m=1}^{\varpi(i)} y_{i,m}/y_{i,k}$. We can therefore write

$$\bar{y}_{i,k} = \frac{y_{i,k}}{\Sigma(x_i)} \overline{\Sigma(x_i)} \quad \text{and} \quad \bar{y}_{j,\ell} = \frac{y_{j,\ell}}{\Sigma(x_j)} \overline{\Sigma(x_j)}. \tag{2.12}$$

In the second formula above, $1 \leqslant j \leqslant n$ and $1 \leqslant \ell \leqslant \varpi(j)$. Then,

$$\{\bar{y}_{i,k}, \bar{y}_{j,\ell}\}^\varpi \stackrel{(2.12)}{=} \left\{ \frac{y_{i,k}}{\Sigma(x_i)} \overline{\Sigma(x_i)}, \frac{y_{j,\ell}}{\Sigma(x_j)} \overline{\Sigma(x_j)} \right\}^\varpi \stackrel{(2.11)}{=} \frac{y_{i,k} y_{j,\ell}}{\Sigma(x_i x_j)} \left\{ \overline{\Sigma(x_i)}, \overline{\Sigma(x_j)} \right\}^\varpi$$

$$\stackrel{(2.12)}{=} \bar{y}_{i,k} \bar{y}_{j,\ell} \frac{\left\{ \overline{\Sigma(x_i)}, \overline{\Sigma(x_j)} \right\}^\varpi}{\overline{\Sigma(x_i x_j)}} \stackrel{(2.8)}{=} \bar{y}_{i,k} \bar{y}_{j,\ell} \frac{\left\{ \Sigma(\tilde{x}_i), \Sigma(\tilde{x}_j) \right\}^\varpi}{\Sigma(\tilde{x}_i \tilde{x}_j)}$$

$$= \bar{y}_{i,k} \bar{y}_{j,\ell} \Sigma \frac{\{\tilde{x}_i, \tilde{x}_j\}}{\tilde{x}_i \tilde{x}_j}, \tag{2.13}$$

where we have used in the last step that Σ is a Poisson morphism. It follows that

K is a Poisson morphism

$$\iff \{\tilde{x}_i, \tilde{x}_j\} = a_{i,j} \tilde{x}_i \tilde{x}_j \text{ for all } 1 \leqslant i, j \leqslant n$$

$$\stackrel{(2.13)}{\iff} \{\bar{y}_{i,k}, \bar{y}_{j,\ell}\}^\varpi = a_{i,j} \bar{y}_{i,k} \bar{y}_{j,\ell} \text{ for all } 1 \leqslant i, j \leqslant n, \; 1 \leqslant k \leqslant \varpi(i),$$

$$1 \leqslant \ell \leqslant \varpi(j)$$

$$\iff K^\varpi \text{ is a Poisson morphism.} \qquad \square$$

Propositions 2.2 and 2.3 lead at once to the following corollary, which is the inverse implication in Theorem 1.1.

Corollary 2.4 *For any weight vector ϖ for Γ_n and for any $\gamma \in \mathbb{F}^*$, the graph $\gamma \Gamma_n^\varpi$ has the KP property.*

3 The 3-dimensional Case

In this section we prove the direct implication of Theorem 1.1 in case the skew-symmetric graph Γ has $n = 3$ vertices. We do not need to prove this in case $n = 2$ because there is only one non-trivial skew-symmetric graph with two vertices, which is the graph $\gamma \Gamma_2$, with $\gamma \in \mathbb{F}^*$, for which we know from Proposition 2.2 that it has the KP property.

3.1 The Known Cases

Let $\Gamma = (S, A)$ be any non-trivial skew-symmetric graph with three vertices, $S = \{1, 2, 3\}$. By assumption, it has at least one arc, which we may suppose to be an arc between the vertices 1 and 2, that is $a_{1,2} \neq 0$. Since, as we have seen, the KP property is preserved by a rescaling of A, we may suppose that $a_{1,2} = 1$; let us denote

$\alpha := a_{1,3}$ and $\beta := a_{2,3}$. We list in Table 1 the values of α and β for which we know that the corresponding graph has the KP property, because it is isomorphic to Γ_3 or to a cloning of Γ_2 (Corollary 2.4). The last three graphs are isomorphic, but it will be convenient to consider them all. For future use (Section 4.2), notice also that these graphs Γ are characterized amongst all non-trivial 3-vertex graphs as follows: Γ has

1. Either a single arc (in which case the graph is disconnected);
2. Or two arcs, both starting from – or ending in – the same vertex;
3. Or three arcs which do not form a circuit.

We will prove that there are no other three-vertex graphs $\Gamma = (S, A)$, with $a_{1,2} = 1$, which have the KP property.

3.2 Computing Efficiently the Poisson Brackets

Since the Kahan map of a Lotka-Volterra system is given by rational functions, which are already quite complicated in dimension 3, we explain here how the condition that the Kahan morphism is a Poisson morphism leads to necessary conditions that are computable by hand, and which will actually be sufficient for our purposes. Since we will use our method also in dimension 4, we will explain it for any skew-symmetric graph Γ with $n > 2$ vertices. As before, the adjacency matrix of Γ is denoted by A. We first write down the basic equations and introduce some notation. We write (2.3) in two different ways (recall that we have set $\varepsilon = 1$). On the one hand, we write it as a linear system in $\tilde{x}_1, \ldots, \tilde{x}_n$,

$$M\tilde{x}^T = x^T . \tag{3.1}$$

Notice that every entry of M is an affine function of $x_1, \ldots, x_n$. For a point $P \in \mathbb{F}^n$ we denote by $M(P)$ the evaluation of M at P and by $\tilde{P}$ the image of P under the

Table 1 The six cases of non-trivial three-vertex graphs Γ, with $a_{1,2} = 1$, which we already know to have the Kahan-Poisson property

(α, β)	$\Gamma = \Delta^\varpi$	Δ	ϖ
(0,0)			(1,1,1)
(0,-1)			(2,1)
(1,0)			(1,2)
(1,-1)			(1,1,1)
(1,1)			(1,1,1)
(-1,-1)			(1,1,1)

Every arc has value 1

Kahan map, i.e., $\tilde{P} = (\tilde{x}_1(P), \tilde{x}_2(P), \tilde{x}_3(P))$. On the other hand, it is easy to see that (2.3) can also be written as $\mathcal{L}_k = 0$, for $k = 1, 2, \dots, n$, where $\mathcal{L}_k$ is defined by

$$\mathcal{L}_k := \left(\sum_{j=1}^{n} a_{k,j} x_j - 1 \right) \tilde{x}_k + x_k \sum_{j=1}^{n} a_{k,j} \tilde{x}_j + x_k . \tag{3.2}$$

Indeed, (3.2) is the k-th entry of $x^T - M\tilde{x}^T$. We denote the column vector whose k-th entry is $\mathcal{L}_k$ by $\mathcal{L}$ and write $\mathcal{L}(\tilde{P})$ for $\mathcal{L}$ with the functions $\tilde{x}_1, \dots, \tilde{x}_n$ evaluated at P. Each entry of $\mathcal{L}(\tilde{P})$ is also an affine function of $x_1, \dots, x_n$.

We can now explain how to explicitly compute, for given i, j and P, the condition $\{\tilde{x}_i, \tilde{x}_j\}(P) = a_{i,j} \tilde{x}_i(P) \tilde{x}_j(P)$. We do this in four different steps.

Step 1: The image point $\tilde{P}$.

If $M(P)$ is non-singular, i.e., $\det M(P) \neq 0$, the Kahan map is defined at P and its image $\tilde{P}$ is computed from

$$M(P)\tilde{P}^T = P^T . \tag{3.3}$$

Notice that $\det M(P)$ depends (linearly) on the entries $a_{k,\ell}$ of A, so for a given P it may be zero for some values of these entries $a_{k,\ell}$. The computations that follow are then not valid for these values; as we will see, it is important to keep track of these values. In the steps which follow, we assume that $\det M(P) \neq 0$.

Step 2: Computation of the Poisson brackets $\{\tilde{x}_j, x_\ell\}(P)$, $\ell = 1, \dots, n$.

We compute the Poisson brackets $\{\tilde{x}_k, x_\ell\}(P)$ for $k, \ell = 1, \dots, n$. This can easily be done directly from (3.1) without solving the latter for $\tilde{x}$, as follows. Let us denote by $\{M, x_\ell\}$ the matrix obtained by taking the Poisson bracket of every entry of M with x_ℓ, and similarly for the column vector $\{\tilde{x}^T, x_\ell\}$. Then it follows from (3.1), using the Leibniz rule, that $\{M, x_\ell\} \tilde{x}^T + M \{\tilde{x}^T, x_\ell\} = \{x^T, x_\ell\}$, so that, at P,

$$M(P) \left\{ \tilde{x}^T, x_\ell \right\}(P) = \left\{ x^T, x_\ell \right\}(P) - \{M, x_\ell\}(P)\tilde{P}^T . \tag{3.4}$$

This gives a linear system for the brackets $\{\tilde{x}_k, x_\ell\}(P)$, $k = 1, \dots, n$. Notice that the matrix governing the linear system is again $M(P)$, so that these brackets are uniquely determined from (3.4) (recall that we have assumed that $\det M(P) \neq 0$). Also, the right hand side of (3.4) is equal to $\left\{ \mathcal{L}(\tilde{P}), x_\ell \right\}(P)$, since $\mathcal{L}_k$ is the k-th entry of $x^T - M\tilde{x}^T$ (see (3.1)). It means that, in order to compute the right hand side of (3.4), we can start from the (2.3) defining the Kahan map, evaluate the functions $\tilde{x}_1, \dots, \tilde{x}_n$ at P, and then take the Poisson bracket at P of the remaining affine functions in $x_1, \dots, x_n$ with x_ℓ. Doing this for $\ell = 1, \dots, n$ and solving the resulting linear system, we find the brackets $\{\tilde{x}_k, x_\ell\}(P)$ for $k, \ell = 1, \dots, n$. They are rational functions of the entries of the adjacency matrix A of Γ.

Step 3: Computation of the Poisson brackets $\{\tilde{x}_k, \tilde{x}_j\}(P)$.

The Poisson brackets $\{\tilde{x}_k, \tilde{x}_j\}(P)$ for $k = 1, \dots, n$ are computed in a quite similar way, using the n Poisson brackets which were computed in Step 2 (recall that j is

fixed). In this step we take the Poisson bracket of (3.1) with $\tilde{x}_j$ at P to obtain, as in Step 2,

$$M(P)\left\{\tilde{x}^T, \tilde{x}_j\right\}(P) = \left\{x^T, \tilde{x}_j\right\}(P) - \left\{M, \tilde{x}_j\right\}(P)\tilde{P}^T = \left\{\mathcal{L}(\tilde{P}), \tilde{x}_j\right\}(P) . \quad (3.5)$$

Notice that, again, the defining matrix of the linear system is $M(P)$ and that, again, the right hand side can easily be computed from the equations defining the Kahan map, where the functions $\tilde{x}_1, \ldots, \tilde{x}_n$ are evaluated at P; it is here that one needs the Poisson brackets $\left\{\tilde{x}_j, x_\ell\right\}$, for $\ell = 1, 2, \ldots, n$, which were computed in Step 2. Solving the resulting linear system, we find the brackets $\left\{\tilde{x}_k, \tilde{x}_j\right\}(P)$ for $k = 1, \ldots, n$. Again, they are rational functions of the entries of the adjacency matrix A of Γ.

Step 4: The Poisson morphism condition(s).

From the previous step we know $\left\{\tilde{x}_i, \tilde{x}_j\right\}(P)$ and we can now write down explicitly the condition that

$$\left\{\tilde{x}_i, \tilde{x}_j\right\}(P) = a_{i,j}\tilde{x}_i(P)\tilde{x}_j(P) ,$$

which is a sufficient condition for the Kahan morphism to be a Poisson morphism. Since the left hand side is a rational function of the entries of the adjacency matrix A, we get a rational condition on these entries. If the condition is not satisfied, we can conclude that the graph Γ does not have the KP property. This is how we will use this condition in what follows.

3.3 The 3-dimensional Case

We now prove the direct implication in Theorem 1.1 in the three-dimensional case. We assume that $\Gamma = (S, A)$ is a three-vertex graph, with $a_{1,2} = 1$. We need to show that when Γ has the KP property, which is equivalent to saying that

$$\left\{\tilde{x}_i, \tilde{x}_j\right\} = a_{i,j}\tilde{x}_i\tilde{x}_j , \quad \text{for } 1 \leqslant i < j \leqslant 3 , \quad (3.6)$$

then Γ is one of the graphs Γ^ϖ in Table 1. We will do this by computing (3.6) at some well-chosen points (and for some particular values of i, j), using the method of the previous subsection.

To do this, it is helpful to represent the six values from the first column in Table 1 as points in the plane, as indicated in Fig. 2. One sees from the figure that these six points lie on (one or two of) the lines $\alpha = 1$, $\beta = -1$, $\alpha = \beta$. This will guide us in the proof as follows: we will first show that for points (α, β) on one of these three lines, the corresponding graph can only have the KP property if it is one of the three special points on that line. Secondly, we will show that for points (α, β) not lying on any of these lines, the corresponding graph cannot have the KP property.

We start with the line $\alpha = 1$, so we suppose that $(\alpha, \beta) = (1, \beta)$, and we take $P := (1, -1, -1)$. The condition which we will compute is $\{\tilde{x}_1, \tilde{x}_3\}(P) = \tilde{x}_1(P)\tilde{x}_3(P)$,

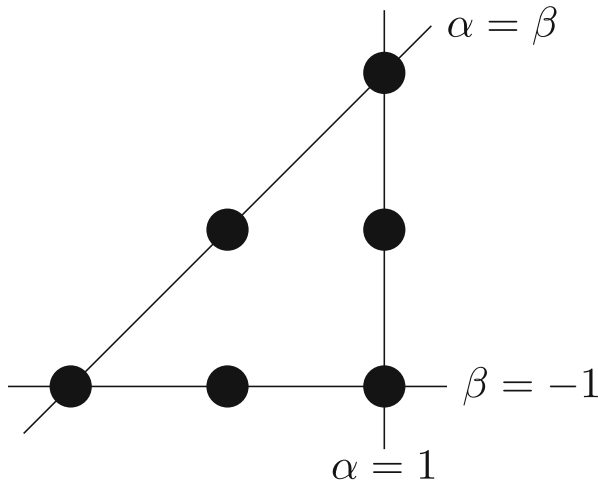**Fig. 2** Viewed as points in the plane, the six values (α, β) for which we know that the corresponding 3-vertex graph has the KP property lie on three lines, each of which contains three of the points

so that $i = 1$ and $j = 3$. Since $a_{1,2} = a_{1,3} = 1$ and $a_{2,3} = \beta$, the Kahan map is in this case defined by the following linear system:

$$
\begin{aligned}
\tilde{x}_1 - x_1 &= \tilde{x}_1(x_2 + x_3) + x_1(\tilde{x}_2 + \tilde{x}_3) , \\
\tilde{x}_2 - x_2 &= \tilde{x}_2(\beta x_3 - x_1) + x_2(\beta \tilde{x}_3 - \tilde{x}_1) , \\
\tilde{x}_3 - x_3 &= \tilde{x}_3(-x_1 - \beta x_2) - x_3(\tilde{x}_1 + \beta \tilde{x}_2) .
\end{aligned}
\tag{3.7}
$$

Following Step 1 of the method, we write these equations as $M\tilde{x}^T = x^T$, where

$$
M = \begin{pmatrix} 1 - x_2 - x_3 & -x_1 & -x_1 \\ x_2 & 1 + x_1 - \beta x_3 & -\beta x_2 \\ x_3 & \beta x_3 & 1 + x_1 + \beta x_2 \end{pmatrix} .
$$

Then

$$
M(P) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 + \beta & \beta \\ -1 & -\beta & 2 - \beta \end{pmatrix} , \quad M(P)^{-1} = \frac{1}{8} \begin{pmatrix} 4 & 2 & 2 \\ 2(1 - \beta) & 5 - 3\beta & 1 - 3\beta \\ 2(\beta + 1) & 1 + 3\beta & 5 + 3\beta \end{pmatrix} ,
$$

so that $\tilde{P} = -\frac{1}{2}(0, 1 - \beta, 1 + \beta)$. Note that $\det M(P) = 8$, so that $\tilde{P}$ is defined for any value of β. Evaluating $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ in (3.7) at P, leads to

$$
\mathcal{L}(\tilde{P}) = \frac{1}{2} \begin{pmatrix} 0 \\ (1 - \beta)(x_1 + (2 + \beta)x_2 - \beta x_3 + 1) \\ (1 + \beta)(x_1 + \beta x_2 + (2 - \beta)x_3 + 1) \end{pmatrix} .
$$

It follows that $\left\{ \mathcal{L}(\tilde{P}), x_\ell \right\} (P)$ is, for $\ell = 1, 2, 3$, respectively given by

$$
\begin{pmatrix} 0 \\ 1 - \beta \\ 1 + \beta \end{pmatrix} , \quad \frac{1 + \beta}{2} \begin{pmatrix} 0 \\ -(1 - \beta)^2 \\ (1 - \beta)^2 - 2 \end{pmatrix} \quad \text{and} \quad \frac{\beta - 1}{2} \begin{pmatrix} 0 \\ 2 - (1 + \beta)^2 \\ (1 + \beta)^2 \end{pmatrix} .
$$

According to Step 2, we get the Poisson brackets $\{\tilde{x}_k, x_\ell\}\,(P)$ by multiplying these three vectors with the inverse of $M(P)$. We display only the brackets $\{x_\ell, \tilde{x}_1\}\,(P)$ since they are the only ones needed to compute $\left\{\mathcal{L}(\tilde{P}), \tilde{x}_1\right\}(P)$ and to finish the computation:

$$\{x_1, \tilde{x}_1\}\,(P) = -\frac{1}{2}\,, \quad \{x_2, \tilde{x}_1\}\,(P) = \frac{1+\beta}{4}\,, \quad \{x_3, \tilde{x}_1\}\,(P) = \frac{1-\beta}{4}\,.$$

Using these values we get

$$\left\{\mathcal{L}(\tilde{P}), \tilde{x}_1\right\}(P) = \frac{\beta(\beta^2 - 1)}{4}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\,.$$

According to Step 3, we get the Poisson brackets $\{\tilde{x}_k, \tilde{x}_1\}\,(P)$ by multiplying this vector with the inverse of $M(P)$. The only bracket we need is $\{\tilde{x}_1, \tilde{x}_3\}\,(P)$, which is found to be equal to $\frac{1}{8}\beta(1 - \beta^2)$. Since $\tilde{x}_1(P) = 0$ the condition that $\{\tilde{x}_1, \tilde{x}_3\}\,(P) = \alpha\tilde{x}_1(P)\tilde{x}_3(P)$ reduces to the condition on β that $\beta(1 - \beta)(1 + \beta) = 0$. We have therefore shown that if (α, β) is on the line $\alpha = 1$, the corresponding graph can only have the KP property if $\beta \in \{-1, 0, 1\}$, which corresponds precisely to the three values for which we know that the corresponding graph has the KP property.

We next consider the line $\beta = -1$ and we take $P := (1, -1, 1)$. We have that

$$M(P) = \begin{pmatrix} 2-\alpha & -1 & -\alpha \\ -1 & 3 & -1 \\ \alpha & -1 & \alpha+2 \end{pmatrix}\,, \quad \text{so that} \quad \tilde{P} = \frac{1}{2}\begin{pmatrix} 1+\alpha \\ 0 \\ 1-\alpha \end{pmatrix}\,.$$

Again, $\det M(P) = 8$ and $\tilde{P}$ is defined for any value of α. The computations of the brackets are very similar to the previous case and lead to a similar result: since $\{\tilde{x}_1, \tilde{x}_2\}\,(P) = \frac{a}{8}(1 - a^2)$ and $\tilde{x}_2(P) = 0$, we may conclude as in the first case from the equality $\{\tilde{x}_1, \tilde{x}_2\}\,(P) = \tilde{x}_1(P)\tilde{x}_2(P)$ that if (α, β) is on the line $\beta = -1$, the corresponding graph can only have the KP property if $\alpha \in \{-1, 0, 1\}$, which corresponds again precisely to the three values for which we know that the corresponding graph has the KP property.

We move to the case that (α, β) is on the line $\alpha = \beta$. We choose $P = (-\frac{1}{2}, 1, \frac{1}{2})$. A new phenomenon is now that $\det M(P)$ depends on α, since

$$M(P) = \frac{1}{2}\begin{pmatrix} -\alpha & 1 & \alpha \\ 2 & 1-\alpha & -2\alpha \\ \alpha & \alpha & \alpha+2 \end{pmatrix}\,, \quad \det M(P) = -\frac{1}{2}(\alpha + 1)\,.$$

It means that what follows is not valid for $\alpha = -1$, but that is not a problem since $(-1, -1)$ is precisely one of the cases for which we know that the corresponding graph has the KP property. Thus, under this assumption, we may continue as before and compute the Poisson brackets to find that $\{\tilde{x}_1, \tilde{x}_2\}\,(P) = \frac{\alpha-1}{4}\left((\alpha - 1)^2 + 5\right)$. The equality $\{\tilde{x}_1, \tilde{x}_2\}\,(P) = \tilde{x}_1(P)\tilde{x}_2(P)$ then evaluates to $\alpha^2(\alpha - 1) = 0$, showing that if $\alpha = \beta$, with $\alpha \neq -1$, then the graph corresponding to (α, β) can only have the KP property if $\alpha = 0$ or $\alpha = 1$, as was to be shown.

We finally consider the case where (α, β) does not belong to any of the three lines of Fig. 2, i.e., we suppose that $\alpha \neq \beta$, that $\alpha \neq 1$ and that $\beta \neq -1$. We show that

the corresponding graph cannot have the KP property. As in the previous cases we use the method of Section 3.2. The formulas are slightly more complicated, so we present the computation in some more detail.

We choose the point $P := (-1, 1, 1)$. Then

$$\{x_1, x_2\}(P) = -1 , \quad \{x_1, x_3\}(P) = -\alpha , \quad \{x_2, x_3\}(P) = \beta .$$

Also,

$$M(P) = \begin{pmatrix} -\alpha & 1 & \alpha \\ 1 & -\beta & -\beta \\ \alpha & \beta & 1 - \alpha + \beta \end{pmatrix} , \quad \text{with } \delta := \det M(P) = (\alpha - 1)(\beta + 1) ,$$

and

$$M(P)^{-1} = \frac{1}{\delta} \begin{pmatrix} (\alpha - 1)\beta & (\alpha - 1)(\beta + 1) & (\alpha - 1)\beta \\ \alpha - \alpha\beta - \beta - 1 & -\alpha(\beta + 1) & \alpha(1 - \beta) \\ (\alpha + 1)\beta & \alpha(\beta + 1) & \alpha\beta - 1 \end{pmatrix} .$$

It follows that $\tilde{P}^T = M(P)^{-1}(-1, 1, 1)^T = (1, -1, 1)^T$, so that

$$\mathcal{L}(\tilde{P}) = \begin{pmatrix} -1 + \alpha x_1 + x_2 + \alpha x_3 \\ 1 + x_1 + \beta x_2 - \beta x_3 \\ -1 - \alpha x_1 - \beta x_2 + (1 - \alpha + \beta)x_3 \end{pmatrix} ,$$

and

$$\left\{ \mathcal{L}(\tilde{P}), x_j \right\}(P) = \begin{pmatrix} 1 + \alpha^2 & -\alpha(1 + \beta) & \beta - \alpha^2 \\ (1 - \alpha)\beta & \beta^2 - 1 & \beta^2 - \alpha \\ (1 - \alpha)(\alpha - \beta) & (1 + \beta)(\alpha - \beta) & \alpha^2 - \beta^2 \end{pmatrix} .$$

Multiplying the latter matrix with $M(P)^{-1}$ we obtain the matrix of Poisson brackets $\{\tilde{x}_i, x_j\}(P)$ of which we only need the first line, which yields the following Poisson brackets:

$$\{\tilde{x}_1, x_1\}(P) = \frac{2\beta}{\beta + 1} , \quad \{\tilde{x}_1, x_2\}(P) = -1 , \quad \{\tilde{x}_1, x_3\}(P) = \frac{2\beta^2 - \alpha\beta - \alpha}{\beta + 1} .$$

Indeed, they are sufficient to compute the Poisson bracket $\left\{ \mathcal{L}(\tilde{P}), \tilde{x}_1 \right\}(P)$, which is given by

$$\left\{ \mathcal{L}(\tilde{P}), \tilde{x}_1 \right\}(P) = \frac{1}{\beta + 1} \begin{pmatrix} \alpha & 1 & \alpha \\ 1 & \beta & -\beta \\ -\alpha & -\beta & 1 - \alpha + \beta \end{pmatrix} \begin{pmatrix} -2\beta \\ \beta + 1 \\ \alpha + \alpha\beta - 2\beta^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2\alpha\beta + \alpha^2 \\ \beta(2\beta - 1 - \alpha) \\ (\alpha - \beta)(2\beta - \alpha + 1) \end{pmatrix} . \tag{3.8}$$

We can now compute $\{\tilde{x}_3, \tilde{x}_1\}(P)$ as the third entry of $\{\tilde{x}^T, \tilde{x}_1\}(P) = M(P)^{-1} \left\{ \mathcal{L}(\tilde{P}), \tilde{x}_1 \right\}(P)$, i.e., as the product of the third line of $M(P)^{-1}$ and (3.8). After some simplifications, we get

$$\{\tilde{x}_3, \tilde{x}_1\}(P) = \frac{1}{\delta}(\alpha - 1)(\beta + 1)(\alpha - 2\beta) = \alpha - 2\beta ,$$

so that
$$\{\tilde{x}_3, \tilde{x}_1\}(P) + \alpha \tilde{x}_1(P)\tilde{x}_3(P) = 2(\alpha - \beta) \neq 0 \, ,$$
since it was supposed that $\alpha \neq \beta$. This shows our claim.

4 The Higher-dimensional Case

In this section we prove Theorem 1.1 when Γ has at least four vertices. To do this, we will use reduction, which amounts to removing vertices from the graph. In Section 4.1 we recall reduction and show that the KP property is preserved by reduction. This is used in Section 4.2 to prove Theorem 1.1 when Γ has four vertices, and in Section 4.3 to prove Theorem 1.1 when Γ has more than four vertices.

4.1 Reduction

Lotka-Volterra systems admit a class of natural reductions, which we first recall. Let $\Gamma = (S, A)$ be a graph with vertex set $S = \{1, 2, \ldots, n\}$, where $n > 1$. Let $S' \subset S$ be a proper, non-empty subset and denote $m := \#S'$. We denote by $\Gamma' = (S', A')$ the induced subgraph of Γ. The natural inclusion map $\Gamma' \to \Gamma$ is a graph morphism, hence induces a morphism of Lotka-Volterra systems $LV(\Gamma') \to LV(\Gamma)$ (see [4, Prop. 3.2]). If we denote the natural coordinates on these Lotka-Volterra systems respectively by $y = (y_1, \ldots, y_m)$ and $x = (x_1, \ldots, x_n)$, then the induced morphism $\iota : \mathbb{F}(y) \to \mathbb{F}(x)$ is an injective Poisson morphism; if we denote by τ the unique strictly increasing function $\tau : \{1, \ldots, m\} \to \{1, \ldots, n\}$ which takes values in S', then $\iota(y_i) = x_{\tau(i)}$, for $i = 1, \ldots, m$. Let us denote by $K : \mathbb{F}(x) \to \mathbb{F}(x)$ and by $K' : \mathbb{F}(y) \to \mathbb{F}(y)$ the Kahan morphisms of $LV(\Gamma)$, respectively of $LV(\Gamma')$. Then K' is the restriction of K to $\mathbb{F}(y)$, i.e., the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbb{F}(y) & \xrightarrow{\ K'\ } & \mathbb{F}(y) \\
{\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\
\mathbb{F}(x) & \xrightarrow[\ K\ ]{} & \mathbb{F}(x)
\end{array}
\tag{4.1}
$$

To see this, it suffices to consider (2.3) which defines K and notice that upon setting $x_j = \tilde{x}_j = 0$ for all $j \in S \setminus S'$, only the equations corresponding to $i \in S'$ remain and are the equations which define the Kahan morphism K'.

Suppose now that Γ has the KP property, so that K is a Poisson morphism. Since ι is an injective Poisson morphism, we may conclude as in the first part of the proof of Proposition 2.3 that K' is also a Poisson morphism. We state this as the following result.

Proposition 4.1 *Suppose that Γ is a skew-symmetric graph which has the KP property. Then any induced subgraph Γ' of Γ has the KP property.* □

One consequence of this proposition is immediate: if Γ is a connected graph which satisfies the KP property, then up to a sign all arcs have the same value.

We will also use the following lemma, which characterizes the graphs Γ_n. Recall that a *tournament* is an (unvalued) graph having (precisely) one arc between any pair of different vertices.

Lemma 4.2 *Suppose that Γ is a tournament with n vertices, having the property that none of its 3-vertex induced subgraphs (triangles) is a circuit. Then $\Gamma \simeq \Gamma_n$.*

Proof Recall that, by definition, Γ_n has vertices $1, 2, \ldots, n$ and has an arc from i to j (with value 1) when $i < j$. For two vertices v, w of $\Gamma = (S, A)$, let us write $v < w$ if there is an arc from v to w. Then $<$ is a total order relation on S: skew-symmetry is clear, as well as the fact that any pair of vertices is comparable (since Γ is a tournament), so we only need to prove transitivity. Let $u, v, w \in S$ and suppose that $u < v$ and $v < w$. Then there is an arc from u to v and an arc from v to w, so there cannot be an arc from w to u, since otherwise the subgraph induced by $\{u, v, w\}$ would be a circuit. Therefore, there is an arc from u to w and $u < w$. Since all total orders on a set of n elements are isomorphic, Γ is isomorphic to Γ_n (by a unique isomorphism). □

As an immediate corollary of the lemma, we find that if Γ is a tournament which has the KP property, then $\Gamma \simeq \Gamma_n$, where n is the number of vertices of Γ.

4.2 The 4-dimensional Case

We use Proposition 4.1 to prove Theorem 1.1 when the (connected) graph Γ has four vertices. In view of Proposition 2.3, we may suppose that Γ is irreducible. Finally, we may assume in view of the comment following Proposition 4.1 that all arcs of Γ have value 1, i.e., that Γ is not valued. We show that the only such graph Γ which has the KP property property is Γ_4.

We know from Proposition 4.1 that if we remove a vertex from Γ then the remaining graph Γ' should have the KP property, so according to Section 3.1 it is trivial or it has either

(I)   A single arc (in which case the graph is disconnected);
(II)   Two arcs, both starting from – or ending in – the same vertex;
(III) Three arcs which do not form a circuit.

We first show that any connected irreducible 4-vertex graph having one of these three properties is – up to a reversal of the direction of all arcs – isomorphic to one of the three graphs in Fig. 3.

Since Γ is connected, Γ contains at least one vertex of degree two or three. More precisely, there are the following three (disjoint) possibilities:

(i)   All vertices of Γ have degree three;
(ii) Γ has a vertex of degree three and three vertices of degree one;
(iii) Γ has a vertex of degree 2.

We first consider the case (i). Then Γ is a tournament which does not contain a circuit. According to Lemma 4.2, $\Gamma \simeq \Gamma_4$. It corresponds to the first graph in Fig. 3.
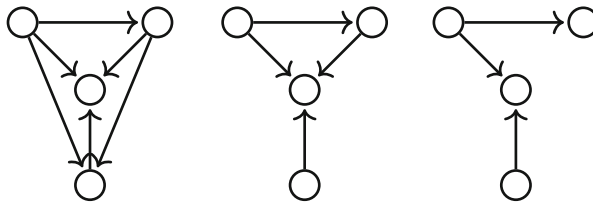
Fig. 3 Up to isomorphism and modulo a reversal of all arcs, there are only three connected irreducible 4-vertex graphs for which every 3-vertex induced subgraph has the KP property. The first one pictured is Γ_4

We next consider the case (ii), so Γ has one vertex of degree three and three vertices of degree one. If one removes any vertex of degree one, the resulting 3-vertex graph must be of type (II), hence the three arcs must be pointing toward the vertex of degree three, or away from it. In any case, Γ is reducible.

We now consider the case (iii), in which we will need to consider several subcases. By assumption, Γ has a vertex v of degree two. We call Γ' the graph obtained by removing v from Γ (together with the arcs incident with v). Since Γ is connected, Γ' is non-trivial, so it is either of type (I), (II) or (III). We analyse each of them separately. We start with type (III) and distinguish three cases, according to whether or not v is connected to the unique vertex with in and outdegree 1 (called b in Fig. 4). In each one of these cases, there is a unique way to add the other arc(s) incident to v; the latter arcs are indicated as a dotted arc. It is clear from the figure that each one of these cases is reducible: the two vertices which are not connected have the same (in and out) neighbors. We now consider the case in which Γ' is of type (II). Modulo a reversal of the direction of all arcs, we may assume that both arcs of Γ' are ending in the same vertex b. There are again three possible cases: if there is an arc between b and v it must be from v to b, in view of (II), and we may assume by symmetry that the other arc is between c and v; the direction of the arc between c and v is irrelevant, up to isomorphism. This gives the first case in Fig. 5. If there is no arc between b and v, both arcs incident with v must either start from v or end in v, which leads to the other two cases in Fig. 5. The first case is the second graph in Fig. 3, while the other two cases are reducible.
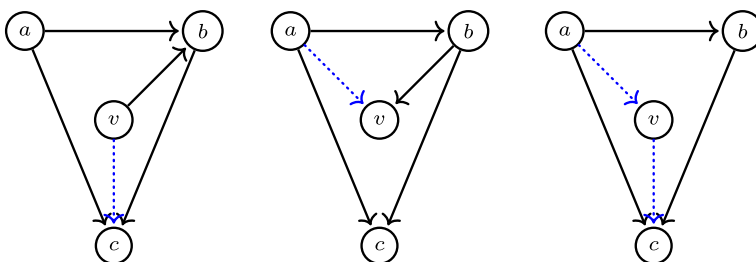

Fig. 4 When $\Gamma' = \Gamma_3$ and v has order two, there are only three graphs Γ satisfying (III). Each one of them is reducible
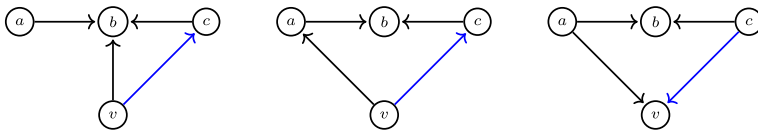
Fig. 5 When Γ' is of type (II) there are up to isomorphism and reversal of all arcs only three possibilities. The second and third ones are reducible, while the first one is the second graph in Fig. 3

The final case to be considered is when Γ' is of type (I). Let us call c the isolated vertex in Γ'; since Γ is connected, there must be an arc between v and c, which by reversing all arcs of Γ may be assumed to be from c to v. Let the labeling of the other vertices be such that the unique arc in Γ' is from a to b. Then, in view of (II), the third arc of Γ must be from a to v. Then Γ is the third graph in Fig. 3.

We have now found all possible connected irreducible 4-vertex graphs for which every 3-vertex induced subgraph has the KP property. To show that Γ_4, which is the first graph in Fig. 3, is the only one having the KP property, we need to show that none of the other two graphs in Fig. 3 has the KP property. We do this by using the method described in Section 3.2. We do this only for one of the graphs, as both graphs are very similar, and hence also the computations to be done.

We consider the last graph in Fig. 3 and label the vertices as follows:

$$\boxed{1} \longrightarrow \boxed{2} \longleftarrow \boxed{3} \longrightarrow \boxed{4}.$$
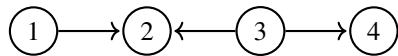
We choose the point $P := (-1, 1, -1, 1)$. Then $M(P)$ and its inverse are given by

$$M(P) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M(P)^{-1} = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

It follows that $\tilde{P} = (-1, -1, 1, 1)$. Evaluating $\tilde{x}_1, \ldots, \tilde{x}_4$ at P we get

$$\mathcal{L}(\tilde{P}) = \begin{pmatrix} 1 - x_2 \\ 1 + x_1 + x_2 + x_3 \\ -1 + x_2 + x_3 + x_4 \\ -1 - x_3 \end{pmatrix},$$
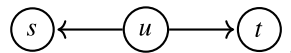
so that

$$\{\tilde{x}_1, x_1\}(P) = \{\tilde{x}_1, x_3\}(P) = 0 \, , \, \{\tilde{x}_1, x_2\}(P) = -3 \, , \, \{\tilde{x}_1, x_4\}(P) = -2 \, .$$

From these values we find that $\{\tilde{x}_1, \tilde{x}_2\}(P) = 3$, so that $\{\tilde{x}_1, \tilde{x}_2\}(P) - \tilde{x}_1(P)\tilde{x}_2(P) = 2$, which proves that the last graph in Figure 3 does not have the KP property.

4.3 The higher-dimensional case

We are now ready to prove Theorem 1.1 for graphs with more than four vertices.

We proceed by contradiction: we assume that Γ is a graph with $n > 4$ vertices which is not a tournament, is connected, irreducible and has the KP property, and show that this leads to a contradiction. First notice that Γ has at least two vertices at distance 2; this is easily seen by considering a shortest chain between any pair of non-adjacent vertices: the first and third vertex of such a chain are at distance 2. Let s and t be two vertices at distance 2 and let u be any vertex adjacent to both s and t. According to (II), the arcs between u on the one hand and s and t on the other hand must both be starting from u or ending in u. By reversing all arcs if needed, we may assume that the subgraph of Γ induced by the vertices s, t and u is given by

$$\boxed{s} \longleftarrow \boxed{u} \longrightarrow \boxed{t} \, .$$

Since Γ is irreducible, s and t cannot have the same neighbors. By the symmetry in s and t, we may suppose that t has an (in or out) neighbor, say v, which is not an (in or out) neighbor of s. In fact, in either case, v cannot be adjacent to s because of (II), applied to the subgraph induced by $\{s, t, v\}$. This leads to three cases, depending on the direction of the arc between v and t and on whether or not there is an arc between u and v; notice that if there is an arc between u and v, it must go from u to v, again because of (II), applied to the subgraph induced by $\{s, u, v\}$. The three cases are displayed in Fig. 6.

The first graph in Fig. 6 is isomorphic to the third graph in Fig. 3, while the other two are isomorphic to the second graph in that figure. We have shown that these graphs do not have the KP property, leading to a contradiction. We have therefore proven Theorem 1.1 for all n.

5 Deformed Lotka-Volterra Systems

In this section we consider deformations of Lotka-Volterra systems, associated with diagonal Poisson brackets which are deformed by constants. We first introduce and study these deformed Poisson brackets, which are associated with augmented graphs, and define the KP property for such graphs. We then characterize these graphs showing that they are closely related to the graphs Γ_n and their clonings.
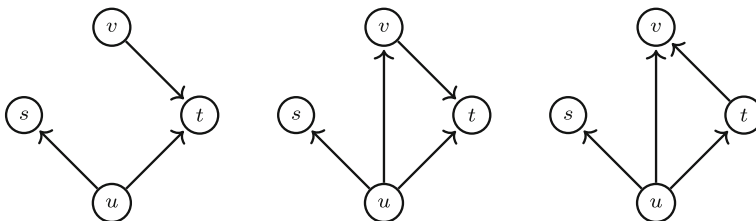

Fig. 6 Under the above assumptions, Γ must contain an induced subgraph of the above type

5.1 Deformed Diagonal Poisson Brackets

We consider in this section deformations of diagonal Poisson brackets with constant terms; they lead to Hamiltonian systems which are deformations of Lotka-Volterra systems, see [5, 6]. We will study these in the next subsection.

Definition 5.1 A Poisson bracket $\{\cdot, \cdot\}_b$ on $\mathbb{F}(x) = \mathbb{F}(x_1, \ldots, x_n)$ is said to be a *diagonal Poisson bracket, deformed by constants*, or simply, a *deformed diagonal Poisson bracket*, if for all $1 \leqslant i, j \leqslant n$, the Poisson brackets $\{x_i, x_j\}_b$ are of the form $\{x_i, x_j\}_b = a_{i,j} x_i x_j + b_{i,j}$, where $a_{i,j}, b_{i,j} \in \mathbb{F}$. The matrix $B = (b_{i,j})$ is called a *deformation matrix of A (or of $\{\cdot, \cdot\}$)*.

The conditions on a matrix B for it to be a deformation matrix of a diagonal Poisson structure are given in the following proposition.

Proposition 5.2 *Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two $n \times n$ skew-symmetric matrices. Then B is a deformation matrix of A if and only if*

$$b_{i,j}(a_{i,k} + a_{j,k}) = 0, \quad \text{for all} \quad 1 \leqslant i, j, k \leqslant n \quad \text{with} \quad i \neq j \neq k \neq i. \quad (5.1)$$

Proof We define a biderivation $\{\cdot, \cdot\}_b$ of $\mathbb{F}(x)$, by $\{x_i, x_j\}_b := a_{i,j} x_i x_j + b_{i,j}$ for $1 \leqslant i, j \leqslant n$. By definition, B is a deformation matrix of A if and only if $\{\cdot, \cdot\}_b$ satifies the Jacobi identity, i.e.,

$$\left\{\{x_i, x_j\}_b, x_k\right\}_b + \circlearrowleft (i, j, k) = 0, \quad \text{for all} \quad 1 \leqslant i < j < k \leqslant n,$$

where $\circlearrowleft (i, j, k)$ means cyclic permutation of the indices i, j, k. Using the fact that $\{\cdot, \cdot\}_b$ is a biderivation and that $\{\cdot, \cdot\}$ is a Poisson bracket, this is equivalent to

$$0 = a_{i,j}\{x_i x_j, x_k\}_b + \circlearrowleft (i, j, k) = a_{i,j} b_{j,k} x_i + a_{i,j} b_{i,k} x_j + \circlearrowleft (i, j, k)$$
$$= a_{k,i} b_{i,j} x_k + a_{j,k} b_{j,i} x_k + \circlearrowleft (i, j, k) = b_{i,j}(a_{k,i} + a_{k,j}) x_k + \circlearrowleft (i, j, k),$$

which in turn amounts to the condition that $b_{i,j}(a_{i,k} + a_{j,k}) = 0$ whenever the indices i, j, k are different, which is precisely (5.1). □

Condition (5.1) can be stated equivalently by saying that the (i, j)-th entry $b_{i,j}$ of B is zero whenever there exists an index k, different from i and j, with $a_{i,k} + a_{j,k} \neq 0$. It follows that, given A, there are two types of pairs of distinct indices (i, j), depending on whether or not $a_{i,k} + a_{j,k} = 0$ for all k, different from i and j; in the positive case, one can assign any value to $b_{i,j}$, while that value must be zero in the negative case, for the constants $b_{i,j}$ to define a deformation matrix of A.

The condition that B is a deformation matrix of A can easily be read off from the skew-symmetric graph $\Gamma = (S, A)$, associated to A. Given three different vertices $i, j, k \in S$, we say that k is an *opposite neighbor* of i and j if the arcs from i to k and from j to k have opposite values, $a_{i,k} + a_{j,k} = 0$; the vertices i and j are said to have *opposite neighborhoods* if every other vertex k is an opposite neighbor of i and

j. On a picture, representing Γ, we will add a dashed arc from i to j when i and j have opposite neighborhoods and, say, $i < j$. By the above, this indicates that if one puts any values at the positions in the B-matrix which correspond to dashed arcs, and zeros at all other positions, then B is a deformation matrix of A and all deformation matrices of A are obtained in this way. We call any triplet $\Delta = (S, A, B)$ with B a deformation matrix of A an *augmentation of* Γ and refer to Δ as an *augmented graph*. A dashed arc from i to j may be labeled with the value $b_{i,j}$ but that will not be needed in what follows.

Example 5.3 Recall that for the graph Γ_n there is an arc from i to j if and only if $i < j$. The vertices 1 and n have opposite neighborhoods and are the only vertices with this property. Therefore, the only possible non-zero entries of the deformation matrix B are $b_{1,n} = -b_{n,1}$. See the first picture in Fig. 7 below for the case of $n = 6$.

Example 5.4 We denote by C_n the graph with n vertices $S = \{1, 2, \ldots, n\}$ and an arc of value 1 from i to $i + 1 \mod n$ for $i \in S$. When $n = 3$, any two vertices have opposite neighborhoods; when $n = 4$, the vertices 1 and 3 have opposite neighborhoods, as well as the vertices 2 and 4; when $n > 4$ no two vertices have opposite neighborhoods. See the second picture in Fig. 7 for the case of $n = 4$.

We show in the next lemma how the augmentations of a graph and of its clonings are related.

Lemma 5.5 *Let Γ be a skew-symmetric graph and let ϖ be a weight vector for Γ. Two vertices (i, k) and (j, ℓ) of the cloned graph Γ^ϖ have opposite neighborhoods if and only if the following two conditions are fulfilled:*

(1)   *The vertices i and j have opposite neighborhoods in Γ;*
(2)   *If there is an arc between i and j in Γ, then $\varpi(i) = \varpi(j) = 1$.*

Proof As before, we let $\Gamma = (S, A)$ with $A = (a_{i,j})$. Let i, j be two vertices of Γ and let us assume that the above conditions (1) and (2) are satisfied. Let (r, s) be a
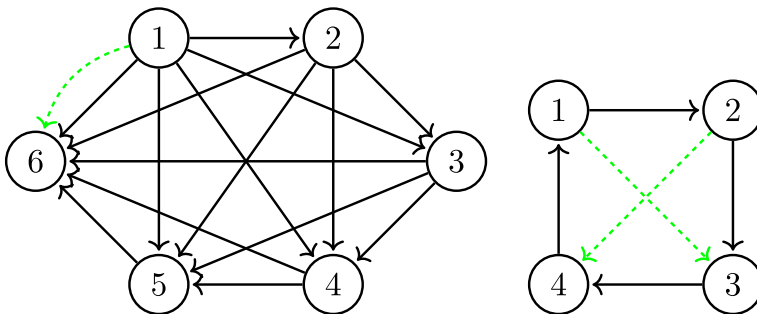

Fig. 7 Augmented graphs of Γ_6 and C_4. The dashed arcs join vertices with opposite neighborhoods

vertex of Γ^{ϖ}, which is different from some given distinct vertices (i, k) and (j, ℓ) of Γ^{ϖ}. Then

$$\text{There is an arc } (i, k) \to (r, s) \text{ in } \Gamma^{\varpi} \text{ with value } a \neq 0$$
$$\Longleftrightarrow \text{There is an arc } i \to r \text{ in } \Gamma \text{ with value } a \neq 0$$
$$\Longleftrightarrow \text{There is an arc } j \to r \text{ in } \Gamma \text{ with value } -a \neq 0$$
$$\Longleftrightarrow \text{There is an arc } (j, \ell) \to (r, s) \text{ in } \Gamma^{\varpi} \text{ with value } -a \neq 0 \ ,$$

which means that (r, s) is an opposite neighbor to (i, k) and (j, ℓ); this shows that (i, k) and (j, ℓ) have opposite neighborhoods. The second equivalence is a direct consequence of (1) when $r \neq i, j$, but needs some explanation when $r = i$ or $r = j$: it is clearly valid when there is no arc between i and j, but when there is an arc between i and j then according to (2), $\varpi(i) = \varpi(j) = 1$, so that in fact $r \neq i$ and $r \neq j$. This shows that the conditions (1) and (2) are sufficient.

We now show that these conditions are also necessary. If (1) does not hold, then there exists a vertex r of Γ which is not an opposite neighbor of i and j. Then for any k, ℓ, the vertex $(r, 1)$ is not an opposite neighbor of (i, k) and (j, ℓ), so that (i, k) and (j, ℓ) do not have opposite neighborhoods. If (2) does not hold, there is an arc from i to j, with value $a \neq 0$, but $\varpi(i) \cdot \varpi(j) > 1$, say $\varpi(i) > 1$. Let (i, m) be a vertex of Γ^{ϖ}, with $k \neq m$. Then there is an arc from (i, m) to (j, ℓ) with value $a \neq 0$, but there is no arc between (i, k) and (i, m). It follows again that (i, k) and (j, ℓ) do not have opposite neighborhoods.   □

Example 5.6 The previous lemma, applied to the weighted graph (Γ_n, ϖ), shows that Γ_n^{ϖ} can only have a pair of vertices with opposite neighborhoods when $\varpi(1) = \varpi(n) = 1$; in this case, $(1, 1)$ and $(n, 1)$ have opposite neighborhoods and is the only pair of vertices with this property. See Fig. 8.

5.2 The Kahan-Poisson Property for Deformations of Lotka-Volterra Systems

We show in this subsection that the deformations with constant terms of the Lotka-Volterra systems LV(Γ) for which their Kahan map is a Poisson map with respect to the corresponding deformed Poisson structure, are precisely those for which their
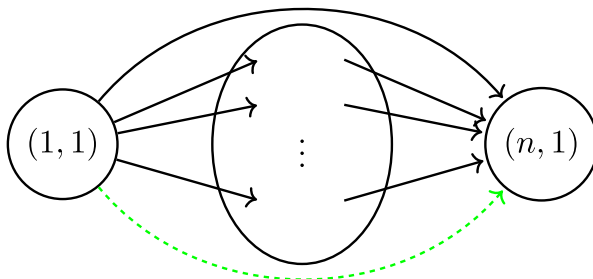

Fig. 8 When $\varpi(1) = \varpi(n) = 1$ the vertices $(1, 1)$ and $(n, 1)$ of Γ_n^{ϖ} are the only vertices with opposite neighborhoods

underlying graph Γ has the KP property. According to Theorem 1.1, this means that every connected component of Γ is isomorphic to $\gamma \Gamma_n$ for some $\gamma \in \mathbb{F}^*$ and $n \in \mathbb{N}^*$.

We first recall from [3] the recipe of the Kahan discretization of a general class of systems of differential equations which covers the deformations of Lotka-Volterra systems that we consider. To this end, consider a system of differential equations on $\mathbb{F}^n$ of the form

$$\dot{x}_i = Q_i(x) + c_i , \quad i = 1, 2, \ldots, n , \tag{5.2}$$

where $x = (x_1, x_2, \ldots, x_n)$, Q_i is a quadratic form and $c_i \in \mathbb{F}$. If we denote by B_i the bilinear form, corresponding to Q_i, so that $Q_i(x) = B_i(x, x)$, then the *Kahan discretization of* (5.2) is given by

$$\frac{\tilde{x}_i - x_i}{2\varepsilon} = B_i(\tilde{x}, x) + c_i , \quad i = 1, 2, \ldots, n . \tag{5.3}$$

Solving (5.3) linearly for $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ we get a family of birational maps on $\mathbb{F}^n$, parametrized by the step size ε. As in the undeformed case, for fixed ε we call the corresponding map the *Kahan map*. The corresponding endomorphism of the field of rational functions $\mathbb{F}(x)$, defined for $i = 1, \ldots, n$ by $K_b(x_i) := \tilde{x}_i$, is called the *Kahan morphism*.

We now introduce the deformed Lotka-Volterra systems which we will study. Let $\Delta = (S, A, B)$ be an augmented graph, whose associated Poisson bracket on $\mathbb{F}(x)$ is denoted by $\{\cdot, \cdot\}_b$. Recall that it is defined by $\{x_i, x_j\}_b := a_{i,j} x_i x_j + b_{i,j}$ for $1 \leqslant i, j \leqslant n$. Taking $H := x_1 + x_2 + \cdots + x_n$ as Hamiltonian, the Hamiltonian vector field $\{\cdot, H\}_b$ is given by the following system of differential equations:

$$\dot{x}_i = \sum_{j=1}^{n} a_{i,j} x_i x_j + c_i , \quad i = 1, 2, \ldots, n , \tag{5.4}$$

where the parameters c_i are related with $B = (b_{i,j})$ by $c_i = \sum_{j=1}^{n} b_{i,j}$. It is a deformation of the Lotka-Volterra system (1.1) and is of the above form (5.2). Using (5.3), the Kahan map of (5.4) is implicitly given by

$$\frac{\tilde{x}_i - x_i}{\varepsilon} = \tilde{x}_i \sum_{j=1}^{n} a_{i,j} x_j + x_i \sum_{j=1}^{n} a_{i,j} \tilde{x}_j + 2c_i , \quad i = 1, 2, \ldots, n . \tag{5.5}$$

Definition 5.7 An augmented graph Δ is said to have the *KP property* if the Kahan map (5.5) with step size $\varepsilon = 1$ is a Poisson map with respect to the Poisson bracket $\{\cdot, \cdot\}_b$.

We first prove an analog of Proposition 2.3 for augmented graphs. Let $\Gamma = (S, A)$ be a connected skew-symmetric graph with vertex set $S = \{1, 2, \ldots, n\}$ and let ϖ be a weight vector for Γ. Let $\Delta^\varpi = (S^\varpi, A^\varpi, B^\varpi)$ be an augmented graph of Γ^ϖ and let $\Delta = (S, A, B)$ be the augmented graph of Γ, defined for $1 \leqslant i, j \leqslant n$ by $b_{i,j} := \sum_{k=1}^{\varpi(i)} \sum_{\ell=1}^{\varpi(j)} b^\varpi_{(i,k),(j,\ell)}$; notice that, according to Lemma 5.5, this defines indeed an augmented graph of Γ. The Poisson brackets on $\mathbb{F}(x)$ and $\mathbb{F}(y)$ associated with Δ and Δ^ϖ are denoted by $\{\cdot, \cdot\}_b$ and $\{\cdot, \cdot\}_b^\varpi$ respectively. The Kahan maps on $\mathbb{F}(x)$ and $\mathbb{F}(y)$ with $\varepsilon = 1$ are denoted by K_b and K_b^ϖ; for $u \in \mathbb{F}(x)$ and $v \in \mathbb{F}(y)$ we

also write $\tilde{u}$ for $K_b(u)$ and $\bar{v}$ for $K_b^\varpi(v)$. The decloning map is, as before, denoted by Σ.

Proposition 5.8 *With the above definitions and notations, the following diagram of fields and field morphisms is commutative:*

$$
\begin{array}{ccc}
(\mathbb{F}(x), \{\cdot, \cdot\}_b) & \xrightarrow{\ K_b\ } & (\mathbb{F}(x), \{\cdot, \cdot\}_b) \\
{\scriptstyle \Sigma}\downarrow & & \downarrow{\scriptstyle \Sigma} \\
(\mathbb{F}(y), \{\cdot, \cdot\}_b^\varpi) & \xrightarrow[\ K_b^\varpi\ ]{} & (\mathbb{F}(y), \{\cdot, \cdot\}_b^\varpi)
\end{array}
\tag{5.6}
$$

The vertical arrow Σ is a Poisson morphism. If K_b^ϖ is a Poisson morphism, then K_b also; if Γ is isomorphic to $\gamma\Gamma_n$ for some $n \in \mathbb{N}^*$ and some $\gamma \in \mathbb{F}^*$, so that K_b is a Poisson morphism, then K_b^ϖ is also a Poisson morphism.

Proof The commutativity of the diagram is shown in exactly the same way as in the proof of Proposition 2.3. For $1 \leqslant i < j \leqslant n$, we have that

$$
\left\{\Sigma(x_i), \Sigma(x_j)\right\}_b^\varpi = \sum_{k=1}^{\varpi(i)} \sum_{\ell=1}^{\varpi(j)} \left\{y_{i,k}, y_{j,\ell}\right\}_b^\varpi = \sum_{k=1}^{\varpi(i)} \sum_{\ell=1}^{\varpi(j)} \left(a_{i,j} y_{i,k} y_{j,\ell} + b_{(i,k),(j,\ell)}^\varpi\right)
$$
$$
= a_{i,j}\Sigma(x_i)\Sigma(x_j) + b_{i,j} = \Sigma\left\{x_i, x_j\right\}_b ,
$$

which shows that Σ is a Poisson morphism. Using the commutativity of (5.6) and the fact that Σ is a morphism of Poisson fields, we get

$$
\Sigma\widetilde{\left\{x_i, x_j\right\}}_b = \overline{\Sigma\left\{x_i, x_j\right\}}_b = \overline{\left\{\Sigma(x_i), \Sigma(x_j)\right\}}_b^\varpi
$$

and

$$
\Sigma\left\{\tilde{x}_i, \tilde{x}_j\right\}_b = \left\{\Sigma(\tilde{x}_i), \Sigma(\tilde{x}_j)\right\}_b^\varpi = \left\{\overline{\Sigma(x_i)}, \overline{\Sigma(x_j)}\right\}_b^\varpi .
$$

It follows that, if K_b^ϖ is a Poisson morphism, then $\Sigma\widetilde{\left\{x_i, x_j\right\}}_b = \Sigma\left\{\tilde{x}_i, \tilde{x}_j\right\}_b$, which implies, by the injectivity of Σ, that $\widetilde{\left\{x_i, x_j\right\}}_b = \left\{\tilde{x}_i, \tilde{x}_j\right\}_b$. We have hereby shown that if K_b^ϖ is a Poisson morphism, then K_b also.

Suppose now that $\Gamma = \gamma\Gamma_n$ where $n \in \mathbb{N}^*$ and $\gamma \in \mathbb{F}^*$. It was shown in [5, Theorem 5.8] that the Kahan map of Γ_n is a Poisson map, so that K_b is a Poisson morphism. We show that K_b^ϖ is also a Poisson morphism. According to Example 5.6, if $\varpi(1) \cdot \varpi(n) > 1$ then the only deformation matrix of $\gamma\Gamma_n^\varpi$ is the zero matrix, and there is nothing to prove. We therefore suppose that $\varpi(1) = \varpi(n) = 1$ and denote $b := b_{(1,1),(n,1)}^\varpi = -b_{(n,1),(1,1)}^\varpi$, which is the only entry of B^ϖ which is possibly non-zero. We show as in the proof of Proposition 2.3 that, for any $i \in S$ and for any $1 \leqslant k \leqslant \varpi(i)$,

$$
\Sigma(x_i)/y_{i,k} \text{ is a Casimir of } \{\cdot, \cdot\}_b^\varpi \text{ and an invariant of } K_b^\varpi . \tag{5.7}
$$

For $i = 1$ or $i = n$ this is obvious because then $k = 1$ and $\Sigma(x_1) = y_{1,1}$ and $\Sigma(x_n) = y_{n,1}$. When $1 < i < n$ we have that $\left\{y_{i,k}, \cdot\right\}_b^\varpi = \left\{y_{i,k}, \cdot\right\}^\varpi$ and so $y_{i,m}/y_{i,k}$ and $\Sigma(x_i)/y_{i,k}$ are Casimirs of $\{\cdot, \cdot\}_b^\varpi$ as well. The fact that $y_{i,m}/y_{i,k}$, and

hence $\Sigma(x_i)/y_{i,k}$, is an invariant of K_b^ϖ for $1 < i < n$ is shown in exactly the same way as in the proof of Proposition 2.3. For $1 \leqslant i < j \leqslant n$, $1 \leqslant k \leqslant \varpi(i)$ and $1 \leqslant \ell \leqslant \varpi(j)$ the computation of the Poisson brackets $\{\bar{y}_{i,k}, \bar{y}_{j,\ell}\}_b^\varpi$ can therefore be done as in (2.13), yielding formally the same result, to wit

$$\{\bar{y}_{i,k}, \bar{y}_{j,\ell}\}_b^\varpi = \bar{y}_{i,k}\bar{y}_{j,\ell}\Sigma\frac{\{\tilde{x}_i, \tilde{x}_j\}_b}{\tilde{x}_i\tilde{x}_j} . \tag{5.8}$$

When $i = 1$ and $j = n$, $\{\tilde{x}_i, \tilde{x}_j\}_b = \{\tilde{x}_1, \tilde{x}_n\}_b = a_{1,n}\tilde{x}_1\tilde{x}_n + b$ and $\bar{y}_{i,k} = \bar{y}_{1,1} = \Sigma(x_1)$ and $\bar{y}_{j,\ell} = y_{n,1} = \Sigma(x_n)$ since $\varpi(1) = \varpi(n) = 1$, so that (5.8) becomes

$$\{\bar{y}_{1,1}, \bar{y}_{n,1}\}_b^\varpi = a_{1,n}\bar{y}_{1,1}\bar{y}_{n,1} + b = \overline{\{y_{1,1}, y_{n,1}\}_b^\varpi} ,$$

as wanted. Otherwise, $\{\tilde{x}_i, \tilde{x}_j\}_b = a_{i,j}\tilde{x}_i\tilde{x}_j$, so that (5.8) becomes

$$\{\bar{y}_{i,k}, \bar{y}_{j,\ell}\}_b^\varpi = a_{i,j}\bar{y}_{i,k}\bar{y}_{j,\ell} = \overline{\{y_{i,k}, y_{j,\ell}\}_b^\varpi} ,$$

which finishes the proof that K_b^ϖ is a Poisson morphism. □

We use the above proposition and Theorem 1.1 to show that the KP property is preserved under deformation.

Proposition 5.9 *Let Γ be a connected skew-symmetric graph. Then the following statements are equivalent:*

(i)   *Γ has the KP property;*
(ii) *All augmented graphs Δ of Γ have the KP property;*
(iii) *Some augmented graph Δ of Γ has the KP property.*

Proof We first prove that (i) implies (ii). If Γ has the KP property, then we know from Theorem 1.1 that Γ is isomorphic to $\gamma\Gamma_n^\varpi$ for some $n \in \mathbb{N}^*$, some $\gamma \in \mathbb{F}^*$ and some weight vector on Γ_n. Let Δ be any augmented graph of Γ. According to Proposition 5.8, the Kahan morphism K_b^ϖ is also a Poisson morphism. This shows that Δ also has the KP property. The proof that (ii) implies (iii) is trivial, because any graph can be considered as an augmented graph of itself with the zero deformation matrix. Suppose now that $\Delta = (S, A, B)$ is an augmented graph of a skew-symmetric graph Γ and that Δ has the KP property. Let K_b be the Kahan morphism of Δ (with $\varepsilon = 1$), which is a Poisson morphism. It is clear from (5.5) and (2.3) that by setting $b_{i,j} := 0$ for all $i, j \in S$ in K_b, we get the Kahan morphism K of Γ (with $\varepsilon = 1$). Since K_b is a Poisson morphism,

$$\{K_b(x_i), K_b(x_j)\}_b = a_{i,j}K_b(x_i)K_b(x_j) + b_{i,j} , \quad \text{for all} \quad i, j \in S .$$

Setting $b_{i,j} = 0$, the right hand side of the above becomes $a_{i,j}K(x_i)K(x_j)$ while the left hand side becomes

$$\{K_b(x_i), K_b(x_j)\}_b \big|_{b=0} = \{K_b(x_i), K_b(x_j)\} \big|_{b=0} = \{K(x_i), K(x_j)\} .$$

Therefore $\{K(x_i), K(x_j)\} = a_{i,j}K(x_i)K(x_j)$ for all $i, j \in S$, which means that Γ has the KP property as well. □

In combination with Theorem 1.1, Proposition 5.9 leads at once to Theorem 1.2.

It also follows from the proposition (or from the theorem) by a simple rescaling argument that an augmented graph has the KP property if and only if the Kahan map of an augmented graph is a Poisson map for some value of $\varepsilon \in \mathbb{F}^*$. Indeed, when ε is given weight -1, while giving a weight 1 to all x_i and a weight 2 to the parameters c_i, the defining (5.5) of the Kahan map become homogeneous, and homotheties of quadratic Poisson structures are Poisson maps, as we already recalled. In particular, the Kahan map with step size ε of an augmented graph $\Delta = (S, A, B)$ is a Poisson map, if and only if the Kahan map with step size 1 of $(S, A, \varepsilon^2 B)$ is a Poisson map, i.e., $(S, A, \varepsilon^2 B)$ has the KP property; in view of Proposition 5.9, this is equivalent to (S, A, B) having the KP property.

Another consequence of the proposition is that the final statement in Proposition 5.8 can be reformulated as an if and only if, so that Proposition 5.8 is a generalization of Proposition 2.3. Indeed, when K_b is a Poisson morphism, so that Δ has the KP property, then Γ has the KP property (by Proposition 5.9), hence also Γ^ϖ (by Proposition 2.3), and hence also Δ^ϖ (again by Proposition 5.9), so that K_b^ϖ is a Poisson morphism. It follows that, in the notations of Proposition 5.8, K_b is a Poisson morphism if and only K_b^ϖ is a Poisson morphism.

6 Liouville and Superintegrability

We have shown in the previous sections that the only connected skew-symmetric graphs Γ which have the KP property are of the form $\Gamma = \gamma \Gamma_n^\varpi$, where $\gamma \in \mathbb{F}^*$, $n \in \mathbb{N}^*$ and ϖ is a weight vector on Γ_n; also, that the only augmented graphs Δ which have the KP property are augmented graphs of $\Gamma = \gamma \Gamma_n^\varpi$. We now show that the corresponding Lotka-Volterra systems LV(Γ) and deformed Lotka-Volterra systems LV(Δ) are both Liouville integrable and superintegrable, and that their Kahan discretizations are both Liouville integrable and superintegrable as well. By a simple rescaling argument, already used several times above, we may assume that $\gamma = 1$, so we will consider in what follows only Γ_n^ϖ and its augmentations.

We first fix the notation and the context. Let $n \in \mathbb{N}^*$, let Γ_n^ϖ be a cloned graph of Γ_n, with $\varpi(1) = \varpi(n) = 1$, and let $b \in \mathbb{F}$. We denote by Δ_n the augmented graph of Γ_n, where the deformation matrix B has as only possible non-zero entries $b_{1,n} = -b_{n,1} := b$. Similarly, Δ_n^ϖ denotes the augmented graph of Γ_n^ϖ, where the deformation matrix B^ϖ has as only possible non-zero entries $b_{(1,1),(n,1)}^\varpi = -b_{(n,1),(1,1)}^\varpi := b$. We consider the fields $\mathbb{F}(x)$ and $\mathbb{F}(y)$, where $x = (x_1, \ldots, x_n)$ and $y = (y_{1,1}, y_{2,1}, y_{2,2}, \ldots, y_{n,1})$, as before.

We consider on $\mathbb{F}(x)$ the Poisson bracket $\{\cdot, \cdot\}_b$, associated with Δ_n, and on $\mathbb{F}(y)$ the Poisson bracket $\{\cdot, \cdot\}_b^\varpi$, associated with Δ_n^ϖ.

Since we will take in this section a more geometrical point of view, we view $\mathbb{F}(x)$ and $\mathbb{F}(y)$ as the field of (rational) functions on $\mathbb{F}^n$, respectively on $\mathbb{F}^{|\varpi|}$; the Poisson structure on $\mathbb{F}^n$ and on $\mathbb{F}^{|\varpi|}$ corresponding to $\{\cdot, \cdot\}_b$ and $\{\cdot, \cdot\}_b^\varpi$ will respectively be denoted by π_b and π_b^ϖ. The standard Lotka-Volterra Hamiltonians on $\mathbb{F}^n$ and on $\mathbb{F}^{|\varpi|}$, which are always the sum of all coordinates, are denoted by H and H^ϖ.

In order to show the integrability of the Hamiltonian system $(\mathbb{F}^{|\varpi|}, \pi_b^{\varpi}, H^{\varpi})$ and of its Kahan discretization, we first recall from [5] the integrability of the deformed Lotka-Volterra system $(\mathbb{F}^n, \pi_b, H)$ and its Kahan discretization. For $1 \leqslant \ell \leqslant \left[\frac{n-1}{2}\right]$, consider the following rational functions:

$$
F_\ell := \begin{cases}
\left(x_1 + x_2 + \cdots + x_{2\ell-1} + \dfrac{\beta}{x_n}\right) \dfrac{x_{2\ell+1} x_{2\ell+3} \ldots x_n}{x_{2\ell} x_{2\ell+2} \ldots x_{n-1}} \;, & \text{if } n \text{ is odd,} \\[3ex]
\left(x_1 + x_2 + \cdots + x_{2\ell} + \dfrac{\beta}{x_n}\right) \dfrac{x_{2\ell+2} x_{2\ell+4} \ldots x_n}{x_{2\ell+1} x_{2\ell+3} \ldots x_{n-1}} \;, & \text{if } n \text{ is even,}
\end{cases}
\tag{6.1}
$$

and let $G_\ell := \jmath(F_\ell)$, where $\jmath : \mathbb{F}(x) \to \mathbb{F}(x)$ is the involutive field automorphism, defined by $\jmath(x_i) := x_{n+1-i}$, for $i = 1, \ldots, n$. Together with the Hamiltonian H, this yields exactly $n - 1$ different rational functions: for example, when n is odd then all F_ℓ and G_ℓ are different, except for $F_1 = G_1$. The following facts were obtained in [5]:

(1) The $n - 1$ rational functions F_ℓ, G_ℓ and H are first integrals of $(\mathbb{F}^n, \pi_b, H)$;
(2) They are *independent*, i.e., their differentials are independent on an open dense subset of $\mathbb{F}^n$;
(3) The rank $\mathrm{Rk}\,\pi_b$ of the Poisson structure π_b is n when n is even, otherwise it is $n - 1$;
(4) The first integrals F_ℓ are *in involution*, i.e., commute for the Poisson bracket;
(5) The first integrals F_ℓ, G_ℓ and H are invariants of the Kahan discretization of $(\mathbb{F}^n, \pi_b, H)$.

Items (1) and (2) say that $(\mathbb{F}^n, \pi_b, H)$ is *superintegrable*, i.e., has $n - 1$ independent first integrals, where n is the dimension of the ambient space. The items (1) – (4) imply that the independent first integrals F_ℓ are in involution and that, with the Hamiltonian, their number is $n - \frac{1}{2}\mathrm{Rk}\,\pi_b$, which is exactly the number required for the *Liouville integrability* of $(\mathbb{F}^n, \pi_b, H)$; for example, when n is odd, $\mathrm{Rk}\,\pi_b = n - 1$ and we have $(n + 1)/2$ functions $F_1, \ldots, F_{(n-1)/2}, H$ which are in involution. Combined with (5) and the fact that the Kahan map is a Poisson map one gets from it that the Kahan discretization of $(\mathbb{F}^n, \pi_b, H)$ is both superintegrable and Liouville integrable, with as invariants the first integrals of the continuous system.

We use these five properties, the properties of the decloning map and of the Poisson structure π_b^{ϖ} to prove the integrability the Hamiltonian system $(\mathbb{F}^{|\varpi|}, \pi_b^{\varpi}, H^{\varpi})$ and its Kahan discretization. As we have already seen in the proof of Proposition 5.8, for any $1 \leqslant i \leqslant n$ and $1 < k \leqslant \varpi(i)$, $y_{i,k}/y_{i,1}$ is a Casimir function of π_b^{ϖ}, yielding $|\varpi| - n$ different Casimir functions, which are clearly independent. The rank of π_b^{ϖ} is therefore at most n, i.e., at most $n - 1$ when n is odd, and at most n when n is even. In fact, we have equality. To see this, consider the *decloning map* $S : \mathbb{F}^{|\varpi|} \to \mathbb{F}^n$ corresponding to decloning morphism Σ, i.e., $\Sigma = S^*$, which is a dominant (actually surjective) Poisson map, since Σ is an injective Poisson morphism. It follows that the rank of π_b^{ϖ} is also bounded from below by the rank of π_b, which is $n - 1$ when n is odd, and n when n is even, and so we have equality.

The cited properties of the decloning map S also imply that the $n - 1$ functions $S^*(F_\ell)$, $S^*(G_\ell)$ and H^{ϖ} are independent first integrals of $(\mathbb{F}^{|\varpi|}, \pi_b^{\varpi}, H^{\varpi})$ and that they are in involution. Combined with the Casimir functions we get $n - 1 + |\varpi| - n = |\varpi| - 1$ different functions; from the simple form of the Casimir functions, it is

clear that the former first integrals are independent from the Casimir functions, so that we have $|\varpi| - 1 = \dim \mathbb{F}^{|\varpi|} - 1$ independent first integrals, which shows that $(\mathbb{F}^{|\varpi|}, \pi_b^{\varpi}, H^{\varpi})$ is superintegrable.

Since S is a Poisson map, the independent functions $S^*(F_\ell)$ and $H^{\varpi} = S^*(H)$ are in involution; together with the $|\varpi| - n$ Casimir functions, we get $(n + 1)/2 + |\varpi| - n = |\varpi| - \frac{1}{2}\mathrm{Rk}\,\pi_b^{\varpi}$ independent functions, including the Hamiltonian, in involution, which shows that $(\mathbb{F}^{|\varpi|}, \pi_b^{\varpi}, H^{\varpi})$ is Liouville integrable.

Moreover, the $|\varpi| - 1$ independent first integrals are invariants of K_b^{ϖ}. For the Casimir functions $y_{i,k}/y_{i,1}$, we have already seen this in the proof of Proposition 5.8. For $1 \leqslant \ell \leqslant \left[\frac{n-1}{2}\right]$, the commutativity of (5.6) implies that

$$K_b^{\varpi}(\Sigma(F_\ell)) = \Sigma(K_b(F_\ell)) = \Sigma(F_\ell) \,,$$

where we have used in the last step that F_ℓ is an invariant of K_b, $K_b(F_\ell) = F_\ell$. Since H^{ϖ} is linear, it is also an invariant of K_b^{ϖ}.

Summing up, and combined with the integrability results in the non-deformed case, it leads to the following proposition.

Proposition 6.1 *Let $n \in \mathbb{N}^*$ and let ϖ be any weight vector on Γ_n. Suppose that Δ_n^{ϖ} is any augmented graph of Γ_n^{ϖ}.*

1. *The Lotka-Volterra system $LV(\Gamma_n^{\varpi})$ and its Kahan discretization are superintegrable and Liouville integrable;*
2. *The deformed Lotka-Volterra system $LV(\Delta_n^{\varpi})$ and its Kahan discretization are superintegrable and Liouville integrable.*

The Lotka-Volterra and deformed Lotka-Volterra systems having Kahan discretizations which are integrable with respect to the original Poisson structure are therefore characterized by the KP property.

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